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Research article

Existence of Sobolev regular solutions for the incompressible flow of liquid crystals in three dimensions

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Abstract: This paper considers a simplified three dimensional Ericksen-Leslie System for nematic liquid crystal flows in the unbounded domain $\Omega := \mathbb{R}^+ \times \mathbb{R}^2$ or the smooth bounded domain Ω. The hydrodynamic system consists of the Navier-Stokes type equations for the fluid velocity coupled with a convective Ginzburg-Landau type equation for the averaged molecular orientation. We first establish the global existence of Sobolev regular solution with finite energies in Sobolev space $H^s(\Omega) \times H^s(\Omega)$, where the index s of the Sobolev space can be any large fixed integer, but $s \neq +\infty$. Then we give an asymptotic expansions of a family of Sobolev regularity solutions for such system in Ω.

Keywords: nematic liquid crystal flow; global Sobolev solution; asymptotic expansion
Mathematics Subject Classification: 35A01, 35Q31

1. Introduction and main results

In this paper, we consider the simplified Ericksen-Leslie system, it models the hydrodynamics of nematic liquid crystals in dimension three. It takes the following form:

$$\begin{aligned}
&\partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla P = -\kappa\nabla\cdot(\nabla d \odot \nabla d),\\
&\nabla\cdot u = 0,\\
&\partial_t d + u\cdot\nabla d = \gamma\Big(\triangle d - f(d)\Big),
\end{aligned} \tag{1.1}$$

where $(t,x)\in\mathbb{R}^+\times\Omega$ with $\Omega=\mathbb{R}^+\times\mathbb{R}^2$ (or Ω being a bounded smooth domain in $\mathbb{R}^3$), $u(t,x):\mathbb{R}^+\times\Omega\to\mathbb{R}^3$ is the fluid velocity, $P(t,x):\mathbb{R}^+\times\Omega\to\mathbb{R}$ stands for the pressure in the fluid, $d(t,x):\mathbb{R}^+\times\Omega\to\mathbb{R}^3$ represents the director field for the averaged macro-scopic molecular orientations. The constants ν, κ and γ are positive constants representing the viscosity of the fluid, the competition between kinetic and potential energy, and the microscopic elastic relaxation time for the molecular orientation field, respectively. The symmetric 3×3 matrix $\nabla d \odot \nabla d$ denotes the Ericksen stress tensor whose (i, j)-th entry is given by $\nabla_i d \cdot \nabla_j d$ for $1 \leq i,\ j \leq 3$. The divergence free condition in the second equations of problem (1.1) guarantees the incompressibility of the fluid. The vector valued nonlinear function $f(d)$ is the gradient of certain smooth scalar potential function $\mathcal{F}(d) : \mathbb{R}^3 \to \mathbb{R}$ such that $f(d) = \nabla_d \mathcal{F}(d)$, we take $\mathcal{F}$ to be the Ginzburg-Landau approximation

$$\mathcal{F}(d) = \frac{1}{4\varepsilon^2}(|d|^2 - 1)^2, \quad f(d) = \frac{1}{\varepsilon^2}(|d|^2 - 1)d, \quad \varepsilon > 0.$$

It is simply the Ginzburg-Landau approximation of the constraint $|d|$ for small ε as given in [1, 2].

The pressure takes the form

$$\Delta P(t, x) = -\nabla \cdot \Big[u \cdot \nabla u + \kappa \nabla \cdot (\nabla d \odot \nabla d) \Big]. \tag{1.2}$$

Liquid crystals are substances that exhibit a phase of matter that has properties between those of a conventional liquid, and those of a solid crystal. It may flow like a liquid, but its molecules may be oriented in a crystal-like way. In the 1960s, Ericksen [3] and Leslie [4] established the hydrodynamic theory of liquid crystals. The Ericksen-Lislie theory describes the liquid crystal flow, including the velocity vector u and direction vector d of the fluid. Since the general Ericksen-Leslie system is very complicated, a simplified model of the Ericksen-Leslie system [1, 2, 5] is meanfully to be considered, and there are large diffculties even in the well-posedness theory. One of difficulties is from the three dimension Navier-Stokes type equations in the first equations in (1.1). Another difficulties is from the harmonic maps in the second equations in (1.1) with the constraint condition $|d| = 1$.

When the direction vector $d \equiv 0$, Eq (1.1) is reduced into the famous three dimension incompressible Navier-Stokes equations

$$\begin{aligned} &\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla P = 0, \\ &\nabla \cdot u = 0. \end{aligned} \tag{1.3}$$

The question of whether a solution of the 3D incompressible Navier-Stokes equations can develop a finite time singularity from smooth initial data with finite energy is one of the Millennium Prize problems [6]. In 1934, Leray [7] showed that the 3D incompressible Navier-Stokes Eq (1.3) admit global-forward-in-time weak solutions of the initial value problem. Caffarelli et al. [8] established a ε-regularity criterion for Eq (1.3). After that, Lin [9] gave a new and simpler proof for the result of Caffarelli, Kohn and Nirenberg. Buckmaster and Vicol [10] proved that the non-uniqueness of solutions for Eq (1.3) in H^s-space for some small $s > 0$. One can see [11–18] for more results on this equations.

