AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(1): 1620–1635.
DOI: 10.3934/math.2023082
Received: 18 August 2022
Revised: 08 October 2022
Accepted: 10 October 2022
Published: 24 October 2022

Research article

Applications of Hölder-İşcan inequality for n-times differentiable (s, m)-convex functions

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Abstract: In this work, Hölder-Isçan inequality is used for the class of n-times differentiable (s, m)-convex functions. The outcomes are new Hermite-Hadamard type inequalities and modified integrals are estimated by better bounds. Special cases are deduced as the existing results from literature. Furthermore, some applications to arithmetic, geometric and logarithmic means are also presented.

Keywords: (s, m)-convex function; n-times differentiable functions; Hermite-Hadamard inequalities; Hölder-Isçan inequality; means
Mathematics Subject Classification: 26D07, 26D15, 26E70

1. Introduction

In recent years, convexity theory has gained special attention by many researchers because of it engrossing properties and expedient characterizations. It has many applications in fields like biology, numerical analysis and statistics (see [1–4]). Mathematical inequalities are extensively studied with all type of convex functions (see [1, 3, 11, 13, 14, 16]). One of the fundamental inequality is Hermite-Hadamard inequality. It has been discussed via different types of convexities and became the center of attention for many researchers. Recently, in 2016, Khan et al. have discussed generalizations of Hermite-Hadamard type for MT-convex functions [26]. In 2017, Khan et al. studied some new inequalities of Hermite-Hadamard types [27]. In 2019, Khurshid et al. have utilized conformable fractional integrals via preinvex functions [28]. In 2020, Khan et al. have discussed Hermite-Hadamard type inequalities via quantum calculus involving green function [29], Mohammed et al. have established a new version of Hermite-Hadamard inequality for Riemann-Liouville fractional integrals [30], Han et al. used fractional integral to generalize Hermite-Hadamard inequality for convex functions [31], Zhao et al. utilized harmonically convex functions to generalized fractional integral inequalities of Hermite-Hdamrd type [32], Awan et al. presented new inequalities of Hermite-Hdamard type for n-polynomial harmonically convex functions [33]. In 2022, Khan et al. introduced some new versions of Hermite-Hadamard integral inequalities in fuzzy fractional calculus for generalized pre-invex functions via fuzzy-interval-valued settings [34]. This reflects the importance of Hermite Hadamard type inequalities among current research.

In [9], s-convex function is given as,

Definition 1.1. A real valued function χ is called s-convex function on $\mathbb{R}$, if

$$\chi\big(\varsigma\rho + (1-\varsigma)\gamma\big) \leq \varsigma^s\chi(\rho) + (1-\varsigma)^s\chi(\gamma),$$

for each $\rho, \gamma \in \mathbb{R}$ and $\varsigma \in (0,1)$ where $s \in (0,1]$.

In [10], m-convexity is discussed as,

Definition 1.2. A real valued function χ defined on $[0,b]$ is said to be a m-convex function for $m \in [0,1]$, if

$$\chi\big(\varsigma\rho + m(1-\varsigma)\gamma\big) \leq \varsigma\chi(\rho) + m(1-\varsigma)\chi(\gamma),$$

holds for all $\rho, \gamma \in [0,b]$ and $\varsigma \in [0,1]$.

(s,m)-convexity in [17] is discussed as,

Definition 1.3. A function $\chi : [0,b] \longrightarrow \mathbb{R}$, $b > 0$ is said to be a (s,m)-convex function in the second sense where $s, m \in (0,1]^2$, if

$$\chi\big(\varsigma\rho + m(1-\varsigma)\gamma\big) \leq \varsigma^s\chi(\rho) + m(1-\varsigma)^s\chi(\gamma),$$

holds provided that all $\rho, \gamma \in [0,b]$ and $\varsigma \in [0,1]$.
Equivalent definition for (s,m)–convex functions:
Let $\rho, \alpha, \gamma \in [0,b]$, $\rho < \alpha < \gamma$

$$\chi(\alpha) \leq \left(\frac{\gamma-\alpha}{\gamma-\rho}\right)^s \chi(\rho) + m\left(\frac{\alpha-\rho}{\gamma-\rho}\right)^s \chi(\gamma). \tag{1.1}$$

Hölder-İşcan Inequality [5]:
Let $p > 1$, χ and ψ be real valued functions defined on $[\rho,\gamma]$ and $|\chi|^p, |\psi|^q$ are integrable functions on interval $[\rho,\gamma]$

$$\begin{aligned}\int_\rho^\gamma |\chi(\omega)\psi(\omega)|d\omega \leq \tfrac{1}{\gamma-\rho}\Big(\int_\rho^\gamma (\gamma-\omega)|\chi(\omega)|^p d\omega\Big)^{\frac{1}{p}}\Big(\int_\rho^\gamma (\gamma-\omega)|\psi(\omega)|^q d\omega\Big)^{\frac{1}{q}}\\ +\tfrac{1}{\gamma-\rho}\Big(\int_\rho^\gamma (\omega-\rho)|\chi(\omega)|^p d\omega\Big)^{\frac{1}{p}}\Big(\int_\rho^\gamma (\omega-\rho)|\psi(\omega)|^q d\omega\Big)^{\frac{1}{q}},\end{aligned} \tag{1.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Following lemma is useful to obtain our main results.

Lemma 1.4. *[8] For $n \in \mathbb{N}$, let $\chi : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be n-times differentiable mapping on U°, where $\rho, \gamma \in U^\circ$, $\rho < \gamma$ and $\chi^n \in L[\rho, \gamma]$, we have following identity*

$$\sum_{\nu=0}^{n-1} (-1)^\nu \left(\frac{\chi^{(\nu)}(\gamma)\gamma^{\nu+1} - \chi^{(\nu)}(\rho)\rho^{\nu+1}}{(\nu+1)!} \right) - \int_\rho^\gamma \chi(\omega) d\omega = \frac{(-1)^{n+1}}{n!} \int_\rho^\gamma \omega^n \chi^{(n)}(\omega) d\omega, \tag{1.3}$$

where an empty set is understood to be nil.

In this paper, Hölder-İşcan inequality is used to modify inequalities involving functions having s-convex or s-concave derivatives at certain powers. The purpose of this paper is to establish some generalized inequalities for n-times differentiable (s, m)-convex functions. Applications of these inequalities to means are also discussed. Means are defined as,

Let $0 < \rho < \gamma$,

$$A(\rho, \gamma) = \frac{\rho + \gamma}{2},$$

$$G(\rho, \gamma) = \sqrt{\rho\gamma},$$

$$L_p(\rho, \gamma) = \Big(\frac{\gamma^{p+1} - \rho^{p+1}}{(p+1)(\gamma - \rho)} \Big)^{\frac{1}{p}},$$

where $p \neq 0, -1$ and $\rho \neq \gamma$.

2. Main results

2.1. Modified inequalities for n-times differentiable convex functions

Theorem 2.1. *For any positive integer n, let $\chi : U \subseteq (0, \infty) \to \mathbb{R}$ be n-times differentiable mapping on U°, where $\rho, \gamma \in U^\circ$ with $\rho < \gamma$. If $\chi^{(n)} \in L[\rho, \gamma]$ and $|\chi^{(n)}|^q$ for $q > 1$ is (s, m)-convex on interval $[\rho, \gamma]$ then*

$$\begin{aligned} &\left| \sum_{\nu=0}^{n-1} (-1)^\nu \left(\tfrac{\chi^{(\nu)}(\gamma)\gamma^{\nu+1} - \chi^{(\nu)}(\rho)\rho^{\nu+1}}{(\nu+1)!} \right) - \int_\rho^\mu \chi(\omega) d\omega \right| \\ &\leq \tfrac{1}{n!}(\gamma - \rho)^{\frac{1}{q}} \left(\begin{array}{c} [\gamma L_{np}^{np}(\rho, \gamma) - L_{np+1}^{np+1}(\rho, \gamma)]^{\frac{1}{p}} \Big[\frac{|\chi^n(\gamma)|^q}{(s+2)(s+1)} + \frac{m|\chi^n(\rho)|^q}{(s+2)} \Big]^{\frac{1}{q}} \\ +[L_{np+1}^{np+1}(\rho, \gamma) - \rho L_{np}^{np}(\rho, \gamma)]^{\frac{1}{p}} \Big[\frac{|\chi^n(\gamma)|^q}{(s+2)} + \frac{m|\chi^n(\rho)|^q}{(s+1)(s+2)} \Big]^{\frac{1}{q}} \end{array} \right), \end{aligned} \tag{2.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since $|\chi^n|^q$ is (s, m)-convex by using inequality (1.1) for $\rho < \omega < \gamma$, using Lemma 1.4 and Hölder-Işcan inequality (1.2),

