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Research article

Ćirić-Reich-Rus type weakly contractive mappings and related fixed point results in modular-like spaces with application

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Abstract: In this article, we define a new space named the modular-like space with its related concepts to prove the existence of a fixed point and a point of coincidence for mappings on this space. Also, we defined Ćirić-Reich-Rus type weakly contractive mappings on modular-like spaces and discussed some conditions that guarantee the existence of the fixed points for these kind of mappings. Some examples are also provided to elaborate the usability of our main results. It is worth mentioning that a modular-like space is a generalization of a modular space, thus our theorems are more general and applicable than the fixed point theorems on modular spaces.

Keywords: modular space; fixed point; point of coincidence; integral equation
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1. Introduction

The fixed point theory is rapidly growing because it has several applications in different branches of science (for details see [9, 10, 13, 15, 37, 38, 45–47, 49]). In 1922, Polish mathematician Banach stated and proved the famous "Banach contraction principle" [9] which is one of the leading results in fixed point theory. This famous theorem has been generalized by several authors in different spaces with various conditions (see [34–36, 39, 40]). Khamsi et al. [20, 22, 23] generalized this result in a modular space. This work has been extended by several author's in different directions. On the other hand, Kuaket and Kumam [25], Mongkolkeha and Kumam [31–33] presented some fixed points on this space with operators satisfying generalized contraction or nonexpansive mappings on convex modular spaces [26]. Later on, Khamsi et al. [4, 21] explored new fixed point results for nonexpansive mappings and an asymptotic point-wise nonexpansive mapping on modular spaces. Further, on the basis of this result in [2], a likeness of DeMarr's that is mean common fixed point of two symmetric Banach mappings on modular spaces is obtained. Almost all researchers focus on key properties of a modular, convexity and Fatou property [3, 11, 22, 27–30].

Rhoades [42] introduced the notion of φ-contractive mappings and obtained some common fixed point theorems (for further study, see [16, 41]). Recently, $\mathfrak{g}$-interpolation over Ćirić-Reich-Rus type contraction and weakly contraction are defined [14].

In late 19^{th} century, fixed point theory is shown to be successful in challenging problems and has contributed significantly to many real-world problems, various strong fixed point results are proved under strong assumptions. Particularly, in modular spaces, some of these assumptions can lead to have some induced norms. For example, some assumptions do not often hold in practice or can lead to some reformulations as a particular problem in normed vector spaces. A recent trend of research has been dedicated to studying the fundamentals of fixed point theorems and relaxing the assumptions (used to prove these results) with the ambition of relaxing the convexity of the modular spaces further.

It is a well-known fact that a mapping which satisfies the Banach contraction principle is necessarily a continuous mapping. Therefore, it was natural to wonder that in a complete metric space, a discontinuous mapping which satisfies somewhat similar contractive conditions may have a fixed point. Kannan [16] answered to this problem positively by introducing a new type of contraction. In 2018, Karapinar [17] introduced the concept of interpolation Kannan-type contraction, this concept appealed to many researchers to investigate and generalized the interpolation type contractive mappings in various contractions like interpolative Ćirić-Reich-Rus type contraction [41], interpolative Hardy Rogers type contraction [18, 19]. Several fixed point results are proved using these generalized interpolative type contractions in metric spaces and Branciari distance spaces.

In this paper, we define a new notion "modular- like space" which is a generalization of a modular space (for details, see [5, 6, 8, 24, 34, 44]) without using the concept of the convexity, which was a very strong condition used in modular space. We state and prove Banach fixed point theorem in modular-like space without using the extra assumptions. After that, we provide some sufficient conditions for the existence of a fixed point or a point of coincidence for some mappings which satisfy $\mathfrak{g}$-Hardy Rogers type contraction or $\mathfrak{g}$-ϕ-Hardy Rogers type contraction, or Ćirić-Reich-Rus type contraction without using strong assumptions on the modular-like space. This paper starts with Section 2, which includes some new definitions, two handy lemmas, and four main theorems on modular-like spaces. In Theorem 2.21, we generalizes interpolation Ćirić-Reich-Rus type contraction mapping and obtain fixed point of involved contraction in modular-like space without strong assumption of convexity as in modular spaces. Further, we provide some examples to support our idea of modular-like space and obtained results. Finally, in Section 3, an application of our main result to the integral equation is provided.

2. Modular-like spaces and related fixed point results

In this section, we provide a definition of generalized modular-like metric space in order to explain the connection between this definition and our new definition "modular- like space", further we provide suitable examples, remarks, and lemmas, which are required to obtain our main results.

Definition 2.1. [48] Let X be a non-empty set. A function $D : (0, +\infty) \times X \times X \to [0, +\infty]$ is said to be a generalized modular-like metric on X, if it satisfies the following axioms:

(1) If $D_\lambda(x, y) = 0$ for some $\lambda > 0$, then $x = y$ for all $x, y \in X$;

(2) $D_\lambda(x, y) = D_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$;

(3) There exists $C > 0$ such that, if $(x, y) \in X \times X$, $(x_n) \subset X$ with $\lim_{n\to+\infty} D_\lambda(x_n, x) = 0$ for some $\lambda > 0$, then $D_\lambda(x, y) \leq C \limsup_{n\to+\infty} D_\lambda(x_n, y)$.

The pair (X, D) is said to be a generalized modular-like metric space.

Now, we are presenting the definition of a modular-like space.

Definition 2.2. A modular-like space is a linear space X together with a function ϱ (called a modular-like space (X, ϱ)) which assigns a real number $\varrho(u)$ to every u belongs X satisfying the following axioms:

(1) $\varrho(u) = 0$ implies $u = 0$,

(2) $\varrho(-u) = \varrho(u)$,

(3) $\varrho(au + bv) \leq \varrho(u) + \varrho(v)$, for every $a, b \in [0, 1]$ such that $a + b = 1$.

Note that, a modular-like space on X satisfies all properties of a modular except that $\varrho(0)$ may be positive, for further explanation, see the following example.

Example 2.3. Let

$$\varrho(u) = \begin{cases} 2, & \text{if } u = 0, \\ 1, & \text{otherwise}, \end{cases}$$

where $u \in \mathbb{R}$. Clearly, $(\mathbb{R}, \varrho)$ is a modular-like space, but as $\varrho(u) = 2$ whenever $u = 0$, then $(\mathbb{R}, \varrho)$ is not a modular space.

For a modular-like space (X, ϱ), a function w_ϱ on $\mathbb{R}^+$ is said a growth function (for more details, see [12]) and is defined as:

$$w_\varrho(x) = \inf\{w \in \mathbb{R} : \varrho(xu) \leq w\varrho(u) : u \in X,\ 0 < \varrho(u)\}.$$

Moreover, ϱ is said to satisfy the Fatou property, if

$$\varrho(x - y) \leq \liminf_n \varrho(x_n - y),$$

whenever (x_n) is a sequence of X and ϱ-convergent to $x \in X$ and $y \in X$.

Next, we show that a general modular-like metric may be induced with a modular-like. Let (X, ϱ) be a modular-like space. Define $D : (0, +\infty) \times X \times X \to [0, +\infty]$ as

$$D_\lambda(x, y) = \varrho(\frac{x - y}{\lambda}),$$

where $\lambda \in (0, +\infty)$. If ϱ satisfies the Fatou property, then $D_\lambda(x, y)$ has all axioms of Definition 2.1 (for more details see [48]).

Note that, we will not use the Fatou property for the modular-like space in our results, so our modular-like does not induce a modular-like metric. In our results, we suppose that $w_\varrho(2) < +\infty$. To ease the notation, we use X instead of modular-like space (X, ϱ).