When the direction vector $d \not\equiv 0$ and $f(d) = |\nabla d|^2 d$, there is a constraint condition $|d| = 1$. This makes the Ericksen-Leslie system more complicated. For this case, Lin ea al. [19] and Hong [20] established the existence of global weak solutions for this system in $\mathbb{R}^2$. Lin and Wang [21] obtained the global existence of a weak solution for the case of three dimension. After that, Huang ea al. [22] showed that two examples of non-trivial solutions to this 3D system with finite time singularity in the unit ball centered at 0.

In order to relax the constraint condition $|d| = 1$ for the Dirichlet energy

$$E(d) = \frac{1}{2} \int_\Omega |\nabla d|^2 dx,$$

Lin and Liu [1, 2] used the Ginzburg-Landau energy

$$E(d) = \frac{1}{2}\int_{\Omega}\Big(|\nabla d|^2 + \frac{1}{2\varepsilon^2}(1-|d|^2)^2\Big)dx$$

to replace the Dirichlet energy, then they [1] obtained the global existence of the solution of (1.1) in dimension two and three, after that, they [2] got partial regularity of weak solutions to this system. Furthermore, they [23] proved existence of solutions for system (1.1) and also analyzed the limits of weak solutions of it as $\varepsilon \to 0$. Specially, by setting the term $f(d) = 0$, Hu and Wang [24] proved that all the global weak solutions constructed in [1] be equal to the unique strong solution. Cavaterra et al. [25] established the existence of optimal boundary controls for this system in two dimension. We point out that any solution convergence to the equilibrium state (constants or functions with infinite energy in the unbounded domain) admits the infinite energy. To the author's knowledgement, there is few result on the global finite energy solution with regularity for this system in the unbounded domain or the smooth bounded domain.

In this paper, we will prove the global existence of Sobolev regular solutions for the three-dimensional incompressible flow of liquid crystals in the unbounded domain and the smooth bounded domain. Meanwhile, we give the explicit asymptotic expansion formula of Sobolev regular solutions for system (1.1).

1.1. The case of unbounded domain $\Omega = \mathbb{R}^+ \times \mathbb{R}^2$

We supplement the three-dimensional incompressible nematic liquid crystals (1.1) with the initial data

$$u(0,x) = u_0(x), \quad d(0,x) = d_0(x), \quad x \in \Omega, \tag{1.4}$$

and the boundary condition

$$u(t,x) = 0, \quad d(t,x) = d_0(x), \quad \forall (t,x) \in \mathbb{R}^+ \times \partial\Omega. \tag{1.5}$$

We assume that the initial data (1.4) satisfies the following conditions

$$\begin{aligned} &\nabla \cdot u_0(x) = 0, \quad x \in \Omega, \\ &u_0(x)|_{x\in\partial\Omega} = 0. \end{aligned} \tag{1.6}$$

We now state one of main results in this paper.

Theorem 1.1. *Let the parameter $\nu, \gamma, \kappa, \varepsilon > 0$ in (1.1). For any fixed constant $s \geq 1$, if the initial data (1.4) of the three dimension nematic liquid crystal flow (1.1) satisfies the condition (1.6), and there exists a small constant $\delta \in (0,1)$ such that*

$$\begin{aligned} &\|u_0(x)\|_{H^{s+2}(\Omega)} \lesssim \delta, \\ &\|d_0(x)\|_{H^{s+2}(\Omega)} \lesssim \delta, \end{aligned}$$

then the three dimension nematic liquid crystal flow (1.1) with the boundary conditions (1.5) admit a global Sobolev regular solution with finite energy

$$(u(t,x), d(t,x)) \in \mathbb{C}((0,+\infty); H^s(\Omega) \times H^s(\Omega)).$$

Moreover, it holds

$$\sup_{t\in(0,+\infty)}\left(\|u(t,x)\|_{H^s(\Omega)}+\|d(t,x)\|_{H^s(\Omega)}\right)\lesssim\delta,$$
$$\sup_{t\in(0,+\infty)}\|P(t,x)\|_{H^s(\Omega)}\lesssim\delta,$$

for any $(t,x)\in(0,\infty)\times\Omega$. *Here, the pressure is given by (1.2).*

As an application, we have the following asymptotic expansion of solutions.

Corollary 1.1. *Let the parameter* $\nu,\gamma,\kappa,\varepsilon>0$ *in (1.1). Assume that the integers* $p>0, 2<q<2(p+1)$ *and the parameter* $0<\delta\ll 1$. *Assume that the small initial data (1.4) satisfies the condition (1.6). Then the three dimension nematic liquid crystal flow (1.1) in the unbounded domain* $\mathbb{R}^+\times\mathbb{R}^2$ *admit an explicit expansion of global Sobolev regular solution with finite energy as follows*

$$u^*(t,x)=\left(u_1^{(0)}(t,x),u_2^{(0)}(t,x),u_3^{(0)}(t,x)\right)+u_0(x)+\mathcal{R}_1(t,x),$$
$$d^*(t,x)=\left(d_1^{(0)}(t,x),d_2^{(0)}(t,x),d_3^{(0)}(t,x)\right)+d_0(x)+\mathcal{R}_2(t,x),$$

