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Research article

On existence theorems for coupled systems of quadratic Hammerstein-Urysohn integral equations in Orlicz spaces

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Abstract: We present two existence theorems for a general system of functional quadratic Hammerstein-Urysohn integral equations in arbitrary Orlicz spaces L_φ, namely when the generating N-functions fulfill Δ' and Δ_3-conditions. The studied system contains many integral equations as special cases such as the Chandrasekhar equations, which have significant applications in technology and different disciplines of science. Our analysis is concerned with the fixed point approach and a measure of noncompactness.

Keywords: measure of noncompactness; Orlicz spaces; coupled systems of integral equations; compact in measure; Δ'; Δ_3-conditions
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1. Introduction

The physical and biological models are often formulated in the structure of differential and integral equations or their systems. The systems of integral equations have various interests in numerous fields of science like nuclear physics [1], heat conduction [2], electromagnetic [3], diffusion equations [4], and multimedia processing [5].

Here, we introduce two separate existence theorems for the coupled system:

$$\begin{cases} x(t) = g_1(t) + f_1\Big(t,\ y(t),\ \lambda \cdot V_1 y(t) \int_a^b \mathcal{K}(t,s) h_1(s, y(s))\, ds,\ \lambda \cdot G_1 y(t) \int_a^b u_1(t,s,y(s))\, ds\Big) \\ y(t) = g_2(t) + f_2\Big(t,\ x(t),\ \lambda \cdot V_2 x(t) \int_a^b \mathcal{K}(t,s) h_2(s, x(s))\, ds,\ \lambda \cdot G_2 x(t) \int_a^b u_2(t,s,x(s))\, ds\Big) \end{cases} \tag{1.1}$$

in Orlicz spaces $L_\varphi(J)$, $J = [a, b]$ (i.e., when the generating N-functions fulfill the Δ'-condition and Δ_3-condition), where $G_i, V_i,\ i = 1, 2$ are general operators acting on some Orlicz spaces.

In the literature, the authors inspected the coupled systems of integral equations in various functions spaces such as in the space $C(I)$ (cf. [6–9]) or in Banach algebras (cf. [10, 11], for example), where the authors utilize assumptions stronger than those in this article. Furthermore, the L_p-solutions for the coupled systems were examined in [12, 13] with polynomial growth on the studied functions. We omit these restrictions and extend these results to study the problem (1.1) with the assistance of the approach presented in [14] which isn't a Banach algebra by utilizing appropriate and different triple Orlicz spaces $(L_\varphi, L_{\varphi_1}, L_{\varphi_2})$.

The solutions in Orlicz spaces allow us to consider operators with generalized growth conditions (of exponential growth, for instance). Then, we do not expect the solutions will be continuous and it is superior to look for the solutions in Orlicz spaces. This is roused by some mathematical models in physics, (cf. [15, 16]). For example, the thermodynamical problem

$$x(t) + \int_J k(t,s)e^{x(s)}\,ds = 0$$

has exponential nonlinearities and it was inspected in Orlicz spaces in [17].

However, The Hammerstein and Urysohn integral equations have been examined in Orlicz spaces L_φ (cf. [18–20]) and in generalized Orlicz spaces (cf. [21, 22]).

Besides, the quadratic integral equations were inspected in Banach-Orlicz algebra [23] and in arbitrary Orlicz spaces in [14, 24] by using the methods of Darbo's fixed point theorems connected with a proper measure of noncompactness under a general set of assumptions see also [25, 26]. The technique of measures of noncompactness was applied in studying various types of integral equations see for example [27–29]. Our outcomes cover and unify these cases as particular cases of problem (1.1).

An interesting particular system of (1.1) is the coupled systems of generalized Chandrasekhar equations

$$\begin{cases} x(t) = a_1(t) + g_1(t, y(t)) \int_0^t \frac{t}{t+s} f_1(s, y(s))\,ds, \\ y(t) = a_2(t) + g_2(t, x(t)) \int_0^t \frac{t}{t+s} f_2(s, x(s))\,ds, \end{cases} \tag{1.2}$$

which have been inspected in the space of continuous functions in [30] and in Banach algebra [31, 32], see also [33, 34] for some generalizations of the system (1.2). In particular, the general Chandrasekhar integral equation of the form

$$x(t) = 1 + \lambda \cdot x(t) \int_0^1 \frac{t}{t+s} \psi(s) \log(1 + \sqrt{x(s)})\,ds$$

performs scattering through a homogeneous semi-infinite plane atmosphere and it is beneficial to look for discontinuous solutions for the non-homogeneous atmosphere (cf. [30, 35]), then, at that point, the solutions in Orlicz spaces are requesting (see also [23] for some details). Those equations are often applicable in the kinetic theory of gases, traffic theory, the theory of neutron transport, the theory of radiative transfer, and mathematical physics (cf. [36, 37]).

This article is instigated to investigate the presence of a.e. monotonic solutions of a general system of functional quadratic Hammerstein-Urysohn integral equations in arbitrary Orlicz spaces in two separate cases under a general set of assumptions. We extend and generalize the outcomes in the former literature from continuous solutions to discontinuous ones.

2. Notation and auxiliary facts

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$, and $J = [a, b] \subset \mathbb{R}$.

In the sequel, we need the next concepts from Orlicz spaces.

A convex and continuous function $M : \mathbb{R}^+ \to \mathbb{R}^+$, is said to be a N-function if it is even and if it fulfills both $\lim_{u\to 0} \frac{M(u)}{u} = 0$ and $\lim_{u\to\infty} \frac{M(u)}{u} = \infty$. Equivalently, M is N-function if and only if it takes the form $M(u) = \int_0^{|u|} p(t)\, dt$, $\forall u \in \mathbb{R}$, where $p : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing, right-continuous function and positive for $t > 0$, which fulfills the conditions $p(0) = 0$, $\lim_{t\to\infty} p(t) = \infty$.

If $q : \mathbb{R}^+ \to \mathbb{R}^+$ be the right-inverse of p, that is, if $q(s) = \sup\{t : p(t) \leq s\}$, $\forall s \in \mathbb{R}^+$, then $N : \mathbb{R}^+ \to \mathbb{R}^+$ given by $N(v) = \int_0^{|v|} q(s)\, ds$, $\forall v \in \mathbb{R}$, is also N-function, and M and N are called mutually complementary.

Now, let $\mathbb{L}_{\mathbb{X}} = L_M(J) \times L_M(J)$ be the Banach space of all ordered pairs (x, y), $x, y \in L_M(J)$ with the norm

$$\|(x, y)\|_{\mathbb{X}} = \|x\|_M + \|y\|_M,$$

where $L_M = L_M(J)$ be the *Orlicz space* of all measurable functions $m : J \to \mathbb{R}$ with the Luxemburg norm

$$\|m\|_M = \inf_{\epsilon>0} \left\{ \int_a^b M\left(\frac{m(s)}{\epsilon}\right) ds \leq 1 \right\}.$$

It is obvious that $(\mathbb{L}_{\mathbb{X}}, \|\cdot\|_{\mathbb{X}})$ is a Banach space.

Let $\mathbb{E}_{\mathbb{X}} = E_M(J) \times E_M(J)$ be the closure in $\mathbb{L}_{\mathbb{X}}$, where $E_M = E_M(J)$ be the closure in $L_M(J)$ of the set of all bounded functions and having equi-absolutely continuous norms (equiintegrability) i.e.,

$$\lim_{\delta\to 0} \sup_{measD<\delta} \sup_{x\in E_M} \|x \cdot \chi_D\|_M = 0,$$

where "meas" and χ_D refer to the Lebesgue measure and the characteristic function of a measurable subset $D \subset J$, respectively.

Definition 2.1. [16] The N-function M fulfills the Δ'-condition if, $\exists\, K,\ t_0 \geq 0$ s.t. $M(ts) \leq KM(t)M(s)$ for $t, s \geq t_0$.

Definition 2.2. [16] The N-function M fulfills the Δ_3-condition if, $\exists\, K,\ t_0 \geq 0$ s.t. $tM(t) \leq M(Kt)$ for $t \geq t_0$.

Definition 2.3. [16] Suppose that a function $f : J \times \mathbb{R} \to \mathbb{R}$ fulfills Carathéodory conditions i.e., it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in J$. Then to every measurable function x, we will call F_f by the superposition operator generated by the function f s.t.

$$F_f(x)(t) = f(t, x(t)), \quad t \in J.$$

Lemma 2.4. *([16, Theorem 17.5]) Suppose that a function $f : J \times \mathbb{R} \to \mathbb{R}$ fulfills Carathéodory conditions. Then*

$$M_2(f(s, x)) \leq a(s) + bM_1(x),$$

where $b \geq 0$ and $a \in L^1$, if and only if the operator F_f acts from L_{M_1} to L_{M_2}.

