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Research article

Uniqueness and existence of positive periodic solutions of functional differential equations

Jiaqi Xu and Chunyan Xue*

School of Applied Science, Beijing Information Science & Technology University, Beijing, 100192, China

* **Correspondence:** Email: xue_chunyan@126.com; Tel: +86 13241838376.

Abstract: In this paper, some new findings on the uniqueness and existence of positive periodic solutions to first-order functional differential equations are presented. These equations have wide applications in a variety of fields. The most important feature of our argument is that we use the theory of Hilbert's metric to prove the uniqueness of the positive periodic solution when $q = -1$ and $-1 < q < 0$. In addition, we also investigate the existence results of positive periodic solutions by applying a fixed point theorem for completely continuous maps in a cone. Two examples demonstrate our findings.

Keywords: functional differential equation; positive periodic solution; uniqueness and existence; Hilbert's metric; fixed point theorem
Mathematics Subject Classification: 34K13

1. Introduction

We view the equation

$$y'(t) = -a(t)y(t) + g(t)y^q(t - \psi(t)), \tag{1.1}$$

where $a(t)$, $\psi(t)$ and $g(t) > 0$ are continuous function T-periodic functions and $\int_0^T a(u)du > 0$ $(T > 0)$. Such functional differential equations arise in ecological models, such as the dynamic disease model [1], population dynamics model [2], population model [3], and the Nicholson blowflies model [4].

In 1750, one of the earliest functional differential equation problems was the Euler's problem of finding a curve so that it resembles a shrinking line. In the past 50 years, many mathematicians are familiar with first-order functional differential equations (see [5–17]). They researched the same and critical question of whether these equations can support positive periodic solutions. Jiang, Wei and

Zhang [8], Cheng and Zhang [15], and Wang [16] proved the existence of positive periodic solution for the first-order functional differential in their papers. In particular, Wang [16] studied the following equation

$$x'(t) = a(t)g(x(t))x(t) - \lambda b(t)f(x(t - \tau(t))). \tag{1.2}$$

Using a famous result of the fixed point index, the author derived the existence results of the positive periodic solution to Eq (1.2) (for any $x > 0$, $f(x) > 0$). Among others, the writer proved the connection between the open intervals (eigenvalue intervals) of the parameter λ and the asymptotic behaviors of the quotient $\frac{f(x)}{x}$ (at $x \to 0$ and $x \to \infty$) so that Eq (1.2) admits zero, one and multiple positive solutions. However, the criteria for the uniqueness of the positive periodic solution of Eq (1.2) have not been established.

In [6], Liu and Li analyzed the existence of the positive periodic solutions to the equation

$$y'(t) = -a(t)y(t) + \lambda h(t)f(y(t - \tau(t))), \tag{1.3}$$

where $f : [0, +\infty) \to [0, +\infty)$ is continuous with $f(0) = 0$. The authors first proved the existence of the positive periodic solution to Eq (1.3) by applying the eigenvalue theory in cones. In addition, by employing the theory of α-concave operator, they derived an excellent result regarding the uniqueness of the periodic solution to Eq (1.3). However, it is difficult to prove the uniqueness of positive periodic solution for Eq (1.1) when $q = -1$ and $-1 < q < 0$.

Inspired by the pieces above, in our paper, we will construct two results for the uniqueness of positive periodic solutions to Eq (1.1) by employing the method of Hilbert's metric, which is completely different from that used in [5–17]. We establish uniqueness results, especially when $q = -1$ and $-1 < q < 1$ for Eq (1.1).

Hilbert [18] first considered Hilbert's metric on the foundations of geometry in 1895. Three noncollinear points have been modeled algebraically, in which the length of one side is equal to the sum of the other two sides. In 1957, Birkhoff [19] proved several extensions of Jentzsch's theorem on integral equations with positive kernels by employing Hilbert's metric. He gives some applications to the projective contraction theorem and simply Jentzsch's theorem. The earlier paper of Klein [20] also presented particular examples of Hilbert's metric. In the past time, the applications of Hilbert's metric were mentioned by plenty of authors, see [21–23].

