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Research article

Energy conservation for the compressible ideal Hall-MHD equations

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Abstract: In this paper, we study the regularity and energy conservation of the weak solutions for compressible ideal Hall-magnetohydrodynamic (Hall-MHD) system, where $(t, x) \in (0, T) \times \mathbb{T}^d (d \geq 1)$. By exploring the special structure of the nonlinear terms in the model, we obtain the sufficient conditions for the regularity of the weak solutions for energy conservation. Our main strategy relies on commutator estimates.

Keywords: compressible ideal Hall-MHD system; regularity; energy conservation
Mathematics Subject Classification: 35B65, 35D30, 35Q35, 76W05

1. Introduction

In this work, we consider the energy conservation for weak solutions of the compressible ideal Hall-magnetohydrodynamic (Hall-MHD) equations

$$\begin{cases} \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p - j \times b = 0, \\ \partial_t b + d_I \nabla \times \left(\dfrac{j \times b}{\rho}\right) - \nabla \times (u \times b) = 0, \\ \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \mathrm{div} b = 0, \end{cases} \tag{1.1}$$

where $\rho > 0$ is the fluid density, u is the velocity field, p is the pressure and b is the magnetic field. ρ and p are scalars, d_I represents the Hall coefficient and $j = \nabla \times b$.

The energy conservation of weak solutions of the Euler equations and the MHD equations is a hot topic in recent decades. For the 3D Euler equations, Onsager [22] put forward famous Onsager's conjecture in 1949, that is, the weak solution with Hölder continuity of exponent $\delta > \frac{1}{3}$ can guarantee the conservation of energy, but the weak solution of $\delta \leq \frac{1}{3}$ is not necessary. In 1994, Eyink [12] proved the first part of this conjecture by means of Fourier series expansion. Constantin et al. [7] considered the conservation of energy when $u \in B^{\alpha}_{3,\infty}, \alpha > \frac{1}{3}$ in the periodic domain. In 2008, Cheskidov et al. [6] improved the previous results by using Littlewood-Paley decomposition. Concerning the second part of conjecture, the first proof of the existence of a square summable weak solution that does not preserve the energy is due to Scheffer in his pioneering paper [24]. A different proof was later given by Shnirelman in [23]. Recently, non-conservation solutions for the 3D incompressible Euler equations have been constructed up to the critical $\frac{1}{3}$ regularity in [5, 17]. For the compressible and incompressible Euler equations and Navier-Stokes equations, we can refer to Feireisl [14], Chen and Yu [9] and Akramov et al. [1] etc. The energy conservation for the incompressible Euler equations in bounded domains can be referred to [2, 3, 11, 21].

Concerning the energy conservation for the MHD equations, there have been a few results. Recently, Gao et al. [16] considered the local energy equation of weak solutions for the incompressible MHD equations. Guo and Tan [15] studied the energy conservation equation of the 3D incompressible MHD equations from the longitudinal and transverse terms on the basis of the energy dissipation method. Wu and Tan [27] showed that the regularity of weak solutions of the nonhomogeneous incompressible MHD equations in Besov space is sufficient to ensure the total energy conservation. Wang and Liu [25] obtained the energy conservation for weak solutions of the compressible non-resistive magnetohydrodynamic flows in a bounded domain $\Omega \subset \mathbb{R}^3$. For the ideal MHD equations, Caflisch et al. [8] proved the energy conservation in a periodic domain with no boundary effect. Kang and Lee [18] obtained the energy and cross-helicity conservation to the ideal MHD equations in the whole space. Later on, Yu [28] improved the pervious results by using the special structure of the nonlinear terms in the ideal MHD equations. Wang and Zuo [26] and Zhang [29] proved the energy conservation of weak solutions to the 3D case in a bounded domain. Bie et al. [4] studied the energy conservation of the compressible ideal MHD equations in periodic domain by using commutator estimation.

The energy conservation for the Hall-MHD equations has also attracted the attention of many researchers. Dumas and Sueur [10] studied the energy identity and magneto-helicity identity for the incompressible Hall-MHD equations in the whole space $\mathbb{R}^3$. Kang et al. [19] obtained the energy conservation for the nonhomogeneous incompressible ideal Hall-MHD equations in the periodic domain $\mathbb{T}^d$. Recently, Kang et al. [20] further considered the energy conservation of impressible Hall-MHD equations in a bounded domain.

A natural question then is when the energy conservation holds true for the compressible ideal Hall-MHD Eq (1.1). Compared with the compressible Euler equations, the ideal Hall-MHD equations have higher nonlinearity in view of the coupling of the velocity field u and the magnetic field b. Moreover, the handling of the Hall term $d_I \nabla \times \left(\frac{j \times b}{\rho}\right)$ brings us some difficulties. Specifically, we need to deal with the following item

$$\int_0^T \int_{\mathbb{T}^d} \varphi(u \times b)^{\epsilon} \cdot j^{\epsilon} dxdt$$

(see (3.8) below for details). In this paper, by using the commutator estimation similar to that used in [7] and the regularization method used in [14], we obtain sufficient conditions for energy conservation of system (1.1). Our result establishes local energy conservation for weak solutions to system (1.1) under the additional assumption that the velocity field u satisfies the condition div$u \in L^1$.

Before giving the result of this paper, we define the pressure potential as

$$P(\rho) = \rho \int_1^{\rho} \frac{p(r)}{r^2} dr. \tag{1.2}$$

The main result of this paper is stated as follows.

Theorem 1.1. *Let the space dimension $d \geq 1$ and (ρ, u, p, b) be a solution of system* (1.1) *in the sense of distributions. Assume*

$$\begin{aligned} & u, b \in B_3^{\alpha,\infty}((0,T) \times \mathbb{T}^d), \quad \rho, \rho u, j \in B_3^{\beta,\infty}((0,T) \times \mathbb{T}^d), \\ & 0 < \underline{\rho} \leq \rho \leq \overline{\rho} \quad a.e. \ in \ (0,T) \times \mathbb{T}^d \end{aligned}$$

for some constants $\underline{\rho}, \overline{\rho}$ such that

$$\beta > \max\left\{1 - 2\alpha, \frac{1}{2}(1-\alpha)\right\}.$$

Assume further that

$$\mathrm{div} u \in L^1((0,T) \times \mathbb{T}^d), \quad p \in C[\underline{\rho}, \overline{\rho}]. \tag{1.3}$$

Then the energy is locally conserved, that is

$$\begin{aligned} & \partial_t\Big(\frac{1}{2}\rho|u|^2 + \frac{1}{2}|b|^2 + P(\rho)\Big) + \mathrm{div}\Big[\frac{1}{2}\rho|u|^2 + P(\rho) + p(\rho))\Big] \\ & - \mathrm{div}[(u \times b) \times b] + \mathrm{div}\Big[d_I\Big(\frac{j \times b}{\rho} \times b\Big)\Big] = 0 \end{aligned} \tag{1.4}$$

in the sense of distributions on $(0,T) \times \mathbb{T}^d$.