Lemma 2.5. *[24] Let f fulfill Carathéodory conditions. If the superposition operator $F_f : L_{M_1} \to E_{M_2}$, then it is continuous.*

For multiplications of operators, we have:

Lemma 2.6. *([38, Theorem 10.2]) Let φ_1, φ_2 and φ be arbitrary N-functions. The following hypotheses are equivalent:*

(1) For every functions $u \in L_{\varphi_1}$ and $w \in L_{\varphi_2}$, $u \cdot w \in L_{\varphi}$.
(2) There exists a constant $k > 0$ s.t. for all measurable functions u, w, we have $\|uw\|_{\varphi} \leq k\|u\|_{\varphi_1}\|w\|_{\varphi_2}$.
(3) There exists numbers $C > 0$, $u_0 \geq 0$ s.t. for all $s, t \geq u_0$ $\varphi\left(\frac{st}{C}\right) \leq \varphi_1(s) + \varphi_2(t)$.
(4) $\limsup_{t\to\infty} \frac{\varphi_1^{-1}(t)\varphi_2^{-1}(t)}{\varphi(t)} < \infty$.

For functions, which shall fulfill the above lemma, we have:

Lemma 2.7. *([16, p. 223]) If there exist complementary N-functions Q_1 and Q_2 s.t. the inequalities*

$$Q_1(\alpha u) < \varphi^{-1}[\varphi_1(u)]$$

$$Q_2(\alpha u) < \varphi^{-1}[\varphi_2(u)]$$

hold, then for every functions $u \in L_{\varphi_1}(I)$ and $w \in L_{\varphi_2}$, $u \cdot w \in L_{\varphi}(I)$. If moreover φ fulfills the Δ_2-condition, then it is sufficient that the inequalities

$$Q_1(\alpha u) < \varphi_1[\varphi^{-1}(u)]$$

$$Q_2(\alpha u) < \varphi_2[\varphi^{-1}(u)]$$

hold.

Let $S = S(J)$ be the set of measurable (in Lebesgue sense) functions on J. Identifying the functions equal almost everywhere the set S is furnished with the metric

$$d(x, y) = \inf_{a>0}[a + meas\{s \ : \ |x(s) - y(s)| \geq a\}],$$

and we obtain a complete metric space. Moreover, the convergence in measure on J is identical to the convergence with respect to the metric d (Proposition 2.14 in [39]).

Let X be a bounded subset of $L_M(J)$. Assume that there is a family of subsets $(\Omega_c)_{0\leq c\leq b-a}$ of the interval J s.t. meas$\Omega_c = c$ for every $c \in [0, b - a]$, and for every $x \in X$, $x(t_1) \geq x(t_2)$, $(t_1 \in \Omega_c, t_2 \notin \Omega_c)$. Such a family is equimeasurable (cf. [40]) and then the set X is compact in measure in $L_M(J)$. It is clear, that by putting $\Omega_c = [0, c) \cup Z$ or $\Omega_c = [0, c) \setminus Z$, where Z is a set with measure zero, this family contains nonincreasing functions (possibly except for a set Z).

We will refer to the functions from this family “a.e. nonincreasing” functions. This is the situation, when we select an integrable and nonincreasing *function* y and all functions equal a.e. to y fulfill the above condition. Thus, we can write that the *elements* from $L_M(J)$ belong to this class of functions.

Theorem 2.8. *[14] Let $X \subset L_M$ be a bounded set consisting of functions which are a.e. nondecreasing (or a.e. nonincreasing) on the interval J. Then X is compact in measure in L_M.*

Corollary 2.9. *Let $U \subset \mathbb{L}_{\mathbb{X}}$ be a bounded set. For every $x, y \in L_M$ are a.e. nonincreasing (or a.e. nondecreasing) on J, then $u = (x, y) \in U$ is a.e. nonincreasing (or a.e. nondecreasing) on J and the set U is compact in measure in $\mathbb{L}_{\mathbb{X}}$.*

Next, suppose that $(E, \|\cdot\|_E)$ be an arbitrary Banach space with zero element θ. Denote by $B_r = \{x \in E : \|x\|_E \leq r\},\ r > 0$ and the symbol $B_r(E)$ points out the space. If $X \subset E$, then $\bar{X}$ and $convX$ refer to the closure and convex closure of X, respectively. The symbols $\mathcal{M}_E$ and $\mathcal{N}_E$ refer to the family of all nonempty and bounded subsets and the subfamily of all relatively compact subsets of E, respectively.

Definition 2.10. [41] A mapping $\mu : \mathcal{M}_E \rightarrow [0,\ \infty)$ points to a measure of noncompactness (MNC) in E if it fulfills:

(i) $\mu(Y) = 0 \iff Y \in \mathcal{N}_E$.
(ii) $Y \subset X \implies \mu(Y) \leq \mu(X)$.
(iii) $\mu(\bar{Y}) = \mu(convY) = \mu(Y)$.
(iv) $\mu(\lambda Y) = |\lambda|\, \mu(Y),\ for\ \lambda \in \mathbb{R}$.
(v) $\mu(Y + X) \leq \mu(Y) + \mu(X)$.
(vi) $\mu(Y \bigcup X) = \max\{\mu(Y),\ \mu(X)\}$.
(vii) If $Y_n \neq \emptyset$ is a sequence of closed and bounded subsets of E s.t. $Y_{n+1} \subset Y_n$, $n = 1, 2, 3, \cdots$, and $\lim_{n\to\infty} \mu(Y_n) = 0$, then the set $Y_\infty = \bigcap_{n=1}^{\infty} Y_n \neq \emptyset$.

An example of such MNC is the Hausdorff MNC $\beta_H(X)$, where $\emptyset \neq X \subset E$ be bounded set (cf. [41]) is given by

$$\beta_H(X) = \inf\{r > 0 : \text{ there exists a finite subset Y of E s.t. } x \subset Y + B_r \}.$$

Definition 2.11. ([39, Definition 3.9], [42]) For any $\epsilon > 0$, the measure of equiintegrability of bounded subsets $\emptyset \neq X \in L_M$ is defined as:

$$c(X) = \lim_{\epsilon\to 0} \sup_{mesD\leq\epsilon} \sup_{x\in X} \|x \cdot \chi_D\|_{L_M(J)},$$

where χ_D refers to the characteristic function of a measurable subset $D \subset J$.

Lemma 2.12. *[24, 42] Let $\emptyset \neq X \subset E_M$ be bounded and compact in measure set. Then*

$$\beta_H(X) = c(X).$$

Definition 2.13. For any $\epsilon > 0$, the measure of equiintegrability of bounded sets $\emptyset \neq U = (X, Y) \subset \mathbb{L}_{\mathbb{X}}$, where $X, Y \subset L_M$ is defined as:

$$\begin{aligned} c(U) &= c(X, Y) = c(X) + c(Y) \\ &= \lim_{\varepsilon\to 0}\sup \sup_{mesD\leq\varepsilon} \sup_{x\in X} \|x \cdot \chi_D\|_M + \lim_{\varepsilon\to 0}\sup \sup_{mesD\leq\varepsilon} \sup_{y\in Y} \|y \cdot \chi_D\|_M, \end{aligned}$$

where χ_D refers to the characteristic function of a measurable subset $D \subset J$.

Corollary 2.14. *Let $\emptyset \neq U \subset \mathbb{L}_{\mathbb{X}}$ be a bounded and compact in measure set. Then*

$$\beta_H(U) = c(U).$$

Theorem 2.15. *Let $Q \neq \emptyset$ be a convex, bounded, and closed subset of $\mathbb{L}_{\mathbb{X}}$. Suppose $H : Q \to Q$ is a continuous operator and there exists a constant $k \in [0, 1)$ with*

$$\mu(H(U)) \leq k\mu(U)$$

for any $\emptyset \neq U \subset Q$. Then H has at least one fixed point in Q.

3. Main results

Now, we present our outcomes in demonstrating the presence of monotonic L_φ-solutions for the coupled systems (1.1). Allow us to define the operator H as following

$$H(x, y)(t) = (H_1 y(t),\ H_2 x(t)),\ t \in J,$$

where

$$H_1 y = g_1 + F_{f_1}\Big(y,\ \mathcal{V}_1(y), U_1(y)\Big),\ \ H_2 x = g_2 + F_{f_2}\Big(x,\ \mathcal{V}_2(x),\ U_2(x)\Big),$$

$$F_{f_i}\Big(z,\ \mathcal{V}_i(z),\ U_i(z)\Big) = f_i\Big(t,\ z,\ \mathcal{V}_i(z),\ U_i(z)\Big),\ \mathcal{V}_i(z) = \lambda \cdot V_i(z) \cdot \mathcal{A}_i(z),\ U_i(z) = \lambda \cdot G_i(z) A_i(z),$$

and

$$A_i(z)(t) = \int_a^b u_i(t, s, z(s))\, ds,\ \mathcal{A}_i(z) = \mathcal{K}_0 F_{h_i}(z),\ \ \mathcal{K}_0(z)(t) = \int_a^b \mathcal{K}(t, s) z(s)\, ds,$$

s.t. F_{h_i} are the superposition operator as in Definition 2.3 and $V_i, G_i,\ i = 1, 2$ are general operators.