$$\begin{aligned} |\chi^n(\omega)|^q &\leq \left| \chi^n(\tfrac{\omega - \rho}{\gamma - \rho}\gamma + m\tfrac{\gamma - \omega}{\gamma - \rho}\rho) \right|^q \\ &\leq (\tfrac{\omega - \rho}{\gamma - \rho})^s |\chi^n(\gamma)|^q + m(\tfrac{\gamma - \omega}{\gamma - \rho})^s |\chi^n(\rho)|^q, \end{aligned}$$

$$\begin{aligned} &\left| \sum_{\nu=0}^{n-1} (-1)^\nu \left(\tfrac{\chi^{(\nu)}(\gamma)\gamma^{\nu+1} - \chi^{(\nu)}(\rho)\rho^{\nu+1}}{(\nu+1)!} \right) - \int_\rho^\gamma \chi(\omega) d\omega \right| \\ &\leq \tfrac{1}{n!} \int_\rho^\gamma \omega^n \left| \chi^{(n)}(\omega) \right| d\omega, \end{aligned}$$

$$\leq \frac{1}{n!}\frac{1}{\gamma-\rho}\left\{\begin{array}{l}\left(\int\limits_{\rho}^{\gamma}(\gamma-\omega)\omega^{np}d\omega\right)^{\frac{1}{p}}\left(\int\limits_{\rho}^{\gamma}(\gamma-\omega)\left|\chi^{(n)}(\omega)\right|^{q}d\omega\right)^{\frac{1}{q}}+\\ \left(\int\limits_{\rho}^{\gamma}(\omega-\rho)\omega^{np}d\omega\right)^{\frac{1}{p}}\left(\int\limits_{\rho}^{\gamma}(\omega-\rho)\left|\chi^{(n)}(\omega)\right|^{q}d\omega\right)^{\frac{1}{q}}\end{array}\right\},$$

$$\begin{aligned}&\leq \tfrac{1}{n!}\tfrac{1}{\gamma-\rho}\left(\int\limits_{\rho}^{\gamma}(\gamma-\omega)\omega^{np}d\omega\right)^{\frac{1}{p}}\left(\int\limits_{\rho}^{\gamma}(\gamma-\omega)\left[(\tfrac{\omega-\rho}{\gamma-\rho})^{s}|\chi^{n}(\gamma)|^{q}+m(\tfrac{\gamma-\omega}{\gamma-\rho})^{s}|\chi^{n}(\rho)|^{q}\right]d\omega\right)^{\frac{1}{q}}\\ &+\tfrac{1}{n!}\tfrac{1}{\gamma-\rho}\left(\int\limits_{\rho}^{\gamma}(\omega-\rho)\omega^{np}d\omega\right)^{\frac{1}{p}}\left(\int\limits_{\rho}^{\gamma}(\omega-\rho)\left[(\tfrac{\omega-\rho}{\gamma-\rho})^{s}|\chi^{n}(\gamma)|^{q}+m(\tfrac{\gamma-\omega}{\gamma-\rho})^{s}|\chi^{n}(\rho)|^{q}\right]d\omega\right)^{\frac{1}{q}},\end{aligned} \tag{2.2}$$

Let

$$\begin{aligned}I_1 &= \left[\int\limits_{\rho}^{\gamma}(\gamma-\omega)\omega^{np}d\omega\right]^{\frac{1}{p}}=\left[\int\limits_{\rho}^{\gamma}(\gamma\omega^{np}-\omega^{np+1})d\omega\right]^{\frac{1}{p}}\\ &=(\gamma-\rho)^{\frac{1}{p}}\left[\gamma\left(\tfrac{\gamma^{np+1}-\rho^{np+1}}{(\gamma-\rho)(np+1)}\right)-\left(\tfrac{\gamma^{np+2}-\rho^{np+2}}{(\gamma-\rho)(np+2)}\right)\right]^{\frac{1}{p}}=(\gamma-\rho)^{\frac{1}{p}}\left[\gamma L_{np}^{np}(\rho,\gamma)-L_{np+1}^{np+1}(\rho,\gamma)\right]^{\frac{1}{p}},\\ I_2 &= \left[\int\limits_{\rho}^{\gamma}(\omega-\rho)\omega^{np}dt\right]^{\frac{1}{p}}=\left[\int\limits_{\rho}^{\gamma}(\omega^{np+1}-\rho\omega^{np})d\omega\right]^{\frac{1}{p}}\\ &=(\gamma-\rho)^{\frac{1}{p}}\left[\left(\tfrac{\gamma^{np+2}-\rho^{np+2}}{(\gamma-\rho)(np+2)}\right)-\rho\left(\tfrac{\gamma^{np+1}-\rho^{np+1}}{(\gamma-\rho)(np+1)}\right)\right]^{\frac{1}{p}}=(\gamma-\rho)^{\frac{1}{p}}\left[L_{np+1}^{np+1}(\rho,\gamma)-\rho L_{np}^{np}(\rho,\gamma)\right]^{\frac{1}{p}},\end{aligned}$$

$$I_3=\int\limits_{\rho}^{\gamma}(\gamma-\omega)(\omega-\rho)^{s}d\omega=(\gamma-\omega)\tfrac{(\omega-\rho)^{s+1}}{s+1}\Big|_{\rho}^{\gamma}+\int\limits_{\rho}^{\gamma}\tfrac{(\omega-\rho)^{s+1}}{s+1}d\omega=\tfrac{(\gamma-\rho)^{s+2}}{(s+1)(s+2)},$$

$$I_4=\int\limits_{\rho}^{\gamma}(\gamma-\omega)^{s+1}d\omega=\tfrac{(\gamma-\rho)^{s+2}}{s+2},\ I_5=\int\limits_{\rho}^{\gamma}(\omega-\rho)^{s+1}d\omega=\tfrac{(\gamma-\rho)^{s+2}}{s+2},$$

$$I_6=\int\limits_{\rho}^{\gamma}(\omega-\rho)(\gamma-\omega)^{s}d\omega=(\omega-\rho)\tfrac{(\gamma-\omega)^{s+1}}{(s+1)}\Big|_{\rho}^{\gamma}+\int\limits_{\rho}^{\gamma}\tfrac{(\gamma-\omega)^{s+1}}{(s+1)}d\omega=\tfrac{(\gamma-\rho)^{s+2}}{(s+1)(s+2)}.$$

Substituting integrals $I_1, I_2, I_3, I_4, I_5, I_6$ in inequality (2.2) we have,

$$\begin{aligned}&\left|\sum_{\nu=0}^{n-1}(-1)^{\nu}\left(\tfrac{\chi^{(\nu)}(\gamma)\gamma^{\nu+1}-\chi^{(\nu)}(\rho)\rho^{\nu+1}}{(\nu+1)!}\right)-\int\limits_{\rho}^{\gamma}\chi(\omega)d\omega\right|\leq\\ &\tfrac{1}{n!(\gamma-\rho)}\left(\begin{array}{l}(\gamma-\rho)^{\frac{1}{p}}[\gamma L_{np}^{np}(\rho,\gamma)-L_{np+1}^{np+1}(\rho,\gamma)]^{\frac{1}{p}}\left[(\gamma-\rho)^{2}\left(\tfrac{|\chi^{n}(\gamma)|^{q}}{(s+2)(s+1)}+\tfrac{m|\chi^{n}(\rho)|^{q}}{(s+2)}\right)\right]^{\frac{1}{q}}\\ +(\gamma-\rho)^{\frac{1}{p}}[L_{np+1}^{np+1}(\rho,\gamma)-\rho L_{np}^{np}(\rho,\gamma)]^{\frac{1}{p}}\left[(\gamma-\rho)^{2}\left(\tfrac{|\chi^{n}(\gamma)|^{q}}{(s+2)}+\tfrac{m|\chi^{n}(\rho)|^{q}}{(s+1)(s+2)}\right)\right]^{\frac{1}{q}}\end{array}\right)\\ &=\tfrac{(\gamma-\rho)^{\frac{1}{p}-1+\frac{2}{q}}}{n!}\left(\begin{array}{l}[\gamma L_{np}^{np}(\rho,\gamma)-L_{np+1}^{np+1}(\rho,\gamma)]^{\frac{1}{p}}\left[\tfrac{|\chi^{n}(\gamma)|^{q}}{(s+2)(s+1)}+\tfrac{m|\chi^{n}(\rho)|^{q}}{(s+2)}\right]^{\frac{1}{q}}\\ +[L_{np+1}^{np+1}(\rho,\gamma)-\rho L_{np}^{np}(\rho,\gamma)]^{\frac{1}{p}}\left[\tfrac{|\chi^{n}(\gamma)|^{q}}{(s+2)}+\tfrac{m|\chi^{n}(\rho)|^{q}}{(s+1)(s+2)}\right]^{\frac{1}{q}}\end{array}\right)\\ &=\tfrac{1}{n!}(\gamma-\rho)^{\frac{1}{q}}\left(\begin{array}{l}[\gamma L_{np}^{np}(\rho,\gamma)-L_{np+1}^{np+1}(\rho,\gamma)]^{\frac{1}{p}}\left[\tfrac{|\chi^{n}(\gamma)|^{q}}{(s+2)(s+1)}+\tfrac{m|\chi^{n}(\rho)|^{q}}{(s+2)}\right]^{\frac{1}{q}}\\ +[L_{np+1}^{np+1}(\rho,\gamma)-\rho L_{np}^{np}(\rho,\gamma)]^{\frac{1}{p}}\left[\tfrac{|\chi^{n}(\gamma)|^{q}}{(s+2)}+\tfrac{m|\chi^{n}(\rho)|^{q}}{(s+1)(s+2)}\right]^{\frac{1}{q}}\end{array}\right).\end{aligned}$$