Furthermore, we research the existence of the positive periodic solution to the following equation:

$$y'(t) = -a(t)y(t) + g(t)f(y(t - \psi(t))), \tag{1.4}$$

where $a(t)$, $\psi(t)$ and $g(t) > 0$ are T-periodic functions and $\int_0^T a(u)du > 0$ ($T > 0$). Comparing with Wang [16] and Liu-Li [17], we here employ a completely different technique to treat Eq (1.4), for detail to see the proof of Theorem 4.1.

In Section 2, we discuss a large number of necessary definitions and lemmas related to Hilbert's metric, which are necessary to prove our main point. In Section 3, we state and prove the uniqueness results of the positive periodic solution to Eq (1.1) by applying the theory of Hilbert's metric. In Section 4, based on the fixed point theorem in a cone, we prove the existence of the positive periodic solution to Eq (1.4). In the last section, the results of our study are illustrated by two examples.

2. Preliminaries

In this section, we review plenty of crucial lemmas and definitions, which will help us to prove our primary results.

Definition 2.1. (See Definition 1.1.1 of [24]) Let P be a convex closed set, and $P \neq \emptyset$. The following conditions must be met for P to be considered a cone:
(i) if $a \in P$, $\lambda > 0$, then $\lambda a \in P$;
(ii) if $a \in P$ and $-a \in P$, then $a = 0$, i.e., $0 \in P$.
A cone P defines a partial ordering in X by $a \leq b$ $(a,\ b \in P)$ if and only if $b - a \in P$.

Let the set of all the interior points of P be P°.
We identify that

$$M(a/b) = \inf\{\xi | a \leq \xi b\},\ m(a/b) = \sup\{\zeta | \zeta b \leq a\},\ a,\ b \in P^{\circ}.$$

There is no doubt in our minds that

$$m(a/b)b \leq a \leq M(a/b)b. \tag{2.1}$$

Definition 2.2. (See Definition 2.2 of [25]) Hilbert's metric is described in P° by

$$d(a, b) = ln\{M(a/b)/m(a/b)\}.$$

Lemma 2.1. *(See [25]) $d(a, b)$ meets the following requirements: $d(a, b) = 0$ is equal to $a = \kappa b,\ \kappa > 0$.*

Lemma 2.2. *(See Lemma 2.2 of [25]) If $a, b \in P^{\circ}$, then $d(\mu a, \nu b) = d(a, b)$ for all $\mu > 0$ and $\nu > 0$.*

Lemma 2.3. *(See Lemma 3.1 of [26]) Set the norm is monotonic with respect to P (i.e., for any $u, v \in P \cap U$ $0 \leq u \leq v \Rightarrow \|u\| \leq \|v\|$),*
(i) $0 < m(u/v) \leq 1 \leq M(u/v) < +\infty$;
(ii) $\|u - v\| \leq 2(e^{d(u,v)} - 1)$.

Lemma 2.4. *(See [26]) If the norm is monotonic with respect to P, then $(P \cap U, d)$ is a Banach space, where $U = \{x | x \in P,\ \|x\| = 1\}$.*

Definition 2.3. (See Definition 3.1 of [25]) If $A : P \to P$, we say that A is nonnegative, and if $A : P^{\circ} \to P^{\circ}$, A is considered positive.

Definition 2.4. (See Definition 3.6 of [25]) If $A : P \to P$, $a, b \in P$ and $a \leq b \Rightarrow Aa \leq Ab$ $(Aa \geq Ab)$, A is said to be increasing (decreasing).

Definition 2.5. (See Definition 3.2 of [25]) If A is positive and $A(\mu a) = \mu^{\alpha} Aa$ $(for\ all\ a \in P^{\circ}, \mu > 0)$, A is said to be positive homogeneous of degree α in P°.

Lemma 2.5. *(See [27]) Supposed that X is a Banach space, $A : X \to X$ is an operator. There is*

$$d(Au, Av) \leq \theta d(u, v), \quad u,\ v \in X,\ 0 < \theta < 1.$$

Then there exists a unique fixed point $u_0 \in X$ of T. Alternatively,

$$Au_0 = u_0.$$

Lemma 2.6. *We make the following assumptions:*
(1) the norm is monotonic with respect to P,
(2) $A : P^\circ \to P^\circ$ is an operator and positive homogeneous of degree q in P°, where $|q| \in (0, 1)$,
(3) A is increasing ($0 < q < 1$) or decreasing ($-1 < q < 0$).
Then A has a unique fixed point in P°.