Definition 3.1. By a solution of the coupled systems (1.1) we mean the ordered pair $u = (x, y) \in \mathbb{L}_{\mathbb{X}}$ s.t. $x, y \in L_\varphi$, where u fulfills the coupled systems (1.1).

We will inspect the solvability of the system (1.1) in two different cases.

3.1. The case of Δ'-condition

Let $\varphi, \varphi_1, \varphi_2$ be N-functions and that M and N are complementary N-functions. Moreover, for $i = 1, 2$, we write the set of assumptions:

(G1) There exists a constant $k_1 > 0$ s.t. for every $u \in L_{\varphi_1}$ and $w \in L_{\varphi_2}$ we have $\|uw\|_\varphi \leq k_1 \|u\|_{\varphi_1} \|w\|_{\varphi_2}$.

(G2) $f_i(t, x, y, z) :\ J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be measurable in t and continuous in x, y and z for almost all t. There exist constants $\alpha_j \geq 0,\ j = 1, 2, 3$ and $a_i \in L_\varphi$ s.t.

$$|f_i(t, x, y, z)| \leq a_i(t) + \alpha_1 \|x\|_\varphi + \alpha_2 \|y\|_\varphi + \alpha_3 \|z\|_\varphi.$$

Moreover, assume that $f_i(t, x, y, z)$ are positive and nondecreasing with respect to each variable separately.

(G3) $G_i, V_i : L_\varphi \to L_{\varphi_1}$, take continuously E_φ into E_{φ_1}. Assume that, for any $z \in E_\varphi$, we have $G_i(z), V_i(z) \in E_{\varphi_1}$ and there exist constants $b_0 > 0,\ c_0 > 0$ s.t.

$$|G_i(z)| \leq b_0 \|z\|_\varphi,\ \ |V_i(z)| \leq c_0 \|z\|_\varphi.$$

Moreover, suppose that G_i, V_i take the set of all a.e. nondecreasing functions into itself.

(G4) $h_i(t,z): J\times\mathbb{R}\to\mathbb{R}$ fulfill Carathéodory conditions and assume that

$$|h_i(t,z)| \le c^*(t) + R_1(|z|)$$

for $t\in J$ and $z\in\mathbb{R}$, where $c^*\in E_N$ and R_1 is nonnegative, nondecreasing, continuous function defined on $\mathbb{R}^+$. Moreover, $h_i(t,z)$ are assumed to be nondecreasing with respect to each variable separately.

(C1) $g_i\in E_\varphi(J)$ are nondecreasing a.e. on J.

(C2) $u_i(t,s,x): J\times J\times\mathbb{R}\to\mathbb{R}$ fulfill Carathéodory conditions (i.e., it is measurable in (t,s) for any $x\in\mathbb{R}$ and continuous in x for almost all $t,s\in J$. Further, $u_i(t,s,x)$ are assumed to be nondecreasing with respect to each variable separately.

(C3) $|u_i(t,s,x)|\le K(t,s)(b^*(s)+R_2(|x|)$ for $t,s\in J$ and $x\in\mathbb{R}$, where $b^*\in E_N$ and R_2 is nonnegative, nondecreasing, continuous function defined on $\mathbb{R}^+$, and $K(t,s)\ge 0$ for $t,s\in J$.

(C4) Let N fulfills the Δ'-condition and suppose that there exist $\omega,\ \gamma,\ u_0\ \ge 0$ for which

$$N(\omega(R_i(u)))\le\gamma\varphi(u)\le\gamma M(u)\ \text{ for }\ u\ge u_0.$$

(K1) $s\to K(t,s)\in L_M$ and $s\to\mathcal{K}(t,s)\in L_M$ for a.e. $t\in J$.

(K2) $K,\mathcal{K}\in E_M(J^2)$, and $t\to K(t,s)\in E_{\varphi_2}(J)$, $t\to\mathcal{K}(t,s)\in E_{\varphi_2}(J)$ for a.e. $s\in J$.

(K3) $\int_a^b\mathcal{K}(t_1,s)\,ds\ge\int_a^b\mathcal{K}(t_2,s)\,ds$, for $t_1,t_2\in J$ with $t_1<t_2$.

Theorem 3.2. *Let the assumptions (G1)–(G4), (C1)–(C4), and (K1)–(K3) be fulfilled. If*

$$\Big(\alpha_1+2\alpha_2k_1|\lambda|c_0\cdot\|\mathcal{K}\|_M(\|c^*\|_N+R_1(1))+2\alpha_3k_1|\lambda|b_0\cdot\|K\|_M(\|b^*\|_N+R_2(1))\Big)\ <\ 1,$$

then there exists a number $\rho>0$ s.t. for all $\lambda\in\mathbb{R}$ with $|\lambda|<\rho$ there exists a solution $u\in\mathbb{E}_{\mathbb{X}}$ of the coupled systems (1.1) which is a.e. nondecreasing on J.

Proof. **Step I.** In what follows, let $i=1,2$. First, by assumption (G2), (G4) and Lemma 2.4, we have that the operators F_{f_i} act from E_φ into itself and $F_{h_i}: E_\varphi\to E_N$.

Next, we need to demonstrate, that the operators H_i map the unit ball in E_φ into the space E_φ continuously. Taking into account supposition (G3) and Lemma 2.6, it is adequate to inspect that property for the operators A_i and $\mathcal{A}_i$.

Since N is N-function fulfilling Δ'-condition and by (C3), (G4), we are able to utilize [16, Theorem 19.1]. From this, there exist constants C,C^* (not depending on the kernels) s.t. for any measurable $T\subset J$ and $u=(x,y)\in\mathbb{L}_{\mathbb{X}}$ with $\|u\|_{\mathbb{X}}=\|x\|_\varphi+\|y\|_\varphi\le 1$, where $x,y\in L_\varphi$, we have

$$\|\mathcal{A}_1(y)\chi_T\|_{\varphi_2}\le C^*\|\mathcal{K}\chi_{T\times J}\|_M\ \text{ and }\ \|A_1(y)\chi_T\|_{\varphi_2}\le C\|K\chi_{T\times J}\|_M. \tag{3.1}$$

Now, by the Hölder inequality and the assumptions (C3), (G4), we get

$$|\mathcal{A}_1y(t)|\le\|\mathcal{K}(t,s)\|\cdot|c^*(s)+R_1(|y(s)|)|\quad\text{and}\ \ |A_1y(t)|\le\|K(t,s)\|\cdot|b^*(s)+R_2(|y(s)|)|$$

for $t,s\in J$. Put $\mathcal{P}(t)=2\|\mathcal{K}(t,\cdot)\|_M$ and $k(t)=2\|K(t,\cdot)\|_M$ for $t\in J$. As $\mathcal{K},K\in E_M(J^2)$ these functions are integrable on J. By the assumptions (K1) and (K2) about the kernels $\mathcal{K},K$ (cf. [20]) we obtain that

$$\|\mathcal{A}_1(y)(t)\|\le\mathcal{P}(t)\cdot(\|c^*\|_N+\|R_1(|y(\cdot)|)\|_N)\ \text{ for a.e. } t\in J,$$

and

$$\|A_1(y)(t)\| \leq k(t) \cdot (\|b^*\|_N + \|R_2(|y(\cdot)|)\|_N), \quad \text{for a.e. } t \in J.$$

Whence for arbitrary measurable subset T of J and $y \in E_\varphi$

$$\|\mathcal{A}_1(y)\chi_T\|_{\varphi_2} \leq \|\mathcal{P}\chi_T\|_{\varphi_2} \cdot (\|c^*\|_N + \|R_1(|y(\cdot)|)\|_N),$$

and

$$\|A_1(y)\chi_T\|_{\varphi_2} \leq \|k\chi_T\|_{\varphi_2} \cdot (\|b^*\|_N + \|R_2(|y(\cdot)|)\|_N).$$

Finally if t is s.t. $\mathcal{K}(t,\cdot) \in E_M,\ K(t,\cdot) \in E_M$, and $y \in E_\varphi$ we have

$$\int_T \|\mathcal{K}(t,s)h_1(s,y(s))\|\, ds \leq 2\|\mathcal{K}(t,\cdot)\chi_T\|_M \cdot (\|c^*\|_N + \|R_1(|y(\cdot)|)\|_N) \ \text{ for a.e. } \ t \in J,$$

and

$$\int_T \|u_1(t,s,y(s))\|\, ds \leq 2\|K(t,\cdot)\chi_T\|_M \cdot (\|b^*\|_N + \|R_2(|y(\cdot)|)\|_N) \ \text{ for a.e. } \ t \in J.$$

From this, it follows that $\mathcal{A}_1, A_1 : B_1(E_\varphi) \to E_{\varphi_2}$.