which is required inequality (2.1). □

For $n = 1$ inequality (2.1) becomes,

$$\left|\left(\frac{\chi(\gamma)\gamma-\chi(\rho)\rho}{\gamma-\rho}\right)-\frac{1}{\gamma-\rho}\int_{\rho}^{\gamma}\chi(\omega)d\omega\right|\leq$$
$$(\gamma-\rho)^{\frac{1}{q}-1}\left(\begin{array}{l}\left[\gamma L_p^p(\rho,\gamma)-L_{p+1}^{p+1}(\rho,\gamma)\right]^{\frac{1}{p}}\left[\frac{|\chi'(\gamma)|^q}{(s+1)(s+2)}+\frac{m|\chi'(\rho)|^q}{(s+2)}\right]^{\frac{1}{q}}\\+\left[L_{p+1}^{p+1}(\rho,\gamma)-\rho L_p^p(\rho,\gamma)\right]^{\frac{1}{p}}\left[\frac{m|\chi'(\rho)|^q}{(s+1)(s+2)}+\frac{|\chi'(\gamma)|^q}{(s+2)}\right]^{\frac{1}{q}}\end{array}\right). \tag{2.3}$$

Remark 2.2. For $s = 1$ and $m = 1$ our resulting inequality (2.1) becomes the inequality (2) of [5].

Theorem 2.3. *For $n \in \mathbb{N}$, let $\chi : U \subseteq (0,\infty) \to \mathbb{R}$ be n-times differentiable mapping on U°, where, $\rho,\gamma \in U^\circ$, $\rho < \gamma$, $\chi^{(n)} \in L[\rho,\gamma]$ and $|\chi^{(n)}|^q$ for $q > 1$, is (s,m)-convex on interval $[\rho,\gamma]$ then following inequality holds*

$$\left|\sum_{\nu=0}^{n-1}(-1)^{\nu}\left(\frac{\chi^{(\nu)}(\gamma)\gamma^{\nu+1}-\chi^{(\nu)}(\rho)\rho^{\nu+1}}{(\nu+1)!}\right)-\int_{\rho}^{\gamma}\chi(\omega)d\omega\right|\leq$$
$$\frac{1}{s^{\frac{1}{q}}n!}\left(\frac{1}{2}\right)^{\frac{1}{p}}(\gamma-\rho)^{\frac{2}{p}-1}\left(\begin{array}{l}\left(\begin{array}{l}\frac{|\chi^{(n)}(\gamma)|^q}{(\gamma-\rho)^{s-1}}\left[-L_{nq+2}^{nq+2}(\rho,\gamma)+(\rho+\gamma)L_{nq+1}^{nq+1}(\rho,\gamma)-\rho\gamma L_{nq}^{nq}(\rho,\gamma)\right]+\\\frac{m|\chi^{(n)}(\rho)|^q}{(\gamma-\rho)^{s-1}}\left[L_{nq+2}^{nq+2}(\rho,\gamma)-2\gamma L_{nq+1}^{nq+1}(\rho,\gamma)+\gamma^2L_{nq}^{nq}(\rho,\gamma)\right]\end{array}\right)^{\frac{1}{q}}+\\\left(\begin{array}{l}\frac{|\chi^{(n)}(\gamma)|^q}{(\gamma-\rho)^{s-1}}\left[L_{nq+2}^{nq+2}(\rho,\gamma)-2\rho L_{nq+1}^{nq+1}(\rho,\gamma)+\rho^2L_{nq}^{nq}(\rho,\gamma)\right]+\\\frac{m|\chi^{(n)}(\rho)|^q}{(\gamma-\rho)^{s-1}}\left[-L_{nq+2}^{nq+2}(\rho,\gamma)+(\rho+\gamma)L_{nq+1}^{nq+1}(\rho,\gamma)-\rho\gamma L_{nq}^{nq}(\rho,\gamma)\right]\end{array}\right)^{\frac{1}{q}}\end{array}\right). \tag{2.4}$$

Proof. Since $|\chi^{(n)}|^q$ for $q > 1$ is (s,m)-convex on $[\rho,\gamma]$, by using Lemma 1.4 and Hölder-İşcan inequality (1.2), since $s \in (0,1]$, this fact can be used for $\omega,\rho,\gamma \in U \subseteq (0,\infty)$,

$$(\omega-\rho)^s < \frac{(\omega-\rho)}{s}, (\gamma-\omega)^s < \frac{(\gamma-\omega)}{s}$$

$$\left|\sum_{\nu=0}^{n-1}(-1)^{\nu}\left(\frac{\chi^{(\nu)}(\gamma)\gamma^{\nu+1}-\chi^{(\nu)}(\rho)\rho^{\nu+1}}{(\nu+1)!}\right)-\int_{\rho}^{\gamma}\chi(\omega)d\omega\right|$$
$$\leq\frac{1}{n!}\int_{\rho}^{\gamma}1.\omega^n\left|\chi^{(n)}(\omega)\right|d\omega,$$
$$\leq\frac{1}{n!}\frac{1}{(\gamma-\rho)}\left(\begin{array}{l}\left[\left(\int_{\rho}^{\gamma}(\gamma-\omega)d\omega\right)^{\frac{1}{p}}\left(\int_{\rho}^{\gamma}(\gamma-\omega)\omega^{nq}\left|\chi^{(n)}(\omega)\right|^qd\omega\right)^{\frac{1}{q}}\right]+\\\left[\left(\int_{\rho}^{\gamma}(\omega-\rho)d\omega\right)^{\frac{1}{p}}\left(\int_{\rho}^{\gamma}(\omega-\rho)\omega^{nq}\left|\chi^{(n)}(\omega)\right|^qd\omega\right)^{\frac{1}{q}}\right]\end{array}\right),$$

$$\leq\frac{1}{n!}\frac{1}{(\gamma-\rho)}\left(\int_{\rho}^{\gamma}(\gamma-\omega)d\omega\right)^{\frac{1}{p}}\left(\int_{\rho}^{\gamma}(\gamma-\omega)\omega^{nq}\left[\left(\frac{\omega-\rho}{\gamma-\rho}\right)^s|\chi^n(\gamma)|^q+m\left(\frac{\gamma-\omega}{\gamma-\rho}\right)^s|\chi^n(\rho)|^q\right]dt\right)^{\frac{1}{q}}+$$
$$\frac{1}{n!}\frac{1}{(\gamma-\rho)}\left(\int_{\rho}^{\gamma}(\omega-\rho)dt\right)^{\frac{1}{p}}\left(\int_{\rho}^{\gamma}(\omega-\rho)\omega^{nq}\left[\left(\frac{\omega-\rho}{\gamma-\rho}\right)^s|\chi^n(\gamma)|^q+m\left(\frac{\gamma-\omega}{\gamma-\rho}\right)^s|\chi^n(\rho)|^q\right]dx\right)^{\frac{1}{q}},$$

$$\leq \frac{1}{s^{\frac{1}{q}}n!}\frac{1}{(\gamma-\rho)}\left(\int_{\rho}^{\gamma}(\gamma-\omega)d\omega\right)^{\frac{1}{p}}\left(\int_{\rho}^{\gamma}(\gamma-\omega)\omega^{nq}\left[\frac{(\omega-\rho)}{(\gamma-\rho)^{s}}|\chi^{n}(\gamma)|^{q}+\frac{m(\gamma-\omega)}{(\gamma-\rho)^{s}}|\chi^{n}(\rho)|^{q}\right]d\omega\right)^{\frac{1}{q}}$$
$$+\frac{1}{s^{\frac{1}{q}}n!}\frac{1}{(\gamma-\rho)}\left(\int_{\rho}^{\gamma}(\omega-\rho)d\omega\right)^{\frac{1}{p}}\left(\int_{\rho}^{\gamma}(\omega-\rho)\omega^{nq}\left[\frac{(\omega-\rho)}{(\gamma-\rho)^{s}}|\chi^{n}(\gamma)|^{q}+\frac{m(\gamma-\omega)}{(\gamma-\rho)^{s}}|\chi^{n}(\rho)|^{q}\right]d\omega\right)^{\frac{1}{q}}, \tag{2.5}$$