for all $x=(x_1,x_2,x_3)\in\mathbb{R}^+\times\mathbb{R}^2$, *where*

$$u_1^{(0)}(t,x)=d_1^{(0)}(t,x):=\delta(1-e^{-t})x_1^q x_2^{2p+1}x_3^{2p+1}e^{-(x_1^{2(p+1)}+x_2^{2(p+1)}+x_3^{2(p+1)})},$$
$$u_2^{(0)}(t,x)=d_2^{(0)}(t,x):=\delta 2^{-1}(p+1)^{-1}(1-e^{-t})(q-2(p+1)x_1^{2(p+1)-q})x_1^{q-1}x_3^{2p+1}e^{-(x_1^{2(p+1)}+x_2^{2(p+1)}+x_3^{2(p+1)})},$$
$$u_3^{(0)}(t,x)=d_3^{(0)}(t,x):=-\delta(p+1)^{-1}(1-e^{-t})(q-2(p+1)x_1^{2(p+1)-q})x_1^{q-1}x_2^{2p+1}e^{-(x_1^{2(p+1)}+x_2^{2(p+1)}+x_3^{2(p+1)})},$$

and the remainder term $\mathcal{R}_k(t,x)\in H^s(\Omega)$ *with* $s\geq 1$ *and* $k=1,2$ *satisfies*

$$\mathcal{R}_k(0,x)=0,\quad \nabla\cdot\mathcal{R}_1(t,x)=0,$$
$$\mathcal{R}_k(t,x)|_{x\in\partial\Omega}=0,\quad \sup_{t\in(0,+\infty)}\|\mathcal{R}_k(t,x)\|_{H^s(\Omega)}\sim O(\delta^2).$$

Furthermore, one can see that

$$u^*(t,x)|_{x\in\partial\Omega}=0,\quad d^*(t,x)|_{x\in\partial\Omega}=d_0(x),\quad \forall(t,x)\in\mathbb{R}^+\times\partial\Omega.$$

Moreover, the pressure is determined by

$$\Delta P^*(t,x)=-\nabla\cdot\left[u^*\cdot\nabla u^*+\kappa\nabla\cdot(\nabla d^*\odot\nabla d^*)\right].$$

Remark 1.1. *Corollary 1.1 is derived directly from the proof of Theorem 1.1. From Corollary 1.1, we can observe that Eq (1.1) admit a global Sobolev regular solution with finite energy by chosing different constants p and q. Moreover, if the vector functions*

$$u^{(0)}(t,x)=\left(u_1^{(0)}(t,x),u_2^{(0)}(t,x),u_3^{(0)}(t,x)\right)$$

and

$$d^{(0)}(t,x)=\left(d_1^{(0)}(t,x),d_2^{(0)}(t,x),d_3^{(0)}(t,x)\right)$$

satisfy the conditions

$$\begin{cases} \nabla \cdot u^{(0)}(t,x) = 0, \\ u^{(0)}(0,x) = 0, \\ \|u^{(0)}\|_{H^s} \lesssim \delta_0, \\ u^{(0)}(t,x)|_{x\in\partial\Omega} = 0, \end{cases}$$

and

$$\begin{cases} d^{(0)}(0,x) = 0, \\ \|d^{(0)}\|_{H^s} \lesssim \delta_0, \\ d^{(0)}(t,x)|_{x\in\partial\Omega} = 0. \end{cases}$$

Moreover, for any fixed $s \geq 1$ *and* $(t,x) \in \Omega$ *and* $i, j = 1, 2, 3$*, it also needs the conditions*

$$\sum_{k=0}^{s} \|\partial_{x_i}^k u_j^{(0)}(t,x)\|_{L^\infty} \lesssim \delta_0,$$

and

$$\sum_{k=0}^{s} \|\partial_{x_i}^k d_j^{(0)}(t,x)\|_{L^\infty} \lesssim \delta_0,$$

then we can use the proof of Theorem 1.1 to construct more global Sobolev regular solutions with finite energy of (1.1) in the unbounded domain $\mathbb{R}^+ \times \mathbb{R}^2$.

1.2. The case of smooth bounded domain $\Omega \subset \mathbb{R}^3$

We have the following results.

Theorem 1.2. *Let the parameter* $\nu, \gamma, \kappa, \varepsilon > 0$ *in (1.1) and* $\frac{\gamma}{\varepsilon^2} < 1$*. For any fixed constant* $s \geq 1$*, if the initial data (1.4) of the three dimension nematic liquid crystal flow (1.1) satisfies the conditions (1.6), and there exists a small constant* $0 < \delta \ll \min\{1, \nu, \gamma\}$ *such that*

$$\|u_0(x)\|_{H^{s+2}(\Omega)} \lesssim \delta,$$
$$\|d_0(x)\|_{H^{s+2}(\Omega)} \lesssim \delta,$$

then the three dimension nematic liquid crystal flow (1.1) with the boundary conditions (1.5) admit a global Sobolev regular solution with finite energy

$$(u(t,x), d(t,x)) \in \mathbb{C}((0,+\infty); H^s(\Omega) \times H^s(\Omega)).$$

Moreover, it holds

$$\sup_{t\in(0,+\infty)} \Big(\|u(t,x)\|_{H^s(\Omega)} + \|d(t,x)\|_{H^s(\Omega)}\Big) \lesssim \delta,$$
$$\sup_{t\in(0,+\infty)} \|P(t,x)\|_{H^s(\Omega)} \lesssim \delta,$$

for any $(t,x) \in (0,\infty) \times \Omega$*. Here, the pressure is given by (1.2).*

In particularly, if the domain $\Omega := ([0,T])^3$ (a regular polyhedron) with the finite constant $T > 0$, we have the following asymptotic expansion of solutions.