2. Preliminaries

In this section, we firstly recall some properties of the Besov space $B_p^{\alpha,\infty}(\Omega)$, where $\alpha \in (0,1)$ and $\Omega = (0,T) \times \mathbb{T}^d$. The said Besov space comprises those functions ω for which the norm

$$||\omega||_{B_p^{\alpha,\infty}(\Omega)} := ||\omega||_{L^p(\Omega)} + \sup_{\xi \in \Omega} \frac{||\omega(\cdot + \xi) - \omega||_{L^p(\Omega \cap (\Omega - \xi))}}{|\xi|^\alpha} \tag{2.1}$$

is finite (here $\Omega - \xi = \{x - \xi : x \in \Omega\}$).

Let $J \in C_c^\infty(R^N)$ for $N = d$ or $N = d + 1$ (according to the choice of Ω) be a standard mollifying kernel and set

$$J^\epsilon(x) = \frac{1}{\epsilon^N} J(\frac{x}{\epsilon}),$$

with the notation $\omega^\epsilon = J^\epsilon * \omega$.

Next, we introduce two lemmas about the properties of mollifiers.

Lemma 2.1. ([13]) *If* $1 \le p < \infty$ *and* $f \in L^p_{loc}(R^+ \times \Omega)$*, then we get*

$$f^\epsilon \to f \quad in\ L^p_{loc}(R^+ \times \Omega).$$

Lemma 2.2. ([27]) *Let* $\omega \in B^{\alpha,\infty}_p(\Omega)$ *with* $1 \le p \le \infty$*, and* $\alpha \ge 0$*. Then we have*

$$\|\omega^\epsilon - \omega\|_{L^p(\Omega)} \le C\epsilon^\alpha \|\omega\|_{B^{\alpha,\infty}_p(\Omega)} \tag{2.2}$$

and

$$\|\nabla\omega^\epsilon\|_{L^p(\Omega)} \le C\epsilon^{\alpha-1}\|\omega\|_{B^{\alpha,\infty}_p(\Omega)}. \tag{2.3}$$

Finally, let us recall the following two commonly used formulas for curl, that is,

$$\nabla \times (A \times B) = (B \cdot \nabla)A - (A \cdot \nabla)B + (\nabla \cdot B)A - (\nabla \cdot B)A, \tag{2.4}$$

$$\operatorname{div}(A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B). \tag{2.5}$$

3. Proof of Theorem 1.1

We mollify the momentum Eq $(1.1)_1$ in time and space, and the corresponding symbols are described in Section 2,

$$\partial_t(\rho u)^\epsilon + \operatorname{div}(\rho u \otimes u)^\epsilon + \nabla p^\epsilon(\rho) - (j \times b)^\epsilon = 0. \tag{3.1}$$

Take a sequence $p_\delta \in C^\infty[\underline{\rho}, \overline{\rho}]$ which converges uniformly to $p \in C[\underline{\rho}, \overline{\rho}]$, that is, for each $\delta > 0$,

$$\|p - p_\delta\|_{L^\infty} \le \delta.$$

According to the definition of P in (1.2), we give the definition of P_δ as follows,

$$P_\delta = \rho \int_1^\rho \frac{p_\delta(r)}{r^2} dr. \tag{3.2}$$

It follows from (3.1) that

$$\partial_t(\rho u)^\epsilon + \operatorname{div}(\rho u \otimes u)^\epsilon + \nabla(p^\epsilon_\delta(\rho)) - (j \times b)^\epsilon = \nabla[p^\epsilon_\delta(\rho) - p^\epsilon(\rho)].$$

Let $\varphi \in C^\infty_0((0,T) \times \mathbb{T}^d)$ be a test function. Multiplying with φu^ϵ and integrating in time and space give

$$\begin{aligned}&\int_0^T \int_{\mathbb{T}^d} \partial_t(\rho u)^\epsilon \cdot \varphi u^\epsilon dxdt + \int_0^T \int_{\mathbb{T}^d} \operatorname{div}(\rho u \otimes u)^\epsilon \cdot \varphi u^\epsilon dxdt - \int_0^T \int_{\mathbb{T}^d} (j \times b)^\epsilon \cdot \varphi u^\epsilon dxdt \\ &+ \int_0^T \int_{\mathbb{T}^d} \nabla(p^\epsilon_\delta(\rho)) \cdot \varphi u^\epsilon dxdt = \int_0^T \int_{\mathbb{T}^d} \nabla[p^\epsilon_\delta(\rho) - p^\epsilon(\rho)] \cdot \varphi u^\epsilon dxdt. \end{aligned} \tag{3.3}$$

Take ϵ>0 small enough so that supp $\varphi \in (\epsilon, T - \epsilon) \times \mathbb{T}^d$. We use an appropriate commutator, as

$$\int_0^T \int_{\mathbb{T}^d} \partial_t(\rho^\epsilon u^\epsilon) \cdot \varphi u^\epsilon dxdt + \int_0^T \int_{\mathbb{T}^d} \operatorname{div}((\rho u)^\epsilon \otimes u^\epsilon) \cdot \varphi u^\epsilon dxdt$$

$$+\int_0^T\int_{\mathbb{T}^d}\varphi(u\times b)^\epsilon\cdot j^\epsilon dxdt+\int_0^T\int_{\mathbb{T}^d}\nabla(p_\delta(\rho^\epsilon))\cdot\varphi u^\epsilon dxdt$$
$$\stackrel{\text{def}}{=} I_1^\epsilon+I_2^\epsilon+I_3^\epsilon+I_4^\epsilon+I_5^\epsilon, \tag{3.4}$$

where

$$\begin{aligned}
I_1^\epsilon &= \int_0^T\int_{\mathbb{T}^d}\partial_t(\rho^\epsilon u^\epsilon-(\rho u)^\epsilon)\cdot\varphi u^\epsilon dxdt,\\
I_2^\epsilon &= \int_0^T\int_{\mathbb{T}^d}\operatorname{div}((\rho u)^\epsilon\otimes u^\epsilon-(\rho u\otimes u)^\epsilon)\cdot\varphi u^\epsilon dxdt,\\
I_3^\epsilon &= \int_0^T\int_{\mathbb{T}^d}[(j\times b)^\epsilon\cdot u^\epsilon+(u\times b)^\epsilon\cdot j^\epsilon]\varphi dxdt,\\
I_4^\epsilon &= \int_0^T\int_{\mathbb{T}^d}\nabla[p_\delta(\rho^\epsilon)-p_\delta^\epsilon(\rho)]\cdot\varphi u^\epsilon dxdt,\\
I_5^\epsilon &= \int_0^T\int_{\mathbb{T}^d}\nabla[p_\delta^\epsilon(\rho)-p^\epsilon(\rho)]\cdot\varphi u^\epsilon dxdt.
\end{aligned}$$