Proof. If $0 < q < 1$, then T is an increasing operator and positive homogeneous of degree q. By Eq (2.1), we have

$$[m(a/b)]^q Ab \le Aa \le [M(a/b)]^q Ab.$$

It shows that

$$M(Aa/Ab) \le [M(a/b)]^q,\ m(Aa/Ab) \ge [m(a/b)]^q,$$

thus

$$\begin{aligned} d(Aa, Ab) &= ln\{M(Aa/Ab)/m(Aa/Ab)\} \\ &\le qln\{M(a/b)/m(a/b)\} \\ &= qd(a, b). \end{aligned} \tag{2.2}$$

Let $A_1 x = \dfrac{Aa}{\|Aa\|}$ $(a \in P)$. Obviously, $\|A_1 x\| = 1$, then $A_1 : P \cap U \to P \cap U$. By Lemma 2.2, for any $a,\ b \in P \cap U$,

$$\begin{aligned} d(A_1 a, A_1 b) &= d\left(\frac{Aa}{\|Aa\|}, \frac{Ab}{\|Ab\|}\right) \\ &= d(Aa, Ab) \\ &\le qd(a, b). \end{aligned}$$

Next, we will show that the $(P \cap U, d)$ is a Banach space.

Let $\{a_n\}$ be a Cauchy sequence in $(P \cap U, d)$, for all $\varepsilon > 0$, $N_1 > 0$, we have

$$d(a_n, a_m) < \varepsilon\ (m, n > N_1).$$

There exists $0 < \iota < 1$ so that

$$ln\frac{1+\iota}{1-\iota} < \varepsilon.$$

Since Lemma 2.3, we know that

$$M(a_n/a_m) \to 1,\ m(a_n/a_m) \to 1\ (n,\ m \to +\infty).$$

Then there exists $N_2 > 0$ so that

$$1 - \iota < m(a_n/a_m) \le 1,\ 1 \le M(a_n/a_m) < 1 + \iota\ (n, m > N_2).$$

Thus

$$(1 - \iota)a_m \le a_n \le (1 + \iota)a_m\ . \tag{2.3}$$

In addition, by Lemma 2.3, there is no doubt about the fact that

$$\|a_m - a_n\| \le 2(e^{d(a_m,a_n)} - 1), \tag{2.4}$$

if $m \to n$, then $\|a_m - a_n\| \to 0$. There exists $a_0 \in X$ so that $a_n \to a_0$ $(n \to \infty)$. Let $n \to \infty$ and m be not changed for (2.3). Then we have

$$(1-\iota)a_m \le a_0 \le (1+\iota)a_m \ (m > N_2),$$

hence $a_0 \in P^\circ$. Obviously, $\|a_0\| = 1$, then $a_0 \in P \cap U$. We see that

$$m(a_0/a_m) \ge 1-\iota,\ M(a_0/a_m) \le 1+\iota \ (m > N_2).$$

Then for $m \ge max\{N_1, N_2\}$, we have

$$d(a_0, a_m) = ln\Big[\frac{M(a_0/a_m)}{m(a_0/a_m)}\Big] \le ln\frac{1+\iota}{1-\iota} < \varepsilon.$$

That is to say $d(a_0, a_m) \to 0$ $(m \to \infty)$. Therefore $(P \cap U, d)$ is a Banach space.

Since Lemma 2.5, A_1 is a contraction mapping with a unique fixed point a_1 $(a_1 \in P \cap U)$, i.e., $A_1 a_1 = a_1$.

Set $a^* = \|Aa_1\|^{\frac{1}{1-q}} a_1$. Then $a^* \in P^\circ$, and

$$Aa^* = \|Aa_1\|^{\frac{q}{1-q}} Aa_1 = \|Aa_1\|^{\frac{q}{1-q}+1} A_1 a_1 = a^*.$$

Hence there is a fixed point a^* of A in P°. In addition, if there exists $b^* \in P^\circ$ so that $Ab^* = b^*$. Equation (2.2) shows that

$$d(a^*, b^*) = d(Aa^*, Ab^*) \le qd(a^*, b^*),$$

then $d(a^*, b^*) = 0$. By Lemma 2.1, $a^* = \kappa b^*, \lambda > 0$. In fact,

$$a^* = Aa^* = A(\kappa b^*) = \kappa^q Ab^* = \kappa^q b^*,$$

so $\kappa = 1$ and $a^* = b^*$. To put it another way, a^* is a unique fixed point for $0 < q < 1$.