Definition 2.4. Suppose X is a modular-like space. Then a sequence (u_n) in $B \subseteq X$ is said to be:

(1) ϱ-convergent to a point $u \in B$, if $\lim_{n\to+\infty} \varrho(u_n - u) = \varrho(0)$, and denoted by $u_n \to u$.

(2) ϱ-Cauchy sequence, if $\lim_{n,m\to+\infty} \varrho(u_n - u_m)$ exists and is finite.

When ϱ is modular then the definition of ϱ-convergence is the same as the convergence in a modular space.

Remark 2.5. Let $\varrho(u) = 1$ for each $u \in \mathbb{R}$ and $u_n = 1$ for all $n \in \mathbb{N}$. Then it is easy to see that $u_n \to 0$ because $1 = \varrho(u_n - 0) = \varrho(0) = 1$ and in the same way, we have $u_n \to 1$. Hence, in modular-like spaces the limit of a convergent sequence is not necessarily unique.

Definition 2.6. A subset B of a modular-like space X is known as:

(1) ϱ-closed, if it contains all limits of ϱ-convergent sequences.

(2) ϱ-complete, if each ϱ-Cauchy sequence in B is ϱ-convergent and ϱ-convergent to a point of B and $\lim_{n\to+\infty} \varrho(u_n - u) = \varrho(0) = \lim_{n,m\to+\infty} \varrho(u_n - u_m)$, for each $n, m \in \mathbb{N} \bigcup \{0\}$.

Note that, every ϱ-closed subset of a ϱ-complete modular-like space is ϱ-complete.

Example 2.7. A functional ϱ, for each real number u satisfying $\varrho(u) = u^2$, is a modular-like. Now, the conditions (1) and (2) clearly hold. Since square is a convex function, condition (3) is satisfied, too. It is easy to show, $(\mathbb{R}, \varrho)$ is a ϱ-complete modular-like space.

The following lemmas are handy tools, which will be used in the sequel.

Lemma 2.8. *For any sequence (u_n) in X such that $u_n \to u$ and for some $v \in X$. The following inequality holds:*

$$\frac{\varrho(u-v)}{\omega_\varrho(2)} - \varrho(0) \le \liminf_n \varrho(u_n - v) \le \limsup_n \varrho(u_n - v) \le \omega_\varrho(2)\varrho(u-v) + \omega_\varrho(2)\varrho(0).$$

Proof. Using the definition of the modular-like and the growth function ω_ϱ at real number 2, we have

$$\begin{aligned}
\varrho(u-v) = \varrho(u - u_n + u_n - v) &= \varrho(2(\frac{u-u_n}{2} + \frac{u_n - v}{2})) \\
&\le \omega_\varrho(2)\varrho(\frac{u-u_n}{2} + \frac{u_n - v}{2}) \\
&\le \omega_\varrho(2)(\varrho(u-u_n) + \varrho(u_n - v)).
\end{aligned}$$

This implies

$$\frac{\varrho(u-v)}{\omega_\varrho(2)} - \varrho(u-u_n) \le \varrho(u_n - v).$$

Since $\lim \varrho(u_n - u) = \varrho(0)$, by taking $\liminf$ on the above inequality, then the left hand side of the above inequality is satisfied. By the same argument, we will have the right hand side too. □

Note that, if $\varrho(0) = 0$, then the inequality of Lemma 2.8 changes to the simple inequality:

$$\frac{\varrho(u-v)}{\omega_\varrho(2)} \leq \liminf_n \varrho(u_n - v) \leq \limsup_n \varrho(u_n - v) \leq \omega_\varrho(2)\varrho(u-v).$$

Further, Rus [43] defined a collection of non-decreasing, positive real valued functions, we will denote this class by Ψ, and is defined as:

Definition 2.9. [43] Suppose that Ψ, represents the collection of all nondecreasing functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\sum_{k=0}^{+\infty} \psi^k(x) < +\infty$ for all $x > 0$. Then, there are two important properties $\psi(x) < x$ for each $x > 0$ and $\psi(0) = 0$.

Now, we are going to prove the following lemmas, which is essential for our main results.

Lemma 2.10. *Let (u_n) be a sequence in X, satisfying*

$$\varrho(u_{n+1} - u_n) \leq \psi(\varrho(u_n - u_{n-1})), \tag{2.1}$$

for all $n \in \mathbb{N}$, where $\psi \in \Psi$. Then, for each $m, n \in \mathbb{N} \bigcup \{0\}$, $\lim\limits_{n,m\to+\infty} \varrho(u_n - u_m) = 0$.

Proof. We suppose that $0 < \varrho(u_1 - u_0)$. By using the condition (2.1) for u_n, n times, we get

$$\varrho(u_{n+1} - u_n) \leq \psi^n(\varrho(u_1 - u_0)).$$

Without less of generality, we suppose that $m > n$, so $m = n + p$, for $p \in \mathbb{N}$, using the above inequality and triangle inequality of the modular-like space, we have

$$\begin{aligned}
\varrho(u_n - u_m) &= \varrho(u_n - u_{n+1} + u_{n+1} - u_{n+p}) = \varrho(2(\frac{u_n - u_{n+1}}{2} + \frac{u_{n+1} - u_{n+p}}{2})) \\
&\leq \omega_\varrho(2)\varrho(\frac{u_n - u_{n+1}}{2} + \frac{u_{n+1} - u_{n+p}}{2}) \\
&\leq \omega_\varrho(2)[\varrho(u_n - u_{n+1}) + \varrho(u_{n+1} - u_{n+p})] \\
&\leq \omega_\varrho(2)[\varrho(u_n - u_{n+1}) + \omega_\varrho(2)\varrho(u_{n+1} - u_{n+2}) + \omega_\varrho(2)\varrho(u_{n+2} - u_{n+p})] \\
&\ \vdots \\
&\leq \omega_\varrho(2)\psi^n(\varrho(u_1 - u_0)) + \omega_\varrho^2(2)\psi^{n+1}(\varrho(u_1 - u_0)) + \ldots + \omega_\varrho^p(2)\psi^{n+p-1}(\varrho(u_1 - u_0)) \\
&\leq \omega_\varrho^p(2) \sum_{k=n}^{m-1} \psi^k(\varrho(u_1 - u_0)).
\end{aligned}$$

Since the series is convergent and $\sum_{k=n}^{m-1} \psi^k(\varrho(u_1 - u_0))$ is converging to zero. Thus,

$$\lim_{n,m\to+\infty} \varrho(u_n - u_m) = 0.$$

□

Now we can state one of our main results, which is an equivalent of Banach's theorem in a modular-like space. We would like to highlight that the convexity of ϱ and Fatou property are not used in our results.

Theorem 2.11. *Let X be a ϱ-complete modular-like space and consider a mapping $g : X \to X$ be a mapping which satisfies $\varrho(gx - gy) \leq k\varrho(x - y)$, for every $x, y \in X$ and some $k \in [0, 1)$. Then g has a unique fixed point.*

Proof. Take $x_0, x_1 \in X$. We know from our assumption that for every $n \geq 1$, there exists a sequence $x_{n+1} \in X$ such that $x_{n+1} = gx_n$ and

$$\varrho(x_{n+1} - x_n) \leq k\varrho(x_n - x_{n-1}).$$

Lemma 2.10 implies that $\lim_{n,m\to+\infty} \rho(x_m - x_n) = 0$. Since X is a ϱ-complete space, there exists $x \in X$ such that $\lim_{n\to+\infty} \varrho(x_n - x) = \varrho(0) = \lim_{m,n\to+\infty} \rho(x_m - x_n) = 0$.