In what follows, we handle each item in (3.4). For the first integral term of the left hand side of (3.4), it follows that

$$\int_0^T\int_{\mathbb{T}^d}\partial_t(\rho^\epsilon u^\epsilon)\cdot\varphi u^\epsilon dxdt=\int_0^T\int_{\mathbb{T}^d}\Big(\varphi\rho_t^\epsilon|u^\epsilon|^2+\frac{1}{2}\varphi\rho^\epsilon\partial_t(|u^\epsilon|^2)\Big)dxdt. \tag{3.5}$$

For the second term, we mollify the continuity Eq $(1.1)_3$ as

$$\partial_t\rho^\epsilon+\operatorname{div}(\rho u)^\epsilon=0, \tag{3.6}$$

and compute

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d}\operatorname{div}((\rho u)^\epsilon\otimes u^\epsilon)\cdot\varphi u^\epsilon dxdt\\
&=-\int_0^T\int_{\mathbb{T}^d}((\rho u)^\epsilon\otimes u^\epsilon):\nabla(\varphi u^\epsilon)dxdt\\
&=-\int_0^T\int_{\mathbb{T}^d}\varphi((\rho u)^\epsilon\otimes u^\epsilon):\nabla(u^\epsilon)dxdt-\int_0^T\int_{\mathbb{T}^d}((\rho u)^\epsilon\otimes u^\epsilon):(\nabla\varphi\otimes u^\epsilon)dxdt\\
&=-\frac{1}{2}\int_0^T\int_{\mathbb{T}^d}\varphi(\rho u)^\epsilon\cdot\nabla(|u^\epsilon|^2)dxdt-\int_0^T\int_{\mathbb{T}^d}((\rho u)^\epsilon\cdot\nabla\varphi)|u^\epsilon|^2dxdt\\
&=\frac{1}{2}\int_0^T\int_{\mathbb{T}^d}\operatorname{div}(\varphi(\rho u)^\epsilon)|u^\epsilon|^2dxdt-\int_0^T\int_{\mathbb{T}^d}((\rho u)^\epsilon\cdot\nabla\varphi)|u^\epsilon|^2dxdt\\
&=\frac{1}{2}\int_0^T\int_{\mathbb{T}^d}(\nabla\varphi\cdot(\rho u)^\epsilon+\varphi\operatorname{div}(\rho u)^\epsilon)|u^\epsilon|^2dxdt-\int_0^T\int_{\mathbb{T}^d}((\rho u)^\epsilon\cdot\nabla\varphi)|u^\epsilon|^2dxdt\\
&=-\frac{1}{2}\int_0^T\int_{\mathbb{T}^d}\varphi\rho_t^\epsilon|u^\epsilon|^2dxdt-\frac{1}{2}\int_0^T\int_{\mathbb{T}^d}(\nabla\varphi\cdot(\rho u)^\epsilon)|u^\epsilon|^2dxdt.
\end{aligned} \tag{3.7}$$

For the third integral of the left hand side of (3.4), we first mollify the Eq $(1.1)_2$ and multiply b^ϵ as

$$\frac{1}{2}\partial_t(|b^\epsilon|^2) + d_I\nabla\times\left(\frac{j\times b}{\rho}\right)^\epsilon\cdot b^\epsilon - \nabla\times(u\times b)^\epsilon\cdot b^\epsilon = 0.$$

Then using (2.5), we get

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d}\varphi(u\times b)^\epsilon\cdot j^\epsilon dxdt\\
&=\int_0^T\int_{\mathbb{T}^d}\varphi\nabla\times(u\times b)^\epsilon\cdot b^\epsilon dxdt - \int_0^T\int_{\mathbb{T}^d}\varphi\mathrm{div}((u\times b)^\epsilon\times b^\epsilon)dxdt\\
&=\int_0^T\int_{\mathbb{T}^d}\varphi\nabla\times(u\times b)^\epsilon\cdot b^\epsilon dxdt + \int_0^T\int_{\mathbb{T}^d}\nabla\varphi\cdot((u\times b)^\epsilon\times b^\epsilon)dxdt\\
&=\frac{1}{2}\int_0^T\int_{\mathbb{T}^d}\varphi\partial_t(|b^\epsilon|^2)dxdt + \int_0^T\int_{\mathbb{T}^d}\varphi d_I\nabla\times\left(\frac{j\times b}{\rho}\right)^\epsilon\cdot b^\epsilon dxdt\\
&\quad+\int_0^T\int_{\mathbb{T}^d}\nabla\varphi\cdot[(u\times b)^\epsilon\times b^\epsilon]dxdt\\
&=-\frac{1}{2}\int_0^T\int_{\mathbb{T}^d}\varphi_t|b^\epsilon|^2dxdt + \int_0^T\int_{\mathbb{T}^d}\nabla\varphi\cdot[(u\times b)^\epsilon\times b^\epsilon]dxdt\\
&\quad+\int_0^T\int_{\mathbb{T}^d}\varphi d_I\Big[\mathrm{div}\Big(\Big(\frac{j\times b}{\rho}\Big)^\epsilon\times b^\epsilon\Big)\Big]dxdt + \int_0^T\int_{\mathbb{T}^d}\varphi d_I\Big(\frac{j\times b}{\rho}\Big)^\epsilon\cdot j^\epsilon dxdt\\
&=-\frac{1}{2}\int_0^T\int_{\mathbb{T}^d}\varphi_t|b^\epsilon|^2dxdt + \int_0^T\int_{\mathbb{T}^d}\nabla\varphi\cdot[(u\times b)^\epsilon\times b^\epsilon]dxdt\\
&\quad-\int_0^T\int_{\mathbb{T}^d}d_I\Big(\Big(\frac{j\times b}{\rho}\Big)^\epsilon\times b^\epsilon\Big)\cdot\nabla\varphi dxdt + \int_0^T\int_{\mathbb{T}^d}\varphi d_I\Big(\frac{j\times b}{\rho}\Big)^\epsilon\cdot j^\epsilon dxdt.
\end{aligned}\tag{3.8}$$

For the fourth integral, due to the chain rule and the mollified mass Eq (3.6), we observe that

$$\partial_tP_\delta(\rho^\epsilon) + u^\epsilon\nabla P_\delta(\rho^\epsilon) + P'_\delta(\rho^\epsilon)\rho^\epsilon\mathrm{div}u^\epsilon = P'_\delta(\rho^\epsilon)\mathrm{div}(\rho^\epsilon u^\epsilon - (\rho u)^\epsilon).$$

By the definition of P_δ in (3.2), we have

$$\rho^\epsilon P'_\delta(\rho^\epsilon) = P_\delta(\rho^\epsilon) + p_\delta(\rho^\epsilon),$$