If $-1 < q < 0$, then A is decreasing. Based on Eq (2.1), we have

$$A[M(a/b)b] \le Aa \le A[m(a/b)b],$$

that is

$$[M(a/b)]^q Ab \le Aa \le [m(a/b)]^q Ab,$$

which illustrates that

$$M(Aa/Ab) \le [m(a/b)]^q,\ m(Aa/Ab) \ge [M(a/b)]^q.$$

Thus

$$\begin{aligned} d(Aa, Ab) &\le ln\{[m(a/b)]^q/[M(a/b)]^q\} \\ &\le qln\{m(a/b)/M(a/b)\} \\ &\le (-q)d(a,b) \end{aligned} \tag{2.5}$$

$$= |q|d(a,b).$$

Let $A_2a = \dfrac{Aa}{\|Aa\|}, a \in P$. Then $A_2 : P \cap U \to P \cap U$, and for all $a,\ b \in P \cap U$,

$$d(A_2a, A_2b) \leq |q|d(a,b).$$

Banach's contraction mapping theorem indicates that there is a fixed point of A_2, i.e., there exists $a_2 \in P \cap U$ such that $A_2a_2 = a_2$. Now we assert that there is a fixed point a_* of A. Set $a_* = \|Aa_2\|^{\frac{1}{1-q}}a_2$. Then

$$\begin{aligned}
Aa_* &= A\left(\|Aa_2\|^{\frac{1}{1-q}}a_2\right)\\
&= \|Aa_2\|^{\frac{q}{1-q}}Aa_2\\
&= \|Aa_2\|^{\frac{q}{1-q}+1}A_2a_2\\
&= \|Aa_2\|^{\frac{1}{1-q}}a_2\\
&= a_*.
\end{aligned}$$

Furthermore, if there exists $b_* \in S$ so that $Tb_* = b_*$. Then

$$d(a_*, b_*) = d(Aa_*, Ab_*) \leq |q|d(a_*, b_*).$$

Consequently, $d(a_*, b_*) = 0$, i.e., $a_* = \delta b_*$ $(\delta > 0)$. Then we have

$$a_* = Aa_* = A(\delta b_*) = \delta^q Ab_* = \delta^q b_*.$$

Obviously, $\delta = 1$ and $a_* = b_*$. All in all, A has a unique fixed point a_* for $-1 < q < 0$. □

Definition 2.6. (See Definition 3.3 of [25]) Assume that $A : P^\circ \to P^\circ$, we clarify the projective diameter $\Lambda(T)$ of A by

$$\Lambda(T) = \sup\{d(Au, Av)|u, v \in P^\circ\}.$$

Definition 2.7. (See Definition 3.4 of [25]) Suppose that $A : P^\circ \to P^\circ$, we define the contraction ratio $r(A)$ of A by

$$r(A) = \inf\{\xi|d(Aa, Ab) \leq \xi d(a,b),\ a, b \in P^\circ\}.$$

Lemma 2.7. *(See Theorem 3.2 of [25]) Assume that A : $P^\circ \to P^\circ$, then*

$$r(A) = tanh\frac{1}{4}\Lambda(A).$$

Lemma 2.8. *(See Theorem 1 of [25]) Let A be a monotone decreasing operator and satisfy*

$$A(\mu x) = \mu^{-a}A(x) \quad for\ x \in P^\circ,\ \mu > 0.$$

Then the contraction ratio $r(A) \leq a$.

Lemma 2.9. *(See Theorem 4.2 of [25]) If $X = C[0,T]$ and $P = \{u(x)|u(x) \geq 0\ in\ 0 \leq x \leq T\}$, then $\{P \cap U, d\}$ is Banach space.*

3. Uniqueness of the positive periodic solution

In this section, we consider the uniqueness of the positive periodic solution to Eq (1.1) by employing the method of Hilbert's metric.