Next, we will show that $\mathcal{A}_1, A_1 : B_1(E_\varphi) \to E_{\varphi_2}$ are continuous. Let $y_n, y_0 \in B_1(E_\varphi)$ be s.t. $\|y_n - y_0\|_\varphi \to 0$ as $n \to \infty$. Suppose, contrary to our claim, that $\mathcal{A}_1,\ A_1$ are not continuous and the $\|\mathcal{A}_1(y_n) - \mathcal{A}_1(y_0)\|_{\varphi_2},\ \|A_1(y_n) - A_1(y_0)\|_{\varphi_2}$ do not converge to zero, then there exist $\varepsilon > 0$ and a subsequence (y_{n_j}) s.t.

$$\|\mathcal{A}_1(y_{n_j}) - \mathcal{A}_1(y_0)\|_{\varphi_2} > \varepsilon,\ \ \|A_1(y_{n_j}) - A_1(y_0)\|_{\varphi_2} > \varepsilon,\ \ j = 1, 2, \cdots \tag{3.2}$$

and the subsequence is a.e. convergent to y_0. Since (y_n) is a subset of the ball, then the sequence $\left(\int_a^b \varphi(|y_n(t)|)dt\right)$ is bounded. As the space E_φ is regular, the balls are norm-closed in L_1 so the sequence $\left(\int_a^b |y_n(t)|dt\right)$ is also bounded.

Moreover, by (C3), (C4) and (G4) there exist $r,\ \omega,\ \gamma,\ u_0 > 0$, s.t. (cf. [16, p. 196])

$$\begin{aligned}
\|R_i(|y(\cdot)|)\|_N &= \frac{1}{\omega}\|\omega R_i(|y(\cdot)|)\|_N \\
&\leq \frac{1}{\omega}\inf_{r>0}\left\{\int_J N(\omega R_i(|y(t)|)/r)dt \leq 1\right\} \\
&\leq \frac{1}{\omega}\left(1 + \int_a^b N(\omega R_i(|y(t)|))dt\right) \\
&\leq \frac{1}{\omega}\left(1 + N(\omega R_i(u_0))(b-a) + \gamma\int_a^b \varphi(|y(t)|)dt\right),
\end{aligned}$$

whenever $y \in L_\varphi$ with $\|y\|_\varphi \leq 1$. Thus

$$\begin{aligned}
\int_T \|\mathcal{K}(t,s)h_1(s,y_n(s))\|\, ds &\leq 2\|\mathcal{K}(t,\cdot)\chi_T\|_M \cdot (\|c^*\|_N + \|R_1(|y_n(\cdot)|)\|_N) \\
&\leq 2\|\mathcal{K}(t,\cdot)\chi_T\|_M\left(\|c^*\|_N + \frac{1}{\omega}\left(1 + N(\omega R_1(u_0))(b-a) + \ \gamma\int_a^b \varphi(|y_n(t)|)dt\right)\right),
\end{aligned}$$

and

$$\int_T \|u_1(t,s,y_n(s))\|\,ds \;\leq\; 2\|K(t,\cdot)\chi_T\|_M\cdot\left(\|b^*\|_N+\frac{1}{\omega}\Big(1+N(\omega R_2(u_0))(b-a)+\gamma\int_a^b\varphi(|y_n(t)|)dt\Big)\right),$$

and then the sequences $(\|\mathcal{K}(t,s)h_1(s,y_n(s))\|)$ and $(\|u_1(t,s,y_n(s))\|)$ are equiintegrable on J for a.e. $t\in J$. By the continuity of $u_1(t,s,\cdot)$ and $h_1(t,\cdot)$ we get

$$\lim_{j\to\infty}\mathcal{K}(t,s)h_1(s,y_{n_j}(s))=\mathcal{K}(t,s)h_1(s,y_0(s))\quad\text{and}\quad\lim_{j\to\infty}u_1(t,s,y_{n_j}(s))=u_1(t,s,y_0(s))$$

for a.e. $s\in J$. Now, applying the Vitali convergence theorem we obtain that

$$\lim_{j\to\infty}\mathcal{A}_1(y_{n_j})(t)=\mathcal{A}_1(y_0)(t),\ \text{ and }\ \lim_{j\to\infty}A_1(y_{n_j})(t)=A_1(y_0)(t)\quad\text{for a.e. } t\in J. \tag{3.3}$$

From (3.1) we have $A_1(y_{n_j})$ and $\mathcal{A}_1(y_{n_j})$ are subsets of E_{φ_2} and then (3.3) contradicts the inequality (3.2). Since A_1 and $\mathcal{A}_1$ are continuous between indicated spaces. By our assumption (G3) the operators G_1 and V_1 are continuous from $B_1(E_\varphi)$ into E_{φ_1} and then by (G1) the operators $U_1,\mathcal{V}_1: B_1(E_\varphi)\to E_\varphi$ are continuous. At last, by the assumption (C1) the operator $H_1:\ B_1(E_\varphi)\to E_\varphi$ is continuous.

Similarly for $x\in E_\varphi$, we have that the operator $H_2:\ B_1(E_\varphi)\to E_\varphi$ is continuous.

Then for $u\in\mathbb{E}_{\mathbb{X}},\ \|u\|_{\mathbb{X}}\leq 1$, we have

$$\begin{aligned}\lim_{n\to\infty}Hu_n(t) &= \lim_{n\to\infty}(H_1y_n(t),\ H_2x_n(t))\\ &= (\lim_{n\to\infty}H_1y_n(t),\ \lim_{n\to\infty}H_2x_n(t))\\ &= (H_1y(t),\ H_2x(t))=Hu(t),\end{aligned}$$

which infers that, the operator $H: B_1(\mathbb{E}_{\mathbb{X}})\to\mathbb{E}_{\mathbb{X}}$ is continuous.

Step II. We will construct the invariant ball, which our operator acts i.e.,

$$B_1(\mathbb{E}_{\mathbb{X}})=\{u=(x,y)\in\mathbb{L}_{\mathbb{X}}: x,y\in E_\varphi,\ \|u\|_{\mathbb{X}}\leq 1\}.$$

Let $u=(x,y)$ be arbitrary elements from $B_1(\mathbb{E}_{\mathbb{X}}),\ x,y\in E_\varphi$, then by utilizing the formula (3.1) and for sufficiently small λ (i.e., $|\lambda|<\rho$), where

$$\rho=\frac{1-\|g_1\|_\varphi-\|g_2\|_\varphi-\|a_1\|_\varphi-\|a_2\|_\varphi-\alpha_1}{\alpha_2k_1c_0\cdot C^*\cdot\|\mathcal{K}\|_M+\alpha_3k_1b_0\cdot C\cdot\|K\|_M},$$

we have

$$\begin{aligned}\|H_1y\|_\varphi &\leq \|g_1\|_\varphi+\|f_1(t,\ y,\ \mathcal{V}_1(y),\ U_1(y))\|_\varphi\\ &\leq \|g_1\|_\varphi+\|a_1\|_\varphi+\ \alpha_1\|y\|_\varphi+\alpha_2\|\mathcal{V}_1y\|_\varphi+\alpha_3\|U_1y\|_\varphi\\ &\leq \|g_1\|_\varphi+\|a_1\|_\varphi+\ \alpha_1\|y\|_\varphi+\alpha_2\|\lambda V_1(y)\cdot\mathcal{A}_1(y)\|_\varphi+\alpha_3\|\lambda G_1(y)\cdot A_1(y)\|_\varphi\\ &\leq \|g_1\|_\varphi+\|a_1\|_\varphi+\ \alpha_1\|y\|_\varphi+\alpha_2k_1|\lambda|\|V_1(y)\|_{\varphi_1}\cdot\|\mathcal{A}_1(y)\|_{\varphi_2}\\ &\quad+\alpha_3k_1|\lambda|\|G_1(y)\|_{\varphi_1}\cdot\|A_1(y)\|_{\varphi_2}\\ &\leq \|g_1\|_\varphi+\|a_1\|_\varphi+\ \alpha_1\|y\|_\varphi+\alpha_2k_1|\lambda|c_0\|y\|_\varphi\cdot\left\|\int_a^b\mathcal{K}(t,s)h_1(s,y(s))\,ds\right\|_{\varphi_2}\end{aligned}$$

$$
\begin{aligned}
&+\alpha_3 k_1 b_0 |\lambda| \|y\|_\varphi \cdot \left\| \int_a^b u(t,s,y(s))\, ds \right\|_{\varphi_2} \\
\leq\ & \|g_1\|_\varphi + \|a_1\|_\varphi + \ \alpha_1 \|y\|_\varphi + \alpha_2 k_1 |\lambda| c_0 \|y\|_\varphi \cdot C^* \cdot \|\mathcal{K}\|_M + \alpha_3 k_1 b_0 |\lambda| \|y\|_\varphi \cdot C \cdot \|K\|_M.
\end{aligned}
$$