After these preparations, we can compute the fourth integral as

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d}\varphi u^\epsilon\cdot\nabla p_\delta(\rho^\epsilon)dxdt\\
&=-\int_0^T\int_{\mathbb{T}^d}\nabla\varphi\cdot u^\epsilon p_\delta(\rho^\epsilon)dxdt - \int_0^T\int_{\mathbb{T}^d}\varphi p_\delta(\rho^\epsilon)\mathrm{div}u^\epsilon dxdt\\
&=-\int_0^T\int_{\mathbb{T}^d}\nabla\varphi\cdot u^\epsilon p_\delta(\rho^\epsilon)dxdt - \int_0^T\int_{\mathbb{T}^d}\varphi\left[\rho^\epsilon P'_\delta(\rho^\epsilon) - P_\delta(\rho^\epsilon)\right]\mathrm{div}u^\epsilon dxdt\\
&=-\int_0^T\int_{\mathbb{T}^d}\nabla\varphi\cdot u^\epsilon p_\delta(\rho^\epsilon)dxdt + \int_0^T\int_{\mathbb{T}^d}\varphi\left[\partial_tP_\delta(\rho^\epsilon) + u^\epsilon\cdot\nabla P_\delta(\rho^\epsilon) + P_\delta(\rho^\epsilon)\mathrm{div}u^\epsilon\right]dxdt
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^T \int_{\mathbb{T}^d} \varphi P'_\delta(\rho^\epsilon)\mathrm{div}(\rho^\epsilon u^\epsilon - (\rho u)^\epsilon) dxdt \\
&= - \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot u^\epsilon p_\delta(\rho^\epsilon) dxdt - \int_0^T \int_{\mathbb{T}^d} \varphi_t P_\delta(\rho^\epsilon) dxdt - \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot P_\delta(\rho^\epsilon) u^\epsilon dxdt \\
&\quad - \int_0^T \int_{\mathbb{T}^d} \varphi P'_\delta(\rho^\epsilon)\mathrm{div}(\rho^\epsilon u^\epsilon - (\rho u)^\epsilon) dxdt.
\end{aligned} \tag{3.9}
$$

Thus, combining (3.4), (3.5) and (3.7)–(3.9) yields

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d} \varphi_t \Big(\frac{1}{2}\rho^\epsilon |u^\epsilon|^2 + \frac{1}{2}|b^\epsilon|^2 + P_\delta(\rho^\epsilon)\Big) dxdt \\
&\quad + \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot \Big[\frac{1}{2}(\rho u)^\epsilon |u^\epsilon|^2 + p_\delta(\rho^\epsilon) u^\epsilon + P_\delta(\rho^\epsilon) u^\epsilon\Big] dxdt \\
&\quad - \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot [(u\times b)^\epsilon \times b^\epsilon] dxdt + \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot d_I \Big[\Big(\frac{j\times b}{\rho}\Big)^\epsilon \times b^\epsilon\Big] dxdt \\
&= - \sum_{i=1}^5 I_i^\epsilon - \int_0^T \int_{\mathbb{T}^d} \varphi P'_\delta(\rho^\epsilon)\mathrm{div}(\rho^\epsilon u^\epsilon - (\rho u)^\epsilon) dxdt + \int_0^T \int_{\mathbb{T}^d} \varphi d_I \Big(\frac{j\times b}{\rho}\Big)^\epsilon \cdot j^\epsilon dxdt.
\end{aligned} \tag{3.10}
$$

To prove Theorem 1.1, we need to show that the left side of (3.10) converges to the left side of (1.4) as first ϵ and then δ tend to zero. In fact, for each fixed $\delta > 0$, when $\epsilon \to 0$, using the standard properties of mollification, we have the following limit:

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d} \varphi_t \Big(\frac{1}{2}\rho^\epsilon |u^\epsilon|^2 + \frac{1}{2}|b^\epsilon|^2 + P_\delta(\rho^\epsilon)\Big) dxdt \\
&\quad + \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot \Big[\frac{1}{2}(\rho u)^\epsilon |u^\epsilon|^2 + p_\delta(\rho^\epsilon) u^\epsilon + P_\delta(\rho^\epsilon) u^\epsilon\Big] dxdt \\
&\quad - \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot [(u\times b)^\epsilon \times b^\epsilon] dxdt + \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot d_I \Big[\Big(\frac{j\times b}{\rho}\Big)^\epsilon \times b^\epsilon\Big] dxdt \\
&\to \int_0^T \int_{\mathbb{T}^d} \varphi_t \Big(\frac{1}{2}\rho |u|^2 + \frac{1}{2}|b|^2 + P_\delta(\rho)\Big) dxdt \\
&\quad + \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot \Big[\frac{1}{2}\rho |u|^2 + p_\delta(\rho) + P_\delta(\rho)\Big] u dxdt \\
&\quad - \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot [(u\times b) \times b] dxdt + \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot d_I \Big[\Big(\frac{j\times b}{\rho}\Big) \times b\Big] dxdt.
\end{aligned} \tag{3.11}
$$

Here, we only give the proof of the following limit :

$$
\int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot P_\delta(\rho^\epsilon) u^\epsilon dxdt \to \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot P_\delta(\rho) u dxdt \tag{3.12}
$$

as $\epsilon \to 0$. To prove (3.12), by the definition of P_δ in (3.2), we set

$$
g'(r) = \frac{p_\delta(r)}{r^2}.
$$

By $p_\delta(\rho) \in C^\infty[\underline{\rho}, \overline{\rho}]$, we can obtain

$$g'(r) \in C(0, \overline{\rho}],\ \ g(r) \in C(0, \overline{\rho}]$$

and

$$P_\delta(\rho) = \rho(g(\rho) - g(1)),$$
$$P_\delta(\rho^\epsilon) = \rho^\epsilon(g(\rho^\epsilon) - g(1)).$$