Corollary 1.2. *Let the parameter $\nu, \gamma, \kappa, \varepsilon > 0$ in (1.1) and $\frac{\gamma}{\varepsilon^2} < 1$. Assume that the integers $p > 1$ and $q > 2$, and the parameter $0 < \delta \ll 1$. The three dimension nematic liquid crystal flow (1.1) in the smooth bounded domain Ω admits an explicit expansion of Sobolev regular solutions with finite energy as follows*

$$u^*(t,x) = \Big(u_1^{(0)}(t,x), u_2^{(0)}(t,x), u_3^{(0)}(t,x)\Big) + \mathcal{R}_1(t,x) + u_0(x),$$
$$d^*(t,x) = \Big(d_1^{(0)}(t,x), d_2^{(0)}(t,x), d_3^{(0)}(t,x)\Big) + \mathcal{R}_1(t,x) + d_0(x),$$

where $\forall x = (x_1, x_2, x_3) \in \Omega := ([0,T])^3$, and

$$\begin{aligned} u_1^{(0)}(t,x) = d_1^{(0)}(t,x) &= \delta(1-e^{-t})x_1^q(x_1-T)^q g(x_2)g(x_3)e^{-r(x_1,x_2,x_3)}, \\ u_2^{(0)}(t,x) = d_2^{(0)}(t,x) &= \delta(1-e^{-t})x_2^q(x_2-T)^q g(x_1)g(x_3)e^{-r(x_1,x_2,x_3)}, \\ u_3^{(0)}(t,x) = d_3^{(0)}(t,x) &= -2\delta(1-e^{-t})x_3^q(x_3-T)^q g(x_1)g(x_2)e^{-r(x_1,x_2,x_3)}, \end{aligned}$$

with

$$g(y) := y^{q-1}(y-T)^{q-1}(2y-T)\Big(q - 2(p+1)y^{2(p+1)}(y-T)^{2(p+1)}\Big),$$
$$r(x_1,x_2,x_3) := \sum_{k=1}^{3} x_k^{2(p+1)}(x_k-T)^{2(p+1)},$$

and the remainder term $\mathcal{R}_k(t,x) \in H^s(\Omega)$ with $s \geq 1$ and $k = 1,2$ satisfies

$$\begin{aligned} &\mathcal{R}_k(0,x) = 0, \quad \nabla \cdot \mathcal{R}_1(t,x) = 0, \\ &\mathcal{R}_k(t,x)|_{x\in\partial\Omega} = 0, \quad \sup_{t\in(0,+\infty)} \|\mathcal{R}_k(t,x)\|_{H^s(\Omega)} \sim O(\delta^2). \end{aligned}$$

Furthermore, one can see that

$$u^*(t,x)|_{x\in\partial\Omega} = 0, \quad d^*(t,x)|_{x\in\partial\Omega} = d_0(x), \quad \forall (t,x) \in \mathbb{R}^+ \times \partial\Omega.$$

Moreover, the pressure is determined by

$$\Delta P^*(t,x) = -\nabla \cdot \Big[u^* \cdot \nabla u^* + \kappa \nabla \cdot (\nabla d^* \odot \nabla d^*)\Big].$$

The organization of this paper is as follows. In Section 2, we show how to choose a suitable initial approximation functions, which lead to the dissipative structure of linearized system. After that, we give the existence of global time-decay Sobolev solution for the linearized equations of the first approximation step. In Section 3, we establish the general approximation step for the construction of the Nash-Moser iteration scheme. This method has been used in [26–31]. For the general Nash-Moser implicit function theorem, we refer to the seminal papers of Nash [32], Moser [33] and Hörmander [34]. This last section will show how to construct a global Sobolev solution for the 3D nematic liquid crystal flow (1.1) by using the proof of convergence for the Nash-Moser iteration scheme.

2. The first approximation step

Throughout this paper, let $\Omega := \mathbb{R}^+ \times \mathbb{R}^2$ or a smooth bounded domain. We denote the usual norms of $\mathbb{L}^2(\Omega)$ and the Sobolev space $\mathbb{H}^s(\Omega)$ by $\|\cdot\|_{\mathbb{L}^2}$ and $\|\cdot\|_{\mathbb{H}^s}$, respectively. The norm of the Sobolev space

$H^s(\Omega) := (\mathbb{H}^s(\Omega))^3$ is denoted by $\|\cdot\|_{H^s}$. The symbol $a \lesssim b$ means that there exists a positive constant C such that $a \leq Cb$. We denote by $(a, b, c)^T$ the column vector in $\mathbb{R}^3$. The letter C with subscripts is used to denote dependencies stands for a positive constant that might change its value at each occurrence.

The proof of Theorem 1.2 is the same with Theorem 1.1 by a small modification of relationship of parameters ν, γ, ε. Thus we give a proof process including both Theorems 1.1 and 1.2.

We introduce a family of smooth operators possessing the following properties.