On the other hand, from our assumption, it is true that for every $x_n = gx_{n-1}$, we have that

$$\varrho(gx_n - gx) \leq k\varrho(x_n - x),$$

which implies that $\lim_{n\to+\infty} \varrho(x_{n+1} - gx) = \lim_{n\to+\infty} \varrho(gx_n - gx) = 0$ and by Lemma 2.8, we have

$$\frac{\varrho(x - gx)}{\omega_\varrho(2)} - \varrho(0) \leq \liminf_n \varrho(x_{n+1} - gx) = 0,$$

this implies that $\varrho(x - gx) = 0$, and by using the contraction, uniqueness of fixed point can be proved. □

From now and onward, X will represent a ϱ-complete modular-like space, $\mathfrak{T}, \mathfrak{g} : X \to X$ are mappings such that $\mathfrak{T}X \subseteq \mathfrak{g}X$ and $\mathfrak{g}X$ is ϱ-closed. Now, we are going to define some new contractions for mappings $\mathfrak{T}, \mathfrak{g} : X \to X$ and $\mathfrak{T}x = \mathfrak{g} = z$ defined on modular-like spaces and prove that these mappings have a point of coincidence $z \in X$ and $x \in X$ as a coincidence point.

Definition 2.12. A mapping $\mathfrak{T}$ satisfies $\mathfrak{g}$-Hardy Rogers type contraction, if there exist $\psi \in \Psi$ and $\alpha, \eta, \omega \in (0, 1)$ that

$$\begin{aligned}\varrho(\mathfrak{T}u - \mathfrak{T}v) \leq \psi\Big(&[\varrho(\mathfrak{g}u - \mathfrak{g}v)]^\alpha[\varrho(\mathfrak{g}u - \mathfrak{T}u)]^\eta[\varrho(\mathfrak{g}v - \mathfrak{T}v)]^\omega \\ &[\frac{\varrho(\mathfrak{g}u - \mathfrak{T}v) + \varrho(\mathfrak{g}v - \mathfrak{T}u)}{4w_\varrho(2)}]^{1-\alpha-\eta-\omega}\Big),\end{aligned} \tag{2.2}$$

for all $u, v \in X$ with $\mathfrak{T}u \neq \mathfrak{g}u, \mathfrak{T}v \neq \mathfrak{g}v, \mathfrak{g}u \neq \mathfrak{g}v$.

Theorem 2.13. *Let $\mathfrak{T}$ be a $\mathfrak{g}$-Hardy Rogers type contraction on a ϱ-complete modular-like space X. Then $\mathfrak{T}$ and $\mathfrak{g}$ have a point of coincidence.*

Proof. Consider any $u_0 \in X$. Since $\mathfrak{T}X \subseteq \mathfrak{g}X$, inductively we can find a sequence (u_n) such that $\mathfrak{g}(u_{n+1}) = \mathfrak{T}(u_n)$ for all $n \in \mathbb{N}\bigcup\{0\}$. If $\mathfrak{g}u_n = \mathfrak{T}u_n$ for some n, then $\mathfrak{T}u_n$ is a point of coincidence for $\mathfrak{g}$ and $\mathfrak{T}$. Assume that $\mathfrak{g}u_n \neq \mathfrak{T}u_n$, for all n. By replacing u and v with u_n and u_{n+1} in (2.2) and using $\mathfrak{g}u_{n+1} = \mathfrak{T}u_n$, we obtain

$$\begin{aligned}\varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n+1}) \leq \psi(&[\varrho(\mathfrak{g}u_n - \mathfrak{g}u_{n+1})]^\alpha[\varrho(\mathfrak{g}u_n - \mathfrak{T}u_n)]^\eta[\varrho(\mathfrak{g}u_{n+1} - \mathfrak{T}u_{n+1})]^\omega \\ &[\frac{\varrho(\mathfrak{g}u_n - \mathfrak{T}u_{n+1}) + \varrho(\mathfrak{g}u_{n+1} - \mathfrak{T}u_n)}{4w_\varrho(2)}]^{1-\alpha-\eta-\omega})\end{aligned}$$

$$= \psi([\varrho(\mathfrak{T}u_{n-1} - \mathfrak{T}u_n)]^{\alpha}[\varrho(\mathfrak{T}u_{n-1} - \mathfrak{T}u_n)]^{\eta}[\varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n+1})]^{\omega}$$
$$[\frac{\varrho(\mathfrak{T}u_{n-1} - \mathfrak{T}u_{n+1}) + \varrho(\mathfrak{T}u_n - \mathfrak{T}u_n)}{4w_{\varrho}(2)}]^{1-\alpha-\eta-\omega})$$
$$\leq \psi([\varrho(\mathfrak{T}u_{n-1} - \mathfrak{T}u_n)]^{\alpha+\eta}[\varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n+1})]^{\omega}$$
$$[\frac{\varrho(\mathfrak{T}u_{n-1} - \mathfrak{T}u_n) + \varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n+1})}{2}]^{1-\alpha-\eta-\omega}),$$

i.e.,

$$\varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n+1}) \leq \psi([\varrho(\mathfrak{T}u_{n-1} - \mathfrak{T}u_n)]^{\alpha+\eta}[\varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n+1})]^{\omega}$$
$$[\frac{\varrho(\mathfrak{T}u_{n-1} - \mathfrak{T}u_n) + \varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n+1})}{2}]^{1-\alpha-\eta-\omega}). \tag{2.3}$$

Since $\psi(x) < x$, for all $x > 0$, we obtain

$$\varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n+1}) < [\varrho(\mathfrak{T}u_{n-1} - \mathfrak{T}u_n)]^{\alpha+\eta}[\varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n+1})]^{\omega}$$
$$[\frac{\varrho(\mathfrak{T}u_{n-1} - \mathfrak{T}u_n) + \varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n+1})}{2}]^{1-\alpha-\eta-\omega}. \tag{2.4}$$

Suppose that $\varrho(\mathfrak{T}u_{n-1} - \mathfrak{T}u_n) < \varrho(\mathfrak{T}u_{n+1} - \mathfrak{T}u_n)$ for some $n \geq 1$. Then,

$$\frac{\varrho(\mathfrak{T}u_{n-1} - \mathfrak{T}u_n) + \varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n+1})}{2} < \varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n+1}).$$

From the inequality (2.4), we infer

$$\varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n+1}) < [\varrho(\mathfrak{T}u_{n-1} - \mathfrak{T}u_n)]^{\alpha+\eta}[\varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n+1})]^{1-\alpha-\eta},$$

which implies that

$$[\varrho(\mathfrak{T}u_{n+1} - \mathfrak{T}u_n)]^{\alpha+\eta} < [\varrho(\mathfrak{T}u_{n-1} - \mathfrak{T}u_n)]^{\alpha+\eta}.$$

Hence,

$$\varrho(\mathfrak{T}u_{n+1} - \mathfrak{T}u_n) < \varrho(\mathfrak{T}u_{n-1} - \mathfrak{T}u_n),$$

which is a contradiction. Thus, for all $n \geq 1$,

$$\varrho(\mathfrak{T}u_{n+1} - \mathfrak{T}u_n) \leq \varrho(\mathfrak{T}u_{n-1} - \mathfrak{T}u_n).$$

Using above inequality and (2.3), we have

$$\varrho(\mathfrak{T}u_{n+1} - \mathfrak{T}u_n) \leq \psi(\varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n-1})). \tag{2.5}$$

By Lemma 2.10, we have $\lim_{n,m\to+\infty} \varrho(\mathfrak{T}u_n - \mathfrak{T}u_m) = 0,$ consequently, the sequences $(\mathfrak{T}u_n)$ and $(\mathfrak{g}u_n)$ are ϱ-Cauchy. So there is z belongs to X which

$$\lim \varrho(\mathfrak{T}u_n - z) = \varrho(0) = \lim \varrho(\mathfrak{g}u_{n+1} - z) = \varrho(\mathfrak{T}u_n - \mathfrak{T}u_{n+p}) = 0. \tag{2.6}$$