So,

$$\begin{aligned}
&P_\delta(\rho^\epsilon)u^\epsilon - P_\delta(\rho)u \\
&= (P_\delta(\rho^\epsilon) - P_\delta(\rho))u^\epsilon + P_\delta(\rho)(u^\epsilon - u) \\
&= [\rho^\epsilon(g(\rho^\epsilon) - g(1)) - \rho(g(\rho) - g(1))]u^\epsilon + P_\delta(\rho)(u^\epsilon - u) \\
&= [\rho^\epsilon(g(\rho^\epsilon) - g(\rho)) + g(\rho)(\rho^\epsilon - \rho) + g(1)(\rho - \rho^\epsilon)]u^\epsilon + P_\delta(\rho)(u^\epsilon - u) \\
&= [g'(\xi)(\rho^\epsilon - \rho)\rho^\epsilon + g(\rho)(\rho^\epsilon - \rho) + g(1)(\rho - \rho^\epsilon)]u^\epsilon + P_\delta(\rho)(u^\epsilon - u),
\end{aligned} \tag{3.13}$$

we use the differential mean value theorem in the last equation, where ξ is a function between ρ and ρ^ϵ. According to (3.13), we get

$$\begin{aligned}
&\Big|\int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot P_\delta(\rho^\epsilon)u^\epsilon dxdt - \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot P_\delta(\rho)udxdt\Big| \\
&\le \int_0^T \int_{\mathbb{T}^d} |\nabla\varphi(P_\delta(\rho^\epsilon)u^\epsilon - P_\delta(\rho)u)|dxdt \\
&\le C(\underline{\rho}, \overline{\rho}, T, \mathbb{T}^d)\|\varphi\|_{C^1}\big(\|\rho^\epsilon - \rho\|_{L^3}\|\rho^\epsilon\|_{L^3}\|u^\epsilon\|_{L^3} + \|\rho^\epsilon - \rho\|_{L^3}\|u^\epsilon\|_{L^3} \\
&\quad + \|\rho^\epsilon - \rho\|_{L^3}\|u^\epsilon\|_{L^3} + \|u^\epsilon - u\|_{L^3}\big) \\
&\le C(\underline{\rho}, \overline{\rho}, T, \mathbb{T}^d)\|\varphi\|_{C^1}\big(\epsilon^\beta\|\rho\|^2_{B_3^{\beta,\infty}}\|u\|_{B_3^{\alpha,\infty}} + \epsilon^\beta\|\rho\|_{B_3^{\beta,\infty}}\|u\|_{B_3^{\alpha,\infty}} \\
&\quad + \epsilon^\beta\|\rho\|_{B_3^{\beta,\infty}}\|u\|_{B_3^{\alpha,\infty}} + \epsilon^\alpha\|u\|_{B_3^{\alpha,\infty}}\big) \\
&\to 0
\end{aligned}$$

as $\epsilon \to 0$, which completes the proof of (3.12).

Next, we will show that the right hand side of (3.11) converges to

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d} \varphi_t\Big(\frac{1}{2}\rho|u|^2 + \frac{1}{2}|b|^2 + P(\rho)\Big)dxdt + \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot \Big[\frac{1}{2}\rho|u|^2 + p(\rho) + P(\rho)\Big]udxdt \\
&- \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot [(u \times b) \times b]dxdt + \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot d_I\Big[\Big(\frac{j \times b}{\rho}\Big) \times b\Big]dxdt
\end{aligned} \tag{3.14}$$

as $\delta \to 0$.

From the choice of p_δ, we have

$$\Big|\int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot (p_\delta(\rho) - p(\rho))udxdt\Big| \le C\|\varphi\|_{C^1}\|p_\delta - p\|_{L^\infty}\|u\|_{L^3} \le C(\varphi, u)\delta. \tag{3.15}$$

For the terms containing $P_\delta(\rho)$, notice that

$$|P_\delta(\rho)-P(\rho)| \le \rho\int_1^\rho \frac{|p_\delta(r)-p(r)|}{r^2}dr \le \|p_\delta-p\|_{L^\infty}\rho\Big|\int_1^\rho \frac{1}{r^2}dr\Big| \le (1+\rho)\delta.$$

Hence we can estimate

$$\Big|\int_0^T\int_{\mathbb{T}^d}\varphi_t(P_\delta(\rho)-P(\rho))dxdt\Big| \le C\|\varphi\|_{C^1}(1+\|\rho\|_{L^1})\delta \le C(\varphi)\delta, \tag{3.16}$$

$$\Big|\int_0^T\int_{\mathbb{T}^d}\nabla\varphi\cdot(P_\delta(\rho)-P(\rho))udxdt\Big| \le C\|\varphi\|_{C^1}\|1+\rho\|_{L^\infty}\|u\|_{L^3}\delta \le C(\varphi,u)\delta. \tag{3.17}$$

Combining with (3.15)–(3.17), when $\delta\to 0$, we obtain the right side of (3.11) converges to (3.14). Thus, according to (3.10), we need to show that

$$I_i^\epsilon(i=1,2,3,4,5),\ \int_0^T\int_{\mathbb{T}^d}\varphi P_\delta'(\rho^\epsilon)\mathrm{div}(\rho^\epsilon u^\epsilon-(\rho u)^\epsilon)dxdt$$

and

$$\int_0^T\int_{\mathbb{T}^d}\varphi d_I\Big(\frac{j\times b}{\rho}\Big)^\epsilon\cdot j^\epsilon dxdt$$

converge to zero as first ϵ and then δ tend to zero.

First it is easy to check

$$\int_0^T\int_{\mathbb{T}^d}\varphi d_I\Big(\frac{j\times b}{\rho}\Big)^\epsilon\cdot j^\epsilon dxdt \to \int_0^T\int_{\mathbb{T}^d}\varphi d_I\Big(\frac{j\times b}{\rho}\Big)\cdot jdxdt = 0 \tag{3.18}$$

as $\epsilon\to 0$. For I_1^ϵ, we observe that

$$\begin{aligned}\rho^\epsilon u^\epsilon-(\rho u)^\epsilon &= (\rho^\epsilon-\rho)(u^\epsilon-u)-\int_{-\epsilon}^\epsilon\int_{\mathbb{T}^d\cap B(0,\epsilon)}J^\epsilon(\tau,\xi)(\rho(t-\tau,x-\xi)\\ &\quad -\rho(t,x))(u(t-\tau,x-\xi)-u(t,x))d\xi d\tau\\ &\overset{\text{def}}{=} I_{11}^\epsilon+I_{12}^\epsilon.\end{aligned} \tag{3.19}$$

Therefore,

$$I_1^\epsilon = \int_0^T\int_{\mathbb{T}^d}\partial_t I_{11}^\epsilon\cdot\varphi u^\epsilon dxdt+\int_0^T\int_{\mathbb{T}^d}\partial_t I_{12}^\epsilon\cdot\varphi u^\epsilon dxdt. \tag{3.20}$$