It is well known that Eq (1.1) is equal to the following equation

$$y(t) = \int_t^{t+T} D(t,s)g(s)y^q(s-\psi(s))ds, \tag{3.1}$$

where

$$D(t,s) = \frac{e^{\int_t^s a(u)du}}{e^{\int_0^T a(u)du} - 1}, \ s \in [t, t+T].$$

Let

$$X = \{g(y)|g(y) \ is \ continuous, \ and \ g(y+T) = g(y), T > 0\}.$$

Then X is a real Banach space.
Set

$$P = \{u|u \in X, u \geq 0\}.$$

Then P is a cone of X.

Define the interior of P by

$$P^\circ = \{u|u \in X, u > 0\}$$

and

$$S = P \cap U,$$

where $U = \{u|u \in X, \ ||u|| = 1\}$.

The norm in X is defined by

$$||u|| = \sup_{0\leq t\leq T} |u(t)|.$$

Theorem 3.1. *Suppose that* $0 < |q| < 1$. *Then Eq (1.1) has a unique positive periodic solution.*

Proof. We define the operator $T : P^\circ \to P^\circ$ by

$$Ty(t) = \int_t^{t+T} D(t,s)g(s)y^q(s-\psi(s))ds, \ t \in [0,T]. \tag{3.2}$$

As we all know, the solutions of Eq (1.1) are equal to the fixed points of

$$Ty(t) = y(t), \ t \in [0,T]. \tag{3.3}$$

For any $\mu > 0$,

$$\begin{aligned} T(\mu y)(t) &= \mu^q \int_t^{t+T} D(t,s)g(s)y^q(s-\psi(s))ds \\ &= \mu^q Ty(t). \end{aligned} \tag{3.4}$$

Together with Definition 2.5, we note that the operator T is positive homogeneous of degree q in P°.

If $y_1 < y_2$ and $y_1, y_2 \in P^\circ$, then $y_1^q < y_2^q$ for $0 < q < 1$. Besides, $D(t,s)$ and $g(t)$ are positive, thus

$$Ty_1 - Ty_2 < 0.$$

Then T is an increasing mapping.

For any $y_1, y_2 \in P$ and $y_1 > y_2 > 0$, we have

$$\|y_1\| = \sup_{0<t<T} |y_1(t)| > \sup_{0<t<T} |y_2(t)| = \|y_2\|.$$

Thus the norm is monotonic with respect to P by Lemma 2.3.

By Lemma 2.6, T has a unique fixed point in P°, which implies that Eq (1.1) has a unique positive periodic solution in P° for $0 < q < 1$.

If $-1 < q < 0$, $y_1^q > y_2^q$, then $Ty_1 - Ty_2 > 0$, that is T is decreasing.

Combine the above proof, T has a unique fixed point in P°, that is to say, Eq (1.1) has a unique positive periodic solution in P° for $-1 < q < 0$.

To sum up, Eq (1.1) has a unique positive periodic solution for $0 < |q| < 1$.

□

Theorem 3.2. *Consider the following equation*

$$Ty(t) = \int_t^{t+T} D(t,s)g(s)y^{-1}(s-\psi(s))ds = y(t). \tag{3.5}$$

Then Eq (3.5) has one and only one solution.

Proof. For all $\mu > 0$,

$$T(\mu y)(t) = \mu^{-1} Ty(t), \tag{3.6}$$

then T is positive homogeneous of degree -1 in P°. For any $y_1, y_2 \in P^\circ$, and $y_1 < y_2$, we have

$$\begin{aligned} &Ty_1 - Ty_2 \\ &= \int_t^{t+T} D(t,s)g(s)[y_1^{-1}(s-\psi(s)) - y_2^{-1}(s-\psi(s))]ds. \end{aligned} \tag{3.7}$$

Obviously, we find that $y_1^{-1} > y_2^{-1}$, and $D(t,s) > 0$, $g(s) > 0$. Then $Ty_1 - Ty_2 > 0$, which means T is a monotone decreasing operator. Lemma 2.8 shows us that $r(T) \le 1$. By Lemma 2.4, we note that $r(T) \le tanh\frac{1}{4}\Lambda(T)$. However, the range of $r(T)$ is $(-1, 1)$, thus $r(T) < 1$.