Since $\mathfrak{g}X$ is a ϱ-closed set, there exists $w \in \mathfrak{g}X$ such that $z = \mathfrak{g}w$. We claim that z is a point of coincidence for $\mathfrak{g}$ and $\mathfrak{T}$, i.e., $z = \mathfrak{g}w = \mathfrak{T}w$. By (2.2), we obtain

$$\varrho(\mathfrak{T}w - \mathfrak{T}u_n) \leq \psi([\varrho(\mathfrak{g}w - \mathfrak{g}u_n)]^{\alpha}[\varrho(\mathfrak{g}w - \mathfrak{T}w)]^{\eta}[\varrho(\mathfrak{T}u_n - \mathfrak{g}u_n)]^{\omega}$$

$$[\frac{\varrho(\mathfrak{g}w - \mathfrak{T}u_n) + \varrho(\mathfrak{g}u_n - \mathfrak{T}w)}{4w_\varrho(2)}]^{1-\alpha-\eta-\omega})$$
$$< [\varrho(\mathfrak{g}w - \mathfrak{g}u_n)]^\alpha[\varrho(\mathfrak{g}w - \mathfrak{T}w)]^\eta[\varrho(\mathfrak{T}u_n - \mathfrak{g}u_n)]^\omega$$
$$[\frac{\varrho(\mathfrak{g}w - \mathfrak{T}u_n) + \varrho(\mathfrak{g}u_n - \mathfrak{T}w)}{4w_\varrho(2)}]^{1-\alpha-\eta-\omega}. \tag{2.7}$$

And by Lemma 2.8, we have $\frac{\varrho(\mathfrak{T}w-z)}{w_\varrho(2)} \leq \limsup_{n\to+\infty} \varrho(\mathfrak{T}w - \mathfrak{T}u_n)$, and $\limsup_{n\to+\infty} \varrho(\mathfrak{T}u_n - \mathfrak{g}u_n) \leq w_\varrho^2(2)\varrho(0)$. These and (2.7) implies that

$$\frac{\varrho(\mathfrak{T}w - z)}{w_\varrho(2)} \leq \limsup_{n\to+\infty}([\varrho(z - \mathfrak{g}u_n)]^\alpha[\varrho(z - \mathfrak{T}w)]^\eta[w_\varrho^2(2)\varrho(0)]^\omega$$
$$[\frac{\varrho(z - \mathfrak{T}u_n) + \varrho(\mathfrak{g}u_n - z)}{4w_\varrho(2)}]^{1-\alpha-\eta-\omega}).$$

Using above inequality and (2.6), we get $\varrho(\mathfrak{T}w - z) = 0$. Therefore,

$$\mathfrak{T}w = z = \mathfrak{g}w.$$

□

Now, we will illustrate our result with the help of the following example.

Example 2.14. Suppose that the self mapping $\mathfrak{T}, \mathfrak{g}$ defined on a set $X = \{1, 2, 3, 4\}$ as following:

$$\mathfrak{T}1 = \mathfrak{T}4 = 3, \mathfrak{T}2 = \mathfrak{T}3 = 4,$$

and

$$\mathfrak{g}1 = 1, \mathfrak{g}2 = 4, \mathfrak{g}3 = 2, \mathfrak{g}4 = 3.$$

Such that $\mathfrak{T}X \subseteq \mathfrak{g}X$ and $\mathfrak{g}X$ is ϱ-closed. Let the modular-like $\varrho : \mathbb{R} \to \mathbb{R}$ be

$$\varrho(x) = \begin{cases} 0.5, & \text{if } x \in \{-1, 0, 1\}, \\ 1, & \text{otherwise.} \end{cases}$$

As (X, ϱ) is ϱ-complete. Also, suppose that $\psi(x) = \frac{5^x-1}{5^x+1}$ and $\alpha = \eta = \omega = \frac{1}{3}$. Since $\mathfrak{T}2 = \mathfrak{g}2 = 4$, and $\mathfrak{T}4 = \mathfrak{g}4 = 3$, so we must show that the inequality (2.2) is satisfied just for $x, y \in \{1, 3\}$. We have

$$\varrho(\mathfrak{T}1 - \mathfrak{T}3) \leq \psi\Big([\varrho(\mathfrak{g}1 - \mathfrak{g}3)]^{\frac{1}{3}}[\varrho(\mathfrak{g}1 - \mathfrak{T}1)]^{\frac{1}{3}}[\varrho(\mathfrak{g}3 - \mathfrak{T}3)]^{\frac{1}{3}}$$
$$[\frac{\varrho(\mathfrak{g}1 - \mathfrak{T}3) + \varrho(\mathfrak{g}3 - \mathfrak{T}1)}{4w_\varrho(2)}]^0\Big), \tag{2.8}$$

i.e., $0.5 = \varrho(-1) \leq \psi(\varrho^{\frac{1}{3}}(-1)\varrho^{\frac{2}{3}}(-2)) = 0.5640$, so $\mathfrak{T}$ is a $\mathfrak{g}$-Hardy Rogers type. So, the mappings $\mathfrak{T}$ and $\mathfrak{g}$ have a point of coincidence. The points of coincidence for mappings $\mathfrak{T}$ and $\mathfrak{g}$ are 3 and 4.

The previous example leads us to the following remark.

Remark 2.15. In Theorem 2.13, $\mathfrak{T}$ and $\mathfrak{g}$ do not have a fixed point, just as $\mathfrak{T}$ and $\mathfrak{g}$ accept points of coincidence which are not necessarily unique.

We use the notation Φ to represent the set of all nondecreasing functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\phi(x) < x, \quad \lim_{s \to x^+} \phi(s) < x \quad \text{and} \quad \lim_{x \to +\infty} (x - \omega_\varrho(2)\phi(x)) = +\infty.$$

Definition 2.16. A mapping $\mathfrak{T}$ is said to be a $\mathfrak{g}$-ϕ-Hardy Rogers type contraction, if there exist $\alpha, \eta, \omega, \nu, \gamma, \zeta \in \mathbb{R}^+$ with $\alpha + \eta + \omega + \nu + \gamma + \zeta = 1$ such that

$$\begin{aligned} \varrho(\mathfrak{T}u - \mathfrak{T}v) \leq \phi\Big(&[\varrho(\mathfrak{g}u - \mathfrak{g}v)]^\alpha [\varrho(\mathfrak{g}u - \mathfrak{T}u)]^\eta [\varrho(\mathfrak{g}v - \mathfrak{T}v)]^\omega \\ &[\frac{\varrho(\mathfrak{g}u - \mathfrak{T}v) + \varrho(\mathfrak{g}v - \mathfrak{T}u)}{2}]^\nu [\varrho(\mathfrak{g}u - \mathfrak{g}u)]^\gamma [\varrho(\mathfrak{g}v - \mathfrak{g}v)]^\zeta\Big), \end{aligned} \tag{2.9}$$

for all $u, v \in X$ with $\mathfrak{T}u \neq \mathfrak{g}u$, $\mathfrak{T}v \neq \mathfrak{g}v$ and $\mathfrak{g}u \neq \mathfrak{g}v$.

Theorem 2.17 is more applicable and general than Theorem 2.13, because the conditions on $\phi \in \Phi$ are weaker than ones on $\psi \in \Psi$ involved in Theorem 2.13.