$$I_1=\int_{\rho}^{\gamma}(\gamma-\omega)d\omega=\frac{(\gamma-\rho)^2}{2}$$
$$I_2=\int_{\rho}^{\gamma}(\gamma-\omega)(\omega-\rho)\omega^{nq}d\omega=\gamma\frac{\omega^{nq+1}}{nq+1}-\rho\gamma\frac{\omega^{nq+1}}{nq+1}-\frac{\omega^{nq+3}}{nq+3}+\frac{\rho\omega^{nq+2}}{nq+2}\Big|_{\rho}^{\gamma}$$
$$=-\left(\frac{\gamma^{nq+3}-\rho^{nq+3}}{nq+3}\right)+\rho\left(\frac{\gamma^{nq+2}-\rho^{nq+2}}{nq+2}\right)+\gamma\left(\frac{\gamma^{nq+2}-\rho^{nq+2}}{nq+2}\right)-\rho\gamma\left(\frac{\gamma^{nq+1}-\rho^{nq+1}}{nq+1}\right)$$
$$=(\gamma-\rho)\left[-L_{nq+2}^{nq+2}(\rho,\gamma)+(\rho+\gamma)L_{nq+1}^{nq+1}(\rho,\gamma)-\rho\gamma L_{nq}^{nq}(\rho,\gamma)\right],$$

$$I_3=\int_{\rho}^{\gamma}(\gamma-\omega)^2\omega^{nq}d\omega=\gamma^2\frac{\omega^{nq+1}}{nq+1}+\frac{\omega^{nq+3}}{nq+3}-2\gamma\frac{\omega^{nq+2}}{nq+2}\Big|_{\rho}^{\gamma}$$
$$=\left(\frac{\gamma^{nq+3}-\rho^{nq+3}}{nq+3}\right)-2\gamma\left(\frac{\gamma^{nq+2}-\rho^{nq+2}}{nq+2}\right)+\gamma^2\left(\frac{\gamma^{nq+1}-\rho^{nq+1}}{nq+1}\right)$$
$$=(\gamma-\rho)\left[L_{nq+2}^{nq+2}(\rho,\gamma)-2\gamma L_{nq+1}^{nq+1}(\rho,\gamma)+\gamma^2L_{nq}^{nq}(\rho,\gamma)\right],$$

$$I_4=\int_{\rho}^{\gamma}(\omega-\rho)^2\omega^{nq}d\omega=\frac{\omega^{nq+3}}{nq+3}+\rho^2\frac{\omega^{nq+1}}{nq+1}-2\rho\frac{\omega^{nq+2}}{nq+2}\Big|_{\rho}^{\gamma}$$
$$=\left(\frac{\gamma^{nq+3}-\rho^{nq+3}}{nq+3}\right)+\rho^2\left(\frac{\gamma^{nq+1}-\rho^{nq+1}}{nq+1}\right)-2\rho\left(\frac{\gamma^{nq+2}-\rho^{nq+2}}{nq+2}\right)$$
$$=(\gamma-\rho)\left[L_{nq+2}^{nq+2}(\rho,\gamma)+\rho^2L_{nq}^{nq}(\rho,\gamma)-2\rho L_{nq+1}^{nq+1}(\rho,\gamma)\right].$$

Substituting integrals $I_1, I_2, I_3, I_4, I_5, I_6$ in inequality (2.5) we have,

$$\left|\sum_{\nu=0}^{n-1}(-1)^{\nu}\left(\frac{\chi^{(\nu)}(\gamma)\gamma^{\nu+1}-\chi^{(\nu)}(\rho)\rho^{\nu+1}}{(\nu+1)!}\right)-\int_{\rho}^{\gamma}\chi(\omega)d\omega\right|\leq\frac{1}{s^{\frac{1}{q}}n!}\left(\frac{1}{2}\right)^{\frac{1}{p}}(\gamma-\rho)^{\frac{2}{p}-1}\times$$
$$\left(\begin{array}{l}\left(\begin{array}{l}\frac{|\chi^{(n)}(\gamma)|^q}{(\gamma-\rho)^s}\left[(\gamma-\rho)(-L_{nq+2}^{nq+2}(\rho,\gamma)+(\rho+\gamma)L_{nq+1}^{nq+1}(\rho,\gamma)-\rho\gamma L_{nq}^{nq}(\rho,\gamma))\right]+\\ \frac{m|\chi^{(n)}(\rho)|^q}{(\gamma-\rho)^s}\left[(\gamma-\rho)(L_{nq+2}^{nq+2}(\rho,\gamma)-2\gamma L_{nq+1}^{nq+1}(\rho,\gamma)+\gamma^2L_{nq}^{nq}(\rho,\gamma))\right]\end{array}\right)^{\frac{1}{q}}\\ +\left(\begin{array}{l}\frac{|\chi^{(n)}(\gamma)|^q}{(\gamma-\rho)^s}\left[(\gamma-\rho)(L_{nq+2}^{nq+2}(\rho,\gamma)-2\rho L_{nq+1}^{nq+1}(\rho,\gamma)+\rho^2L_{nq}^{nq}(\rho,\gamma))\right]+\\ \frac{m|\chi^{(n)}(\rho)|^q}{(\gamma-\rho)^s}\left[(\gamma-\rho)(-L_{nq+2}^{nq+2}(\rho,\gamma)+(\rho+\gamma)L_{nq+1}^{nq+1}(\rho,\gamma)-\rho\gamma L_{nq}^{nq}(\rho,\gamma))\right]\end{array}\right)^{\frac{1}{q}}\end{array}\right),$$

$$=\frac{1}{s^{\frac{1}{q}}n!}\left(\frac{1}{2}\right)^{\frac{1}{p}}(\gamma-\rho)^{\frac{2}{p}-1}\times$$
$$\left(\begin{array}{l}\left(\begin{array}{l}\frac{|\chi^{(n)}(\gamma)|^q}{(\gamma-\rho)^{s-1}}\left[-L_{nq+2}^{nq+2}(\rho,\gamma)+(\rho+\gamma)L_{nq+1}^{nq+1}(\rho,\gamma)-\rho\gamma L_{nq}^{nq}(\rho,\gamma)\right]+\\ \frac{m|\chi^{(n)}(\rho)|^q}{(\gamma-\rho)^{s-1}}\left[L_{nq+2}^{nq+2}(\rho,\gamma)-2\gamma L_{nq+1}^{nq+1}(\rho,\gamma)+\gamma^2L_{nq}^{nq}(\rho,\gamma)\right]\end{array}\right)^{\frac{1}{q}}+\\ \left(\begin{array}{l}\frac{|\chi^{(n)}(\gamma)|^q}{(\gamma-\rho)^{s-1}}\left[L_{nq+2}^{nq+2}(\rho,\gamma)-2\rho L_{nq+1}^{nq+1}(\rho,\gamma)+\rho^2L_{nq}^{nq}(\rho,\gamma)\right]+\\ \frac{m|\chi^{(n)}(\rho)|^q}{(\gamma-\rho)^{s-1}}\left[-L_{nq+2}^{nq+2}(\rho,\gamma)+(\rho+\gamma)L_{nq+1}^{nq+1}(\rho,\gamma)-\rho\gamma L_{nq}^{nq}(\rho,\gamma)\right]\end{array}\right)^{\frac{1}{q}}\end{array}\right).$$

For $n = 1$, Theorem2.3 reduced to the inequality

$$\left|\frac{\gamma\chi(\gamma)-\rho\chi(\rho)}{(\gamma-\rho)} - \frac{1}{(\gamma-\rho)}\int_{\rho}^{\gamma}\chi(\omega)d\omega\right| \leq$$

$$\frac{1}{s^{\frac{1}{q}}}\left(\frac{1}{2}\right)^{\frac{1}{p}}(\gamma-\rho)^{\frac{2}{p}-2}\left(\begin{array}{l}\left(\begin{array}{l}\frac{|\chi^{(1)}(\gamma)|^q}{(\gamma-\rho)^{s-1}}\left[-L_{q+2}^{q+2}(\rho,\gamma)+(\rho+\gamma)L_{q+1}^{q+1}(\rho,\gamma)-\rho\gamma L_q^q(\rho,\gamma)\right]+\\ \frac{m|\chi^{(1)}(\rho)|^q}{(\gamma-\rho)^{s-1}}\left[L_{q+2}^{q+2}(\rho,\gamma)-2\gamma L_{q+1}^{q+1}(\rho,\gamma)+\gamma^2 L_q^q(\rho,\gamma)\right]\end{array}\right)^{\frac{1}{q}}+\\ \left(\begin{array}{l}\frac{|\chi^{(1)}(\gamma)|^q}{(\gamma-\rho)^{s-1}}\left[L_{q+2}^{q+2}(\rho,\gamma)-2\rho L_{q+1}^{q+1}(\rho,\gamma)+\rho^2 L_q^q(\rho,\gamma)\right]+\\ \frac{m|\chi^{(1)}(\rho)|^q}{(\gamma-\rho)^{s-1}}\left[-L_{q+2}^{q+2}(\rho,\gamma)+(\rho+\gamma)L_{q+1}^{q+1}(\rho,\gamma)-\rho\gamma L_q^q(\rho,\gamma)\right]\end{array}\right)^{\frac{1}{q}}\end{array}\right). \tag{2.6}$$

□

Remark 2.4. For $s = 1$ and $m = 1$ our resulting inequality (2.4) becomes the inequality (6) of [5].