Next, we consider the mapping $\hat{T}y = \dfrac{Ty}{\|Ty\|}$, then for $x, y \in P^\circ$,

$$\begin{aligned} d(\hat{T}x, \hat{T}y) &= d(Tx, Ty) \\ &\le r(T)d(x,y). \end{aligned}$$

But $r(T) < 1$ and so $\hat{T}$ is a contraction mapping. By Lemma 2.9, S is a Banach space and $\hat{T}$ has a unique fixed point $y_3 \in S$, i.e., $\hat{T}y_3 = y_3$. Set $\bar{y} = \|Ty_3\|^{\frac{1}{2}} y_3$. Now we will verify $\bar{y}$ is a unique fixed point of T in S.

$$T\bar{y} = T\left(\|Ty_3\|^{\frac{1}{2}} y_3\right)$$

$$
\begin{aligned}
&= \|Ty_3\|^{-\frac{1}{2}} Ty_3 \\
&= \|Ty_3\|^{-\frac{1}{2}+1} T_3 y_3 \\
&= \|Ty_3\|^{\frac{1}{2}} y_3 \\
&= \bar{y}.
\end{aligned}
$$

There is no doubt that $\alpha = 1$ and $\bar{x} = \bar{y}$. So Eq (3.5) has a unique positive periodic solution. □

4. Existence of the positive periodic solution

In this section, we will analyze the existence of the positive periodic solution for Eq (1.4).

Let $X = C[0, T]$ denote a real Banach space with the norm

$$\|y\| = \sup_{0 \le t \le T} |y(t)|.$$

Set

$$\Gamma_r = \{y \in P | \|y\| < r\}, \quad \overline{\Gamma}_r = \{y \in P | \|y\| \le r\}, \quad \partial\overline{\Gamma}_r = \{y \in P | \|y\| = r\}, \quad for\ r > 0.$$

Define a cone P by

$$P = \{y \in X | y(t) \ge 0\}.$$

We specify the operator $A : P \to P$ by

$$Ay(t) = \int_0^T D(t, s) g(s) f(y(s - \psi(s))) ds, \tag{4.1}$$

where

$$D(t, s) = \frac{e^{\int_t^s a(u)du}}{e^{\int_0^T a(u)du} - 1}, \quad for\ s \in [0, T].$$

As is well known, Eq (4.1) has a fixed point $\hat{y} \in P$ ($\hat{y} > 0$) if and only if $\hat{y}$ is the positive periodic solution to Eq (1.4). We verify the existence of the fixed point for Eq (4.1) by employing the fixed point theorem in a cone.

Lemma 4.1. *(See [26]) Let* Γ_1, Γ_2 *be open bounded sets of* P *with* $0 \in \Gamma_1$ *and* $\overline{\Gamma}_1 \subset \Gamma_2$. *Supposed that* $T : P \cap (\overline{\Gamma}_2 \setminus \Gamma_1) \to P$ *is complete continuous, and it satisfies at least one of the following requirements:*

(H_1) If there exists $u_0 \in P \setminus \{0\}$ *so that* $a - Ta \ne du_0$ *for all* $a \in P \cap \partial\Gamma_2$ *and all* $d \ge 0$; $Ta \ne \mu x$ *for all* $a \in P \cap \partial\Gamma_1$ *and all* $\mu \ge 1$.

(H_2) If there exists $u_0 \in P \setminus \{0\}$ *so that* $a - Ta \ne du_0$ *for all* $a \in P \cap \partial\Gamma_1$ *and all* $d \ge 0$; $Ta \ne \mu x$, *for all* $a \in P \cap \partial\Gamma_2$ *and all* $\mu \ge 1$.

Then there is a fixed point of T *in* $P \cap (\Gamma_2 \setminus \overline{\Gamma}_1)$.

Theorem 4.1. *Suppose that*

(1) $g(s)$ *is continuous function, and* $\int_0^T g(s)ds > 0$.

(2) $f(y)$ *is continuous and* $f(y) \ge 0$ *for* $0 \le y < \infty$.

(3) there exists $0 < \alpha < 1$ *so that*

$$0 < \lim_{y \to 0^+} \frac{f(y)}{y^\alpha} \le +\infty,$$

(4) there exists $0 < \alpha^* < 1$ *such that*

$$0 \le \overline{\lim_{y \to +\infty}} \frac{f(y)}{y^{\alpha^*}} < +\infty.$$

As a result, (4.1) has and only has one positive periodic solution.