Theorem 2.17. *Every $\mathfrak{g}$-ϕ-Hardy Rogers type contractive mappings $\mathfrak{T}$ has a point of coincidence.*

Proof. Let $u \in X$, and $\mathfrak{T}X \subseteq \mathfrak{g}X$, define a sequence (u_n) such that $u_0 = u$, and $\mathfrak{g}u_{n+1} = \mathfrak{T}u_n$, for all integer n. If $\mathfrak{g}u_n = \mathfrak{T}u_n$, for some n then $\mathfrak{T}u_n$ is a point of coincidence for $\mathfrak{g}$ and $\mathfrak{T}$.

Assume that $\mathfrak{g}u_n \neq \mathfrak{T}u_n$, for all n. If we show that $D_n(u_0) = diam\{\mathfrak{T}u_0, \mathfrak{T}u_1, \ldots, \mathfrak{T}u_n\}$ is convergent to a finite real point as $n \to +\infty$, then $\{\mathfrak{T}u_n\}$ is a ϱ-Cauchy sequence. For that, first we have to show that $D_n(u_0) = \varrho(\mathfrak{T}u_0 - \mathfrak{T}u_k)$, for some $k = k(n) \in \{0, 1, \ldots, n\}$. Suppose, on contrary that there are positive integers $1 \leq i = i(n) \leq j = j(n)$ such that $D_n(u_0) = \varrho(\mathfrak{T}u_i - \mathfrak{T}u_j) > 0$. From our assumption, by replacing u and v with u_i and u_j in (2.9) and using $\mathfrak{g}u_{n+1} = \mathfrak{T}u_n$, we obtain

$$\begin{aligned} \varrho(\mathfrak{T}u_i - \mathfrak{T}u_j) &\leq \phi([\varrho(\mathfrak{g}u_i - \mathfrak{g}u_j)]^\alpha [\varrho(\mathfrak{g}u_i - \mathfrak{T}u_i)]^\eta [\varrho(\mathfrak{g}u_j - \mathfrak{T}u_j)]^\omega \\ &\quad [\frac{\varrho(\mathfrak{g}u_i - \mathfrak{T}u_j) + \varrho(\mathfrak{g}u_j - \mathfrak{T}u_i)}{2}]^\nu [\varrho(\mathfrak{g}u_i - \mathfrak{g}u_i)]^\gamma [\varrho(\mathfrak{g}u_j - \mathfrak{g}u_j)]^\zeta) \\ &= \phi([\varrho(\mathfrak{T}u_{i-1} - \mathfrak{T}u_{j-1})]^\alpha [\varrho(\mathfrak{T}u_{i-1} - \mathfrak{T}u_i)]^\eta [\varrho(\mathfrak{T}u_{j-1} - \mathfrak{T}u_j)]^\omega \\ &\quad [\frac{\varrho(\mathfrak{T}u_{i-1} - \mathfrak{T}u_j) + \varrho(\mathfrak{T}u_{j-1} - \mathfrak{T}u_i)}{2}]^\nu [\varrho(\mathfrak{T}u_{i-1} - \mathfrak{T}u_{i-1})]^\gamma [\varrho(\mathfrak{T}u_{j-1} - \mathfrak{T}u_{j-1})]^\zeta) \\ &\leq \phi(D_n(u_0)), \end{aligned} \tag{2.10}$$

which implies that

$$D_n(u_0) \leq \phi(D_n(u_0)) < D_n(u_0),$$

which is a contradiction. Thus,

$$D_n(u_0) = \varrho(\mathfrak{T}u_0 - \mathfrak{T}u_k)$$

holds. It is clear that $D_n(u_0)$ is nondecreasing. So, $\lim_{n \to +\infty} D_n(u_0)$ exists, i.e.,

$$\lim_{n \to +\infty} D_n(u_0) = +\infty$$

or

$$\lim_{n \to +\infty} D_n(u_0) < +\infty.$$

Suppose that $D_n(u_0)$ is infinite as $n \to +\infty$, by the triangle inequality, we have

$$\varrho(\mathfrak{T}u_0 - \mathfrak{T}u_k) \le \omega_\varrho(2)(\varrho(\mathfrak{T}u_0 - \mathfrak{T}u_1) + \varrho(\mathfrak{T}u_1 - \mathfrak{T}u_k)).$$

So, by (2.10), we have

$$D_n(u_0) \le \omega_\varrho(2)(\varrho(\mathfrak{T}u_0 - \mathfrak{T}u_1) + \phi(D_n(u_0))).$$

Therefore,

$$(I - \omega_\varrho(2)\phi)(D_n(u_0)) \le \omega_\varrho(2)\varrho(\mathfrak{T}u_0 - \mathfrak{T}u_1) < +\infty,$$

which contradicts that

$$\lim_{x\to+\infty}(I - \omega_\varrho(2)\phi)x = +\infty.$$

Which implies that

$$\lim_{n\to+\infty} D_n(u_0) < +\infty.$$

Now, we have to show that $D_n(u_0)$ is ϱ-bounded as $n \to +\infty$. We shall prove that

$$D = \lim_{n\to+\infty} D_n(u_0) = 0.$$

Let n be any arbitrary integer, and i, j be any positive integers with $i, j \ge n + 1$. Then by inequality (2.10), we have

$$\varrho(\mathfrak{T}u_i - \mathfrak{T}u_j) \le \phi(D_n(u_0)).$$

Suppose that $D > 0$. Hence, we get $D \le \lim_{n\to+\infty} \phi(D_n(u_0)) = \lim_{s\to D^+} \phi(s) < D$ which is a contradiction. Therefore, $D = 0$, i.e., $\lim_{n\to+\infty} D_n(u_0) = \lim_{n,m\to+\infty} \varrho(\mathfrak{T}u_n - \mathfrak{T}u_m) = 0$. This implies $\{\mathfrak{T}u_n\}$ is a ϱ-Cauchy sequence, so $(\mathfrak{g}u_n)$ is a ϱ-Cauchy sequence. Since X is a ϱ-complete space, there exists $z \in X$ such that

$$\lim_{n\to+\infty} \varrho(\mathfrak{T}u_n - z) = \lim_{n\to+\infty} \varrho(\mathfrak{g}u_{n+1} - z) = \varrho(0) = \lim_{n,m\to+\infty} \varrho(\mathfrak{T}u_n - \mathfrak{T}u_m) = 0. \tag{2.11}$$

Since $\mathfrak{g}X$ is a ϱ-closed set, there exists $w \in \mathfrak{g}X$ such that $z = \mathfrak{g}w$. We claim that z is a point of coincidence for $\mathfrak{g}$ and $\mathfrak{T}$. For this, we assume that $z = \mathfrak{g}w \ne \mathfrak{T}w$. Then by (2.9), we obtain

$$\begin{aligned}
\varrho(\mathfrak{T}w - \mathfrak{T}u_n) \le\ & \phi([\varrho(\mathfrak{g}w - \mathfrak{g}u_n)]^\alpha[\varrho(\mathfrak{g}w - \mathfrak{T}w)]^\eta[\varrho(\mathfrak{T}u_n - \mathfrak{g}u_n)]^\omega \\
& [\frac{\varrho(\mathfrak{g}w - \mathfrak{T}u_n) + \varrho(\mathfrak{g}u_n - \mathfrak{T}w)}{2}]^\nu[\varrho(\mathfrak{g}w - \mathfrak{g}w)]^\gamma[\varrho(\mathfrak{g}u_n - \mathfrak{g}u_n)]^\zeta) \\
<\ & [\varrho(\mathfrak{g}w - \mathfrak{g}u_n)]^\alpha[\varrho(\mathfrak{g}w - \mathfrak{T}w)]^\eta[\varrho(\mathfrak{T}u_n - \mathfrak{g}u_n)]^\omega \\
& [\frac{\varrho(\mathfrak{g}w - \mathfrak{T}u_n) + \varrho(\mathfrak{g}u_n - \mathfrak{T}w)}{2}]^\nu[\varrho(\mathfrak{g}w - \mathfrak{g}w)]^\gamma[\varrho(\mathfrak{g}u_n - \mathfrak{g}u_n)]^\zeta.
\end{aligned} \tag{2.12}$$

