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Research article

Fixed point results for generalized contractions in controlled metric spaces with applications

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Abstract: The purpose of this article is to establish some common fixed point results for generalized contractions including some precise control functions of two variables in the setting of controlled metric spaces. As consequences of our leading results, we derive common fixed point and fixed point results for contractions with control functions of one variable and constants. We also discuss controlled metric spaces endowed with a graph and obtain some common fixed point results in this newly introduced space. As an application of our leading result, we examine the solution of a Fredholm type integral equation.

Keywords: fixed point; controlled metric spaces; control function; graph; Fredholm type integral equation
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1. Introduction

The theory of fixed points has worldwide applications in distinct fields of science and Engineering [1–3]. M. Frechet is a principal researcher in this theory who defined the notion of metric space in 1906. Metric space methods have been employed for decades in numerous applications, for example in internet search engines, protein classification and image classification. The best applied fixed point result is the Banach contraction principle that has been extended by either changing the contractive condition or by functioning on a further generalized metric spaces [4–8]. In current years, some novel types of generalized metric spaces were presented and numerous spaces established as hybrids of the foregoing varieties were examined such as rectangular metric space, b-metric space,

bv_s-metric space, extended b-metric space. The key in creating these spaces is to generalize or extend the third axiom of metric space that is, its triangle property. In 1993, Czerwik [9] introduced the notion of b-metric space which extends the metric space by enhancing the triangle equality metric axiom by placing a constant $s \geq 1$ multiplied to the right-hand side, is one of the great applied extensions for metric spaces. Later on, Kamran et al. [10] gave a new kind of extended b-metric spaces by putting a function $\sigma(\xi, \varrho)$ on the place of constant s and this function depends on the parameters used on left-hand side of the triangle inequality. Recently, Mlaiki et al. [11] replaced the constant s by a function $\sigma(\xi, \varrho)$ which act separately on each term in the right-hand side of the triangle inequality and defined controlled metric space. They established Banach contraction principle in the background of this newly introduced space. Subsequently, Lateef [18] obtained Fisher type fixed point result and generalized the leading result of Mlaiki et al. [11]. For more characteristics, we assign the researchers to [12–33]. In this article, we utilize the notion of controlled metric space to establish common fixed point theorems for rational contractive mappings dealing with some precise control functions of two variables in the background of controlled metric space. As an outcome of our pioneering theorems, we derive common fixed point and fixed point theorems for contractive mappings including control functions of one variable and constants. In this way, we generalize the main result of Lateef [18] as well as the leading theorem of Mlaiki et al. [11]. We also discuss controlled metric spaces equipped with a graph and obtain some common fixed point results in this newly introduced space.

2. Preliminaries

In 1993, Czerwik [9] introduced the concept of b-metric space (b-MS) in this way.

Definition 1. ([9])Let $\mathfrak{U}$ be a non empty set and $s \geq 1$. A function $\ell : \mathfrak{U} \times \mathfrak{U} \to [0, \infty)$ is said to be b-metric if following conditions hold:

(ℓ1) $\ell(\xi, \varrho) \geq 0$ and $\ell(\xi, \varrho) = 0$ if and only if $\xi = \varrho$;

(ℓ2) $\ell(\xi, \varrho) = \ell(\varrho, \xi)$;

(ℓ3) $\ell(\xi, \omega) \leq s[\ell(\xi, \varrho) + \ell(\varrho, \omega)]$;

for all $\xi, \varrho, \omega \in \mathfrak{U}$. The pair $(\mathfrak{U}, \ell)$ is said to be a b-metric space (b-MS).

In 2017, Kamran et al. [10] gave the notion of extended b-metric space (EbMS) as follows:

Definition 2. ([10]) Let $\mathfrak{U}$ be a non empty set and $\sigma : \mathfrak{U} \times \mathfrak{U} \to [1, \infty)$. A function $\ell : \mathfrak{U} \times \mathfrak{U} \to [0, \infty)$ is said to be extended b-metric if following conditions hold:

(i) $\ell(\xi, \varrho) \geq 0$ and $\ell(\xi, \varrho) = 0$ if and only if $\xi = \varrho$;

(ii) $\ell(\xi, \varrho) = \ell(\varrho, \xi)$;

(iii) $\ell(\xi, \varrho) \leq \sigma(\xi, \varrho)[\ell(\xi, \omega) + \ell(\varrho, \omega)]$;

for all $\xi, \varrho, \omega \in \mathfrak{U}$, then $(\mathfrak{U}, \ell)$ is called an extended b-metric space (EbMS).

In 2018, a contemporary extended b-metric space was initiated by Mlaiki et al. [11] which is known as controlled metric space as follows:

Definition 3. ([11]) Let $\mathfrak{U}$ be a non empty set and $\sigma : \mathfrak{U} \times \mathfrak{U} \to [1, \infty)$. A function $\ell : \mathfrak{U} \times \mathfrak{U} \to [0, \infty)$ is said to be controlled metric if following conditions hold:

(i) $\ell(\xi, \varrho) = 0$ if and only if $\xi = \varrho$;

(ii) $\ell(\xi, \varrho) = \ell(\varrho, \xi)$;

(iii) $\ell(\xi, \varrho) \leq \sigma(\xi, \omega)\ell(\xi, \omega) + \sigma(\varrho, \omega)\ell(\varrho, \omega)$;

for all $\xi, \varrho, \omega \in \mathfrak{U}$, then $(\mathfrak{U}, \ell, \sigma)$ is said to be a controlled metric space (CMS).

Example 1. Let $\mathfrak{U} = \{0, 1, 2\}$. Define the mapping $\ell : \mathfrak{U} \times \mathfrak{U} \to [0, \infty)$ by

$$\ell(0, 0) = \ell(1, 1) = \ell(2, 2) = 0$$

and

$$\ell(0, 1) = \ell(1, 0) = 1,$$

$$\ell(0, 2) = \ell(2, 0) = \frac{1}{2},$$

$$\ell(1, 2) = \ell(2, 1) = \frac{2}{5}.$$

Define the symmetric control function $\sigma : \mathfrak{U} \times \mathfrak{U} \to [1, \infty)$ by

$$\sigma(0, 0) = \sigma(1, 1) = \sigma(2, 2) = \sigma(0, 2) = 1,$$

$$\sigma(1, 2) = \frac{5}{4}, \ \sigma(0, 1) = \frac{11}{10}.$$

Then $(\mathfrak{U}, \ell, \sigma)$ is CMS.

Theorem 1. *([11]) Let $(\mathfrak{U}, \sigma, \ell)$ be a complete CMS and $\wp : \mathfrak{U} \to \mathfrak{U}$ such that*

$$\ell(\wp\xi, \wp\varrho) \leq \tau(\ell(\xi, \varrho))$$

for all $\xi, \varrho \in \mathfrak{U}$, where $\tau \in [0, 1)$. For $\xi_0 \in \mathfrak{U}$, take $\xi_J = \wp^J \xi_0$. Assume that

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\sigma(\xi_{i+1}, \xi_{i+2})\sigma(\xi_{i+1}, \xi_m)}{\sigma(\xi_i, \xi_{i+1})} < \frac{1}{\tau}.$$

In addition, assume that $\lim_{J \to +\infty} \sigma(\xi_J, \xi)$ and $\lim_{J \to +\infty} \sigma(\xi, \xi_J)$ exist and are finite, then there exists a unique point $\xi^* \in \mathfrak{U}$ such that $\wp\xi^* = \xi^*$.

Lateef [12] obtained the following result in a complete CMS as follows:

Theorem 2. *([12]) Let $(\mathfrak{U}, \sigma, \ell)$ be a complete CMS and $\wp : \mathfrak{U} \to \mathfrak{U}$ be such that*

$$\ell(\wp\xi, \wp\varrho) \leq \tau(\ell(\xi, \wp\xi) + \ell(\varrho, \wp\varrho))$$

for all $\xi, \varrho \in \mathfrak{U}$, where $\tau \in (0, \frac{1}{2})$. For $\xi_0 \in \mathfrak{U}$, take $\xi_J = \wp^J \xi_0$. Suppose that

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\sigma(\xi_{i+1},\xi_{i+2})\sigma(\xi_{i+1},\xi_m)}{\sigma(\xi_i,\xi_{i+1})}<\frac{1}{\tau}.$$

In addition, assume that $\lim_{\jmath\to+\infty}\sigma(\xi_\jmath,\xi)$ and $\lim_{\jmath\to+\infty}\sigma(\xi,\xi_\jmath)$ exist and are finite, then there exists a unique point $\xi^*\in\mathfrak{U}$ such that $\wp\xi^*=\xi^*$.

Later on, Lateef [18] established Fisher type fixed point result in this way.

Theorem 3. *Let* $(\mathfrak{U},\sigma,\ell)$ *be a complete CMS,* $\wp:\mathfrak{U}\to\mathfrak{U}$ *and there exist* $\tau_1,\tau_2\in[0,1)$ *with* $\tau=\tau_1+\tau_2<1$ *such that*

$$\ell(\wp\xi,\wp\varrho)\leq\tau_1\ell(\xi,\varrho)+\tau_2\frac{\ell(\xi,\wp_1\xi)\ell(\varrho,\wp_2\varrho)}{1+\ell(\xi,\varrho)}$$

for all $\xi,\varrho\in\mathfrak{U}$. *For* $\xi_0\in\mathfrak{U}$, *take* $\xi_\jmath=\wp^\jmath\xi_0$. *Suppose that*

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\sigma(\xi_{i+1},\xi_{i+2})\sigma(\xi_{i+1},\xi_m)}{\sigma(\xi_i,\xi_{i+1})}<\frac{1}{\tau}.$$

Moreover, suppose that $\lim_{\jmath\to+\infty}\sigma(\xi_\jmath,\xi)$ and $\lim_{\jmath\to+\infty}\sigma(\xi,\xi_\jmath)$ exist and are finite, then there exists a unique point $\xi^*\in\mathfrak{U}$ such that $\wp\xi^*=\xi^*$.