Applying (2.2), (2.3), integration by parts and Hölder inequality, we get

$$\begin{aligned}&\Big|\int_0^T\int_{\mathbb{T}^d}\partial_t I_{11}^\epsilon\cdot\varphi u^\epsilon dxdt\Big|\\ &=\Big|\int_0^T\int_{\mathbb{T}^d}\partial_t[(\rho^\epsilon-\rho)(u^\epsilon-u)]\cdot\varphi u^\epsilon dxdt\Big|\\ &\le \int_0^T\int_{\mathbb{T}^d}|\varphi_t(\rho^\epsilon-\rho)(u^\epsilon-u)\cdot u^\epsilon|dxdt+\int_0^T\int_{\mathbb{T}^d}|\varphi(\rho^\epsilon-\rho)(u^\epsilon-u)\cdot\partial_t u^\epsilon|dxdt\end{aligned}$$

$$
\begin{aligned}
&\le \|\varphi\|_{C^1}\|\rho^\epsilon-\rho\|_{L^3(\Omega)}\cdot\|u^\epsilon-u\|_{L^3(\Omega)}\cdot\|u^\epsilon\|_{L^3(\Omega)}\\
&\quad+\|\varphi\|_{C^0}\|\rho^\epsilon-\rho\|_{L^3(\Omega)}\cdot\|u^\epsilon-u\|_{L^3(\Omega)}\cdot\|\nabla u^\epsilon\|_{L^3(\Omega)}\\
&\le C\|\varphi\|_{C^1}\epsilon^\beta\epsilon^\alpha\|\rho\|_{B_3^{\beta,\infty}}\|u\|^2_{B_3^{\alpha,\infty}}+C\|\varphi\|_{C^0}\epsilon^\beta\epsilon^\alpha\epsilon^{\alpha-1}\|\rho\|_{B_3^{\beta,\infty}}\|u\|^2_{B_3^{\alpha,\infty}}\to 0 \qquad (3.21)
\end{aligned}
$$

as $\epsilon\to 0$ for any $2\alpha+\beta>1$.

For the second integral of the right hand side of (3.20), using Fubini theorem, (2.2) and (2.3), Hölder inequality and integration by parts, we estimate

$$
\begin{aligned}
&\Big|\int_0^T\int_{\mathbb{T}^d}\partial_t I_{12}^\epsilon\cdot\varphi u^\epsilon dxdt\Big|\\
&=\Big|\int_0^T\int_{\mathbb{T}^d}\partial_t\Big[\int_{-\epsilon}^{\epsilon}\int_{\mathbb{T}^d\cap B(0,\epsilon)}J^\epsilon(\tau,\xi)(\rho(t-\tau,x-\xi)\\
&\quad-\rho(t,x))(u(t-\tau,x-\xi)-u(t,x))d\xi d\tau\Big]\cdot\varphi u^\epsilon dxdt\Big|\\
&=\Big|\int_0^T\int_{\mathbb{T}^d}\int_{-\epsilon}^{\epsilon}\int_{\mathbb{T}^d\cap B(0,\epsilon)}J^\epsilon(\tau,\xi)(\rho(t-\tau,x-\xi)\\
&\quad-\rho(t,x))(u(t-\tau,x-\xi)-u(t,x))\cdot\partial_t(\varphi(t,x)u^\epsilon(t,x))d\xi d\tau dxdt\Big|\\
&\le\int_{-\epsilon}^{\epsilon}\int_{\mathbb{T}^d\cap B(0,\epsilon)}\int_0^T\int_{\mathbb{T}^d}|J^\epsilon(\tau,\xi)(\rho(t-\tau,x-\xi)\\
&\quad-\rho(t,x))(u(t-\tau,x-\xi)-u(t,x))\cdot\varphi_t(t,x)u^\epsilon(t,x)|dxdtd\xi d\tau\\
&\quad+\int_{-\epsilon}^{\epsilon}\int_{\mathbb{T}^d\cap B(0,\epsilon)}\int_0^T\int_{\mathbb{T}^d}|J^\epsilon(\tau,\xi)(\rho(t-\tau,x-\xi)\\
&\quad-\rho(t,x))(u(t-\tau,x-\xi)-u(t,x))\cdot\varphi(t,x)\partial_t u^\epsilon(t,x)|dxdtd\xi d\tau\\
&\le\|\varphi\|_{C^1}\frac{C}{\epsilon^N}\int_{-\epsilon}^{\epsilon}\int_{\mathbb{T}^d\cap B(0,\epsilon)}\|(\rho(t-\tau,x-\xi)-\rho(t,x))\|_{L^3(V)}\\
&\quad\times\|u(t-\tau,x-\xi)-u(t,x)\|_{L^3(V)}\|u^\epsilon(t,x)\|_{L^3(V)}d\xi d\tau\\
&\quad+\|\varphi\|_{C^0}\frac{C}{\epsilon^N}\int_{-\epsilon}^{\epsilon}\int_{\mathbb{T}^d\cap B(0,\epsilon)}\|(\rho(t-\tau,x-\xi)-\rho(t,x))\|_{L^3(V)}\\
&\quad\times\|u(t-\tau,x-\xi)-u(t,x)\|_{L^3(V)}\|\partial_t u^\epsilon(t,x)\|_{L^3(V)}d\xi d\tau\\
&\le C\|\varphi\|_{C^1}\epsilon^\beta\epsilon^\alpha\|\rho\|_{B_3^{\beta,\infty}}\|u\|^2_{B_3^{\alpha,\infty}}+C\|\varphi\|_{C^0}\epsilon^\beta\epsilon^\alpha\epsilon^{\alpha-1}\|\rho\|_{B_3^{\beta,\infty}}\|u\|^2_{B_3^{\alpha,\infty}}\to 0 \qquad (3.22)
\end{aligned}
$$

as $\epsilon\to 0$ for any $2\alpha+\beta>1$, where $V=\Omega\cap(\Omega+(\tau,\xi))$, $|(\tau,\xi)|<\epsilon$.

The estimate for I_2^ϵ is similar,

$$
\begin{aligned}
&(\rho u)^\epsilon\otimes u^\epsilon-(\rho u\otimes u)^\epsilon\\
&=((\rho u)^\epsilon-(\rho u))\otimes(u^\epsilon-u)-\int_{-\epsilon}^{\epsilon}\int_{\mathbb{T}^d\cap B(0,\epsilon)}J^\epsilon(\tau,\xi)((\rho u)(t-\tau,x-\xi)\\
&\quad-(\rho u)(t,x))\otimes(u(t-\tau,x-\xi)-u(t,x))d\xi d\tau\\
&\overset{\text{def}}{=} I_{21}^\epsilon+I_{22}^\epsilon.
\end{aligned}
$$

Thus,

$$I_2^\epsilon = \int_0^T \int_{\mathbb{T}^d} \mathrm{div}(I_{21}^\epsilon) \cdot \varphi u^\epsilon dxdt + \int_0^T \int_{\mathbb{T}^d} \mathrm{div}(I_{22}^\epsilon) \cdot \varphi u^\epsilon dxdt. \tag{3.23}$$

Thanks to (2.2), (2.3), integration by parts and Hölder inequality, we get

$$\begin{aligned}
&\Big| \int_0^T \int_{\mathbb{T}^d} \mathrm{div}(I_{21}^\epsilon) \cdot \varphi u^\epsilon dxdt \Big| \\
&= \Big| \int_0^T \int_{\mathbb{T}^d} \mathrm{div}[((\rho u)^\epsilon - \rho u) \otimes (u^\epsilon - u)] \cdot \varphi u^\epsilon dxdt \Big| \\
&\le \int_0^T \int_{\mathbb{T}^d} |((\rho u)^\epsilon - \rho u) \otimes (u^\epsilon - u) u^\epsilon \cdot \nabla \varphi| dxdt \\
&\quad + \int_0^T \int_{\mathbb{T}^d} |\varphi((\rho u)^\epsilon - \rho u) \otimes (u^\epsilon - u) : \nabla u^\epsilon| dxdt \\
&\le C\|\varphi\|_{C^1} \epsilon^\beta \epsilon^\alpha \|\rho u\|_{B_3^{\beta,\infty}} \|u\|^2_{B_3^{\alpha,\infty}} + C\|\varphi\|_{C^0} \epsilon^\beta \epsilon^\alpha \epsilon^{\alpha-1} \|\rho u\|_{B_3^{\beta,\infty}} \|u\|^2_{B_3^{\alpha,\infty}} \to 0
\end{aligned} \tag{3.24}$$

as $\epsilon \to 0$ for any $2\alpha + \beta > 1$.