Proof. If there exists $\varepsilon_0 > 0$ so that

$$y - Ay \neq 0, \quad for\ y \in P \quad with\ 0 < \|y\| \le \varepsilon_0. \tag{4.2}$$

Otherwise, there is a fixed point in P will be accurate.

By (3), there exist $\eta > 0$ and $\varepsilon_1 > 0$ so that

$$f(y) \ge \eta y^{\alpha}, \quad for\ 0 \le y \le \varepsilon_1.$$

According to the definition of $D(t, s)$, we might as well assume

$$\min_{0 \le t \le T} D(t, s) = K_1$$

and

$$\max_{0 \le t \le T} D(t, s) = K_2,$$

where K_1 and K_2 are positive constants.

Define $B : C[0, T] \to C[0, T]$ by

$$By = \phi, \quad for\ y \in C[0, T],$$

where $\phi(t) \equiv 1$, $\phi \in C[0, T]$. Then it is obvious to prove that $B : P \cap \Gamma_r \to P$ is completely continuous and $\inf\limits_{P \cap \Gamma_r} \|By\| > 0$, $\phi \in P \setminus \{0\}$ with $\|\phi\| = 1$.

Choose

$$\varepsilon_2 = \min \left\{ \varepsilon_0, \varepsilon_1, \left(K_2 \eta \int_0^T g(s) ds \right)^{\frac{1}{1-\alpha}} \right\},$$

and $0 < r \le \varepsilon_2$. Now we verify that

$$y - Ay \neq qBy, \quad for\ y \in P \cap \partial\Gamma_r,\ q \ge 0. \tag{4.3}$$

Indeed, if not, there exist $y_1 \in P \cap \partial\Gamma_r$ and $q_1 \ge 0$ so that $y_1 - Ay_1 = q_1 By_1$. By (4.2), we have $q_1 > 0$. Then $y_1 = q_1 By_1 + Ay_1 \ge q_1 \phi$. Let $q^* = \sup\{q | y_1(s) \ge q\phi(s),\ s \in [0, T]\}$, then $q_1 \le q^* < +\infty$, and $y_1(s) \ge q^* \phi(s) = q^*$. So

$$q^* \le y_1(s) \le \|y_1\| = r \le \left(K_1 \eta \int_0^T g(s) ds \right)^{\frac{1}{1-\alpha}}, \quad s \in [0, T]. \tag{4.4}$$

Then if $t \in [0, T]$, we have

$$y_1(t) = \int_0^T D(t, s) g(s) f(y_1(s - \psi(s))) ds + q_1 \phi(t)$$

$$
\begin{aligned}
&\geq \int_0^T D(t,s)g(s)\eta[y_1(t)]^\alpha ds + q_1 \\
&\geq \int_0^T D(t,s)g(s)\eta(q^*)^\alpha ds + q_1 \\
&\geq K_1\eta(q^*)^\alpha \int_0^T g(s)ds + q_1 \\
&\geq q^* + q_1,
\end{aligned}
$$

which shows that $y_1(t) \geq q^* + q_1$ (for $t \in C[0,T]$). This conflicts withthe definition of q^*. So Eq (4.3) is true.

On the other hand, from (4), there exist $a > 0$ and $y_0 > 0$ such that

$$f(y) \leq ay^{\alpha^*}, \quad for\ y \geq y_0.$$

Then

$$0 \leq f(y) \leq \mu + ay^{\alpha^*}, \quad for\ 0 \leq y < +\infty, \tag{4.5}$$

where $\mu = \max\limits_{0\leq y\leq y_0} f(y)$.