3. Main results

We give our leading result in this way.

Theorem 4. *Let* $(\mathfrak{U},\sigma,\ell)$ *be a complete CMS,* $\wp_1,\wp_2:\mathfrak{U}\to\mathfrak{U}$ *and there exist the mappings* $\tau_1,\tau_2:\mathfrak{U}\times\mathfrak{U}\to[0,1)$ *such that*

(i) $\tau_1(\wp_2\wp_1\xi,\varrho)\leq\tau_1(\xi,\varrho)$ and $\tau_1(\xi,\wp_1\wp_2\varrho)\leq\tau_1(\xi,\varrho)$;

(ii) $\tau_2(\wp_2\wp_1\xi,\varrho)\leq\tau_2(\xi,\varrho)$ and $\tau_2(\xi,\wp_1\wp_2\varrho)\leq\tau_2(\xi,\varrho)$;

(iii) $\tau_1(\xi,\varrho)+\tau_2(\xi,\varrho)<1$;

(iv)

$$\ell(\wp_1\xi,\wp_2\varrho)\leq\tau_1(\xi,\varrho)\ell(\xi,\varrho)+\tau_2(\xi,\varrho)\frac{\ell(\xi,\wp_1\xi)\ell(\varrho,\wp_2\varrho)}{1+\ell(\xi,\varrho)},\tag{3.1}$$

for all $\xi,\varrho\in\mathfrak{U}$. For $\xi_0\in\mathfrak{U}$, a sequence $\{\xi_\jmath\}_{\jmath\geq 0}$ is defined as $\xi_{2\jmath+1}=\wp_1\xi_{2\jmath}$ and $\xi_{2\jmath+2}=\wp_2\xi_{2\jmath+1}$ for each $\jmath\geq 0$. Assume that

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\sigma(\xi_{i+1},\xi_{i+2})\sigma(\xi_{i+1},\xi_m)}{\sigma(\xi_i,\xi_{i+1})}<\frac{1}{\tau},\tag{3.2}$$

where $\frac{\tau_1(\xi_0,\xi_1)}{1-\tau_2(\xi_0,\xi_1)}=\tau$. In addition, suppose that $\lim_{\jmath\to+\infty}\sigma(\xi_\jmath,\xi)$ and $\lim_{\jmath\to+\infty}\sigma(\xi,\xi_\jmath)$ exist and are finite, then there exists a unique point $\xi^*\in\mathfrak{U}$ such that $\wp_1\xi^*\cap\wp_2\xi^*=\xi^*$.

Proof. Let $\xi_0\in\mathfrak{U}$. We construct $\{\xi_\jmath\}$ in $\mathfrak{U}$ by $\xi_{2\jmath+1}=\wp_1\xi_{2\jmath}$ and $\xi_{2\jmath+2}=\wp_2\xi_{2\jmath+1}$ for each $\jmath\geq 0$. From assumption and (3.1) we get

$$\begin{aligned}\ell(\xi_{2\jmath+1},\xi_{2\jmath+2}) &= \ell(\wp_1\xi_{2\jmath},\wp_2\xi_{2\jmath+1})\\ &\leq \tau_1(\xi_{2\jmath},\xi_{2\jmath+1})\ell(\xi_{2\jmath},\xi_{2\jmath+1})\end{aligned}$$

$$
\begin{aligned}
& +\tau_2(\xi_{2J},\xi_{2J+1})\frac{\ell(\xi_{2J},\wp_1\xi_{2J})\ell(\xi_{2J+1},\wp_2\xi_{2J+1})}{1+\ell(\xi_{2J},\xi_{2J+1})} \\
= \;& \tau_1(\wp_2\wp_1\xi_{2J-2},\xi_{2J+1})\ell(\xi_{2J},\xi_{2J+1}) \\
& +\tau_2(\wp_2\wp_1\xi_{2J-2},\xi_{2J+1})\frac{\ell(\xi_{2J},\xi_{2J+1})\ell(\xi_{2J+1},\xi_{2J+2})}{1+\ell(\xi_{2J},\xi_{2J+1})} \\
\le \;& \tau_1(\xi_{2J-2},\xi_{2J+1})\ell(\xi_{2J},\xi_{2J+1})+\tau_2(\xi_{2J-2},\xi_{2J+1})\ell(\xi_{2J+1},\xi_{2J+2}) \\
= \;& \tau_1(\wp_2\wp_1\xi_{2J-4},\xi_{2J+1})\ell(\xi_{2J},\xi_{2J+1})+\tau_2(\wp_2\wp_1\xi_{2J-4},\xi_{2J+1})\ell(\xi_{2J+1},\xi_{2J+2}) \\
\le \;& \tau_1(\xi_{2J-4},\xi_{2J+1})\ell(\xi_{2J},\xi_{2J+1})+\tau_2(\xi_{2J-4},\xi_{2J+1})\ell(\xi_{2J+1},\xi_{2J+2}) \\
\le \;& \\
& . \\
& . \\
& . \\
\le \;& \tau_1(\xi_0,\xi_{2J+1})\ell(\xi_{2J},\xi_{2J+1})+\tau_2(\xi_0,\xi_{2J+1})\ell(\xi_{2J+1},\xi_{2J+2}) \\
= \;& \tau_1(\xi_0,\wp_1\wp_2\xi_{2J-1})\ell(\xi_{2J},\xi_{2J+1})+\tau_2(\xi_0,\wp_1\wp_2\xi_{2J-1})\ell(\xi_{2J+1},\xi_{2J+2}) \\
\le \;& \tau_1(\xi_0,\xi_{2J-1})\ell(\xi_{2J},\xi_{2J+1})+\tau_2(\xi_0,\xi_{2J-1})\ell(\xi_{2J+1},\xi_{2J+2}) \\
& . \\
& . \\
& . \\
\le \;& \tau_1(\xi_0,\xi_1)\ell(\xi_{2J},\xi_{2J+1})+\tau_2(\xi_0,\xi_1)\ell(\xi_{2J+1},\xi_{2J+2}).
\end{aligned}
$$

This implies that

$$\ell(\xi_{2J+1},\xi_{2J+2}) \le \left(\frac{\tau_1(\xi_0,\xi_1)}{1-\tau_2(\xi_0,\xi_1)}\right)\ell(\xi_{2J},\xi_{2J+1}).$$

Similarly,

$$
\begin{aligned}
\ell(\xi_{2J+2},\xi_{2J+3}) = \;& \ell(\wp_2\xi_{2J+1},\wp_1\xi_{2J+2}) = \ell(\wp_1\xi_{2J+2},\wp_2\xi_{2J+1}) \\
\le \;& \tau_1(\xi_{2J+2},\xi_{2J+1})\ell(\xi_{2J+2},\xi_{2J+1}) \\
& +\tau_2(\xi_{2J+2},\xi_{2J+1})\frac{\ell(\xi_{2J+2},\wp_1\xi_{2J+2})\ell(\xi_{2J+1},\wp_1\xi_{2J+1})}{1+\ell(\xi_{2J+2},\xi_{2J+1})} \\
= \;& \tau_1(\xi_{2J+2},\wp_1\wp_2\xi_{2J-1})(\xi_{2J+2},\xi_{2J+1}) \\
& +\tau_2(\xi_{2J+2},\wp_1\wp_2\xi_{2J-1})\frac{\ell(\xi_{2J+2},\xi_{2J+3})\ell(\xi_{2J+1},\xi_{2J+2})}{1+\ell(\xi_{2J+2},\xi_{2J+1})} \\
\le \;& \tau_1(\xi_{2J+2},\xi_{2J-1})(\xi_{2J+2},\xi_{2J+1})+\tau_2(\xi_{2J+2},\xi_{2J-1})\ell(\xi_{2J+2},\xi_{2J+3}) \\
= \;& \tau_1(\xi_{2J+2},\wp_1\wp_2\xi_{2J-3})(\xi_{2J+2},\xi_{2J+1})+\tau_2(\xi_{2J+2},\wp_1\wp_2\xi_{2J-3})\ell(\xi_{2J+2},\xi_{2J+3}) \\
\le \;& \tau_1(\xi_{2J+2},\xi_{2J-3})(\xi_{2J+2},\xi_{2J+1})+\tau_2(\xi_{2J+2},\xi_{2J-3})\ell(\xi_{2J+2},\xi_{2J+3}) \\
& . \\
& . \\
& . \\
\le \;& \tau_1(\xi_{2J+2},\xi_0)(\xi_{2J+2},\xi_{2J+1})+\tau_2(\xi_{2J+2},\xi_0)\ell(\xi_{2J+2},\xi_{2J+3})
\end{aligned}
$$

$$
\begin{aligned}
&= \tau_1(\wp_2\wp_1\xi_{2\jmath},\xi_0)(\xi_{2\jmath+2},\xi_{2\jmath+1}) + \tau_2(\wp_2\wp_1\xi_{2\jmath},\xi_0)\ell(\xi_{2\jmath+2},\xi_{2\jmath+3}) \\
&\leq \tau_1(\xi_{2\jmath},\xi_0)(\xi_{2\jmath+2},\xi_{2\jmath+1}) + \tau_2(\xi_{2\jmath},\xi_0)\ell(\xi_{2\jmath+2},\xi_{2\jmath+3}) \\
&\quad . \\
&\quad . \\
&\quad . \\
&\leq \tau_1(\xi_1,\xi_0)(\xi_{2\jmath+2},\xi_{2\jmath+1}) + \tau_2(\xi_1,\xi_0)\ell(\xi_{2\jmath+2},\xi_{2\jmath+3}) \\
&= \tau_1(\xi_0,\xi_1)(\xi_{2\jmath+2},\xi_{2\jmath+1}) + \tau_2(\xi_0,\xi_1)\ell(\xi_{2\jmath+2},\xi_{2\jmath+3}).
\end{aligned}
$$