By Lemma 2.8, we have

$$\frac{1}{\omega_\varrho(2)}\varrho(\mathfrak{T}w - z) \le \limsup_{n\to+\infty} \varrho(\mathfrak{T}w - \mathfrak{T}u_n),$$

and

$$\limsup_{n\to+\infty} \varrho(\mathfrak{T}u_n - \mathfrak{g}u_n) \le \omega_\varrho^2(2)\varrho(0).$$

Thus, from (2.12), we infer that

$$\frac{1}{\omega_\varrho(2)}\varrho(\mathfrak{T}w-z) \leq \limsup_{n\to+\infty}[\varrho(\mathfrak{g}w-\mathfrak{g}u_n)]^\alpha[\varrho(\mathfrak{g}w-\mathfrak{T}w)]^\eta[\varrho(\mathfrak{T}u_n-\mathfrak{g}u_n)]^\omega$$
$$[\frac{\varrho(\mathfrak{g}w-\mathfrak{T}u_n)+\varrho(\mathfrak{g}u_n-\mathfrak{T}w)}{2}]^\nu[\varrho(\mathfrak{g}w-\mathfrak{g}w)]^\gamma[\varrho(\mathfrak{g}u_n-\mathfrak{g}u_n)]^\zeta \tag{2.13}$$
$$\leq \limsup_{n\to+\infty}[\varrho(z-\mathfrak{g}u_n)]^\alpha[\varrho(z-\mathfrak{T}w)]^\eta[w_\varrho^2(2)\varrho(0)]^\omega$$
$$[\frac{\varrho(z-\mathfrak{T}u_n)+\omega_\varrho(2)\varrho(z-\mathfrak{T}w)}{2}]^\nu[\varrho(0)]^\gamma[\omega_\varrho^2(2)\varrho(0)]^\zeta. \tag{2.14}$$

From above inequality and (2.11) implies that $\varrho(\mathfrak{T}w-z)=0,$ which is a contradiction. Therefore,

$$\mathfrak{T}w = z = \mathfrak{g}w,$$

that is, z is a point of coincidence for $\mathfrak{T}$ and $\mathfrak{g}$ in X. □

Corollary 2.18. *If there exist $\alpha,\eta,\omega,\nu,\gamma,\zeta\in\mathbb{R}^+$ with $\alpha+\eta+\omega+\nu+\gamma+\zeta=1$ such that for $k\in(0,1)$,*

$$\varrho(\mathfrak{T}u-\mathfrak{T}v) \leq k\Big([\varrho(\mathfrak{g}u-\mathfrak{g}v)]^\alpha[\varrho(\mathfrak{g}u-\mathfrak{T}u)]^\eta[\varrho(\mathfrak{g}v-\mathfrak{T}v)]^\omega$$
$$[\frac{\varrho(\mathfrak{g}u-\mathfrak{T}v)+\varrho(\mathfrak{g}v-\mathfrak{T}u)}{2}]^\nu[\varrho(\mathfrak{g}u-\mathfrak{g}u)]^\gamma[\varrho(\mathfrak{g}v-\mathfrak{g}v)]^\zeta\Big),$$

for all $u,v\in X$ with $\mathfrak{T}u\neq\mathfrak{g}u,\ \mathfrak{T}v\neq\mathfrak{g}v$ and $\mathfrak{g}u\neq\mathfrak{g}v$. Then $\mathfrak{g}$ and $\mathfrak{T}$ have a point of coincidence.

Proof. If, we take $\phi(x)=kx$ in Theorem 2.17, the result is concluded. □

Example 2.19. Define a function $\varrho:\mathbb{R}\to\mathbb{R}$ by

$$\varrho(u) = \begin{cases} 6, & u\in\{-1,1\}, \\ 3, & u\neq\{-1,1\}. \end{cases}$$

Let $X=\{0,1,2\}$, then (X,ϱ) is a ϱ-complete modular-like space. Define self mappings $\mathfrak{T}$ and $\mathfrak{g}$ on X by $\mathfrak{g}u=u$ and $\mathfrak{T}$ defined by $\mathfrak{T}(0)=0,\ \mathfrak{T}(1)=2,$ and $\mathfrak{T}(2)=0$. Then, $\mathfrak{T}$ is a $\mathfrak{g}$-ϕ-Hardy Rogers type contraction for $\alpha=0.4$, $\eta=0.4$, $\nu=0.1$, $\omega+\gamma+\zeta=0.1$, and $\phi(x)=0.99x$, it is enough to show that the inequality (2.9) holds, for $u=1$ and $v=2$. Further,

$$\varrho(\mathfrak{T}1-\mathfrak{T}2) \leq \phi\Big([\varrho(\mathfrak{g}1-\mathfrak{g}2)]^\alpha[\varrho(\mathfrak{g}1-\mathfrak{T}1)]^\eta[\varrho(\mathfrak{g}2-\mathfrak{T}2)]^\omega$$
$$[\frac{\varrho(\mathfrak{g}1-\mathfrak{T}2)+\varrho(\mathfrak{g}2-\mathfrak{T}1)}{2}]^\nu[\varrho(\mathfrak{g}1-\mathfrak{g}1)]^\gamma[\varrho(\mathfrak{g}2-\mathfrak{g}2)]^\zeta\Big), \tag{2.15}$$

which implies that

$$\varrho(2) \leq \phi\Big([\varrho(-1)]^\alpha[\varrho(-1)]^\eta[\varrho(2)]^\omega[\frac{\varrho(1)+\varrho(0)}{2}]^\nu[\varrho(0)]^\gamma[\varrho(0)]^\zeta\Big). \tag{2.16}$$

Because,

$$3 \leq 0.99\cdot 6^{0.8}\cdot 3^{0.1}\cdot 4.5^{0.1} = 5.3850.$$

Therefore $\mathfrak{T}$ and g have a point of coincidence $\mathfrak{T}0=g0$.

Definition 2.20. Consider $\alpha, \eta \in (0,1)$ with $\alpha + \eta < 1$ such that

$$\varsigma(\varrho(\mathfrak{T}u - \mathfrak{T}v)) \leq \varsigma(R(u,v)) - \varphi(R(u,v)), \tag{2.17}$$

for all $u, v \in X \backslash Fix(\mathfrak{T})$, where $Fix(\mathfrak{T})$ contains some points of X that do not change by the mapping $\mathfrak{T}$, and $R(u,v) = [\varrho(u-v)]^{\alpha}[\varrho(u - \mathfrak{T}u)]^{\eta}[\varrho(v - \mathfrak{T}v)]^{1-\alpha-\eta}$. Then mapping $\mathfrak{T}$ is called Ćirić-Reich-Rus type weakly contractive mapping.
Where $\varphi : \mathbb{R}^+ \to \mathbb{R}^{\geq 0}$ is a lower semi-continuous function with $\varphi(x) = 0$ if and only if $x = 0$, and $\varsigma : \mathbb{R}^+ \to \mathbb{R}^{\geq 0}$ is a continuous monotone nondecreasing function with $\varsigma(x) = 0$ if and only if $x = 0$.