Similarly, for $x \in E_\varphi$, we have

$$
\|H_2 x\|_\varphi \leq \|g_2\|_\varphi + \|a_2\|_\varphi + \ \alpha_1 \|x\|_\varphi + \alpha_2 k_1 |\lambda| c_0 \|x\|_\varphi \cdot C^* \cdot \|\mathcal{K}\|_M + \alpha_3 k_1 b_0 \|x\|_\varphi |\lambda| \cdot C \cdot \|K\|_M.
$$

Then for $u \in \mathbb{E}_\mathbb{X}$, we have

$$
\begin{aligned}
\|Hu\|_\mathbb{X} &= \|H_1 y\|_\varphi + \|H_2 x\|_\varphi \\
&\leq \|g_1\|_\varphi + \|g_2\|_\varphi + \|a_1\|_\varphi + \|a_2\|_\varphi + \ \alpha_1 (\|x\|_\varphi + \|y\|_\varphi) \\
&\quad + \alpha_2 k_1 |\lambda| c_0 \cdot C^* \cdot \|\mathcal{K}\|_M (\|x\|_\varphi + \|y\|_\varphi) + \alpha_3 k_1 \lambda | b_0 \cdot C \cdot \|K\|_M (\|x\|_\varphi + \|y\|_\varphi) \\
&= \|g_1\|_\varphi + \|g_2\|_\varphi + \|a_1\|_\varphi + \|a_2\|_\varphi \\
&\quad + \alpha_1 \|u\|_\mathbb{X} + \alpha_2 k_1 |\lambda| c_0 \cdot C^* \cdot \|\mathcal{K}\|_M \|u\|_\mathbb{X} + \alpha_3 k_1 |\lambda| b_0 \cdot C \cdot \|K\|_M \|u\|_\mathbb{X} \\
&\leq \|g_1\|_\varphi + \|g_2\|_\varphi + \|a_1\|_\varphi + \|a_2\|_\varphi + \ \alpha_1 \\
&\quad + \rho k_1 \Big(\alpha_2 c_0 \cdot C^* \cdot \|\mathcal{K}\|_M + \alpha_3 b_0 \cdot C \cdot \|K\|_M \Big) \leq 1,
\end{aligned}
$$

whenever $\|u\|_\mathbb{X} \leq 1$. Then we have $H : B_1(\mathbb{E}_\mathbb{X}) \to \mathbb{E}_\mathbb{X}$ is continuous.

Step III. Let $Q_1 \subset B_1(\mathbb{E}_\mathbb{X})$ which contains all functions that are a.e. nondecreasing on J. This set is bounded, nonempty, convex, and closed in $\mathbb{E}_\mathbb{X}$. Moreover, the set Q_1 is compact in measure due to Corollary 2.9.

Step IV. Now, we will show that H preserves the monotonicity of functions. Take $u = (x, y) \in Q_1$, then x and y are a.e. nondecreasing on J and consequently $A_i, \mathcal{A}_i, i = 1, 2$ are a.e. nondecreasing on J thanks for the assumption (C2), (K3) and (G4).

By (G3), the operators $\mathcal{V}_i, U_i, i = 1, 2$ are a.e. nondecreasing on J. Further, F_{f_i}, $i = 1, 2$ are additionally of the same type in virtue of the assumption (G2). Additionally, the assumption (C1) grants us that the operators H_1, H_2 are also a.e. nondecreasing on J. This gives us that $H : Q_1 \to Q_1$ is continuous.

Step V. We will demonstrate that H is a contraction concerning the MNC. Assume that $\emptyset \neq X \subset Q_1$ and let $\varepsilon > 0$ be arbitrary. Then, for an arbitrary $u = (x, y) \in X$ and for a set $D \subset J$, meas$D \leq \varepsilon$, we have

$$
\begin{aligned}
&\|H_1 y \cdot \chi_D\|_\varphi \leq \|g_1 \cdot \chi_D\|_\varphi + \left\| F_{f_1}\Big(y,\ \mathcal{V}_1 y,\ U_1 y\Big) \cdot \chi_D \right\|_\varphi \\
\leq\ & \|g_1 \cdot \chi_D\|_\varphi + \|a_1 \cdot \chi_D\|_\varphi + \alpha_1 \|y \cdot \chi_D\|_\varphi + \alpha_2 \|\mathcal{V}_1 y \cdot \chi_D\|_\varphi + \alpha_3 \|U(x) \cdot \chi_D\|_\varphi \\
\leq\ & \|g_1 \cdot \chi_D\|_\varphi + \|a_1 \cdot \chi_D\|_\varphi + \alpha_1 \|y \cdot \chi_D\|_\varphi + \alpha_2 \|\lambda \cdot V_1(y) \mathcal{A}_1 y \cdot \chi_D\|_\varphi \\
& + \alpha_3 \|\lambda \cdot G_1(y) \cdot A_1(y) \cdot \chi_D\|_\varphi \\
\leq\ & \|g_1 \cdot \chi_D\|_\varphi + \|a_1 \cdot \chi_D\|_\varphi + \alpha_1 \|y \cdot \chi_D\|_\varphi \\
& + \alpha_2 k_1 |\lambda| \cdot \|V_1(y) \cdot \chi_D\|_{\varphi_1} \|\mathcal{A}_1 y \cdot \chi_D\|_{\varphi_2} + \alpha_3 k_1 |\lambda| \cdot \|G_1(y) \cdot \chi_D\|_{\varphi_1} \cdot \|A_1(y) \cdot \chi_D\|_{\varphi_2} \\
\leq\ & \|g_1 \cdot \chi_D\|_\varphi + \|a_1 \cdot \chi_D\|_\varphi + \alpha_1 \|y \cdot \chi_D\|_\varphi + \alpha_2 k_1 |\lambda| c_0 \cdot \|y \cdot \chi_D\|_\varphi \left\| \int_D \mathcal{K}(t,s) h_1(s, y(s))\, ds \right\|_{\varphi_2}
\end{aligned}
$$

$$
\begin{aligned}
& +\alpha_3 k_1|\lambda| b_0 \cdot \|y \cdot \chi_D\|_\varphi \cdot \left\| \int_D u(t,s,x(s))\, ds \right\|_{\varphi_2} \\
\leq\; & \|g_1 \cdot \chi_D\|_\varphi + \|a_1 \cdot \chi_D\|_\varphi + \alpha_1 \|y \cdot \chi_D\|_\varphi + \alpha_2 k_1 |\lambda| c_0 \cdot \|y \cdot \chi_D\|_\varphi \cdot 2\|\mathcal{K}\|_M \cdot \|c^* + R_1(1)\|_N \\
& +\alpha_3 k_1|\lambda| b_0 \cdot \|y \cdot \chi_D\|_\varphi \cdot 2\|K\|_M \cdot \|b^* + R_2(1)\|_N \\
\leq\; & \|g_1 \cdot \chi_D\|_\varphi + \|a_1 \cdot \chi_D\|_\varphi + \alpha_1 \|y \cdot \chi_D\|_\varphi + \alpha_2 k_1 |\lambda| c_0 \cdot \|y \cdot \chi_D\|_\varphi \cdot 2\|\mathcal{K}\|_M (\|c^*\|_N + R_1(1)) \\
& +\alpha_3 k_1|\lambda| b_0 \cdot \|y \cdot \chi_D\|_\varphi \cdot 2\|K\|_M (\|b^*\|_N + R_2(1)).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\|H_2 x \cdot \chi_D\|_\varphi \leq\; & \|g_2 \cdot \chi_D\|_\varphi + \|a_2 \cdot \chi_D\|_\varphi + \alpha_1 \|x \cdot \chi_D\|_\varphi + \alpha_2 k_1 |\lambda| c_0 \cdot \|x \cdot \chi_D\|_\varphi \cdot 2\|\mathcal{K}\|_M (\|c^*\|_N + R_1(1)) \\
& +\alpha_3 k_1|\lambda| b_0 \cdot \|x \cdot \chi_D\|_\varphi \cdot 2\|K\|_M (\|b^*\|_N + R_2(1)).
\end{aligned}
$$