2.2. Hadamard type inequality via (s, m)-convexity

Theorem 2.5. *If function $\chi : [0, b] \longrightarrow \mathbb{R}$, $b > 0$ is a (s,m)-convex function in the second sense where $(s, m) \in (0, 1]^2$, holds provided that all $\rho, \gamma \in [0, b]$ and $\varsigma \in [0, 1]$, then*

$$\begin{aligned}&2^s\chi\left(\frac{\rho+m\gamma}{2}\right) \leq \left[\frac{1}{m\gamma-\rho}\int_{\rho}^{m\gamma}\chi(\omega)d\omega + \frac{m^2}{m\gamma-\rho}\int_{\frac{\rho}{m}}^{\gamma}\chi(l)dl\right]\\ &\leq \frac{\chi(\rho)+m\chi(\gamma)}{s+1} + \frac{\chi(\gamma)+m\chi(\frac{\rho}{m^2})}{s+1}.\end{aligned} \tag{2.7}$$

Proof. A function $\chi : [0, b] \longrightarrow \mathbb{R}$, $b > 0$ is said to be a (s, m)-convex function in the second sense where $s, m \in (0, 1]^2$, if

$$\chi\Big(\varsigma\rho + m(1-\varsigma)\gamma\Big) \leq \varsigma^s\chi(\rho) + m(1-\varsigma)^s\chi(\gamma),$$

holds provided that all $\rho, \gamma \in [0, b]$ and $\varsigma \in [0, 1]$.
Integrating w.r.t ς on $[0, 1]$,

$$\int_0^1\chi(\varsigma\rho + m(1-\varsigma)\gamma)d\varsigma \leq \int_0^1\varsigma^s\chi(\rho)d\varsigma + \int_0^1 m(1-\varsigma)^s\chi(\gamma)d\varsigma,$$

$$= \frac{\varsigma^{s+1}}{s+1}\Big|_0^1\chi(\rho) - m\chi(\gamma)\frac{(1-\varsigma)^{s+1}}{s+1}\Big|_0^1 = \frac{\chi(\rho)+m\chi(\gamma)}{s+1}.$$

$$\int_0^1\chi(\varsigma\rho + m(1-\varsigma)\gamma)d\varsigma \leq \frac{\chi(\rho)+m\chi(\gamma)}{s+1}. \tag{2.8}$$

and

$$\chi(\varsigma\gamma + m(1-\varsigma)\tfrac{\rho}{m^2}) \leq \varsigma^s\chi(\gamma) + m(1-\varsigma)^s\chi(\tfrac{\rho}{m^2}),$$

$$\int_0^1\chi(\varsigma\gamma + m(1-\varsigma)\frac{\rho}{m^2})d\varsigma \leq \frac{\chi(\gamma)+m\chi(\frac{\rho}{m^2})}{s+1}. \tag{2.9}$$

As χ is (s,m)-convex,

$$\chi\left(\frac{\rho+m\gamma}{2}\right)=\chi\left(\frac{\varsigma\rho+(1-\varsigma)m\gamma}{2}+m.\frac{(1-\varsigma)\frac{\rho}{m}+\varsigma\gamma}{2}\right)$$
$$\leq\left(\frac{1}{2}\right)^s\chi(\varsigma\rho+(1-\varsigma)\gamma m)+m\left(\frac{1}{2}\right)^s\chi(\varsigma\gamma+(1-\varsigma)\frac{\rho}{m}),$$

Integrating w.r.t ς over $[0,1]$ and by using (2.8) and (2.9) we get,

$$\begin{aligned}2^s\chi\left(\frac{\rho+m\gamma}{2}\right)&\leq\int_0^1(\chi(\varsigma\rho+(1-\varsigma)\gamma m)\,d\varsigma+m\int_0^1\chi(\varsigma\gamma+(1-\varsigma)\frac{\rho}{m})d\varsigma\\&\leq\frac{\chi(\rho)+m\chi(\gamma)}{s+1}+\frac{\chi(\gamma)+m\chi(\frac{\rho}{m^2})}{s+1}.\end{aligned}\tag{2.10}$$

Substituting in first integral,
$\varsigma\rho+(1-\varsigma)\gamma m=\omega$,

$$\int_0^1\chi(\varsigma\rho+(1-\varsigma)m\gamma)d\varsigma=\frac{1}{\gamma m-\rho}\int_\rho^{\gamma m}\chi(\omega)d\omega.\tag{2.11}$$

Substituting in the second integral,
$\varsigma\gamma+(1-\varsigma)\frac{\rho}{m}=l$,

$$\int_0^1\chi(\varsigma\gamma+(1-\varsigma)\frac{\rho}{m})d\varsigma=\frac{m}{\gamma m-\rho}\int_{\frac{\rho}{m}}^{\gamma}\chi(l)dl,\tag{2.12}$$

Using (2.11) and (2.12) in (2.10) required inequality (2.7) obtained.

□

Remark 2.6. For $s,m=1$ inequality (2.7) becomes classical Hadamard inequality for convex functions.

2.3. Application of Hadamrd type inequality via (s,m)-concavity

Theorem 2.7. *For $n\in\mathbb{N}$, let $\chi:U\subseteq(0,\infty)\to\mathbb{R}$ be n-times differentiable mapping on U°, where, $\rho,\gamma\in U^\circ$, $\rho<\gamma$ and $\chi^{(n)}\in L[\rho,\gamma]$ and $|\chi^{(n)}|^q$ for $q>1$ is (s,m)-concave on interval $[\rho,m\gamma]$, then*

$$\begin{aligned}&\left|\sum_{\nu=0}^{n-1}(-1)^\nu\left(\frac{\chi^{(\nu)}(\gamma)\gamma^{\nu+1}-\chi^{(\nu)}(\rho)\rho^{\nu+1}}{(\nu+1)!}\right)-\int_\rho^{m\gamma}\chi(\omega)d\omega\right|\leq\\&\frac{2^{\frac{s}{q}}(m\gamma-\rho)^{\frac{1}{q}}\left|\chi^{(n)}\left(\frac{\rho+m\gamma}{2}\right)\right|}{n!}\left(\begin{array}{l}\left(\gamma L_{np}^{np}(\rho,m\gamma)-L_{np+1}^{np+1}(\rho,m\gamma)\right)^{\frac{1}{p}}\\+\left(L_{np+1}^{np+1}(\rho,m\gamma)-\rho L_{np}^{np}(\rho,m\gamma)\right)^{\frac{1}{p}}\end{array}\right).\end{aligned}\tag{2.13}$$

Proof. $|\chi^{(n)}|^q$ for $q>1$ is (s,m)-concave then by using Theorem 2.5 we have,

$$\frac{\left|\chi^{(n)}(\rho)\right|^q+m\left|\chi^{(n)}(\gamma)\right|^q}{s+1}+\frac{\left|\chi^{(n)}(\gamma)\right|^q+m\left|\chi^{(n)}(\frac{\rho}{m^2})\right|^q}{s+1}-\frac{m^2}{(m\gamma-\rho)}\int_{\frac{\rho}{m}}^{\gamma}\left|\chi^{(n)}(l)\right|^q dl$$
$$\leq\frac{1}{(m\gamma-\rho)}\int_\rho^{m\gamma}\left|\chi^{(n)}(\omega)\right|^q d\omega\leq 2^s\left|\chi^{(n)}\left(\frac{\rho+m\gamma}{2}\right)\right|^q,$$

$$\int_{\rho}^{m\gamma}\left|\chi^{(n)}(\omega)\right|^{q}d\omega \leq 2^{s}(m\gamma-\rho)\left|\chi^{(n)}\left(\frac{\rho+m\gamma}{2}\right)\right|^{q},$$

$$\frac{1}{(m\gamma-\rho)}\int_{\rho}^{\gamma m}(\gamma-\omega)\left|\chi^{(n)}(\omega)\right|^{q}d\omega \leq \int_{\rho}^{\gamma m}\left|\chi^{(n)}(\omega)\right|^{q}d\omega \leq 2^{s}(m\gamma-\rho)\left|\chi^{(n)}\left(\frac{\rho+m\gamma}{2}\right)\right|^{q},$$

$$\frac{1}{(m\gamma-\rho)}\int_{\rho}^{\gamma m}(\gamma-\omega)\left|\chi^{(n)}(\omega)\right|^{q}d\omega \leq \int_{\rho}^{\gamma m}\left|\chi^{(n)}(\omega)\right|^{q}d\omega \leq 2^{s}(m\gamma-\rho)\left|\chi^{(n)}\left(\frac{\rho+m\gamma}{2}\right)\right|^{q}.$$