Theorem 2.21. *Every Ćirić-Reich-Rus type weakly contractive mapping $\mathfrak{T}$ has a fixed point.*

Proof. For any $u_0 \in X$, consider a sequence (u_n) defined as $u_{n+1} = \mathfrak{T}u_n$, $n \in \mathbb{N}$. If $u_{n+1} = u_n$ holds for some n then u_n is clearly a fixed point of $\mathfrak{T}$ in X. Otherwise, if $u_{n+1} \neq u_n$ for each $n \geq 0$. From (2.17), we have

$$\begin{aligned} \varsigma(\varrho(u_{n+1} - u_n)) &\leq \varsigma([\varrho(u_n - u_{n-1})]^{\alpha}[\varrho(u_n - u_{n+1})]^{\eta}[\varrho(u_{n-1} - u_n)]^{1-\alpha-\eta}) \\ &\quad -\varphi([\varrho(u_n - u_{n-1})]^{\alpha}[\varrho(u_n - u_{n+1})]^{\eta}[\varrho(u_{n-1} - u_n)]^{1-\alpha-\eta}) \\ &\leq \varsigma([\varrho(u_n - u_{n-1})]^{1-\eta}[\varrho(u_n - u_{n-1})]^{\eta}), \end{aligned} \tag{2.18}$$

which can be written as

$$\varrho(u_{n+1} - u_n) \leq [\varrho(u_n - u_{n-1})]^{1-\eta}[\varrho(u_n - u_{n+1})]^{\eta}.$$

After simplification, we get

$$[\varrho(u_{n+1} - u_n)]^{1-\eta} \leq [\varrho(u_n - u_{n-1})]^{1-\eta},$$

and so

$$\varrho(u_{n+1} - u_n) \leq \varrho(u_n - u_{n-1}), \quad \text{for all } n \geq 1.$$

As $(\varrho(u_{n+1} - u_n))$ is a decreasing sequence of positive real numbers. There exists some $c \geq 0$ such that $\lim_{n \to +\infty} \varrho(u_{n+1} - u_n) = c$. From inequality (2.18), we have

$$\varsigma(c) \leq \varsigma(c) - \varphi(c),$$

which implies that $c = 0$. Hence,

$$\lim_{n \to +\infty} \varrho(u_{n+1} - u_n) = 0. \tag{2.19}$$

Now, we have to prove that (u_n) is a ϱ-Cauchy sequence. On contrary suppose that there exists a real number $\epsilon > 0$, for any $k \in \mathbb{N}$, $m_k \geq n_k \geq k$ such that

$$\varrho(u_{m_k} - u_{n_k}) \geq \epsilon, \ \varrho(u_{m_k - 1} - u_{n_k}) < \epsilon. \tag{2.20}$$

From (2.17) and using (2.20), we obtain

$$\varsigma(\epsilon) \leq \varsigma(\varrho(u_{m_k} - u_{n_k})) \leq \varsigma(R(u_{m_k-1}, u_{n_k-1})) - \varphi(R(u_{m_k-1}, u_{n_k-1})),$$

where

$$R(u_{m_k-1}, u_{n_k-1}) = [\varrho(u_{m_k-1} - u_{n_k-1})]^{\alpha}[\varrho(u_{m_k-1} - u_{m_k})]^{\eta}[\varrho(u_{n_k-1} - u_{n_k})]^{1-\alpha-\eta}.$$

Also,

$$\begin{aligned}\varrho(u_{m_k-1}-u_{n_k-1}) &\le \omega_\varrho(2)(\varrho(u_{m_k-1}-u_{n_k})+\varrho(u_{n_k}-u_{n_k-1}))\\ &\le \omega_\varrho(2)(\epsilon+\varrho(u_{n_k-1}-u_{n_k})),\end{aligned}$$

and using (2.19), we conclude $\lim\limits_{k\to+\infty} R(u_{m_k-1},u_{n_k-1}) = 0$. Then $\varsigma(\epsilon) \le \varsigma(0) - \varphi(0) = 0$, which is contradiction with $\epsilon > 0$, thus (u_n) is a ϱ-Cauchy sequence. As X is ϱ-complete, we obtain $z \in X$ such that $\lim\limits_{n\to+\infty} \varrho(u_n - z) = \varrho(u_n - u_{n+1}) = \varrho(0) = 0$. Assume that $\mathfrak{T}z \neq z$, we have for all $n \in \mathbb{N}$,

$$\varsigma(\varrho(u_{n+1}-\mathfrak{T}z)) \le \varsigma(R(u_n,z)) - \varphi(R(u_n,z)), \tag{2.21}$$

where

$$R(u_n,z) = [\varrho(u_n-z)]^\alpha[\varrho(u_{n+1}-u_n)]^\eta[\varrho(z-\mathfrak{T}z)]^{1-\alpha-\eta}.$$

Using (2.19), we get $\lim\limits_{n\to+\infty} R(u_n,z) = 0$, apply limit $n \to +\infty$ in (2.21) and using Lemma 2.8, we have $\varsigma(\frac{1}{\omega_\varrho(2)}\varrho(z-Tz)) \le \varsigma(\varrho(u_{n+1}-Tz)) \le \varsigma(0)-\varphi(0) = 0$, which is a contradiction, thus $\mathfrak{T}z = z$. □

In the following example, we bring an Ćirić-Reich-Rus type weakly contractive mapping which satisfies the conditions of Theorem 2.21.

Example 2.22. Define $\varrho : \mathbb{R} \to \mathbb{R}$ as:

$$\varrho(u) = \begin{cases} 0, & u = 0,\\ 5, & u \in (-1,1)-\{0\}\\ 3, & \text{otherwise.}\end{cases}$$

Let $X = [0,5]$, then (X,ϱ) is a ϱ-complete modular-like space. Suppose $\mathfrak{T} : X \to X$ is defined as:

$$\mathfrak{T}u = \begin{cases} 0, & u \in [0,1),\\ 3, & u \in [1,5].\end{cases}$$

Choose $\varsigma(x) = x^2$, and $\varphi(x) = \frac{x}{3}$ for all $x \in \mathbb{R}^{\ge 0}$, $\alpha = 0.4$ and $\eta = 0.3$.
Now, we are going to satisfy the inequality (2.17). We have two steps.

Step 1. Suppose that $u, v \in [0,1)$ or $u, v \in [1,5]$, then

$$\varsigma(\varrho(\mathfrak{T}u-\mathfrak{T}v)) = \varsigma(\varrho(0)) = 0,$$

If $R(u,v) = 0$ then inequality (2.17) is satisfied on the other hand, if $R(u,v) \neq 0$, then we have $R(u,v) \ge 1$ and therefore

$$\varsigma(R(u,v)) - \varphi(R(u,v)) = R^2(u,v) - \frac{R(u,v)}{3} \ge 0.$$

Thus, the inequality (2.17) is satisfied.

Step 2. Suppose that $u \in [0,1)$ and $v \in [1,5]$, then

$$\varsigma(\varrho(\mathfrak{T}u-\mathfrak{T}v)) = \varsigma(\varrho(3)) = 9,$$

and

$$3.4968 \le R(u,v) \le 5,$$

thus

$$\varsigma(R(u,v)) - \varphi(R(u,v)) \geq 12.22789 - \frac{5}{3} = 10.561223.$$

Hence

$$\varsigma(\varrho(\mathfrak{T}u - \mathfrak{T}v)) \leq \varsigma(R(u,v)) - \varphi(R(u,v)),$$

for all $u, v \in [0,5]\backslash\{0,3\}$. Then, $\mathfrak{T}$ possesses two fixed points 0 and 3.