Then

$$
\begin{aligned}
\|Hu \cdot \chi_D\|_{\mathbb{X}} =\; & \|H_1 y \cdot \chi_D\|_\varphi + \|H_2 x \cdot \chi_D\|_\varphi \\
\leq\; & \|g_1 \cdot \chi_D\|_\varphi + \|a_1 \cdot \chi_D\|_\varphi + \|g_2 \cdot \chi_D\|_\varphi + \|a_2 \cdot \chi_D\|_\varphi + \alpha_1 (\|x \cdot \chi_D\|_\varphi + \|y \cdot \chi_D\|_\varphi) \\
& +2\alpha_2 k_1 |\lambda| c_0 \cdot \|\mathcal{K}\|_M (\|c^*\|_N + R_1(1))(\|x \cdot \chi_D\|_\varphi + \|y \cdot \chi_D\|_\varphi) \\
& +2\alpha_3 k_1 |\lambda| b_0 \cdot \|K\|_M (\|b^*\|_N + R_2(1))(\|x \cdot \chi_D\|_\varphi + \|y \cdot \chi_D\|_\varphi) \\
\leq\; & \|g_1 \cdot \chi_D\|_\varphi + \|a_1 \cdot \chi_D\|_\varphi + \|g_2 \cdot \chi_D\|_\varphi + \|a_2 \cdot \chi_D\|_\varphi \\
& +\alpha_1 \|u \cdot \chi_D\|_{\mathbb{X}} + 2\alpha_2 k_1 |\lambda| c_0 \cdot \|\mathcal{K}\|_M (\|c^*\|_N + R_1(1)) \|u \cdot \chi_D\|_{\mathbb{X}} \\
& +2\alpha_3 k_1 |\lambda| b_0 \cdot \|K\|_M (\|b^*\|_N + R_2(1)) \|u \cdot \chi_D\|_{\mathbb{X}}.
\end{aligned}
$$

Hence, considering that $g_i, a_i \in E_\varphi,\ i = 1, 2$, we have

$$
\lim_{\varepsilon \to 0} \{ \sup_{mes\ D \leq \varepsilon} \ \sup_{u \in X} [\sup\{\|g_1 \chi_D\|_\varphi + \|a_1 \chi_D\|_\varphi + \|g_2 \chi_D\|_\varphi + \|a_2 \chi_D\|_\varphi = 0\}]\}.
$$

Thus by Definition 2.13, we get

$$
c(H(X)) \leq W \cdot c(X),
$$

where

$$
W = \Big(\alpha_1 + 2\alpha_2 k_1 |\lambda| c_0 \cdot \|\mathcal{K}\|_M (\|c^*\|_N + R_1(1)) + 2\alpha_3 k_1 |\lambda| b_0 \cdot \|K\|_M (\|b^*\|_N + R_2(1)) \Big).
$$

Since $\emptyset \neq X \subset Q_1$ is a bounded and compact in measure subset of $\mathbb{E}_{\mathbb{X}}$, we can utilize Corollary 2.14 and get

$$
\beta_H(H(X)) \leq W \cdot \beta_H(X).
$$

Since $W < 1$, we can utilize Theorem 2.15, which achieves the verification. □

3.2. The case of Δ_3-condition

Allow us to consider the case of N-functions fulfilling Δ_3-condition with the growth essentially more fast than a polynomial. Note that the N-function M determines the properties of the Orlicz spaces L_M, and then the less restrictive rate of the growth of this function infers the "worser" properties of the space. By ϑ we will assign the norm of the identity operator from L_φ into L^1 i.e., $\sup\{\|x\|_1 : x \in B_1(L_\varphi)\}$.

Theorem 3.3. *Assume that $\varphi, \varphi_1, \varphi_2$ are N-functions and M and N are complementary N-functions, and that (G1)–(G4), (C1)–(C3), (K1) and (K3) hold true. Additionally, write the following assumptions:*

(C5) (1) N fulfills the Δ_3-condition.
(2) $\mathcal{K} \in E_M(J^2),\ K \in E_M(J^2)$ and $t \to \mathcal{K}(t,s) \in E_{\varphi_2}(J),\ t \to K(t,s) \in E_{\varphi_2}(J)$ for a.e. $s \in J$.
(3) There exist $\beta,\ u_0 > 0$ s.t.

$$R_i(u) \le \beta \frac{M(u)}{u}, \quad \text{for} \quad u \ge u_0, i = 1, 2.$$

(4) φ_2 is a N-function fulfilling

$$\iint_{J^2} \varphi_2(M(|\mathcal{K}(t,s)|))\, dtds < \infty, \quad \iint_{J^2} \varphi_2(M(|K(t,s)|))\, dtds < \infty.$$

(K4) Assume there exists a number $r > 0$ with

$$W' = \left[\alpha_1 + 2C\alpha_2 k_1 c_0 |\lambda| \|\mathcal{K}\|_{\varphi_2 \circ M}\Big(\|c^*\|_N + R_1(r)\Big) + 2C\alpha_3 k_1 b_0 |\lambda| \cdot \|K\|_{\varphi_2 \circ M}\Big(\|b^*\|_N + R_2(r)\Big)\right] < 1,$$

and

$$\begin{aligned} &\|g_1\|_\varphi + \|a_1\|_\varphi + \|g_2\|_\varphi + \|a_2\|_\varphi + \alpha_1 \cdot r + \alpha_2 k_1 c_0 |\lambda| 2C \cdot r \cdot \|\mathcal{K}\|_{\varphi_2 \circ M}\Big(\|c^*\|_N \\ &+ \frac{1}{\omega}\Big(1 + \eta_0 u_0 (b-a)\Big)\Big) + \alpha_3 k_1 b_0 |\lambda| 2C \cdot r \cdot \|K\|_{\varphi_2 \circ M}\Big(\|b^*\|_N + \frac{1}{\omega}\Big(1 + \eta_0 u_0 (b-a)\Big)\Big) \\ &+ \Big(\alpha_2 c_0 \|\mathcal{K}\|_{\varphi_2 \circ M} + \alpha_3 b_0 \|K\|_{\varphi_2 \circ M}\Big)\frac{k_1 |\lambda| 2C \cdot \eta_0 \vartheta}{\omega} r^2 \le r, \end{aligned} \tag{3.4}$$

where $C = (2 + (b-a)(1 + \varphi(1)))$. Then, there exists a number $\rho > 0$ s.t. for all $\lambda \in \mathbb{R}$ with $|\lambda| < \rho$, there exists a solution $u \in \mathbb{E}_{\mathbb{X}}$ of the coupled systems (1.1) which is a.e nondecreasing on J.

Proof. **Step I'.** In this case, we will inspect the operator H on the whole $\mathbb{E}_{\mathbb{X}}$. By [16, Lemma 15.1 and Theorem 19.2] and the assumption $(C5)_{(4)}$:

$$\begin{aligned} \|\mathcal{A}_1(y)\chi_T\|_{\varphi_2} &\le 2 \cdot C \cdot \|\mathcal{K} \cdot \chi_{T \times J}\|_{\varphi_2 \circ M}(\|c^*\|_N + \|R_1(|y(\cdot)|)\|_N) \\ \|A_1(y)\chi_T\|_{\varphi_2} &\le 2 \cdot C \cdot \|K \cdot \chi_{T \times J}\|_{\varphi_2 \circ M}(\|b^*\|_N + \|R_2(|y(\cdot)|)\|_N) \end{aligned}$$

for arbitrary $y \in L_\varphi$ and arbitrary measurable subset T of J. Let us note, that the assumption $(C5)_{(3)}$ implies that, $\exists\ \omega, u_0 > 0$ and $\eta_0 > 1$ s.t. $N(\omega R_i(y)) \le \eta_0 y$ for $y \ge u_0, i = 1, 2$. Thus for $y \in L_\varphi$

$$\begin{aligned} \|R_i(|y(\cdot)|)\|_N &\le \frac{1}{\omega}\left(1 + \int_J N(\omega R_i(|y(s)|)\, ds\right) \\ &\le \frac{1}{\omega}\left(1 + N(\omega R_i(y))(b-a) + \eta_0 \int_J |y(s)|\, ds\right) \\ &\le \frac{1}{\omega}\left(1 + \eta_0 u_0 (b-a) + \eta_0 \int_J |y(s)|\, ds\right),\ i = 1, 2. \end{aligned}$$

The remaining estimations can be derived as in Theorem 3.2 and then we obtain that $\mathcal{A}_1, A_1 : E_\varphi \to E_{\varphi_2}$, so by assumption (G3) and the properties of F_{f_1}, we get $H_1 : E_\varphi \to E_\varphi$. Similarly, for $x \in E_\varphi$, we have $H_2 : E_\varphi \to E_\varphi$. Then for $u \in \mathbb{E}_{\mathbb{X}}$, we have $H : \mathbb{E}_{\mathbb{X}} \to \mathbb{E}_{\mathbb{X}}$.