Lemma 2.1. *[34, 35] There is a family $\{\Pi_\theta\}_{\theta\geq 1}$ of smoothing operators in the space $H^s(\Omega)$ acting on the class of functions such that*

$$\|\Pi_\theta u\|_{H^{s_1}(\Omega)} \leq C\theta^{(s_1-s_2)_+}\|u\|_{H^{s_2}(\Omega)}, \quad \forall s_1,\ s_2 \geq 0, \tag{2.1}$$

$$\|\Pi_\theta u - u\|_{H^{s_1}(\Omega)} \leq C\theta^{s_1-s_2}\|u\|_{H^{s_2}(\Omega)}, \quad 0 \leq s_1 \leq s_2,$$
$$\|\frac{d}{d\theta}\Pi_\theta u\|_{H^{s_1}(\Omega)} \leq C\theta^{(s_1-s_2)_+-1}\|u\|_{H^{s_2}(\Omega)}, \quad \forall s_1,\ s_2 \geq 0,$$

where C is a positive constant and $(s_1 - s_2)_+ := \max(0, s_1 - s_2)$.

In our iteration scheme, we set

$$\theta = N_m = N_0^m, \quad \forall m = 0, 1, 2, \ldots,$$

where N_0 is a fixed positive constant, then by (2.1), it holds

$$\|\Pi_{N_m} u\|_{H^{s_1}(\Omega)} \lesssim N_m^{s_1-s_2}\|u\|_{H^{s_2}(\Omega)}, \quad \forall s_1 \geq s_2. \tag{2.2}$$

We consider the approximation problem of the 3D nematic liquid crystal Eq (1.1) as follows

$$\begin{aligned} \mathcal{L}_1(u, d) &:= \partial_t u - \nu\Delta u + \Pi_{N_m}\Big(u\cdot\nabla u + \nabla P + \kappa\nabla\cdot(\nabla d \odot \nabla d)\Big), \\ \mathcal{L}_2(u, d) &:= \partial_t d - \gamma\triangle d + \Pi_{N_m}\Big(u\cdot\nabla d - \frac{\gamma}{\varepsilon^2}(|d|^2 - 1)d\Big), \end{aligned} \tag{2.3}$$

with the initial data (1.4), the boundary condition (1.5) and the incompressible condition

$$\nabla\cdot u = 0.$$

2.1. The initial approximation function

Let $s \geq 1$ be a fixed finite constant and $0 < \delta_0 < \delta^2 \ll 1$. For any $x \in \Omega$, we choose the initial approximation functions

$$u^{(0)}(t, x) = \Big(u_1^{(0)}(t, x), u_2^{(0)}(t, x), u_3^{(0)}(t, x)\Big) \in H^s(\Omega),$$
$$d^{(0)}(t, x) = \Big(d_1^{(0)}(t, x), d_2^{(0)}(t, x), d_3^{(0)}(t, x)\Big) \in H^s(\Omega),$$

where we require

$$\begin{cases} \nabla\cdot u^{(0)}(t, x) = 0, \\ u^{(0)}(0, x) = 0, \\ \|u^{(0)}\|_{H^s} \lesssim \delta_0, \\ u^{(0)}(t, x)|_{x\in\partial\Omega} = 0, \end{cases} \tag{2.4}$$

and

$$\begin{cases} d^{(0)}(0,x) = 0, \\ \|d^{(0)}\|_{H^s} \lesssim \delta_0, \\ d^{(0)}(t,x)|_{x\in\partial\Omega} = 0. \end{cases} \tag{2.5}$$

Moreover, for any fixed $s \geq 1$, and $\forall t > 0$ and $x \in \Omega$ and $i, j = 1, 2, 3$, it also needs the conditions

$$\sum_{k=0}^{s} \|\partial_{x_i}^k u_j^{(0)}(t,x)\|_{L^\infty} \lesssim \delta_0, \tag{2.6}$$

and

$$\sum_{k=0}^{s} \|\partial_{x_i}^k d_j^{(0)}(t,x)\|_{L^\infty} \lesssim \delta_0, \tag{2.7}$$

and the initial error term

$$\|E^{(0)}\|_{H^s} \lesssim \delta_0, \quad \|\overline{E}^{(0)}\|_{H^s} \lesssim \delta_0, \tag{2.8}$$

where $E^{(0)}$ and $\overline{E}^{(0)}$ denote the error term taking the form

$$\begin{aligned} E^{(0)} &:= \mathcal{L}_1(u^{(0)}, d^{(0)}), \\ \overline{E}^{(0)} &:= \mathcal{L}_2(u^{(0)}, d^{(0)}), \end{aligned}$$

with the vector form

$$\begin{aligned} E^{(0)} &= (E_1^{(0)}, E_2^{(0)}, E_3^{(0)}), \\ \overline{E}^{(0)} &= (\overline{E}_1^{(0)}, \overline{E}_2^{(0)}, \overline{E}_3^{(0)}). \end{aligned}$$

There are two family of explicit examples. In fact, many vector functions can be chosen to satisfy conditions (2.4)–(2.8). We now give two families of exact examples of the initial approximation function satisfying (2.4)–(2.8) for the unbounded domain and the smooth bounded domain.