Example 2.23. Suppose that the self mapping $T : \{1,2,3\} \to \{1,2,3\}$ defined as following:

$$\mathfrak{T}1 = \mathfrak{T}2 = 2, \mathfrak{T}3 = 3.$$

Also, define $\varsigma(x) = 2x$, $\varphi(x) = \frac{x}{10}$ and the modular-like $\varrho : \mathbb{R} \to \mathbb{R}$ by

$$\varrho(x) = \begin{cases} 1, & \text{if } x = 0, \\ 2, & \text{otherwise.} \end{cases}$$

Let $\alpha = 0.5$ and $\eta = 0.4$. We have

$$R(1,1) = \varrho^{0.5}(0)\varrho^{0.5}(-1) = \sqrt{2}.$$

Since

$$Fix\mathfrak{T} = \{2,3\},$$

so it remains to show the inequality (2.17) for $\{1\}$. We have $\varsigma(\varrho(\mathfrak{T}1 - \mathfrak{T}1)) \leq \varsigma(R(1,1)) - \varphi(R(1,1))$ if and only if $\varsigma(1) \leq \varsigma(\sqrt{2}) - \varphi(\sqrt{2})$ if and only if $2 \leq 2\sqrt{2} - \frac{\sqrt{2}}{10} = \frac{19}{10}\sqrt{2}$. Then $\mathfrak{T}$ is called to satisfy $\acute{C}$iri$\acute{c}$-Reich-Rus type weakly contractive and has two fixed points.

From the help of examples, we have the following remarks.

Remark 2.24. If $\mathfrak{T}$ is $\acute{C}$iri$\acute{c}$-Reich-Rus type weakly contraction, $\mathfrak{T}$ accepts a fixed point that is not necessarily unique.

Remark 2.25. In Theorem 2.13, $\mathfrak{T}$ and $\mathfrak{g}$ do not need a fixed point, just as $\mathfrak{T}$ and $\mathfrak{g}$ accept a point of coincidence and is not necessarily unique. Also, it is not necessary that $\mathfrak{T}$ and $\mathfrak{g}$ to be continuous. In Example 2.14, $\mathfrak{g}$ is not continuous because the sequence (2) is convergent to 1 but ($\mathfrak{g}2$) is not convergent to $\mathfrak{g}1$.

3. Application to integral equation

Now, consider a real-valued continuous function α defined on $[a_0, b_0]$ such that

$$\alpha(x) = \beta(x) + \gamma \int_a^b G(x,s)K(s,\alpha(s))ds, \quad x \in [a_0, b_0], \tag{3.1}$$

where γ is a constant, $\mathfrak{K} : [a_0, b_0] \times \mathbb{R} \to [a_0, b_0]$ is lower semi-continuous, and $G : [a_0, b_0] \times [a_0, b_0] \to (0, +\infty)$ is continuous.

For simplicity, we will use $X = \{\alpha \in C[a_0, b_0] : \max_x |\alpha(x)|^2 \geq 1\}$, where $C[a_0, b_0]$ denote all real continuous functions defined on $[a_0, b_0]$, and a modular-like ϱ defined on X as

$$\varrho(\alpha)=\begin{cases}\max\limits_{a_0\le x\le b_0}\mid\alpha(x)\mid^2, & \alpha\neq 0,\\ 1, & \alpha=0.\end{cases}$$

Since X is ϱ-complete and the integral equation defined in (3.1) can be reformulated to show that α is a solution of problem (3.1) if and only if α is a fixed point of $\mathfrak{f}: X\to X$ defined as:

$$\mathfrak{f}\alpha=\beta(x)+\gamma\int_{a_0}^{b_0}G(x,s)\Re(s,\alpha(s))ds.$$

Now, we suppose that the following assumptions are satisfied:

(i) for any $u,v\in X$,

$$|\Re(x,u(x))-\Re(x,v(x))|\le\frac{1}{2}\mid u(x)-v(x)\mid^{0.8}\mid u(x)-\mathfrak{f}u(x)\mid^{0.1}\mid v(x)-\mathfrak{f}v(x)\mid^{0.1},$$

(ii) $\max\limits_{a_0\le x\le b_0}\int_{a_0}^{b_0}G^2(x,z)dz\le\frac{4}{b_0-a_0}$,

(iii) $\mid\gamma\mid^2\le 1$.

Then, the integral equation in (3.1) has a solution in X.

For $\mathfrak{f}u=\mathfrak{f}v$ Eq (2.17) is satisfied, so we suppose that $\mathfrak{f}u\neq\mathfrak{f}v$. By the Cauchy-Schwarz inequality and assumption (i–iii), we have

$$\begin{aligned}\varrho(\mathfrak{f}u-\mathfrak{f}v)&=\max_{a_0\le x\le b_0}\mid\beta(x)+\gamma\int_{a_0}^{b_0}G(x,s)\Re(s,u(s))ds-\beta(x)-\gamma\int_{a_0}^{b_0}G(x,s)\Re(s,v(s))ds\mid^2\\
&=\mid\gamma\mid^2\max_{a_0\le x\le b_0}\mid\int_{a_0}^{b_0}G(x,s)[\Re(s,u(s))-\Re(s,v(s))]ds\mid^2\\
&\le\mid\gamma\mid^2\max_{a_0\le x\le b_0}\{\int_{a_0}^{b_0}G^2(x,s)ds\int_{a_0}^{b_0}[\Re(s,u(s))-\Re(s,v(s))]^2ds\}\\
&=\mid\gamma\mid^2\max_{a_0\le x\le b_0}\int_{a_0}^{b_0}G^2(x,s)ds\\
&\quad\frac{1}{4}\int_{a_0}^{b_0}\mid u(s)-v(s)\mid^{1.6}\mid u(s)-\mathfrak{f}u(s)\mid^{0.2}\mid v(s)-\mathfrak{f}v(s)\mid^{0.2}\,ds\\
&\le\frac{\mid\gamma\mid^2}{b_0-a_0}\int_{a_0}^{b_0}\max_{a_0\le s\le b_0}\mid u(s)-v(s)\mid^{1.6}\mid u(s)-\mathfrak{f}u(s)\mid^{0.2}\mid v(s)-\mathfrak{f}v(s)\mid^{0.2}\,ds\\
&\le\mid\gamma\mid^2[\max_{a_0\le s\le b_0}\mid u(s)-v(s)\mid^2]^{0.8}[\max_{a_0\le s\le b_0}\mid u(s)-\mathfrak{f}u(s)\mid^2]^{0.1}[\max_{a_0\le s\le b_0}\mid v(s)-\mathfrak{f}v(s)\mid^2]^{0.1}\\
&\le[\varrho(u-v)]^{0.8}[\varrho(u-\mathfrak{f}u)]^{0.1}[\varrho(v-\mathfrak{f}v)]^{0.1}.\end{aligned}$$

Theorem 2.21 with $\varphi(u)=0$, $\varsigma(x)=\frac{x}{3}$, $\alpha=0.8$, and $\eta=0.1$, is satisfied i.e.,

$$\begin{aligned}\varsigma(\varrho(\mathfrak{f}u-\mathfrak{f}v))\le{}&\varsigma([\varrho(u-v)]^{0.8}[\varrho(u-\mathfrak{f}u)]^{0.1}[\varrho(v-\mathfrak{f}v)]^{0.1})\\&-\varphi([\varrho(u-v)]^{0.8}[\varrho(u-\mathfrak{f}u)]^{0.1}[\varrho(v-\mathfrak{f}v)]^{0.1}).\end{aligned}$$

Therefore, $\mathfrak{f}$ has a fixed point $w\in X$, i.e., the integral Eq (3.1) has a solution.

4. Conclusions

In this paper, we define a modular-like space and prove some results on this space. This paper opens a door for proving various results on modular-like space. In our main results, we introduced a new method in which ϱ-convergence of the constructed sequence is used. Further, we illustrated each result with the help of a proper and nontrivial example, which shows the validity of our result. Further, as an application, we solved an integral equation with the help of our main theorem.

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Conflict of interest

The authors declare to have no competing interests.

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