This implies that

$$
\ell(\xi_{2\jmath+2},\xi_{2\jmath+3}) \leq \left(\frac{\tau_1(\xi_0,\xi_1)}{1-\tau_2(\xi_0,\xi_1)}\right)\ell(\xi_{2\jmath+1},\xi_{2\jmath+2}) = \tau\ell(\xi_{2\jmath+1},\xi_{2\jmath+2}).
$$

By pursuing in this direction, we get

$$
\begin{aligned}
\ell(\xi_\jmath,\xi_{\jmath+1}) &\leq \tau\ell(\xi_{\jmath-1},\xi_\jmath) \\
&\leq \tau^2\ell(\xi_{\jmath-2},\xi_{\jmath-1}) \\
&\leq \ldots \leq \tau^\jmath\ell(\xi_0,\xi_1).
\end{aligned}
$$

Thus

$$
\ell(\xi_\jmath,\xi_{\jmath+1}) \leq \tau^\jmath\ell(\xi_0,\xi_1). \tag{3.3}
$$

Now for all $\jmath, m \in \mathbb{N}$ with $\jmath < m$, we get

$$
\begin{aligned}
\ell(\xi_\jmath,\xi_m) &\leq \sigma(\xi_\jmath,\xi_{\jmath+1})\ell(\xi_\jmath,\xi_{\jmath+1}) + \sigma(\xi_{\jmath+1},\xi_m)\ell(\xi_{\jmath+1},\xi_m) \\
&\leq \sigma(\xi_\jmath,\xi_{\jmath+1})\ell(\xi_\jmath,\xi_{\jmath+1}) + \sigma(\xi_{\jmath+1},\xi_m)\sigma(\xi_{\jmath+1},\xi_{\jmath+2})\ell(\xi_{\jmath+1},\xi_{\jmath+2}) \\
&\quad +\sigma(\xi_{\jmath+1},\xi_m)\sigma(\xi_{\jmath+2},\xi_m)\ell(\xi_{\jmath+2},\xi_m) \\
&\leq \sigma(\xi_\jmath,\xi_{\jmath+1})\ell(\xi_\jmath,\xi_{\jmath+1}) + \sigma(\xi_{\jmath+1},\xi_m)\sigma(\xi_{\jmath+1},\xi_{\jmath+2})\ell(\xi_{\jmath+1},\xi_{\jmath+2}) \\
&\quad +\sigma(\xi_{\jmath+1},\xi_m)\sigma(\xi_{\jmath+2},\xi_m)\sigma(\xi_{\jmath+2},\xi_{\jmath+3})\ell(\xi_{\jmath+2},\xi_{\jmath+3}) \\
&\quad +\sigma(\xi_{\jmath+1},\xi_m)\sigma(\xi_{\jmath+2},\xi_m)\sigma(\xi_{\jmath+3},\xi_m)\ell(\xi_{\jmath+3},\xi_m) \\
&\leq \cdots \\
&\leq \sigma(\xi_\jmath,\xi_{\jmath+1})\ell(\xi_\jmath,\xi_{\jmath+1}) + \sum_{i=\jmath+1}^{m-2}\left(\prod_{k=\jmath+1}^{i}\sigma(\xi_k,\xi_m)\right)\sigma(\xi_i,\xi_{i+1})\ell(\xi_i,\xi_{i+1}) \\
&\quad + \prod_{i=\jmath+1}^{m-1}\sigma(\xi_i,\xi_m)\ell(\xi_{m-1},\xi_m)
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
\ell(\xi_\jmath,\xi_m) &\leq \sigma(\xi_\jmath,\xi_{\jmath+1})\ell(\xi_\jmath,\xi_{\jmath+1}) + \sum_{i=\jmath+1}^{m-2}\left(\prod_{k=\jmath+1}^{i}\sigma(\xi_k,\xi_m)\right)\sigma(\xi_i,\xi_{i+1})\ell(\xi_i,\xi_{i+1}) \\
&\quad + \left(\prod_{i=\jmath+1}^{m-1}\sigma(\xi_i,\xi_m)\right)\sigma(\xi_{m-1},\xi_m)\ell(\xi_{m-1},\xi_m)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sigma(\xi_{\jmath},\xi_{\jmath+1})\tau^{\jmath}\ell(\xi_0,\xi_1) + \sum_{i=\jmath+1}^{m-2}\left(\prod_{k=\jmath+1}^{i}\sigma(\xi_k,\xi_m)\right)\sigma(\xi_i,\xi_{i+1})\tau^{i}\ell(\xi_0,\xi_1) \\
&\quad + \left(\prod_{i=\jmath+1}^{m-1}\sigma(\xi_i,\xi_m)\right)\sigma(\xi_{m-1},\xi_m)\tau^{m-1}\ell(\xi_0,\xi_1) \\
&= \sigma(\xi_{\jmath},\xi_{\jmath+1})\tau^{\jmath}\ell(\xi_0,\xi_1) + \sum_{i=\jmath+1}^{m-1}\left(\prod_{k=\jmath+1}^{i}\sigma(\xi_k,\xi_m)\right)\sigma(\xi_i,\xi_{i+1})\tau^{i}\ell(\xi_0,\xi_1).
\end{aligned}
$$

Thus

$$
\ell(\xi_{\jmath},\xi_m) \leq \sigma(\xi_{\jmath},\xi_{\jmath+1})\tau^{\jmath}\ell(\xi_0,\xi_1) + \sum_{i=\jmath+1}^{m-1}\left(\prod_{k=\jmath+1}^{i}\sigma(\xi_k,\xi_m)\right)\sigma(\xi_i,\xi_{i+1})\tau^{i}\ell(\xi_0,\xi_1). \tag{3.4}
$$

Let

$$
\Psi_l = \sum_{i=0}^{l}\left(\prod_{k=0}^{i}\sigma(\xi_k,\xi_m)\right)\sigma(\xi_i,\xi_{i+1})\tau^{i}\ell(\xi_0,\xi_1).
$$

From (3.4), we get

$$
\ell(\xi_{\jmath},\xi_m) \leq \ell(\xi_0,\xi_1)[\tau^{\jmath}\sigma(\xi_{\jmath},\xi_{\jmath+1}) + (\Psi_{m-1} - \Psi_{\jmath})]. \tag{3.5}
$$

Since $\sigma(\xi,\varrho) \geq 1$, and by employing the ratio test, $\lim_{\jmath\to+\infty}\Psi_{\jmath}$ exists. Thus $\{\Psi_{\jmath}\}$ is Cauchy sequence. Lastly, letting $\jmath, m \to +\infty$ in (3.5), we get that

$$
\lim_{\jmath,m\to+\infty}\ell(\xi_{\jmath},\xi_m) = 0. \tag{3.6}
$$

Hence, $\{\xi_{\jmath}\}$ is a Cauchy sequence in $(\mathfrak{U},\ell,\sigma)$. As $(\mathfrak{U},\ell,\sigma)$ is complete, so there exists $\xi^* \in \mathfrak{U}$ such that

$$
\lim_{\jmath\to+\infty}\ell(\xi_{\jmath},\xi^*) = 0, \tag{3.7}
$$

that is $\xi_{\jmath} \to \xi^*$ as $\jmath \to +\infty$. Now, by (3.1) and assumption (iii), we have

$$
\begin{aligned}
\ell(\xi^*,\wp_1\xi^*) &\leq \sigma(\xi^*,\xi_{2\jmath+2})\ell(\xi^*,\xi_{2\jmath+2}) + \sigma(\xi_{2\jmath+2},\wp_1\xi^*)\ell(\xi_{2\jmath+2},\wp_1\xi^*) \\
&= \sigma(\xi^*,\xi_{2\jmath+2})\ell(\xi^*,\xi_{2\jmath+2}) + \sigma(\xi_{2\jmath+2},\wp_1\xi^*)\ell(\wp_2\xi_{2\jmath+1},\wp_1\xi^*) \\
&= \sigma(\xi^*,\xi_{2\jmath+2})\ell(\xi^*,\xi_{2\jmath+2}) + \sigma(\xi_{2\jmath+2},\wp_1\xi^*)\ell(\wp_1\xi^*,\wp_2\xi_{2\jmath+1}) \\
&= \sigma(\xi^*,\xi_{2\jmath+2})\ell(\xi^*,\xi_{2\jmath+2}) + \sigma(\xi_{2\jmath+2},\wp_1\xi^*)\left[\begin{array}{c} \tau_1(\xi^*,\xi_{2\jmath+1})\ell(\xi^*,\xi_{2\jmath+1}) \\ +\tau_2(\xi^*,\xi_{2\jmath+1})\frac{\ell(\xi^*,\wp_1\xi^*)\ell(\xi_{2\jmath+1},\xi_{2\jmath+2})}{1+\ell(\xi^*,\xi_{2\jmath+1})} \end{array}\right] \\
&= \sigma(\xi^*,\xi_{2\jmath+2})\ell(\xi^*,\xi_{2\jmath+2}) + \sigma(\xi_{2\jmath+2},\wp_1\xi^*)\left[\begin{array}{c} \tau_1(\xi^*,\xi_{2\jmath+1})\ell(\xi^*,\xi_{2\jmath+1}) \\ +\tau_2(\xi^*,\xi_{2\jmath+1})\frac{\ell(\xi^*,\wp_1\xi^*)\ell(\xi_{2\jmath+1},\xi_{2\jmath+2})}{1+\ell(\xi^*,\xi_{2\jmath+1})} \end{array}\right].
\end{aligned}
$$

Taking $\jmath \to +\infty$ and utilizing (3.7), we get a contradiction to the fact that $\ell(\xi^*,\wp_1\xi^*) > 0$. Thus $\ell(\xi^*,\wp_1\xi^*) = 0$. It implies that $\xi^* = \wp_1\xi^*$. Likewise, we can prove that $\xi^* = \wp_2\xi^*$.Thus, ξ^* is a common fixed point of $\wp_1$ and $\wp_2$. In due course, we prove that ξ^* is unique. Suppose that there exists another point $\xi^{/} \in \mathfrak{U}$ such that $\xi^{/} = \wp_1\xi^{/} = \wp_2\xi^{/}$. It follows from

$$\begin{aligned}\ell(\xi^*,\xi^/) &= \ell(\wp_1\xi^*,\wp_2\xi^/) \le \tau_1(\xi^*,\xi^/)\ell(\xi^*,\xi^/) + \tau_2(\xi^*,\xi^/)\frac{\ell(\xi^*,\wp_1\xi^*)\ell(\varrho,\wp_2\xi^/)}{1+\ell(\xi^*,\xi^/)}\\ &= \tau_1(\xi^*,\xi^/)\ell(\xi^*,\xi^/).\end{aligned}$$