For the unbounded domain $\Omega = \mathbb{R}^+ \times \mathbb{R}^2$, let the integers $p > 1$ and $2 < q < 2(p+1)$. We choose the initial approximation functions of the following form

$$\begin{aligned} u^{(0)}(t,x) &= (u_1^{(0)}(t,x), u_2^{(0)}(t,x), u_3^{(0)}(t,x)), \quad \forall (t,x) \in \mathbb{R}^+ \times \Omega, \\ d^{(0)}(t,x) &= (d_1^{(0)}(t,x), d_2^{(0)}(t,x), d_3^{(0)}(t,x)), \quad \forall (t,x) \in \mathbb{R}^+ \times \Omega, \end{aligned}$$

where

$$\begin{aligned} u_1^{(0)}(t,x) = d_1^{(0)}(t,x) &:= \delta(1-e^{-t})x_1^q x_2^{2p+1} x_3^{2p+1} e^{-(x_1^{2(p+1)} + x_2^{2(p+1)} + x_3^{2(p+1)})}, \\ u_2^{(0)}(t,x) = d_2^{(0)}(t,x) &:= \delta(1-e^{-t})(q - 2(p+1)x_1^{2(p+1)-q})x_1^{q-1} x_3^{2p+1} e^{-(x_1^{2(p+1)} + x_2^{2(p+1)} + x_3^{2(p+1)})}, \\ u_3^{(0)}(t,x) = d_3^{(0)}(t,x) &:= -2\delta(1-e^{-t})(q - 2(p+1)x_1^{2(p+1)-q})x_1^{q-1} x_2^{2p+1} e^{-(x_1^{2(p+1)} + x_2^{2(p+1)} + x_3^{2(p+1)})}. \end{aligned}$$

For the smooth bounded domain $\Omega := ([0,T])^3$ (a regular polyhedron) with the finite constant $T > 0$, we set the integers $p > 1$ and $q > 2$, and

$$\begin{aligned} u_1^{(0)}(t,x) = d_1^{(0)}(t,x) &= \delta(1-e^{-t})x_1^q (x_1 - T)^q g(x_2) g(x_3) e^{-r(x_1,x_2,x_3)}, \\ u_2^{(0)}(t,x) = d_2^{(0)}(t,x) &= \delta(1-e^{-t})x_2^q (x_2 - T)^q g(x_1) g(x_3) e^{-r(x_1,x_2,x_3)}, \\ u_3^{(0)}(t,x) = d_3^{(0)}(t,x) &= -2\delta(1-e^{-t})x_3^q (x_3 - T)^q g(x_1) g(x_2) e^{-r(x_1,x_2,x_3)}, \end{aligned}$$

with

$$g(y) := y^{q-1}(y-T)^{q-1}(2y-T)\Big(q-2(p+1)y^{2(p+1)}(y-T)^{2(p+1)}\Big),$$

$$r(x_1,x_2,x_3) := \sum_{k=1}^{3} x_k^{2(p+1)}(x_k-T)^{2(p+1)}.$$

By straightforward computations, we obtain

$$\nabla\cdot u^{(0)}(t,x) = 0,$$

and

$$u^{(0)}(t,x)|_{x\in\partial\Omega} = 0, \quad \forall t\geq 0,$$
$$d^{(0)}(t,x)|_{x\in\partial\Omega} = 0.$$

Moreover, for every fixed integers p and q, the assumptions (2.4)–(2.8) holds.

Here, the initial approximation pressure satisfies

$$\Delta P^{(0)}(t,x) = -\nabla\cdot\Big[u^{(0)}\cdot\nabla u^{(0)} + \kappa\nabla\cdot(\nabla d^{(0)}\odot\nabla d^{(0)})\Big].$$

2.2. The time-decay of first approximation step

We now construct the first approximation solution denoted by $\big(u^{(1)}(t,\lambda x), d^{(1)}(t,\lambda x)\big)$ of system (2.3). There should be two cases on the parameter λ according to the domain.

Case 2.1. *If the domain is the unbounded domain $\mathbb{R}^+\times\mathbb{R}^2$, then we set the parameter $\lambda>1$.*

Case 2.2. *If the domain is a bounded domain Ω, we set the parameter $\lambda = 1$, meanwhile, we should require $\frac{\lambda}{\varepsilon^2}<1$.*

The first approximation step between the initial approximation function and first approximation solution is denoted by

$$\mathbf{h}^{(1)}(t,\lambda x) := u^{(1)}(t,\lambda x) - u^{(0)}(t,x),$$
$$\mathbf{w}^{(1)}(t,\lambda x) := d^{(1)}(t,\lambda x) - d^{(0)}(t,x).$$

Then we linearize the nonlinear system (2.3) around $(u^{(0)}, d^{(0)})$ to get the linearized operators as follows

$$\begin{aligned}\mathcal{J}_1[u^0,d^{(0)}](\mathbf{h}^{(1)},\mathbf{w}^{(1)}) := \mathbf{h}_t^{(1)} - \nu\lambda^2\Delta\mathbf{h}^{(1)} + \Pi_{N_1}\Big[\lambda\big(u^{(0)}\cdot\nabla\big)\mathbf{h}^{(1)} + \big(\mathbf{h}^{(1)}\cdot\nabla\big)u^{(0)}\\ + \lambda\nabla\Big((\mathcal{D}_{u^{(0)}}P)\mathbf{h}^{(1)} + (\mathcal{D}_{d^{(0)}}P)\mathbf{w}^{(1)}\Big)\\ + \kappa\nabla\cdot(\nabla d^{(0)}\odot\lambda\nabla\mathbf{w}^{(1)}) + \kappa\nabla\cdot(\lambda\nabla\mathbf{w}^{(1)}\odot\nabla d^{(0)})\Big],\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\mathcal{J}_2[u^0,d^{(0)}](\mathbf{h}^{(1)},\mathbf{w}^{(1)}) := \mathbf{w}_t^{(1)} - \gamma\lambda^2\Delta\mathbf{w}^{(1)} + \Pi_{N_1}\Big[\lambda\big(u^{(0)}\cdot\nabla\big)\mathbf{w}^{(1)} + \big(\mathbf{w}^{(1)}\cdot\nabla\big)u^{(0)}\\ + \frac{\gamma}{\varepsilon^2}\Big(|d^{(0)}|^2\mathbf{w}^{(1)} - \mathbf{w}^{(1)}\Big)\Big],\end{aligned} \tag{2.10}$$