Using Lemma 1.4 and Hölder-Îşcan inequality (1.2),

$$\begin{aligned}
&\left|\sum_{\nu=0}^{n-1}(-1)^{\nu}\left(\frac{\chi^{(\nu)}(\gamma)\gamma^{\nu+1}-\chi^{(\nu)}(\rho)\rho^{\nu+1}}{(\nu+1)!}\right)-\int_{\rho}^{\gamma m}\chi(\omega)d\omega\right| \leq \frac{1}{n!}\int_{\rho}^{\gamma m}\omega^{n}\left|\chi^{(n)}(\omega)\right|d\omega,\\
&\leq \frac{1}{n!}\frac{1}{\gamma-\rho}\left\{\begin{array}{l}\left(\int_{\rho}^{\gamma m}(\gamma-\omega)\omega^{np}d\omega\right)^{\frac{1}{p}}\left(\int_{\rho}^{m\gamma}(\gamma-\omega)|\chi^{n}(\omega)|^{q}d\omega\right)^{\frac{1}{q}}+\\ \left(\int_{\rho}^{\gamma m}(\omega-\rho)\omega^{np}d\omega\right)^{\frac{1}{p}}\left(\int_{\rho}^{m\gamma}(\omega-\rho)|\chi^{n}(\omega)|^{q}d\omega\right)^{\frac{1}{q}}\end{array}\right\},\\
&\leq \frac{1}{n!}\frac{1}{\gamma-\rho}\left(\begin{array}{l}\left(\int_{\rho}^{\gamma m}(\gamma-\omega)\omega^{np}d\omega\right)^{\frac{1}{p}}\left(2^{s}(m\gamma-\rho)^{2}\left|\chi^{(n)}\left(\frac{\rho+m\gamma}{2}\right)\right|^{q}\right)^{\frac{1}{q}}+\\ \left(\int_{\rho}^{\gamma m}(\omega-\rho)\omega^{np}d\omega\right)^{\frac{1}{p}}\left(2^{s}(m\gamma-\rho)^{2}\left|\chi^{(n)}\left(\frac{\rho+m\gamma}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\end{array}\right),
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
I_{1} &= \left(\int_{\rho}^{\gamma m}(\gamma-\omega)\omega^{np}d\omega\right)^{\frac{1}{p}} = \left(\gamma\frac{\omega^{np+1}}{np+1}\Big|_{\rho}^{\gamma m}-\frac{\omega^{np+2}}{np+2}\Big|_{\rho}^{\gamma m}\right)^{\frac{1}{p}}\\
&= (m\gamma-\rho)^{\frac{1}{p}}\left(\gamma L_{np}^{np}(\rho,m\gamma)-L_{np+1}^{np+1}(\rho,m\gamma)\right)^{\frac{1}{p}},\\
I_{2} &= \left(\int_{\rho}^{\gamma m}(\omega-\rho)\omega^{np}d\omega\right)^{\frac{1}{p}} = \left(\frac{\omega^{np+2}}{np+2}\Big|_{\rho}^{\gamma m}-\rho\frac{\omega^{np+1}}{np+1}\Big|_{\rho}^{\gamma m}\right)^{\frac{1}{p}}\\
&= (m\gamma-\rho)^{\frac{1}{p}}\left(L_{np+1}^{np+1}(\rho,m\gamma)-\rho L_{np}^{np}(\rho,m\gamma)\right)^{\frac{1}{p}}.
\end{aligned}$$

Substituting integrals I_1, I_2 in inequality (2.14) required inequality (2.13) is obtained.
For $n = 1$ inequality (2.13) becomes,

$$\begin{aligned}
&\left|\frac{\chi(\gamma)\gamma-\rho\chi(\rho)}{(\gamma-\rho)}-\frac{1}{(\gamma-\rho)}\int_{\rho}^{\gamma m}\chi(\omega)d\omega\right| \leq\\
&\frac{2^{\frac{s}{q}}(m\gamma-\rho)^{\frac{1}{q}}\left|\chi^{(1)}\left(\frac{\rho+\gamma}{2}\right)\right|}{1!}\left(\begin{array}{l}\left(\gamma L_{p}^{p}(\rho,m\gamma)-L_{p+1}^{p+1}(\rho,m\gamma)\right)^{\frac{1}{p}}\\ +\left(L_{p+1}^{p+1}(\rho,m\gamma)-\rho L_{p}^{p}(\rho,m\gamma)\right)^{\frac{1}{p}}\end{array}\right).
\end{aligned} \tag{2.15}$$

□

Remark 2.8. For $s = 1$ and $m = 1$ our resulting inequality becomes the inequality obtained in Theorem 4 of [5].

2.4. Applications of newly obtained inequalities to means

Proposition 2.9. *Let $\rho, \gamma \in (0, \infty)$, where $\rho < \gamma$, $q > 1$, $n, i \in \mathbb{N}$ with $i \geq n$,*

$$\begin{aligned} &\left| L_i^i(\rho,\gamma)\left[(i+1)\sum_{\nu=0}^{n-1}\frac{(-1)^\nu P(i,\nu)}{(\nu+1)!}-1\right]\right| \leq \frac{1}{n!}(\gamma-\rho)^{\frac{1}{q}-1} \\ &\times\left(\begin{array}{c} \left[\gamma L_{np}^{np}(\rho,\gamma)-L_{np+1}^{np+1}(\rho,\gamma)\right]^{\frac{1}{p}}\left(\frac{\gamma^{(i-n)q}}{(s+1)(s+2)}+\frac{m\rho^{(i-n)q}}{(s+2)}\right)^{\frac{1}{q}} \\ +\left[L_{np+1}^{np+1}(\rho,\gamma)-\rho L_{np}^{np}(\rho,\gamma)\right]^{\frac{1}{p}}\left(\frac{m\rho^{(i-n)q}}{(s+1)(s+2)}+\frac{\gamma^{(i-n)q}}{(s+2)}\right)^{\frac{1}{q}} \end{array}\right), \end{aligned} \tag{2.16}$$

where

$$P(i,n) = \left\{\begin{array}{ll} i(i-1)...(i-n+1), i > n & \\ n!, & i = n \\ 1, & n = 0 \end{array}\right\}.$$

Proof. Let

$$\chi(\omega) = \omega^i, \left|\chi^{(n)}(\omega)\right|^q = \left|P(i,n)\omega^{i-n}\right|^q$$

Let

$$\begin{aligned} g(\varsigma) &= |P(i,n)(\varsigma\rho + m(1-\varsigma)\gamma|^{(i-n)q} - |P(i,n)\varsigma^s\rho|^{(i-n)q} - |mP(i,n)(1-\varsigma)^s\gamma|^{(i-n)q}, \\ g''(\varsigma) &= P(i,n)((i-n)q)((i-n)q-1)(\varsigma\rho + m(1-\varsigma)\gamma)^{(i-n)q-2}(\rho - m\gamma)^2 - \\ &s(s-1)\varsigma^{s-2}P(i,n)\rho^{(i-n)q} - ms(s-1)(1-\varsigma)^{s-2}P(i,n)\gamma^{(i-n)q}, \end{aligned}$$

$g''(\varsigma) \geq 0$ means g is convex and $g(1) = g(0) = 0$, which omplies $g \leq 0$, hence

$$|P(i,n)(\varsigma\rho + m(1-\varsigma)\gamma)|^{(i-n)q} \leq |P(i,n)\varsigma^s\rho|^{(i-n)q} + |mP(i,n)(1-\varsigma)^s\gamma)|^{(i-n)q}.$$

By using Theorem 2.1 for $|\chi^n(\omega)|^q$ which is (s,m)–convex for $s, m \in (0,1]^2$ inequality (2.16) obtained. □

Remark 2.10. For $s, m = 1$ inequality (2.16) becomes inequality (3) of [5].