Step II'. We will inspect the operator H on the ball $B_r(\mathbb{E}_{\mathbb{X}})$, where r is as in assumption (K4). For arbitrary $u = (x, y) \in B_r(\mathbb{E}_{\mathbb{X}})$, $x, y \in E_\varphi$, and for sufficiently small λ (i.e., $|\lambda| < \rho$), where

$$\rho = \frac{1}{\alpha_2 k_1 c_0 C \|\mathcal{K}\|_{\varphi_2 \circ M}\Big(\|c^*\|_N + \frac{1}{\omega}\Big(1 + \eta_0 u_0(b-a)\Big)\Big) + \alpha_3 k_1 b_0 C \|K\|_{\varphi_2 \circ M}\Big(\|b^*\|_N + \frac{1}{\omega}\Big(1 + \eta_0 u_0(b-a)\Big)\Big)},$$

we have

$$\begin{aligned}
\|H_1 y\|_\varphi \leq\ & \|g_1\|_\varphi + \|a_1\|_\varphi + \alpha_1 \|y\|_\varphi + \alpha_2 k_1 c_0 |\lambda| \cdot \|y\|_\varphi \Big\| \int_J \mathcal{K}(t,s) h_1(s, y(s))\, ds \Big\|_{\varphi_2} \\
& + \alpha_3 k_1 b_0 |\lambda| \cdot \|y\|_\varphi \Big\| \int_J u_1(t, s, y(s))\, ds \Big\|_{\varphi_2} \\
\leq\ & \|g_1\|_\varphi + \|a_1\|_\varphi + \alpha_1 \|y\|_\varphi + \alpha_2 k_1 c_0 |\lambda| 2C \cdot \|y\|_\varphi \|\mathcal{K}\|_{\varphi_2 \circ M} \Big(\|c^*\|_N \\
& + \frac{1}{\omega}\Big(1 + \eta_0 u_0(b-a) + \eta_0 \int_J |y(s)|\, ds\Big)\Big) + \alpha_3 k_1 b_0 |\lambda| 2C \cdot \|y\|_\varphi \|K\|_{\varphi_2 \circ M} \Big(\|b^*\|_N \\
& + \frac{1}{\omega}\Big(1 + \eta_0 u_0(b-a) + \eta_0 \int_J |y(s)|\, ds\Big)\Big) \\
\leq\ & \|g_1\|_\varphi + \|a_1\|_\varphi + \alpha_1 \|y\|_\varphi + \alpha_2 k_1 c_0 |\lambda| 2C \cdot \|y\|_\varphi \|\mathcal{K}\|_{\varphi_2 \circ M} \Big(\|c^*\|_N \\
& + \frac{1}{\omega}\Big(1 + \eta_0 u_0(b-a) + \eta_0 \vartheta \|y\|_\varphi\Big)\Big) + \alpha_3 k_1 b_0 |\lambda| 2C \cdot \|y\|_\varphi \|K\|_{\varphi_2 \circ M} \Big(\|b^*\|_N \\
& + \frac{1}{\omega}\Big(1 + \eta_0 u_0(b-a) + \eta_0 \vartheta \|y\|_\varphi\Big)\Big) \\
\leq\ & \|g_1\|_\varphi + \|a_1\|_\varphi + \alpha_1 \|y\|_\varphi + \alpha_2 k_1 c_0 |\lambda| 2C \cdot \|y\|_\varphi \|\mathcal{K}\|_{\varphi_2 \circ M} \Big(\|c^*\|_N + \frac{1}{\omega}\Big(1 + \eta_0 u_0(b-a)\Big)\Big) \\
& + \alpha_3 k_1 b_0 |\lambda| 2C \cdot \|y\|_\varphi \|K\|_{\varphi_2 \circ M} \Big(\|b^*\|_N + \frac{1}{\omega}\Big(1 + \eta_0 u_0(b-a)\Big)\Big) \\
& + \alpha_2 k_1 c_0 |\lambda| 2C \cdot \|\mathcal{K}\|_{\varphi_2 \circ M} \frac{\eta_0 \vartheta}{\omega} \|y\|_\varphi^2 + \alpha_3 k_1 b_0 |\lambda| 2C \cdot \|y\|_\varphi \|K\|_{\varphi_2 \circ M} \frac{\eta_0 \vartheta}{\omega} \|y\|_\varphi^2.
\end{aligned}$$

Similarly, for $x \in E_\varphi$, we have

$$\begin{aligned}
\|H_2 x\|_\varphi \leq\ & \|g_1\|_\varphi + \|a_1\|_\varphi + \alpha_1 \|x\|_\varphi + \alpha_2 k_1 c_0 |\lambda| 2C \cdot \|x\|_\varphi \|\mathcal{K}\|_{\varphi_2 \circ M} \Big(\|c^*\|_N + \frac{1}{\omega}\Big(1 + \eta_0 u_0(b-a)\Big)\Big) \\
& + \alpha_3 k_1 b_0 |\lambda| 2C \cdot \|x\|_\varphi \|K\|_{\varphi_2 \circ M} \Big(\|b^*\|_N + \frac{1}{\omega}\Big(1 + \eta_0 u_0(b-a)\Big)\Big) \\
& + \alpha_2 k_1 c_0 |\lambda| 2C \cdot \|\mathcal{K}\|_{\varphi_2 \circ M} \frac{\eta_0 \vartheta}{\omega} \|x\|_\varphi^2 + \alpha_3 k_1 b_0 |\lambda| 2C \cdot \|x\|_\varphi \|K\|_{\varphi_2 \circ M} \frac{\eta_0 \vartheta}{\omega} \|x\|_\varphi^2.
\end{aligned}$$

Then, for $u \in B_r(\mathbb{E}_{\mathbb{X}})$, we get

$$\|Hu\|_\varphi \leq \|g_1\|_\varphi + \|a_1\|_\varphi + \|g_2\|_\varphi + \|a_2\|_\varphi + \alpha_1 \|u\|_{\mathbb{X}} + \alpha_2 k_1 c_0 |\lambda| 2C \cdot \|u\|_{\mathbb{X}} \|\mathcal{K}\|_{\varphi_2 \circ M} \Big(\|c^*\|_N$$

$$+\frac{1}{\omega}\Big(1+\eta_0 u_0(b-a)\Big)\Big)+\alpha_3 k_1 b_0|\lambda|2C\cdot\|u\|_{\mathbb{X}}\|K\|_{\varphi_2\circ M}\Big(\|b^*\|_N+\frac{1}{\omega}\Big(1+\eta_0 u_0(b-a)\Big)\Big)$$

$$+\alpha_2 k_1 c_0|\lambda|2C\cdot\|\mathcal{K}\|_{\varphi_2\circ M}\frac{\eta_0\vartheta}{\omega}\|u\|^2_{\mathbb{X}}+\alpha_3 k_1 b_0|\lambda|2C\cdot\|K\|_{\varphi_2\circ M}\frac{\eta_0\vartheta}{\omega}\|u\|^2_{\mathbb{X}}$$

$$\leq \|g_1\|_\varphi+\|a_1\|_\varphi+\|g_2\|_\varphi+\|a_2\|_\varphi+\alpha_1\cdot r+\alpha_2 k_1 c_0|\lambda|2C\cdot r\cdot\|\mathcal{K}\|_{\varphi_2\circ M}\Big(\|c^*\|_N$$

$$+\frac{1}{\omega}\Big(1+\eta_0 u_0(b-a)\Big)\Big)+\alpha_3 k_1 b_0|\lambda|2C\cdot r\cdot\|K\|_{\varphi_2\circ M}\Big(\|b^*\|_N+\frac{1}{\omega}\Big(1+\eta_0 u_0(b-a)\Big)\Big)$$

$$+\Big(\alpha_2 c_0\|\mathcal{K}\|_{\varphi_2\circ M}+\alpha_3 b_0\|K\|_{\varphi_2\circ M}\Big)\frac{k_1|\lambda|2C\cdot\eta_0\vartheta}{\omega}r^2\leq r,$$

where $\|u\|_{\mathbb{X}}\leq r$ and r is a positive number fulfilling (3.4). Then, $H: B_r(\mathbb{E}_{\mathbb{X}})\to B_r(\mathbb{E}_{\mathbb{X}})$ is continuous.

Step III' and **Step IV'** are like those from Theorem 3.2 for a subset $Q_r\subset B_r(E_\varphi)$.