where $\mathcal{D}_{u^{(0)}}$ denotes the Fréchet derivatives on $u^{(0)}$, and by (1.2), it takes the form

$$
\begin{aligned}
\nabla(\mathcal{D}_{u^{(0)}}P)\mathbf{h}^{(1)} &:= -\lambda\nabla\Delta^{-1}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\Big(\partial_{x_{k_2}}h_{k_1}^{(1)}\partial_{x_{k_1}}u_{k_2}^{(0)}+\partial_{x_{k_2}}u_{k_1}^{(0)}\partial_{x_{k_1}}h_{k_2}^{(1)}\Big),\\
\nabla(\mathcal{D}_{d^{(0)}}P)\mathbf{w}^{(1)} &:= -\kappa\lambda\nabla\Delta^{-1}\Big(\nabla\cdot(\lambda\nabla\mathbf{w}^{(1)}\odot\nabla d^{(0)})+\nabla\cdot(\nabla d^{(0)}\odot\lambda\nabla\mathbf{w}^{(1)})\Big)\\
&= -\kappa\lambda\nabla\Delta^{-1}\Big[\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\Big(\partial_{x_{k_1}}^2 d_{k_2}^{(0)}\partial_{x_j}w_{k_2}^{(1)}+\lambda\partial_{x_{k_1}}d_{k_2}^{(0)}\partial_{x_{k_1}}\partial_{x_j}w_{k_2}^{(1)}\\
&\qquad+\partial_{x_{k_1}}^2 w_{k_2}^{(1)}\partial_{x_j}d_{k_2}^{(0)}+\lambda\partial_{x_{k_1}}w_{k_2}^{(1)}\partial_{x_{k_1}}\partial_{x_j}d_{k_2}^{(0)}\Big)\Big].
\end{aligned}
\tag{2.11}
$$

We now consider the linear system

$$
\begin{aligned}
&\mathcal{J}_1[u^0,d^{(0)}](\mathbf{h}^{(1)},\mathbf{w}^{(1)})=\Pi_{N_1}E^{(0)},\\
&\mathcal{J}_2[u^0,d^{(0)}](\mathbf{h}^{(1)},\mathbf{w}^{(1)})=\Pi_{N_1}\overline{E}^{(0)},\\
&\nabla\cdot\mathbf{h}^{(1)}=0,\\
&\mathbf{h}^{(1)}(0,\lambda x)=\mathbf{h}_0^{(1)}(\lambda x),\quad \mathbf{w}^{(1)}(0,\lambda x)=\mathbf{w}_0^{(1)}(\lambda x),
\end{aligned}
\tag{2.12}
$$

and the boundary condition

$$
\begin{aligned}
&\mathbf{h}^{(1)}(t,\lambda x)|_{x\in\partial\Omega}=0,\\
&\mathbf{w}^{(1)}(t,\lambda x)|_{x\in\partial\Omega}=0,
\end{aligned}
\tag{2.13}
$$

from which, the solution of it gives the first approximation step of 3D nematic liquid crystal flow (1.1).

Before we carry out some *a priori* estimates, for $j=1,2,3$, we rewrite equations of (2.12) into a coupled system as follows

$$
\begin{aligned}
&\partial_t h_j^{(1)}-\nu\lambda^2\Delta h_j^{(1)}+\lambda\Pi_{N_1}\sum_{i=1}^{3}u_i^{(0)}\partial_{x_i}h_j^{(1)}+\Pi_{N_1}\sum_{i=1}^{3}h_i^{(1)}\partial_{x_i}u_j^{(0)}\\
&+\lambda\Pi_{N_1}\partial_{x_j}\Big((\mathcal{D}_{u^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{d^{(0)}}P)\mathbf{w}^{(1)}\Big)\\
&+\kappa\lambda\Pi_{N_1}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\Big(\partial_{x_{k_1}}^2 d_{k_2}^{(0)}\partial_{x_j}w_{k_2}^{(1)}+\lambda\partial_{x_{k_1}}d_{k_2}^{(0)}\partial_{x_{k_1}}\partial_{x_j}w_{k_2}^{(1)}\\
&+\partial_{x_{k_1}}^2 w_{k_2}^{(1)}\partial_{x_j}d_{k_2}^{(0)}+\lambda\partial_{x_{k_1}}w_{k_2}^{(1)}\partial_{x_{k_1}}\partial_{x_j}d_{k_2}^{(0)}\Big)=\Pi_{N_1}E_j^{(0)},
\end{aligned}
\tag{2.14}
$$