For the second integral of the right hand side of (3.23), we have

$$\begin{aligned}
&\Big| \int_0^T \int_{\mathbb{T}^d} \mathrm{div}(I_{22}^\epsilon) \cdot \varphi u^\epsilon dxdt \Big| \\
&= \Big| \int_0^T \int_{\mathbb{T}^d} \mathrm{div}\Big[\int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d \cap B(0,\epsilon)} J^\epsilon(\tau,\xi)((\rho u)(t-\tau, x-\xi) - (\rho u)(t,x)) \\
&\quad \otimes (u(t-\tau, x-\xi) - u(t,x)) d\xi d\tau \Big] \cdot \varphi u^\epsilon dxdt \Big| \\
&\le C\|\varphi\|_{C^1} \epsilon^\beta \epsilon^\alpha \|\rho u\|_{B_3^{\beta,\infty}} \|u\|^2_{B_3^{\alpha,\infty}} + C\|\varphi\|_{C^0} \epsilon^\beta \epsilon^\alpha \epsilon^{\alpha-1} \|\rho u\|_{B_3^{\beta,\infty}} \|u\|^2_{B_3^{\alpha,\infty}} \to 0
\end{aligned} \tag{3.25}$$

as $\epsilon \to 0$ for any $2\alpha + \beta > 1$.

Next we deal with I_3^ϵ. Denote

$$\begin{aligned}
g_1 &= (j \times b)^\epsilon \cdot u^\epsilon - (j^\epsilon \times b^\epsilon) \cdot u^\epsilon, \\
g_2 &= (u \times b)^\epsilon \cdot j^\epsilon - (u^\epsilon \times b^\epsilon) \cdot j^\epsilon.
\end{aligned}$$

Thus, one writes

$$I_3^\epsilon = \int_0^T \int_{\mathbb{T}^d} \varphi g_1 dxdt + \int_0^T \int_{\mathbb{T}^d} \varphi g_2 dxdt \overset{\text{def}}{=} I_{31}^\epsilon + I_{32}^\epsilon.$$

Let us calculate I_{31}^ϵ first, and I_{32}^ϵ the same way. Similar to the estimate of I_1^ϵ and I_2^ϵ, we derive

$$\begin{aligned}
&j^\epsilon \times b^\epsilon - (j \times b)^\epsilon \\
&= (j^\epsilon - j) \times (b^\epsilon - b) - \int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d \cap B(0,\epsilon)} J^\epsilon(\tau,\xi) j(t-\tau, x-\xi) \\
&\quad - j(t,x)) \times (b(t-\tau, x-\xi) - b(t,x)) d\xi d\tau \\
&\overset{\text{def}}{=} R_1^\epsilon + R_2^\epsilon.
\end{aligned}$$

So,

$$I_{31}^{\epsilon}=-\int_0^T\int_{\mathbb{T}^d}R_1^{\epsilon}\cdot\varphi u^{\epsilon}dxdt-\int_0^T\int_{\mathbb{T}^d}R_2^{\epsilon}\cdot\varphi u^{\epsilon}dxdt. \tag{3.26}$$

Using (2.2) and Hölder inequality, we get

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{T}^d}R_1^{\epsilon}\cdot\varphi u^{\epsilon}dxdt\right|&=\left|\int_0^T\int_{\mathbb{T}^d}[(j^{\epsilon}-j)\times(b^{\epsilon}-b)]\cdot\varphi u^{\epsilon}dxdt\right|\\&\le C\|\varphi\|_{C^0}\epsilon^{\beta}\epsilon^{\alpha}\|j\|_{B_3^{\beta,\infty}}\|b\|_{B_3^{\alpha,\infty}}\|u\|_{B_3^{\alpha,\infty}}\to 0\end{aligned} \tag{3.27}$$

as $\epsilon\to 0$.

For the second integral of the right hand side of (3.26), we use (2.2), Hölder inequality and Fubini theorem to get that

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{T}^d}R_2^{\epsilon}\cdot\varphi u^{\epsilon}dxdt\right|&=\left|\int_0^T\int_{\mathbb{T}^d}\int_{-\epsilon}^{\epsilon}\int_{\mathbb{T}^d\cap B(0,\epsilon)}J^{\epsilon}(\tau,\xi)(j(t-\tau,x-\xi)-j(t,x))\right.\\&\quad\left.\times(b(t-\tau,x-\xi)-b(t,x))d\xi d\tau\cdot\varphi u^{\epsilon}dxdt\right|\\&\le C\|\varphi\|_{C^0}\epsilon^{\beta}\epsilon^{\alpha}\|j\|_{B_3^{\beta,\infty}}\|b\|_{B_3^{\alpha,\infty}}\|u\|_{B_3^{\alpha,\infty}}\to 0\end{aligned} \tag{3.28}$$

as $\epsilon\to 0$.

For I_4^{ϵ}, we observe that if $p_\delta\in C^{\infty}[\underline{\rho},\overline{\rho}]$, then

$$|p_\delta(h)-p_\delta(h_0)-p_\delta'(h_0)(h-h_0)|\le C(h-h_0)^2$$

for any $h,h_0\in[\underline{\rho},\overline{\rho}]$. Note that the constant C can be chosen independently of h,h_0. Therefore

$$|p_\delta(\rho^{\epsilon}(t,x))-p_\delta(\rho(t,x))-p_\delta'(\rho(t,x))(\rho^{\epsilon}(t,x)-\rho(t,x))|\le C(\rho^{\epsilon}(t,x)-\rho(t,x))^2, \tag{3.29}$$

and similarly,

$$|p_\delta(\rho(t,y))-p_\delta(\rho(t,x))-p_\delta'(\rho(t,x))(\rho(t,y)-\rho(t,x))|\le C(\rho(t,y)-\rho(t,x))^2. \tag{3.30}$$