Step V'. Assume that $\emptyset\neq X\subset Q_r$ and let $\varepsilon>0$ be fixed arbitrary constant. Then, for an arbitrary $x\in X$ and for a set $D\subset J$, $\text{meas}D\leq\varepsilon$, we obtain

$$\|H_1 y\cdot\chi_D\|_\varphi$$

$$\leq \|g_1\cdot\chi_D\|_\varphi+\|a_1\cdot\chi_D\|_\varphi+\alpha_1\|y\cdot\chi_D\|_\varphi+\alpha_2 k_1 c_0|\lambda|\cdot\|y\cdot\chi_D\|_\varphi\Big\|\int_D\mathcal{K}(t,s)h_1(s,y(s))\,ds\Big\|_{\varphi_2}$$

$$+\alpha_3 k_1 b_0|\lambda|\cdot\|y\cdot\chi_D\|_\varphi\Big\|\int_D u_1(t,s,y(s))\,ds\Big\|_{\varphi_2}$$

$$\leq \|g_1\cdot\chi_D\|_\varphi+\|a_1\cdot\chi_D\|_\varphi+\alpha_1\|y\cdot\chi_D\|_\varphi+\alpha_2 k_1 c_0|\lambda|\cdot\|y\cdot\chi_D\|_\varphi\Big\|\int_D\mathcal{K}(\cdot,s)(c^*(s)+R_1(|y(s)|))\,ds\Big\|_{\varphi_2}$$

$$+\alpha_3 k_1 b_0|\lambda|\cdot\|y\cdot\chi_D\|_\varphi\cdot\Big\|\int_D|K(\cdot,s)|(b^*(s)+R_2(|y(s)|))\,ds\Big\|_{\varphi_2}$$

$$\leq \|g_1\cdot\chi_D\|_\varphi+\|a_1\cdot\chi_D\|_\varphi+\alpha_1\|y\cdot\chi_D\|_\varphi+2C\alpha_2 k_1 c_0|\lambda|\cdot\|y\cdot\chi_D\|_\varphi\|\mathcal{K}\|_{\varphi_2\circ M}\Big(\|c^*\|_N+R_1(r)\Big)$$

$$+2C\alpha_3 k_1 b_0|\lambda|\cdot\|y\cdot\chi_D\|_\varphi\cdot\|K\|_{\varphi_2\circ M}\Big(\|b^*\|_N+R_2(r)\Big).$$

Similarly, we have

$$\begin{aligned}\|H_2 x\cdot\chi_D\|_\varphi \leq\ & \|g_2\cdot\chi_D\|_\varphi+\|a_2\cdot\chi_D\|_\varphi+\alpha_1\|x\cdot\chi_D\|_\varphi\\ &+2C\alpha_2 k_1 c_0|\lambda|\cdot\|y\cdot\chi_D\|_\varphi\|\mathcal{K}\|_{\varphi_2\circ M}\Big(\|c^*\|_N+R_1(r)\Big)\\ &+2C\alpha_3 k_1 b_0|\lambda|\cdot\|x\cdot\chi_D\|_\varphi\cdot\|K\|_{\varphi_2\circ M}\Big(\|b^*\|_N+R_2(r)\Big).\end{aligned}$$

Then,

$$\begin{aligned}\|Hu\cdot\chi_D\|_\varphi \leq\ & \|g_1\cdot\chi_D\|_\varphi+\|g_2\cdot\chi_D\|_\varphi+\|a_1\cdot\chi_D\|_\varphi+\|a_2\cdot\chi_D\|_\varphi\\ &+\alpha_1\|u\cdot\chi_D\|_\varphi+2C\alpha_2 k_1 c_0|\lambda|\cdot\|u\cdot\chi_D\|_\varphi\|\mathcal{K}\|_{\varphi_2\circ M}\Big(\|c^*\|_N+R_1(r)\Big)\\ &+2C\alpha_3 k_1 b_0|\lambda|\cdot\|u\cdot\chi_D\|_\varphi\cdot\|K\|_{\varphi_2\circ M}\Big(\|b^*\|_N+R_2(r)\Big).\end{aligned}$$

Similarly, as in Theorem 3.2, we get

$$\beta_H(H(X)) \le \Big[\alpha_1 + 2C\alpha_2 k_1 c_0|\lambda|\|\mathcal{K}\|_{\varphi_2\circ M}\Big(\|c^*\|_N + R_1(r)\Big) + 2C\alpha_3 k_1 b_0|\lambda|\cdot\|K\|_{\varphi_2\circ M}\Big(\|b^*\|_N + R_2(r)\Big)\Big]\beta_H(X).$$

Since

$$\Big[\alpha_1 + 2C\alpha_2 k_1 c_0|\lambda|\|\mathcal{K}\|_{\varphi_2\circ M}\Big(\|c^*\|_N + R_1(r)\Big) + 2C\alpha_3 k_1 b_0|\lambda|\|K\|_{\varphi_2\circ M}\Big(\|b^*\|_N + R_2(r)\Big)\Big] < 1,$$

we can utilize Theorem 2.15, which achieves the verification. □

4. Remarks and examples

Next, we present some remarks and examples about our outcomes.

Remark 4.1. Let M_1 and M_2, be complementary functions for N_1 and N_2, respectively. If $M_1(u) = \exp|u| - |u| - 1$ and $M_2(u) = \frac{u^2}{2} = N_2(u)$. In this case M_1 fulfills the Δ_3-condition and $N_1(u) = (1+|u|)\cdot\ln(1+|u|) - |u|$ fulfills Δ'-condition. If we define a N-function either as $\Psi(u) = M_2[N_1(u)]$ or $\Psi(u) = N_1[M_2(u)]$, then by choosing arbitrary kernel $\mathcal{K}$ from the space L_Ψ we can utilize [16, Theorem 15.4]. Thus $\mathcal{K}_0 x(t) = \int_a^b \mathcal{K}(t,s)x(s)\,ds : L_{M_1} \to L_{M_2}$ is continuous and we shall utilize our existence theorems.

Remark 4.2. We can find the acting and continuity conditions for the operator $G(x) = l(t)\cdot x(t),\ l \in L_\varphi$, between various Orlicz spaces in [16, Theorem 18.2] (cf. our assumption (G3)).

Example 4.3. Let $x = y,\ g_i(t) = 1,\ f_i(t,x,y,z) = y, V_i(z)(t) = z(t),\ h_i(t,z) = z$, then we have the Chandrasekhar equations

$$x(t) = 1 + \lambda\cdot x(t)\int_0^1 \frac{t}{t+s}e^s x(s)\,ds,\quad t\in[0,1],$$

which has been inspected in [25, 36].

Example 4.4. Let $g_i(t) = 1,\ f_i(t,x,y,z) = y, V_i(z)(t) = z(t),\ h_i(t,z) = z,\ i = 1,2$, then we have the coupled systems of Chandrasekhar equations

$$\begin{cases} x(t) = 1 + \lambda\cdot y(t)\int_0^1 \frac{t}{t+s}e^s y(s)\,ds, & t\in[0,1],\\ y(t) = 1 + \lambda\cdot x(t)\int_0^1 \frac{t}{t+s}e^s x(s)\,ds, & t\in[0,1],\end{cases}$$

which have been inspected in [30].

Example 4.5. In case of $g_i(t) = 1,\ f_i(t,x,y,z) = y,\ V_i(z)(t) = z(t),\ i = 1,2$, then we have the coupled systems of generalized Chandrasekhar equations

$$\begin{cases} x(t) = 1 + \lambda\cdot y(t)\int_0^1 \frac{t}{t+s}e^s(c^*(s) + \log(1+\sqrt{y(s)})\,ds, & t\in[0,1],\\ y(t) = 1 + \lambda\cdot x(t)\int_0^1 \frac{t}{t+s}e^s(c^*(s) + \log(1+\sqrt{x(s)})\,ds, & t\in[0,1],\end{cases}$$

where $R_1(\cdot) = \log(1+\sqrt{\cdot})$.

Example 4.6. Let $V_i(x)(t) = q_i(t) \cdot z(t),\ G_i = l_i(t) \cdot z(t),\ i = 1, 2$, then we have the coupled systems

$$\begin{cases} x(t) = g_1(t) + f_1\Big(t,\ y(t),\ \lambda \cdot q_1(t) \cdot y(t) \int_a^b \mathcal{K}(t,s) h_1(s, y(s))\, ds,\ \lambda \cdot l_1 \cdot y(t) \int_a^b u_1(t, s, y(s))\, ds\Big), \\ y(t) = g_2(t) + f_2\Big(t,\ x(t),\ \lambda \cdot q_2(t) \cdot x(t) \int_a^b \mathcal{K}(t,s) h_2(s, x(s))\, ds,\ \lambda \cdot l_2 \cdot x(t) \int_a^b u_2(t, s, x(s))\, ds\Big), \end{cases}$$

where $t \in J$ which is a particular case of the system (1.1).

5. Conclusions

The current article presented two existence theorems for a general coupled system of functional quadratic Hammerstein-Urysohn integral equations in arbitrary Orlicz spaces L_φ. We utilize the analysis concerning the fixed point approach and a proper measure of noncompactness in L_φ, which is not a Banach algebra. The studied problem contains many integral equations as special cases in the available literature such as the Chandrasekhar equations which have significant applications in technology and different disciplines of science.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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