Since $\tau_1(\xi^*,\xi^/) \in [0,1)$, so we have $\ell(\xi^*,\xi^/) = 0$. Thus, we get $\xi^* = \xi^/$, which shows that ξ^* is unique. □

Example 2. Let $\mathfrak{U} = [0,1]$. Now we define $\ell : \mathfrak{U}\times\mathfrak{U} \to [0,\infty)$ by

$$\ell(\xi,\varrho) = (\xi+\varrho)^2\,,$$

where $\sigma(\xi,\varrho) = 2+\xi+\varrho$, for all $\xi,\varrho \in \mathfrak{U}$. Now we define $\wp_1,\wp_2 : \mathfrak{U} \to \mathfrak{U}$ by

$$\wp_1\xi = \frac{\xi}{3} \text{ and } \wp_2\xi = \frac{\xi}{4},$$

for $\xi \in \Re$. Choose $\tau_1,\tau_2 : \mathfrak{U}\times\mathfrak{U} \to [0,1)$ by

$$\tau_1(\xi,\varrho) = \frac{16+\xi+\varrho}{144} \le \text{ and } \tau_2(\xi,\varrho) = \frac{15+\xi+\varrho}{144}.$$

Then evidently,

$$\tau_1(\xi,\varrho) + \tau_2(\xi,\varrho) < 1.$$

Now

$$\tau_1(\wp_2\wp_1\xi,\varrho) = \frac{1}{9} + \frac{\xi}{1726} + \frac{\varrho}{144} \le \frac{16+\xi+\varrho}{144} = \tau_1(\xi,\varrho),$$

and

$$\tau_1(\xi,\wp_1\wp_2\varrho) = \frac{1}{9} + \frac{\xi}{144} + \frac{\varrho}{1726} \le \frac{16+\xi+\varrho}{144} = \tau_1(\xi,\varrho),$$

also,

$$\tau_2(\wp_2\wp_1\xi,\varrho) = \frac{5}{46} + \frac{\xi}{1726} + \frac{\varrho}{144} \le \frac{15+\xi+\varrho}{144} = \tau_1(\xi,\varrho),$$

and

$$\tau_1(\xi,\wp_1\wp_2\varrho) = \frac{5}{46} + \frac{\xi}{144} + \frac{\varrho}{1726} \le \frac{15+\xi+\varrho}{144} = \tau_1(\xi,\wp_1\wp_2\varrho).$$

Take $\xi_0 = 0$, so (3.2) is satisfied. Let $\xi,\varrho \in \mathfrak{U}$. Then

$$\begin{aligned}\ell(\wp_1\xi,\wp_2\varrho) &= \frac{(4\xi+3\varrho)^2}{144} \le \frac{(4\xi+4\varrho)^2}{144}\\ &\le \frac{16+\xi+\varrho}{144}(\xi+\varrho)^2 + \frac{15+\xi+\varrho}{144}\frac{\left(\frac{5\xi}{4}\right)^2\left(\frac{6\varrho}{4}\right)^2}{1+(\xi+\varrho)^2}\\ &= \tau_1(\xi,\varrho)\ell(\xi,\varsigma) + \tau_2(\xi,\varrho)\frac{\ell(\xi,\wp_1\xi)\ell(\varrho,\wp_2\varrho)}{1+\ell(\xi,\varrho)}.\end{aligned}$$

Thus all the assumption of Theorem 4 are satisfied and there exists $\xi^* = 0 \in \mathfrak{U}$ such that $\wp_1\xi^* \cap \wp_2\xi^* = \xi^*$.

By setting $\wp_1 = \wp_2 = \wp$ in Theorem 4, we derive the following result.

Corollary 1. *Let* $(\mathfrak{U}, \sigma, \ell)$ *be a complete CMS,* $\wp : \mathfrak{U} \to \mathfrak{U}$ *and there exist the mappings* $\tau_1, \tau_2 : \mathfrak{U} \times \mathfrak{U} \to [0, 1)$ *such that*

(i) $\tau_1(\wp\xi, \varrho) \leq \tau_1(\xi, \varrho)$ and $\tau_1(\xi, \wp\varrho) \leq \tau_1(\xi, \varrho)$;

(ii) $\tau_2(\wp\xi, \varrho) \leq \tau_2(\xi, \varrho)$ and $\tau_2(\xi, \wp\varrho) \leq \tau_2(\xi, \varrho)$;

(iii) $\tau_1(\xi, \varrho) + \tau_2(\xi, \varrho) < 1$;

(iv)

$$\ell(\wp\xi, \wp\varrho) \leq \tau_1(\xi, \varrho)\ell(\xi, \varrho) + \tau_2(\xi, \varrho)\frac{\ell(\xi, \wp\xi)\ell(\varrho, \wp\varrho)}{1 + \ell(\xi, \varrho)},$$

for all $\xi, \varrho \in \mathfrak{U}$. For $\xi_0 \in \mathfrak{U}$, a sequence $\{\xi_\jmath\}_{\jmath \geq 0}$ is generated as $\xi_{\jmath+1} = \wp\xi_\jmath$ for each $\jmath \geq 0$. Assume that

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\sigma(\xi_{i+1}, \xi_{i+2})\sigma(\xi_{i+1}, \xi_m)}{\sigma(\xi_i, \xi_{i+1})} < \frac{1}{\tau},$$

where $\frac{\tau_1(\xi_0, \xi_1)}{1 - \tau_2(\xi_0, \xi_1)} = \tau$. Additionally, suppose that $\lim_{\jmath \to +\infty} \sigma(\xi_\jmath, \xi)$ and $\lim_{\jmath \to +\infty} \sigma(\xi, \xi_\jmath)$ exist and are finite, then there exists a unique point $\xi^* \in \mathfrak{U}$ such that $\wp\xi^* = \xi^*$.

4. Deduced results

Theorem 5. *Let* $(\mathfrak{U}, \sigma, \ell)$ *be a complete CMS,* $\wp_1, \wp_2 : \mathfrak{U} \to \mathfrak{U}$ *and there exist the mappings* $\beta_1, \beta_2 : \mathfrak{U} \to [0, 1)$ *such that*

(i) $\beta_1(\wp_1\xi) \leq \beta_1(\xi)$ and $\beta_2(\wp_1\xi) \leq \beta_2(\xi)$;

(ii) $\beta_1(\wp_2\xi) \leq \beta_1(\xi)$ and $\beta_2(\wp_2\xi) \leq \beta_2(\xi)$;

(iii) $(\beta_1 + \beta_2)(\xi) < 1$;

(iv)

$$\ell(\wp_1\xi, \wp_2\varrho) \leq \beta_1(\xi)\ell(\xi, \varrho) + \beta_2(\xi)\frac{\ell(\xi, \wp_1\xi)\ell(\varrho, \wp_2\varrho)}{1 + \ell(\xi, \varrho)},$$

for all $\xi, \varrho \in \mathfrak{U}$. For $\xi_0 \in \mathfrak{U}$, a sequence $\{\xi_\jmath\}_{\jmath \geq 0}$ is generated as $\xi_{2\jmath+1} = \wp_1\xi_{2\jmath}$ and $\xi_{2\jmath+2} = \wp_2\xi_{2\jmath+1}$ for each $\jmath \geq 0$. Suppose that

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\sigma(\xi_{i+1}, \xi_{i+2})\sigma(\xi_{i+1}, \xi_m)}{\sigma(\xi_i, \xi_{i+1})} < \frac{1}{\beta},$$

where $\frac{\beta_1(\xi_0)}{1 - \beta_2(\xi_0)} = \beta$. In addition, suppose that $\lim_{\jmath \to +\infty} \sigma(\xi_\jmath, \xi)$ and $\lim_{\jmath \to +\infty} \sigma(\xi, \xi_\jmath)$ exist and are finite, then there exists a unique point $\xi^* \in \mathfrak{U}$ such that $\wp_1\xi^* \cap \wp_2\xi^* = \xi^*$.

Proof. Define $\tau_1, \tau_2 : \mathfrak{U} \times \mathfrak{U} \to [0, 1)$ by

$$\tau_1(\xi, \varrho) = \beta_1(\xi) \quad \text{and} \quad \tau_2(\xi, \varrho) = \beta_2(\xi),$$

for all $\xi, \varrho \in \mathfrak{U}$. Then for all $\xi, \varrho \in \mathfrak{U}$, □

(i) $\tau_1(\wp_2\wp_1\xi,\varrho) = \beta_1(\wp_2\wp_1\xi) \leq \beta_1(\wp_1\xi) \leq \beta_1(\xi) = \tau_1(\xi,\varrho)$ and $\tau_1(\xi,\wp_1\wp_2\varrho) = \beta_1(\xi) = \beta_1(\xi,\varrho)$;

(ii) $\tau_2(\wp_2\wp_1\xi,\varrho) = \beta_2(\wp_2\wp_1\xi) \leq \beta_2(\wp_1\xi) \leq \beta_2(\xi) = \tau_2(\xi,\varrho)$ and $\tau_2(\xi,\wp_1\wp_2\varrho) = \beta_2(\xi) = \beta_2(\xi,\varrho)$;

(iii) $\tau_1(\xi,\varrho) + \tau_2(\xi,\varrho) = \beta_1(\xi) + \beta_2(\xi) < 1$;

(iv)

$$\begin{aligned}\ell(\wp_1\xi,\wp_2\varrho) &\leq \beta_1(\xi)\ell(\xi,\varrho) + \beta_2(\xi)\frac{\ell(\xi,\wp_1\xi)\ell(\varrho,\wp_2\varrho)}{1+\ell(\xi,\varrho)}\\ &= \tau_1(\xi,\varrho)\ell(\xi,\varrho) + \tau_2(\xi,\varrho)\frac{\ell(\xi,\wp_1\xi)\ell(\varrho,\wp_2\varrho)}{1+\ell(\xi,\varrho)}.\end{aligned}$$

By Theorem 4, $\wp_1$ and $\wp_2$ have a unique common fixed point.