Applying convolution with respect to y to (3.30) we get, after invoking Jensen's inequality:

$$|p_\delta^{\epsilon}(\rho(t,x))-p_\delta(\rho(t,x))-p_\delta'(\rho(t,x))(\rho^{\epsilon}(t,x)-\rho(t,x))|\le C(\rho(t,y)-\rho(t,x))^2*_y J^{\epsilon}, \tag{3.31}$$

where $|x-y|\le\epsilon$. According to (3.29) and (3.31), one gets

$$|p_\delta(\rho^{\epsilon}(t,x))-p_\delta^{\epsilon}(\rho(t,x))|\le C(\rho^{\epsilon}(t,x)-\rho(t,x))^2+C(\rho(t,y)-\rho(t,x))^2*_y J^{\epsilon}. \tag{3.32}$$

We estimate

$$\begin{aligned}|I_4^{\epsilon}|&=\left|\int_0^T\int_{\mathbb{T}^d}\nabla[p_\delta(\rho^{\epsilon})-p_\delta^{\epsilon}(\rho)]\cdot\varphi u^{\epsilon}dxdt\right|\\&\le\int_0^T\int_{\mathbb{T}^d}|\varphi[p_\delta(\rho^{\epsilon})-p_\delta^{\epsilon}(\rho)]\mathrm{div}u^{\epsilon}|dxdt+\int_0^T\int_{\mathbb{T}^d}|[p_\delta(\rho^{\epsilon})-p_\delta^{\epsilon}(\rho)]u^{\epsilon}\cdot\nabla\varphi|dxdt\end{aligned}$$

$$\leq C\|\varphi\|_{C^0}\epsilon^{2\beta}\epsilon^{\alpha-1}\|\rho\|^2_{B_3^{\beta,\infty}}\|u\|_{B_3^{\alpha,\infty}} + C\|\varphi\|_{C^1}\epsilon^{2\beta}\epsilon^{\alpha}\|\rho\|^2_{B_3^{\beta,\infty}}\|u\|_{B_3^{\alpha,\infty}} \to 0 \tag{3.33}$$

as $\epsilon \to 0$ for any $2\beta + \alpha > 1$. Next, we show that I_5^ϵ converges to zero as first ϵ and then δ tend to zero,

$$\begin{aligned}
|I_5^\epsilon| &\leq \left| \int_0^T \int_{\mathbb{T}^d} \nabla[p_\delta^\epsilon(\rho) - p^\epsilon(\rho)]\varphi u^\epsilon dxdt \right| \\
&\leq \int_0^T \int_{\mathbb{T}^d} |[p_\delta^\epsilon(\rho) - p^\epsilon(\rho)]\varphi \mathrm{div} u^\epsilon| dxdt + \int_0^T \int_{\mathbb{T}^d} |[p_\delta^\epsilon(\rho) - p^\epsilon(\rho)]\nabla\varphi u^\epsilon| dxdt \\
&\leq C\|\varphi\|_{C^0}\|(p_\delta - p)^\epsilon\|_{L^\infty}\|\mathrm{div} u^\epsilon\|_{L^1} + C\|\varphi\|_{C^1}\|(p_\delta - p)^\epsilon\|_{L^\infty}\|u^\epsilon\|_{L^1} \\
&\leq C\|\varphi\|_{C^0}\|p_\delta - p\|_{L^\infty}\|\mathrm{div} u\|_{L^1} + C\|\varphi\|_{C^1}\|p_\delta - p\|_{L^\infty}\|u\|_{L^1} \\
&\leq 2C\delta \to 0
\end{aligned} \tag{3.34}$$

as $\delta \to 0$. Finally, let us estimate

$$\int_0^T \int_{\mathbb{T}^d} \varphi P_\delta'(\rho^\epsilon)\mathrm{div}(\rho^\epsilon u^\epsilon - (\rho u)^\epsilon) dxdt. \tag{3.35}$$

We use (3.19) to split (3.35) into two parts, so we can estimate the first part as

$$\begin{aligned}
\left| \int_0^T \int_{\mathbb{T}^d} \varphi P_\delta'(\rho^\epsilon)\mathrm{div} I_{11}^\epsilon dxdt \right| &= \left| \int_0^T \int_{\mathbb{T}^d} \varphi P_\delta'(\rho^\epsilon)\mathrm{div}[(\rho^\epsilon - \rho^\epsilon)(u^\epsilon - u^\epsilon)] dxdt \right| \\
&\leq \int_0^T \int_{\mathbb{T}^d} |\nabla\varphi(\rho^\epsilon - \rho^\epsilon)(u^\epsilon - u^\epsilon)P_\delta'(\rho^\epsilon)| dxdt \\
&\quad + \int_0^T \int_{\mathbb{T}^d} |\varphi(\rho^\epsilon - \rho^\epsilon)(u^\epsilon - u^\epsilon)P_\delta''(\rho^\epsilon)\nabla\rho^\epsilon| dxdt \\
&\leq C\|\varphi\|_{C^0}\epsilon^\beta\epsilon^\alpha\|\rho\|_{B_3^{\beta,\infty}}\|u\|_{B_3^{\alpha,\infty}} \\
&\quad + C\|\varphi\|_{C^1}\epsilon^\beta\epsilon^\alpha\epsilon^{\beta-1}\|\rho\|^2_{B_3^{\beta,\infty}}\|u\|_{B_3^{\alpha,\infty}} \to 0
\end{aligned} \tag{3.36}$$

when $\epsilon \to 0$ for any $2\beta + \alpha > 1$. The second part is estimated similarly. Thus, combining (3.18)–(3.28) and (3.33)–(3.36), we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d} \varphi_t\Big(\frac{1}{2}\rho|u|^2 + \frac{1}{2}|b|^2 + P(\rho)\Big) dxdt + \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot \Big[\Big(\frac{1}{2}\rho|u|^2 + p(\rho) + P(\rho)\Big)u\Big] dxdt \\
&\quad - \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot [(u \times b) \times b] dxdt + \int_0^T \int_{\mathbb{T}^d} \nabla\varphi \cdot d_I\Big(\Big(\frac{j \times b}{\rho}\Big) \times b\Big) dxdt \\
&= 0
\end{aligned}$$

as $\epsilon \to 0$, which completes the proof of Theorem 1.1.

4. Conclusions

In this paper, we study the regularity and energy conservation of the weak solutions for compressible ideal Hall-MHD equations, where $(t, x) \in (0, T) \times \mathbb{T}^d (d \geq 1)$. Compared with the compressible Euler equations, the ideal Hall-MHD equations have higher nonlinearity in view of the coupling of the velocity field u and the magnetic field b. Then, by exploring the special structure of the nonlinear terms in the model, we obtain the sufficient conditions for energy conservation under the additional assumption that the velocity field u satisfies the condition $\mathrm{div}u \in L^1$. Our main strategy relies on commutator estimates.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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