Example 2.11. Taking $i = 2$, $n = 1$, $p = q = 2$ in Proposition 2.9, the following is valid:

$$2A(\rho^2,\gamma^2) + G^2(\rho,\gamma) \leq \left(\frac{3}{2\sqrt{6}}\right)\left(\begin{array}{c} \left[A(3\rho^2,\gamma^2)+G^2(\rho,\gamma)\right]^{\frac{1}{2}}\left(\frac{\gamma^2}{(s+1)(s+2)}+\frac{m\rho^2}{(s+2)}\right)^{\frac{1}{2}} \\ +\left[A(\rho^2,3\gamma^2)+G^2(\rho,\gamma)\right]^{\frac{1}{2}}\left(\frac{m\rho^2}{(s+1)(s+2)}+\frac{\gamma^2}{(s+2)}\right)^{\frac{1}{2}} \end{array}\right),$$

where A and G are classical arithmetic and geometric means, respectively.

Proposition 2.12. *Let $\rho, \gamma \in (0, \infty)$, with, $\rho < \gamma$, $q > 1$ and $n \in \mathbb{N}$,*

$$1 \leq (\gamma-\rho)^{\frac{1}{q}-1}\left(\begin{array}{c} \left[\gamma L_p^p(\rho,\gamma)-L_{p+1}^{p+1}(\rho,\gamma)\right]^{\frac{1}{p}}\left[\left(\frac{\gamma^{-q}}{(s+1)(s+2)}+\frac{m\rho^{-q}}{(s+2)}\right)\right]^{\frac{1}{q}}+ \\ \left[L_{p+1}^{p+1}(\rho,\gamma)-\rho L_p^p(\rho,\gamma)\right]^{\frac{1}{p}}\left[\left(\frac{m\rho^{-q}}{(s+1)(s+2)}+\frac{\gamma^{-q}}{(s+2)}\right)\right]^{\frac{1}{q}} \end{array}\right), \tag{2.17}$$

where L is classical logarithmic mean.

Proof.

$$\chi(\omega) = \ln \omega, \left|\chi^{(1)}(\omega)\right|^q = \left|\omega^{-1}\right|^q$$

Let

$$g(\varsigma) = |(\varsigma\rho + m(1-\varsigma)\gamma|^{-q} - |\varsigma^s\rho|^{-q} - |m(1-\varsigma)^s\gamma|^{-q}$$

$$g''(\varsigma) = (-q)(-q-1)(\varsigma\rho + m(1-\varsigma)\gamma)^{-q-2}(\rho - m\gamma)^2$$
$$-s(s-1)\varsigma^{s-2}\rho^{-q} - ms(s-1)(1-\varsigma)^{s-2}\gamma^{-q},$$

$g''(\varsigma) \geq 0$ means g is convex and $g(1) = g(0) = 0$ which implies $g \leq 0$ as

$$|(\varsigma\rho + m(1-\varsigma)\gamma|^{-q} \leq |\varsigma^s\rho|^{-q} + |m(1-\varsigma)^s\gamma|^{-q}.$$

So $\left|\chi^{(1)}(\omega)\right|^q$ is (s, m)-convex. Then by using inequality (2.3) required inequality (2.17) obtained. □

Remark 2.13. For $s, m = 1$ inequality (2.17) becomes (4) of [5].

Example 2.14. For $n = 1$ and $p = q = 2$, Proposition 2.12 gives:

$$1 \leq \frac{1}{\sqrt{6}}\left(\begin{array}{l} \left[A(3\rho^2, \gamma^2) + G^2(\rho, \gamma)\right]^{\frac{1}{p}}\left[\left(\frac{\gamma^{-2}}{(s+1)(s+2)} + \frac{m\rho^{-2}}{(s+2)}\right)\right]^{\frac{1}{2}} + \\ \left[A(\rho^2, 3\gamma^2) + G^2(\rho, \gamma)\right]^{\frac{1}{p}}\left[\left(\frac{m\rho^{-2}}{(s+1)(s+2)} + \frac{\gamma^{-2}}{(s+2)}\right)\right]^{\frac{1}{2}} \end{array}\right).$$

Proposition 2.15. *Let* $\rho, \gamma \in (0, \infty)$, $\rho < \gamma$, $q > 1$, $i \in (-\infty, 0] \cup [1, \infty)\backslash\{-2q, -q\}$ *then*

$$L_{\frac{i}{q}+1}^{\frac{i}{q}+1}(\rho, \gamma) \leq (\gamma - \rho)^{\frac{1}{q}-1}\left(\begin{array}{l} \left[\gamma L_p^p(\rho, \gamma) - L_{p+1}^{p+1}(\rho, \gamma)\right]^{\frac{1}{p}}\left[\left(\frac{\gamma^i}{(s+1)(s+2)} + \frac{m\rho^i}{(s+2)}\right)\right]^{\frac{1}{q}} \\ +\left[L_{p+1}^{p+1}(\rho, \gamma) - \rho L_p^p(\rho, \gamma)\right]^{\frac{1}{p}}\left[\left(\frac{m\rho^i}{(s+1)(s+2)} + \frac{\gamma^i}{(s+2)}\right)\right]^{\frac{1}{q}} \end{array}\right). \tag{2.18}$$

Proof.

$$\chi(t) = \frac{q}{i+q}\omega^{\frac{i}{q}+1}, |\chi'(\omega)|^q = \omega^i$$

Let

$$g(\varsigma) = |(\varsigma\rho + m(1-\varsigma)\gamma|^i - |\varsigma^s\rho|^i - |m(1-\varsigma)^s\gamma|^i,$$

$$g''(\varsigma) = (i)(i-1)(\varsigma\rho + m(1-\varsigma)\gamma)^{i-2}(\rho - m\gamma)^2 - s(s-1)\varsigma^{s-2}\rho^i - ms(s-1)(1-\varsigma)^{s-2}\gamma^i,$$

$g''(\varsigma) \geq 0$ and $g(1) = g(0)$ so $g \leq 0$ and $|\chi'(\omega)|^q$ is (s, m)-convex, by using inequality (2.3) we have (2.18). □

Remark 2.16. For $s, m = 1$ inequality (2.18) becomes (5) of [5].

Example 2.17. For $i = 2$ and $p = q = 2$ Proposition 2.15 reduced to

$$2A(\rho^2, \gamma^2) + G^2(\rho, \gamma) \leq$$
$$\left(\frac{3}{\sqrt{6}}\right)\left(\begin{array}{l} \left[A(3\rho^2, \gamma^2) + G^2(\rho, \gamma)\right]^{\frac{1}{2}}\left[\left(\frac{\gamma^2}{(s+1)(s+2)} + \frac{m\rho^2}{(s+2)}\right)\right]^{\frac{1}{2}} + \\ \left[A(\rho^2, 3\gamma^2) + G^2(\rho, \gamma)\right]^{\frac{1}{2}}\left[\left(\frac{m\rho^2}{(s+1)(s+2)} + \frac{\gamma^2}{(s+2)}\right)\right]^{\frac{1}{2}} \end{array}\right). \tag{2.19}$$

Proposition 2.18. *Let $\rho, \gamma \in (0, \infty)$ with $\rho < \gamma$, $q > 1$ and $n \in \mathbb{N}$ then we have*

$$\left|L_i^i(\rho,\gamma)\left[\sum_{v=0}^{n-1}\frac{(-1)^v P(i,v)}{(v+1)!}-1\right]\right| \le \frac{P(i,n)}{s^{\frac{1}{q}}n!}\left(\frac{1}{2}\right)^{\frac{1}{p}}(\gamma-\rho)^{\frac{2}{p}-1}$$
$$\times\left(\begin{array}{c}\frac{\gamma^{(i-n)q}}{(\gamma-\rho)^{s-1}}\left[-L_{nq+2}^{nq+2}(\rho,\gamma)+(\rho+\gamma)L_{nq+1}^{nq+1}(\rho,\gamma)-\rho\gamma L_{nq}^{nq}(\rho,\gamma)\right]+\\ \frac{m\rho^{(i-n)q}}{(\gamma-\rho)^{s-1}}\left[L_{nq+2}^{nq+2}(\rho,\gamma)-2\gamma L_{nq+1}^{nq+1}(\kappa,\mu)+\mu^2 L_{nq}^{nq}(\rho,\gamma)\right]\end{array}\right)^{\frac{1}{q}} \tag{2.20}$$
$$+\frac{P(i,n)}{s^{\frac{1}{q}}n!}\left(\frac{1}{2}\right)^{\frac{1}{p}}(\gamma-\rho)^{\frac{2}{p}-1}\left(\begin{array}{c}\frac{\gamma^{(i-n)q}}{(\gamma-\rho)^{s-1}}\left[L_{nq+2}^{nq+2}(\rho,\gamma)-2\rho L_{nq+1}^{nq+1}(\rho,\gamma)+\rho^2 L_{nq}^{nq}(\rho,\gamma)\right]+\\ \frac{m\rho^{(i-n)q}}{(\gamma-\rho)^{s-1}}\left[-L_{nq+2}^{nq+2}(\rho,\gamma)+(\rho+\gamma)L_{nq+1}^{nq+1}(\rho,\gamma)-\rho\gamma L_{nq}^{nq}(\rho,\gamma)\right]\end{array}\right)^{\frac{1}{q}}.$$

Proof. Let,

$$\chi(\omega)=\omega^i, \left|\chi^{(n)}(\omega)\right|^q=\left[P(i,n)\omega^{i-n}\right]^q$$

As $|\chi^n(\omega)|^q$ is (s,m)-convex on $(0,\infty)$, therefore by using Theorem 2.3 required inequality (2.20) is obtained. □

Remark 2.19. For $s, m = 1$ inequality (2.20) becomes inequality obtained in Proposition 4 of [5].