Corollary 2. *Let* $(\mathfrak{U},\sigma,\ell)$ *be a complete CMS,* $\wp : \mathfrak{U} \to \mathfrak{U}$ *and there exist the mappings* $\beta_1,\beta_2 : \mathfrak{U} \to [0,1)$ *such that*

(i) $\beta_1(\wp\xi) \leq \beta_1(\xi)$ and $\beta_2(\wp\xi) \leq \beta_2(\xi)$;

(ii) $(\beta_1+\beta_2)(\xi) < 1$;

(iii)

$$\ell(\wp\xi,\wp\varrho) \leq \beta_1(\xi)\ell(\xi,\varrho) + \beta_2(\xi)\frac{\ell(\xi,\wp\xi)\ell(\varrho,\wp\varrho)}{1+\ell(\xi,\varrho)}, \tag{4.1}$$

for all $\xi,\varrho \in \mathfrak{U}$. For $\xi_0 \in \mathfrak{U}$, we set $\frac{\beta_1(\xi_0)}{1-\beta_2(\xi_0)} = \beta$. Assume that

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\sigma(\xi_{i+1},\xi_{i+2})\sigma(\xi_{i+1},\xi_m)}{\sigma(\xi_i,\xi_{i+1})} < \frac{1}{\beta},$$

where $\xi_{j+1} = \wp\xi_j$ for each $j \geq 0$. Furthermore, suppose that $\lim_{j\to+\infty}\sigma(\xi_j,\xi)$ and $\lim_{j\to+\infty}\sigma(\xi,\xi_j)$ exist and are finite, then there exists a unique point $\xi^* \in \mathfrak{U}$ such that $\wp\xi^* = \xi^*$.

Proof. Taking $\wp_1 = \wp_2 = \wp$ in the Theorem 5. □

Theorem 6. *Let* $(\mathfrak{U},\sigma,\ell)$ *be a complete CMS and* $\wp : \mathfrak{U} \to \mathfrak{U}$. *Let there exist* $\beta_1,\beta_2 : \mathfrak{U} \to [0,1)$ *such that*

(i) $\beta_1(\wp^j\xi) \leq \beta_1(\xi)$ and $\beta_2(\wp^j\xi) \leq \beta_2(\xi)$;

(ii) $(\beta_1+\beta_2)(\xi) < 1$;

(iii)

$$\ell(\wp^j\xi,\wp^j\varrho) \leq \beta_1(\xi)\ell(\xi,\varrho) + \beta_2(\xi)\frac{\ell(\xi,\wp^j\xi)\ell(\varrho,\wp^j\varrho)}{1+\ell(\xi,\varrho)},$$

for all $\xi,\varrho \in \mathfrak{U}$ and for some $j \in \mathbb{N}$. For $\xi_0 \in \mathfrak{U}$, we set $\frac{\beta_1(\xi_0)}{1-\beta_2(\xi_0)} = \beta$. Suppose that

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\sigma(\xi_{i+1},\xi_{i+2})\sigma(\xi_{i+1},\xi_m)}{\sigma(\xi_i,\xi_{i+1})} < \frac{1}{\beta},$$

where $\xi_{j+1} = \wp\xi_j$ for each $j \geq 0$. In addition, suppose that $\lim_{j\to+\infty}\sigma(\xi_j,\xi)$ and $\lim_{j\to+\infty}\sigma(\xi,\xi_j)$ exist and are finite, then there exists a unique point $\xi^* \in \mathfrak{U}$ such that $\wp\xi^* = \xi^*$.

Proof. By result 2, we get that $\wp^{\jmath}\xi^* = \xi^*$. Now, as

$$\wp^{\jmath}(\wp\xi^*) = \wp(\wp^{\jmath}\xi^*) = \wp\xi^*,$$

so, $\wp\xi^*$ is a fixed point of $\wp^{\jmath}$. Thus $\wp\xi^* = \xi^*$. Since the fixed point of $\wp^{\jmath}$ is unique, so ξ^* is also a fixed point of $\wp$. □

Corollary 3. *Let* $(\mathfrak{U}, \sigma, \ell)$ *be a complete CMS,* $\wp_1, \wp_2 : \mathfrak{U} \to \mathfrak{U}$ *and there exist* $\gamma_1, \gamma_2 \in [0, 1)$ *with* $\gamma_1 + \gamma_2 < 1$ *such that*

$$\ell(\wp_1\xi, \wp_2\varrho) \leq \gamma_1\ell(\xi, \varrho) + \gamma_2\frac{\ell(\xi, \wp_1\xi)\ell(\varrho, \wp_2\varrho)}{1 + \ell(\xi, \varrho)},$$

for all $\xi, \varrho \in \mathfrak{U}$. For $\xi_0 \in \mathfrak{U}$, a sequence $\{\xi_{\jmath}\}_{\jmath\geq 0}$ is generated as $\xi_{2\jmath+1} = \wp_1\xi_{2\jmath}$ and $\xi_{2\jmath+2} = \wp_2\xi_{2\jmath+1}$ for each $\jmath \geq 0$. Assume that

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\sigma(\xi_{i+1}, \xi_{i+2})\sigma(\xi_{i+1}, \xi_m)}{\sigma(\xi_i, \xi_{i+1})} < \frac{1}{\gamma},$$

where $\frac{\gamma_1}{1-\gamma_2} = \gamma$. In addition, assume that $\lim_{\jmath\to+\infty}\sigma(\xi_{\jmath}, \xi)$ and $\lim_{\jmath\to+\infty}\sigma(\xi, \xi_{\jmath})$ exist and are finite, then there exists a unique point $\xi^* \in \mathfrak{U}$ such that $\wp_1\xi^* \cap \wp_2\xi^* = \xi^*$.

Proof. Taking $\gamma_1(\cdot) = \gamma_1$ and $\gamma_2(\cdot) = \gamma_2$ in Theorem 5. □

Corollary 4. *Let* $(\mathfrak{U}, \sigma, \ell)$ *be a complete CMS,* $\wp : \mathfrak{U} \to \mathfrak{U}$ *and there exist* $\gamma_1, \gamma_2 \in [0, 1)$ *with* $\gamma_1+\gamma_2 < 1$ *such that*

$$\ell(\wp\xi, \wp\varrho) \leq \gamma_1\ell(\xi, \varrho) + \gamma_2\frac{\ell(\xi, \wp\xi)\ell(\varrho, \wp\varrho)}{1 + \ell(\xi, \varrho)},$$

for all $\xi, \varrho \in \mathfrak{U}$. *For* $\xi_0 \in \mathfrak{U}$, *we set* $\frac{\gamma_1}{1-\gamma_2} = \gamma$. *Suppose that*

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\sigma(\xi_{i+1}, \xi_{i+2})\sigma(\xi_{i+1}, \xi_m)}{\sigma(\xi_i, \xi_{i+1})} < \frac{1}{\gamma},$$

where $\xi_{\jmath+1} = \wp\xi_{\jmath}$, for all $\jmath \geq 0$. Furthermore, assume that $\lim_{\jmath\to+\infty}\sigma(\xi_{\jmath}, \xi)$ and $\lim_{\jmath\to+\infty}\sigma(\xi, \xi_{\jmath})$ are finite and exist, then there exists a unique point $\xi^* \in \mathfrak{U}$ such that $\wp\xi^* = \xi^*$.

Proof. Taking $\wp_1 = \wp_2 = \wp$ in Theorem 3. □

Corollary 5. *Let* $(\mathfrak{U}, \sigma, \ell)$ *be a complete CMS,* $\wp : \mathfrak{U} \to \mathfrak{U}$ *and there exist* $\gamma_1, \gamma_2 \in [0, 1)$ *with* $\gamma_1+\gamma_2 < 1$ *such that*

$$\ell(\wp^{\jmath}\xi, \wp^{\jmath}\varrho) \leq \gamma_1\ell(\xi, \varrho) + \gamma_2\frac{\ell(\xi, \wp^{\jmath}\xi)\ell(\varrho, \wp^{\jmath}\varrho)}{1 + \ell(\xi, \varrho)},$$

for all $\xi, \varrho \in \mathfrak{U}$. *For* $\xi_0 \in \mathfrak{U}$, *with* $\frac{\gamma_1}{1-\gamma_2} = \gamma$, *suppose that*

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\sigma(\xi_{i+1}, \xi_{i+2})\sigma(\xi_{i+1}, \xi_m)}{\sigma(\xi_i, \xi_{i+1})} < \frac{1}{\gamma},$$

where $\xi_{\jmath+1} = \wp\xi_{\jmath}$, for each $\jmath \geq 0$. Furthermore, assume that, $\lim_{\jmath\to+\infty}\sigma(\xi_{\jmath}, \xi)$ and $\lim_{\jmath\to+\infty}\sigma(\xi, \xi_{\jmath})$ exist and are finite, then there exists a unique point $\xi^* \in \mathfrak{U}$ such that $\wp\xi^* = \xi^*$.