Choose sufficiently large $R > 0$ so that

$$\frac{\mu}{R} + \frac{a}{R^{1-\alpha^*}} < \frac{1}{K_2\int_0^T g(s)ds}. \tag{4.6}$$

Next, we will verify that

$$y \in P \cap \partial\Gamma_R,\ \theta \geq 1 \Rightarrow Ay \neq \theta y.$$

Indeed, if not, there exist $y_0 \in P \cap \partial\Gamma_R$ and $\theta_0 \geq 1$ such that $Ay_0 = \theta_0 y_0$, then if $t \in [0,T]$, we have

$$
\begin{aligned}
\theta_0 y_0(t) &= \int_0^T D(t,s)g(s)f(y_0(s-\psi(s)))ds \\
&\leq \int_0^T D(t,s)g(s)(\mu + ay_0^{\alpha^*})ds \\
&\leq (\mu + ay_0^{\alpha^*})K_2 \int_0^T g(s)ds.
\end{aligned}
$$

Therefore, $\theta_0 R = \theta_0\|y_0\| \leq (\mu + ay_0^{\alpha^*})K_2\int_0^T g(s)ds$. That is to say,

$$\theta_0 \leq (\frac{\mu}{R} + \frac{a}{R^{1-\alpha^*}})M\int_0^T g(s)ds < 1,$$

which is a conflict to $\theta_0 \geq 1$.

In light of Lemma 4.1, there is a fixed point of A in $P \cap (\Gamma_R \setminus \overline{\Gamma}_r)$. □

5. Conclusions

This study sets out to verify the uniqueness and existence of positive periodic solutions for Eq (1.1). It contributes to our solving of other mathematical problems and practical problems. Our study shows that using the theory of Hilbert's metric to prove the uniqueness of positive periodic solution for Eq (1.1), as well as employing the fixed point theorem in a cone to prove the existence of positive periodic solution for Eq (1.4).

We employ the theory of Hilbert's metric to prove the uniqueness of the positive periodic solution for Eq (1.4) when $f(y(t-\psi(t))) = y^q(t-\psi(t))$. When $0 < q < 1$, $q = -1$ and $-1 < q < 0$, the uniqueness results are verified based on the theory of Hilbert's metric. The major limitation of this study is that it is difficult to directly verify the uniqueness of the positive periodic solution for Eq (1.4) by applying the theory of Hilbert's metric.

6. Two examples

This section illustrates our conclusions in Section 3 with two examples.

Example 6.1. Suppose that X, P, P°, and S have the same meanings with Section 3, we think about the equation

$$v'(t) = -\frac{1}{\pi}v(t) + sin^2t \cdot v^{\frac{1}{3}}(t - \pi/1000),\ t \in [0, 1]. \tag{6.1}$$

Then Eq (6.1) has a unique positive periodic solution.

Proof. In this example, $a(t) = \frac{1}{\pi}$, $h(t) = sin^2t$, $\psi(t) = \pi/1000$, and $q = \frac{1}{3}$.

As we all know, Eq (6.1) has a unique solution if and only if the equation

$$Av(t) = v(t) = \int_0^1 D(t, s)sin^2t \cdot v^{\frac{1}{3}}(s - \pi/1000)ds \tag{6.2}$$

has a fixed point, where

$$D(t, s) = \frac{e^{\frac{s-t}{\pi}}}{e^T - 1},\ s \in [0, 1].$$

It is easy to see that A is an increasing operator which is positive homogeneous of degree $\frac{1}{3}$ in P°. Moreover, the norm is monotonic with respect to P.

In a word, A has a unique fixed point in P°, which shows Eq (6.1) has a unique positive periodic solution.

□

Example 6.2. We examine the following equation

$$u'(t) = (cos^2t)u(t) + u^{-\frac{1}{2}}(t - sint),\ t \in [0, T]. \tag{6.3}$$

Then Eq (6.3) has one and only one positive periodic solution.

Proof. In this equation, $a(t) = cos^2 t$, $h(t) = 1$, $\psi(t) = -sint$, and $q = -\frac{1}{2}$.

We think about

$$Au(t) = \int_0^T D(t,s)u^{-\frac{1}{2}}(s - sins)ds = u(t), \tag{6.4}$$

where $D(t,s) = \frac{e^{\int_s^t cos^2 udu}}{e^{\int_0^T cos^2 udu} - 1}$, $s \in [0, T]$.

Next, we will verify that there is a fixed point of Eq (6.4).

Clearly, A is a decreasing operator and positive homogeneous of degree $-\frac{1}{2}$ in P°. What's more, the norm is monotonic with respect to P°.

As a result, Eq (6.3) has a unique solution in P°. □

Conflict of interest

The authors declare no conflict of interest.

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