Proposition 2.20. *Let $\rho, \gamma \in (0, \infty)$ with $\rho < \gamma$ $q > 1$ and $n \in \mathbb{N}$ then we have,*

$$1 \le \frac{(\gamma-\rho)^{\frac{2}{p}-2}}{s^{\frac{1}{q}}.2^{\frac{1}{p}}}\left(\begin{array}{c}\left(\begin{array}{c}\frac{\gamma^{-q}}{(\gamma-\rho)^{s-1}}\left[-L_{q+2}^{q+2}(\rho,\gamma)+(\rho+\gamma)L_{q+1}^{q+1}(\rho,\gamma)-\rho\gamma L_q^q(\rho,\gamma)\right]+\\ \frac{m\rho^{-q}}{(\gamma-\rho)^{s-1}}\left[L_{q+2}^{q+2}(\rho,\gamma)-2\gamma L_{q+1}^{q+1}(\rho,\gamma)+\gamma^2 L_q^q(\rho,\gamma)\right]\end{array}\right)^{\frac{1}{q}}\\ +\left(\begin{array}{c}\frac{\gamma^{-q}}{(\gamma-\rho)^{s-1}}\left[L_{q+2}^{q+2}(\rho,\gamma)-2\rho L_{q+1}^{q+1}(\rho,\gamma)+\rho^2 L_q^q(\rho,\gamma)\right]+\\ \frac{m\rho^{-q}}{(\gamma-\rho)^{s-1}}\left[-L_{q+2}^{q+2}(\rho,\gamma)+(\rho+\gamma)L_{q+1}^{q+1}(\rho,\gamma)-\rho\gamma L_q^q(\rho,\gamma)\right]\end{array}\right)^{\frac{1}{q}}\end{array}\right), \tag{2.21}$$

Proof.

$$\chi(\omega)=\ln\omega, \left|\chi^{(1)}(\omega)\right|^q=\left[\omega^{-1}\right]^q$$

As $\left|\chi^{(1)}(\omega)\right|^q$ is (s,m)–convex, therefore by using inequality (2.6) required (2.21) obtained. □

Remark 2.21. For $s, m = 1$ inequality (2.21) becomes inequality obtained in Proposition 5 of [5].

Proposition 2.22. *Let $\rho, \gamma \in (0, \infty)$ with $\rho < \gamma$ $q > 1$ and $i \in (-\infty, 0]\backslash\{-2q, q\}$, then*

$$L_{\frac{i}{q}+1}^{\frac{i}{q}+1}(\rho,\gamma) \le \frac{(\gamma-\rho)^{\frac{2}{p}-2}}{s^{\frac{1}{q}}.2^{\frac{1}{p}}}\left(\begin{array}{c}\left(\begin{array}{c}\frac{\gamma^{i}}{(\gamma-\rho)^{s-1}}\left[-L_{q+2}^{nq+2}(\rho,\gamma)+(\rho+\gamma)L_{q+1}^{q+1}(\rho,\gamma)-\rho\gamma L_q^q(\rho,\gamma)\right]+\\ \frac{m\rho^{i}}{(\gamma-\rho)^{s-1}}\left[L_{q+2}^{q+2}(\rho,\gamma)-2\gamma L_{q+1}^{q+1}(\rho,\gamma)+\gamma^2 L_q^q(\rho,\gamma)\right]\end{array}\right)^{\frac{1}{q}}\\ +\left(\begin{array}{c}\frac{\gamma^{i}}{(\gamma-\rho)^{s-1}}\left[L_{q+2}^{q+2}(\rho,\gamma)-2\rho L_{q+1}^{q+1}(\rho,\gamma)+\rho^2 L_q^q(\rho,\gamma)\right]+\\ \frac{m\rho^{m}}{(\gamma-\rho)^{s-1}}\left[-L_{q+2}^{q+2}(\rho,\gamma)+(\rho+\gamma)L_{q+1}^{q+1}(\rho,\gamma)-\rho\gamma L_q^q(\rho,\gamma)\right]\end{array}\right)^{\frac{1}{q}}\end{array}\right). \tag{2.22}$$

Proof.

$$\chi(\omega)=\frac{q}{i+q}\omega^{\frac{i}{q}+1}|\chi^{'}(\omega)|^q=\omega^i$$

$|\chi^{'}(w)|^q$ is (s,m)-convex by using inequality (2.6) required (2.22) obtained. □

For $i = 1$ inequality (2.22) becomes,

$$L_{\frac{1}{q}+1}^{\frac{1}{q}+1}(\rho,\gamma) \leq \frac{(\gamma-\rho)^{\frac{2}{p}-2}}{s^{\frac{1}{q}}.2^{\frac{1}{p}}} \left(\begin{array}{c} \left(\begin{array}{c} \frac{\gamma^1}{(\gamma-\rho)^{s-1}} \left[-L_{q+2}^{q+2}(\rho,\gamma) + (\rho+\gamma)L_{q+1}^{q+1}(\rho,\gamma) - \rho\gamma L_q^q(\rho,\gamma) \right] + \\ \frac{m\rho^1}{(\gamma-\rho)^{s-1}} \left[L_{q+2}^{q+2}(\rho,\gamma) - 2\gamma L_{q+1}^{q+1}(\rho,\gamma) + \gamma^2 L_q^q(\rho,\gamma) \right] \end{array} \right)^{\frac{1}{q}} \\ + \left(\begin{array}{c} \frac{\gamma^1}{(\gamma-\rho)^{s-1}} \left[L_{q+2}^{q+2}(\rho,\gamma) - 2\rho L_{q+1}^{q+1}(\rho,\gamma) + \rho^2 L_q^q(\rho,\gamma) \right] + \\ \frac{m\rho^1}{(\gamma-\rho)^{s-1}} \left[-L_{q+2}^{q+2}(\rho,\gamma) + (\rho+\gamma)L_{q+1}^{q+1}(\rho,\gamma) - \rho\gamma L_q^q(\rho,\gamma) \right] \end{array} \right)^{\frac{1}{q}} \end{array} \right). \tag{2.23}$$

Remark 2.23. For $s, m = 1$ inequality (2.22) becomes inequality obtained in Proposition 6 of [5].

Proposition 2.24. *Let $\rho, \gamma \in (0, \infty)$ with $\rho < \gamma$, $q > 1$ and $i \in [0, 1]$ we have,*

$$\begin{array}{l} L_{\frac{i}{q}+1}^{\frac{i}{q}+1}(\rho,\gamma) \leq \\ \frac{2^{\frac{s}{q}}(m\gamma-\rho)^{\frac{1}{q}}}{1!} A^{\frac{i}{q}}(\rho,\gamma) \left(\begin{array}{c} \left(\gamma L_p^p(\rho, m\gamma) - L_{p+1}^{p+1}(\rho, m\gamma) \right)^{\frac{1}{p}} \\ + \left(L_{p+1}^{p+1}(\rho, m\gamma) - \rho L_p^p(\rho, m\gamma) \right)^{\frac{1}{p}} \end{array} \right). \end{array} \tag{2.24}$$

Proof.

$$\chi(\omega) = \frac{q}{i+q}\omega^{\frac{i}{q}+1}, |\chi'(\omega)|^q = \omega^i.$$

As $|\chi'(\omega)|^q$ is (s, m)-concave by using inequality (2.15) we obtain required inequality (2.24). □

Remark 2.25. For $s, m = 1$ inequality (2.24) becomes the inequality obtained in Proposition 9 of [5].

3. Conclusions

In this paper, Hölder-Isçan inequality is utilized to prove Hermite-Hadamard type inequalities for n-times differentiable (s, m)-convex functions. The method is adequate and provide many generalizations of existing results as shown in remarks. Moreover, many other inequalities can be generalized for other types of convex functions.

Acknowledgments

This research received funding support from the NSRF via the Program Management Unit for Human Resources & Institutional Development, Research and Innovation, (grant number B05F650018)

Conflict of interest

The authors declare no conflict of interest.

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