Proof. Setting $\gamma_1(\cdot) = \gamma_1$ and $\gamma_2(\cdot) = \gamma_2$ in Theorem 6. □

Corollary 6. *Let $(\mathfrak{U}, \sigma, \ell)$ be a complete CMS, $\wp : \mathfrak{U} \to \mathfrak{U}$ and there exists $\gamma_1 \in [0, 1)$ such that*

$$\ell(\wp\xi, \wp\varrho) \leq \gamma_1 \ell(\xi, \varrho),$$

for all $\xi, \varrho \in \mathfrak{U}$. For $\xi_0 \in \mathfrak{U}$, with $\frac{\gamma_1}{1-\gamma_1} = \gamma$, suppose that

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\sigma(\xi_{i+1}, \xi_{i+2})\sigma(\xi_{i+1}, \xi_m)}{\sigma(\xi_i, \xi_{i+1})} < \frac{1}{\gamma},$$

where $\xi_{J+1} = \wp\xi_J$ for each $J \geq 0$. Moreover, assume that, $\lim_{J\to+\infty} \sigma(\xi_J, \xi)$ and $\lim_{J\to+\infty} \sigma(\xi, \xi_J)$ exist and are finitet, then there exists a unique point $\xi^* \in \mathfrak{U}$ such that $\wp\xi^* = \xi^*$.

5. Applications in graphs

Let $(\mathfrak{U}, \sigma, \ell)$ be a CMS and G be a directed graph. Let us represent by G^{-1}, the graph achieved from G by changing the direction of $E(G)$. Hence,

$$E(G^{-1}) = \{(\xi, \varrho) \in \mathfrak{U}\times\mathfrak{U} : (\varrho, \xi) \in E(G)\}.$$

Definition 4. An element $\xi \in \mathfrak{U}$ is claimed to be common fixed point of $(\wp_1, \wp_2)$, if $\wp_1(\xi) = \wp_2(\xi) = \xi$. We shall represent by CFix$(\wp_1, \wp_2)$, the set of all common fixed points of $(\wp_1, \wp_2)$, i.e.

$$CFix(\wp_1, \wp_2) = \{\xi \in \mathfrak{U} : \wp_1(\xi) = \wp_2(\xi) = \xi\}.$$

Definition 5. Suppose that $\wp_1, \wp_2 : \mathfrak{U} \to \mathfrak{U}$ are two mappings on complete CMS $(\mathfrak{U}, \sigma, \ell)$ equipped with a directed graph G. Then $(\wp_1, \wp_2)$ is said to be a G-orbital cyclic pair, if for any $\xi \in \mathfrak{U}$

$$(\xi, \wp_1\xi) \in E(G) \Longrightarrow (\wp_1\xi, \wp_2(\wp_1\xi)) \in E(G),$$

$$(\xi, \wp_2\xi) \in E(G) \Longrightarrow (\wp_2\xi, \wp_1(\wp_2\xi)) \in E(G).$$

Let us consider the following sets

$$\begin{aligned} \mathfrak{U}^{\wp_1} &= \{\xi \in \mathfrak{U} : (\xi, \wp_1\xi) \in E(G)\}, \\ \mathfrak{U}^{\wp_2} &= \{\xi \in \mathfrak{U} : (\xi, \wp_2\xi) \in E(G)\}. \end{aligned}$$

Remark 1. If the pair $(\wp_1, \wp_2)$ be a G-orbital-cyclic pair, then $\mathfrak{U}^{\wp_1} \neq \emptyset \Longleftrightarrow \mathfrak{U}^{\wp_2} \neq \emptyset$.

Proof. Let $\xi_0 \in \mathfrak{U}^{\wp_1}$. Then $(\xi_0, \wp_1\xi_0) \in E(G) \Longrightarrow (\wp_1\xi_0, \wp_2(\wp_1\xi_0)) \in E(G)$. If we represent by $\xi_1 = \wp_1\xi_0$, then we get that $(\xi_1, \wp_2(\xi_1)) \in E(G)$, thus $\mathfrak{U}^{\wp_2} \neq \emptyset$. □

Theorem 7. *Let $(\mathfrak{U}, \sigma, \ell)$ be a complete CMS equipped with a directed graph G and $\wp_1, \wp_2 : \mathfrak{U} \to \mathfrak{U}$ is G-orbital cyclic pair. Assume that there exists $\tau_1 \in [0, 1)$ such that*

(i) $\mathfrak{U}^{\wp_1} \neq \emptyset$;

(ii) for all $\xi \in \mathfrak{U}^{\wp_1}$ and $\varrho \in \mathfrak{U}^{\wp_2}$,

$$\ell(\wp_1\xi, \wp_2\varrho) \leq \tau_1 \max\{\ell(\xi, \varrho), \ell(\xi, \wp_1\xi), \ell(\varrho, \wp_2\varrho)\}; \tag{5.1}$$

(iii) for all $(\xi_\jmath)_{\jmath\in\mathbb{N}} \subset \mathfrak{U}$, one has $(\xi_\jmath, \xi_{\jmath+1}) \in E(G)$,

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\sigma(\xi_{i+1}, \xi_{i+2})\sigma(\xi_{i+1}, \xi_m)}{\sigma(\xi_i, \xi_{i+1})} < \frac{1}{\tau}, \tag{5.2}$$

where $\tau = \frac{\tau_1}{1-\tau_1}$;

(iv) $\wp_1$ and $\wp_2$ are continuous, or for all $(\xi_\jmath)_{\jmath\in\mathbb{N}} \subset \mathfrak{U}$, with $\xi_\jmath \to \xi$ as $\jmath \to +\infty$, and $(\xi_\jmath, \xi_{\jmath+1}) \in E(G)$ for $\jmath \in \mathbb{N}$, we have $\xi \in \mathfrak{U}^{\wp_1} \cap \mathfrak{U}^{\wp_2}$. In these conditions, $\mathrm{CFix}(\wp_1\wp_2) \neq \emptyset$;

(v) for all $\xi \in \mathfrak{U}$, we have $\lim_{\jmath\to+\infty} \sigma(\xi_\jmath, \xi)$ and $\lim_{\jmath\to+\infty} \sigma(\xi, \xi_\jmath)$ exist and finite;

(vi) if $(\xi^*, \xi^/) \in \mathrm{CFix}(\wp_1, \wp_2)$ implies $\xi^* \in \mathfrak{U}^{\wp_1}$ and $\xi^/ \in \mathfrak{U}^{\wp_2}$, then the pair $(\wp_1, \wp_2)$ has a unique common fixed point.

Proof. Let $\xi_0 \in \mathfrak{U}^{\wp_1}$. Thus $(\xi_0, \wp_1\xi_0) \in E(G)$. As the pair $(\wp_1, \wp_2)$ is G-orbital cyclic, we get $(\wp_1\xi_0, \wp_2\wp_1\xi_0) \in E(G)$. Construct ξ_1 by $\xi_1 = \wp_1\xi_0$, we have $(\xi_1, \wp_2\xi_1) \in E(G)$ and from here $(\wp_2\xi_1, \wp_1\wp_2\xi_1) \in E(G)$. Denoting by $\xi_2 = \wp_2\xi_1$, we have $(\xi_2, \wp_1\xi_2) \in E(G)$. Continuing along these lines, we generate a sequence $(\xi_\jmath)_{\jmath\in\mathbb{N}}$ with $\xi_{2\jmath} = \wp_2\xi_{2\jmath-1}$ and $\xi_{2\jmath+1} = \wp_1\xi_{2\jmath}$, such that $\left(\xi_{2\jmath}, \xi_{2\jmath+1}\right) \in E(G)$. We assume that $\xi_\jmath \neq \xi_{\jmath+1}$. If, there exists $\jmath_0 \in \mathbb{N}$, such that $\xi_{\jmath_0} = \xi_{\jmath_0+1}$, then in the view of the fact that $\Delta \subset E(G)$, $\left(\xi_{\jmath_0}, \xi_{\jmath_0+1}\right) \in E(G)$ and thus $\xi^* = \xi_{\jmath_0}$ is a fixed point of $\wp_1$. Now to manifest that $\xi^* \in \mathrm{CFix}(\wp_1, \wp_2)$, we shall discuss these two cases for $\jmath_0$. If $\jmath_0$ is even, then $\jmath_0 = 2\jmath$. Then, $\xi_{2\jmath} = \xi_{2\jmath+1} = \wp_1\xi_{2\jmath}$ and thus, $\xi_{2\jmath}$ is a fixed point of $\wp_1$. Assume that $\xi_{2\jmath} = \xi_{2\jmath+1} = \wp_1\xi_{2\jmath}$ but $\ell(\wp_1\xi_{2\jmath}, \wp_2\xi_{2\jmath+1}) > 0$, and let $\xi = \xi_{2\jmath} \in \mathfrak{U}^{\wp_1}$ and $\varrho = \xi_{2\jmath+1} \in \mathfrak{U}^{\wp_2}$. So

$$\begin{aligned}
0 &< \ell(\xi_{2\jmath+1}, \xi_{2\jmath+2}) = \ell(\wp_1\xi_{2\jmath}, \wp_2\xi_{2\jmath+1}) \\
&\leq \tau_1 \max\left\{\ell(\xi_{2\jmath}, \xi_{2\jmath+1}), \ell(\xi_{2\jmath}, \wp_1\xi_{2\jmath}), \ell(\xi_{2\jmath+1}, \wp_2\xi_{2\jmath+1})\right\} \\
&= \tau_1 \max\left\{\ell(\xi_{2\jmath}, \xi_{2\jmath+1}), \ell(\xi_{2\jmath}, \xi_{2\jmath+1}), \ell(\xi_{2\jmath+1}, \xi_{2\jmath+2})\right\} \\
&= \tau_1 \ell(\xi_{2\jmath+1}, \xi_{2\jmath+2})
\end{aligned}$$

that is contradiction because $\tau_1 < 1$. Hence $\xi_{2\jmath}$ is a fixed point of $\wp_2$ too. Likewise if $\jmath_0$ is odd number, then there exists $\xi^* \in \mathfrak{U}$ such that $\wp_1\xi^* \cap \wp_2\xi^* = \xi^*$. So we assume that $\xi_\jmath \neq \xi_{\jmath+1}$ for all $\jmath \in \mathbb{N}$. Now we shall show that $(\xi_\jmath)_{\jmath\in\mathbb{N}}$ is Cauchy sequence. We have these two possible cases to discuss:

Case 1. $\xi = \xi_{2\jmath} \in \mathfrak{U}^{\wp_1}$ and $\varrho = \xi_{2\jmath+1} \in \mathfrak{U}^{\wp_2}$.

$$\begin{aligned}
0 &< \ell(\xi_{2\jmath+1}, \xi_{2\jmath+2}) = \ell(\wp_1\xi_{2\jmath}, \wp_2\xi_{2\jmath+1}) \\
&\leq \tau_1 \max\left\{\ell(\xi_{2\jmath}, \xi_{2\jmath+1}), \ell(\xi_{2\jmath}, \wp_1\xi_{2\jmath}), \ell(\xi_{2\jmath+1}, \wp_2\xi_{2\jmath+1})\right\} \\
&= \tau_1 \max\left\{\ell(\xi_{2\jmath}, \xi_{2\jmath+1}), \ell(\xi_{2\jmath}, \xi_{2\jmath+1}), \ell(\xi_{2\jmath+1}, \xi_{2\jmath+2})\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \tau_1 \max\left\{\ell(\xi_{2\jmath},\xi_{2\jmath+1}),\ell(\xi_{2\jmath+1},\xi_{2\jmath+2})\right\}\\
&\leq \tau_1\left[\ell(\xi_{2\jmath},\xi_{2\jmath+1})+\ell(\xi_{2\jmath+1},\xi_{2\jmath+2})\right]
\end{aligned}
$$

that is

$$(1-\tau_1)\ell(\xi_{2\jmath+1},\xi_{2\jmath+2}) \leq \tau_1\ell(\xi_{2\jmath},\xi_{2\jmath+1}),$$

which implies

$$\ell(\xi_{2\jmath+1},\xi_{2\jmath+2}) \leq \frac{\tau_1}{1-\tau_1}\ell(\xi_{2\jmath},\xi_{2\jmath+1}). \tag{5.3}$$

Case 2. $\xi = \xi_{2\jmath} \in \mathfrak{U}^{\wp_1}$ and $\varrho = \xi_{2\jmath-1} \in \mathfrak{U}^{\wp_2}$.

$$
\begin{aligned}
0 &< \ell(\xi_{2\jmath+1},\xi_{2\jmath}) = \ell(\wp_1\xi_{2\jmath},\wp_2\xi_{2\jmath-1})\\
&\leq \tau_1 \max\left\{\ell(\xi_{2\jmath},\xi_{2\jmath-1}),\ell(\xi_{2\jmath},\wp_1\xi_{2\jmath}),\ell(\xi_{2\jmath-1},\wp_2\xi_{2\jmath-1})\right\}\\
&= \tau_1 \max\left\{\ell(\xi_{2\jmath},\xi_{2\jmath-1}),\ell(\xi_{2\jmath},\xi_{2\jmath+1}),\ell(\xi_{2\jmath-1},\xi_{2\jmath})\right\}\\
&= \tau_1 \max\left\{\ell(\xi_{2\jmath-1},\xi_{2\jmath}),\ell(\xi_{2\jmath},\xi_{2\jmath+1})\right\}\\
&\leq \tau_1\left[\ell(\xi_{2\jmath-1},\xi_{2\jmath})+\ell(\xi_{2\jmath},\xi_{2\jmath+1})\right]
\end{aligned}
$$

that is

$$(1-\tau_1)\ell(\xi_{2\jmath+1},\xi_{2\jmath}) \leq \tau_1\ell(\xi_{2\jmath},\xi_{2\jmath-1}),$$

which implies

$$\ell(\xi_{2\jmath},\xi_{2\jmath+1}) \leq \frac{\tau_1}{1-\tau_1}\ell(\xi_{2\jmath-1},\xi_{2\jmath}). \tag{5.4}$$

Since $\tau = \frac{\tau_1}{1-\tau_1}$, so we have

$$\ell(\xi_\jmath,\xi_{\jmath+1}) \leq \tau\ell(\xi_{\jmath-1},\xi_\jmath). \tag{5.5}$$

Thus, we have

$$
\begin{aligned}
\ell(\xi_\jmath,\xi_{\jmath+1}) &\leq \tau\ell(\xi_{\jmath-1},\xi_\jmath)\\
&\leq \tau^2\ell(\xi_{\jmath-2},\xi_{\jmath-1})\\
&\leq\\
&\quad .\\
&\quad .\\
&\quad .\\
&\leq \tau^\jmath\ell(\xi_0,\xi_1).
\end{aligned}
$$

For all $\jmath, m \in \mathbb{N}(\jmath < m)$, we have

$$
\begin{aligned}
\ell(\xi_\jmath,\xi_m) &\leq \sigma(\xi_\jmath,\xi_{\jmath+1})\ell(\xi_\jmath,\xi_{\jmath+1}) + \sigma(\xi_{\jmath+1},\xi_m)\ell(\xi_{\jmath+1},\xi_m)\\
&\leq \sigma(\xi_\jmath,\xi_{\jmath+1})\ell(\xi_\jmath,\xi_{\jmath+1}) + \sigma(\xi_{\jmath+1},\xi_m)\sigma(\xi_{\jmath+1},\xi_{\jmath+2})\ell(\xi_{\jmath+1},\xi_{\jmath+2})\\
&\quad +\sigma(\xi_{\jmath+1},\xi_m)\sigma(\xi_{\jmath+2},\xi_m)\ell(\xi_{\jmath+2},\xi_m)\\
&\leq \sigma(\xi_\jmath,\xi_{\jmath+1})\ell(\xi_\jmath,\xi_{\jmath+1}) + \sigma(\xi_{\jmath+1},\xi_m)\sigma(\xi_{\jmath+1},\xi_{\jmath+2})\ell(\xi_{\jmath+1},\xi_{\jmath+2})\\
&\quad +\sigma(\xi_{\jmath+1},\xi_m)\sigma(\xi_{\jmath+2},\xi_m)\sigma(\xi_{\jmath+2},\xi_{\jmath+3})\ell(\xi_{\jmath+2},\xi_{\jmath+3})
\end{aligned}
$$

$$
\begin{aligned}
&\quad +\sigma(\xi_{\jmath+1},\xi_m)\sigma(\xi_{\jmath+2},\xi_m)\sigma(\xi_{\jmath+3},\xi_m)\ell(\xi_{\jmath+3},\xi_m)\\
&\leq \cdots\\
&\leq \sigma(\xi_\jmath,\xi_{\jmath+1})\ell(\xi_\jmath,\xi_{\jmath+1}) + \sum_{i=\jmath+1}^{m-2}\left(\prod_{k=\jmath+1}^{i}\sigma(\xi_k,\xi_m)\right)\sigma(\xi_i,\xi_{i+1})\ell(\xi_i,\xi_{i+1})\\
&\quad +\prod_{i=\jmath+1}^{m-1}\sigma(\xi_i,\xi_m)\ell(\xi_{m-1},\xi_m),
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
\ell(\xi_\jmath,\xi_m) &\leq \sigma(\xi_\jmath,\xi_{\jmath+1})\ell(\xi_\jmath,\xi_{\jmath+1}) + \sum_{i=\jmath+1}^{m-2}\left(\prod_{k=\jmath+1}^{i}\sigma(\xi_k,\xi_m)\right)\sigma(\xi_i,\xi_{i+1})\ell(\xi_i,\xi_{i+1})\\
&\quad +\left(\prod_{i=\jmath+1}^{m-1}\sigma(\xi_i,\xi_m)\right)\sigma(\xi_{m-1},\xi_m)\ell(\xi_{m-1},\xi_m)\\
&\leq \sigma(\xi_\jmath,\xi_{\jmath+1})\tau^\jmath\ell(\xi_0,\xi_1) + \sum_{i=\jmath+1}^{m-2}\left(\prod_{k=\jmath+1}^{i}\sigma(\xi_k,\xi_m)\right)\sigma(\xi_i,\xi_{i+1})\tau^i\ell(\xi_0,\xi_1)\\
&\quad +\left(\prod_{i=\jmath+1}^{m-1}\sigma(\xi_i,\xi_m)\right)\sigma(\xi_{m-1},\xi_m)\tau^{m-1}\ell(\xi_0,\xi_1)\\
&= \sigma(\xi_\jmath,\xi_{\jmath+1})\tau^\jmath\ell(\xi_0,\xi_1) + \sum_{i=\jmath+1}^{m-1}\left(\prod_{k=\jmath+1}^{i}\sigma(\xi_k,\xi_m)\right)\sigma(\xi_i,\xi_{i+1})\tau^i\ell(\xi_0,\xi_1). \qquad (5.6)
\end{aligned}
$$

Let

$$
\Psi_l = \sum_{i=0}^{l}\left(\prod_{k=0}^{i}\sigma(\xi_k,\xi_m)\right)\sigma(\xi_i,\xi_{i+1})\tau^i\ell(\xi_0,\xi_1).
$$

From (5.6), we get

$$
\ell(\xi_\jmath,\xi_m) \leq \ell(\xi_0,\xi_1)[\tau^\jmath\sigma(\xi_\jmath,\xi_{\jmath+1}) + (\Psi_{m-1} - \Psi_\jmath)]. \qquad (5.7)
$$

Now as $\sigma(\xi,\varrho) \geq 1$, and by utilizing ratio test, $\lim_{\jmath\to+\infty}\Psi_\jmath$ exists. Clearly, if we let $\jmath, m \to +\infty$ in (5.7), we get that

$$
\lim_{\jmath,m\to+\infty}\ell(\xi_\jmath,\xi_m) = 0. \qquad (5.8)
$$

Hence, $\{\xi_\jmath\}$ is a Cauchy sequence in $(\mathfrak{U},\ell)$. So there exists $\xi^* \in \mathfrak{U}$ such that

$$
\lim_{\jmath\to+\infty}\ell(\xi_\jmath,\xi^*) = 0. \qquad (5.9)
$$

that is $\xi_\jmath \to \xi^*$ as $\jmath \to +\infty$. It is obvious that

$$
\lim_{\jmath\to+\infty}\xi_{2\jmath} = \lim_{\jmath\to+\infty}\xi_{2\jmath+1} = \xi^*. \qquad (5.10)
$$

As $\wp_1$ and $\wp_2$ are continuous, so we have

$$
\xi^* = \lim_{\jmath\to+\infty}\xi_{2\jmath+1} = \lim_{\jmath\to+\infty}\wp_1(\xi_{2\jmath}) = \wp_1(\xi^*),
$$

$$\xi^* = \lim_{\jmath\to+\infty} \xi_{2\jmath+2} = \lim_{\jmath\to+\infty} \wp_2(\xi_{2\jmath+1}) = \wp_2(\xi^*).$$

Now letting $\xi = \xi^* \in \mathfrak{U}^{\wp_1}$ and $\varrho = \xi_{2\jmath+2} \in \mathfrak{U}^{\wp_2}$, we have

$$\begin{aligned} 0 &< \ell(\wp_1\xi^*, \xi_{2\jmath+2}) = \ell(\wp_1\xi^*, \wp_2(\xi_{2\jmath+1})) \\ &\leq \tau_1 \max\left\{\ell(\xi^*, \xi_{2\jmath+1}), \ell(\xi^*, \wp_1\xi^*), \ell(\xi_{2\jmath+1}, \wp_2(\xi_{2\jmath+1})\right\} \\ &= \tau_1 \max\left\{\ell(\xi^*, \xi_{2\jmath+1}), \ell(\xi^*, \wp_1\xi^*), \ell(\xi_{2\jmath+1}, \xi_{2\jmath+2})\right\}. \end{aligned}$$

Letting $\jmath \to +\infty$ and using (5.10), we can simply conclude that $\ell(\xi^*, \wp_1\xi^*) = 0$. This yields that $\xi^* = \wp_1\xi^*$. Similarly, suppose that $\xi = \xi_{2\jmath+1} \in \mathfrak{U}^{\wp_1}$ and $\varrho = \xi^* \in \mathfrak{U}^{\wp_2}$, we have

$$\begin{aligned} 0 &< \ell(\xi_{2\jmath+2}, \wp_2\xi^*) = \ell(\wp_1(\xi_{2\jmath}), \wp_2\xi^*) \\ &\leq \tau_1 \max\left\{\ell(\xi_{2\jmath}, \xi^*), \ell(\xi_{2\jmath}, \wp_1(\xi_{2\jmath})), \ell(\xi^*, \wp_2\xi^*)\right\} \\ &= \tau_1 \max\left\{\ell(\xi_{2\jmath}, \xi^*), \ell(\xi_{2\jmath}, \xi_{2\jmath+1}), \ell(\xi^*, \wp_2\xi^*)\right\}. \end{aligned}$$

Letting $\jmath \to +\infty$ and using (5.10), we can simply conclude that $\ell(\xi^*, \wp_2\xi^*) = 0$. This yields that $\xi^* = \wp_2\xi^*$. □

Corollary 7. *Let $(\mathfrak{U}, \sigma, \ell)$ be a complete CMS euipped with a directed graph G and $\wp : \mathfrak{U} \to \mathfrak{U}$ is a G-orbital-cyclic. Suppose that there exists $\tau_1 \in [0, 1)$ such that*

(i) $\mathfrak{U}^{\wp} \neq \emptyset$;

(ii) for all $\xi, \varrho \in \mathfrak{U}^{\wp}$, we have

$$\ell(\wp\xi, \wp\varrho) \leq \tau_1 \max\{\ell(\xi, \varrho), \ell(\xi, \wp\xi), \ell(\varrho, \wp\varrho)\};$$

(iii) for all $(\xi_\jmath)_{\jmath\in\mathbb{N}} \subset \mathfrak{U}$ one has $(\xi_\jmath, \xi_{\jmath+1}) \in E(G)$,

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\sigma(\xi_{i+1}, \xi_{i+2})\sigma(\xi_{i+1}, \xi_m)}{\sigma(\xi_i, \xi_{i+1})} < \frac{1}{\tau},$$

where $\tau = \frac{\tau_1}{1-\tau_1}$;

(iv) $\wp$ is continuous, or for each $(\xi_\jmath)_{\jmath\in\mathbb{N}} \subset \mathfrak{U}$, with $\xi_\jmath \to \xi$ as $\jmath \to +\infty$, and $(\xi_\jmath, \xi_{\jmath+1}) \in E(G)$ for $\jmath \in \mathbb{N}$, we have $\xi \in \mathfrak{U}^{\wp}$;

(v) for all $\xi \in \mathfrak{U}$, we have $\lim_{\jmath\to+\infty} \sigma(\xi_\jmath, \xi)$ and $\lim_{\jmath\to+\infty} \sigma(\xi, \xi_\jmath)$ exist and finite, then $\wp$ has a unique fixed point.

Example 3. Let $\mathfrak{U} = \{0, 1, 2, 3, 4\}$. Define $\ell : \mathfrak{U} \times \mathfrak{U} \to [0, +\infty)$ by

$$\ell(\xi, \varrho) = |\xi - \varrho|^2$$

and $\ell : \mathfrak{U} \times \mathfrak{U} \to [1, +\infty)$ by

$$\sigma(\xi, \varrho) = 1 + \xi + \varrho$$

for all $\xi, \varrho \in \mathfrak{U}$. Then $(\mathfrak{U}, \sigma, \ell)$ is complete CMS. Now define $\wp : \mathfrak{U} \to \mathfrak{U}$ by

$$\wp\xi = 0, \text{ for } \xi \in \{0, 1\},$$

and

$$\wp\xi = 1, \text{ for } \xi \in \{2, 3\}.$$

Also define $G = \{(0, 1), (0, 2), (2, 3), (0, 0), (1, 1), (2, 2), (3, 3)\}$, then G is directed graph. Then all assumptions of Corollary 3 are satisfied with $\tau_1 = \frac{1}{3}$ and $\xi^* = 0$ is the unique fixed point of $\wp$.

6. Applications in integral equations

In this section, we investigate the solution of Fredholm-type integral equation

$$\xi(t) = \int_0^1 K(t, s, \xi(t))\, ds, \tag{6.1}$$

for all $t \in [0, 1]$, where $K(t, s, \xi(t))$ is a continuous function from $[0, 1] \times [0, 1]$ into $\mathbb{R}$. Let $\mathfrak{U} = C([0, 1], (-\infty, +\infty))$. Now we define $\ell : \mathfrak{U} \times \mathfrak{U} \to [1,\infty)$ by

$$\ell(\xi, \varrho) = \sup_{t\in[0,1]} \left(\frac{|\xi(t)| + |\varrho(t)|}{2} \right).$$

Then $(\mathfrak{U}, \sigma, \ell)$ is a complete CMS with $\sigma(\xi, \varrho) = 2$.

Theorem 8. *Assume that*

(a) $|K(t, s, \xi(t))| + |K(t, s, \varrho(t))| \leq \tau_1 \left(\sup_{t\in[0,1]} |\xi(t)| + |\varrho(t)|\right) (|\xi(t)| + |\varrho(t)|)$ for some $\tau_1 \to \mathfrak{U} \to [0, 1)$;

(b) $K\left(t, s, \int_0^1 K(t, s, \xi(t))\, ds\right) < K(t, s, \xi(t))$;

for all $t, s \in [0, 1]$. Then the integral equation (6.1) has a unique solution.

Proof. Define $\wp : \mathfrak{U} \to \mathfrak{U}$ by

$$\wp\xi(t) = \int_0^1 K(t, s, \xi(t))\, ds.$$

Then

$$\ell(\wp\xi, \wp\varrho) = \sup_{t\in[0,1]} \left(\frac{|\wp\xi(t)| + |\wp\varrho(t)|}{2} \right).$$

Now

$$
\begin{aligned}
\ell(\wp\xi(t),\wp\varrho(t)) &= \frac{|\wp\xi(t)|+|\wp\varrho(t)|}{2} \\
&= \frac{\left|\int_0^1 K(t,s,\xi(t))\,ds\right| + \left|\int_0^1 K(t,s,\varrho(t))\,ds\right|}{2} \\
&\leq \frac{\int_0^1 |K(t,s,\xi(t))|\,ds + \int_0^1 |K(t,s,\varrho(t))|\,ds}{2} \\
&= \frac{\int_0^1 (|K(t,s,\xi(t))| + |K(t,s,\varrho(t))|)\,ds}{2} \\
&\leq \frac{\int_0^1 \left(\tau_1\left(\sup_{t\in[0,1]}|\xi(t)|+|\varrho(t)|\right)(|\xi(t)|+|\varrho(t)|)\right)ds}{2} \\
&\leq \tau_1\left(\sup_{t\in[0,1]}|\xi(t)|+|\varrho(t)|\right)\ell(\xi(t),\varrho(t)).
\end{aligned}
$$

Also we observe that

$$
\sigma(\xi,\varrho) = \frac{1}{\tau_1\left(\sup_{t\in[0,1]}|\xi(t)|+|\varrho(t)|\right)}.
$$

Thus all the conditions of result 6 are satisfied. Hence Eq (6.1) has a unique solution. □

7. Conclusions

In the current work, we have utilized the notion of controlled metric space and proved common fixed point results of self mappings for generalized contractions involving control functions of two variables. We also established common fixed point results in controlled metric space equipped with a graph. We have derived common fixed point and fixed point results for contractions with control functions of one variable as consequences of our leading result. We also supply a non trivial example to support the obtained results As as application of our prime result, we have investigated the solution of Fredholm type integral equation.

Some related generalizations of such contractions for the multivalued mappings $\wp : \mathfrak{U} \to CB(\mathfrak{U})$ and for fuzzy mappings $\wp : \mathfrak{U} \to \mathcal{F}(\mathfrak{U})$ would be a special field for future work. A distinct way of future study would be to employ our results in the solution of fractional differential inclusions.

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Conflict of interest

The authors declare no conflicts of interests.

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