coupled with

$$
\begin{aligned}
&\partial_t w_j^{(1)}-\gamma\lambda^2\Delta w_j^{(1)}+\lambda\Pi_{N_1}\sum_{i=1}^{3}u_i^{(0)}\partial_{x_i}w_j^{(1)}+\Pi_{N_1}\sum_{i=1}^{3}h_i^{(1)}\partial_{x_i}d_j^{(0)}\\
&+\frac{\gamma}{\varepsilon^2}\Big(|d^{(0)}|^2 w_j^{(1)}-w_j^{(1)}\Big)=\Pi_{N_1}\overline{E}_j^{(0)},
\end{aligned}
\tag{2.15}
$$

with the initial data

$$
h_j^{(1)}(0,\lambda x)=h_{0j}^{(1)}(\lambda x),\quad w_j^{(1)}(0,\lambda x)=w_{0j}^{(1)}(\lambda x),
\tag{2.16}
$$

and the boundary condition (2.13).

We will use the weighted function estimate to get the decay in time of solutions. It is like the Carleman estimate. Let $\psi(x_1)$ be a function defined in $(0,+\infty)$ such that

$$0<\kappa\leq\psi''(x_1)-(\psi'(x_1))^2<+\infty, \tag{2.17}$$

and $e^{-\psi(x_1)}$ is bounded in $(0,+\infty)$. The condition (2.17) implies $\psi''(x_1)\geq\kappa>1$. In fact, there are many functions can satisfy above conditions. For a simple example, we take the function as the form

$$\psi(x_1)=-\ln|\cos(\sqrt{\kappa}x_1)|,\quad x_1\neq 2i\pi+\frac{\pi}{2},\ for\ i\in\mathbb{Z}.$$

We now derive $\mathbb{L}^2$-estimate of solution for the linear systems (2.14) and (2.15).

Lemma 2.2. *If the domain is the unbounded domain $\mathbb{R}^+\times\mathbb{R}^2$, let the parameter $\lambda>1$. If the domain is a bounded domain Ω, let the parameter $\lambda=1$ and $\frac{\gamma}{\varepsilon^2}<1$. Assume that the initial approximation function $(u^{(0)},d^{(0)})$ satisfies conditions (2.4)–(2.8). Then the solution $(\boldsymbol{h}^{(1)}(t,\lambda x),\boldsymbol{w}^{(1)}(t,\lambda x))$ of the linear systems (2.14) and (2.15) satisfies*

$$\begin{aligned}&\sum_{j=1}^{3}\int_\Omega\Big((h_j^{(1)})^2+(w_j^{(1)})^2\Big)dx\\&\lesssim e^{-C_{\nu,\gamma,\lambda,\varepsilon,\delta}t}\sum_{j=1}^{3}\Big(\int_\Omega\Big((h_{0j}^{(1)})^2+(w_{0j}^{(1)})^2\Big)dx+\Pi_{N_1}\int_0^t\int_\Omega\Big((E_j^{(0)})^2+(\overline{E}_j^{(0)})^2\Big)dxdt\Big),\end{aligned}\tag{2.18}$$

where $C_{\nu,\gamma,\lambda,\varepsilon,\delta}$ denotes a positive constant depending on $\nu,\gamma,\lambda,\varepsilon,\delta$.

Proof. Multiplying both sides of equations in (2.14) and (2.15) by $e^{-\psi(x_1)}h_j^{(1)}$ and $e^{-\psi(x_1)}w_j^{(1)}$, respectively, then integrating over Ω by noticing the boundary condition (2.13), for $j=1,2,3$, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega(h_j^{(1)})^2e^{-\psi(x_1)}dx\quad+\quad&\nu\lambda^2\sum_{i=1}^{3}\int_\Omega(\partial_{x_i}h_j^{(1)})^2e^{-\psi(x_1)}dx\\&+\frac{\nu\lambda^2}{2}\int_\Omega\Big(\psi''(x_1)-(\psi'(x_1))^2\Big)(h_j^{(1)})^2e^{-\psi(x_1)}dx\\&+\lambda\Pi_{N_1}\sum_{i=1}^{3}\int_\Omega\Big(u_i^{(0)}\partial_{x_i}h_j^{(1)}\Big)h_j^{(1)}e^{-\psi(x_1)}dx\\&+\Pi_{N_1}\sum_{i=1}^{3}\int_\Omega\Big(h_i^{(1)}\partial_{x_i}u_j^{(0)}\Big)h_j^{(1)}e^{-\psi(x_1)}dx\\&+\lambda\Pi_{N_1}\int_\Omega\partial_{x_j}\Big((\mathcal{D}_{u^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{d^{(0)}}P)\mathbf{w}^{(1)}\Big)h_j^{(1)}e^{-\psi(x_1)}dx\\&+\kappa\lambda\Pi_{N_1}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega\Big((\partial^2_{x_{k_1}}d_{k_2}^{(0)})(\partial_{x_j}w_{k_2}^{(1)})+\lambda(\partial_{x_{k_1}}d_{k_2}^{(0)})(\partial_{x_{k_1}}\partial_{x_j}w_{k_2}^{(1)})\\&+(\partial^2_{x_{k_1}}w_{k_2}^{(1)})(\partial_{x_j}d_{k_2}^{(0)})+\lambda(\partial_{x_{k_1}}w_{k_2}^{(1)})(\partial_{x_{k_1}}\partial_{x_j}d_{k_2}^{(0)})\Big)h_j^{(1)}e^{-\psi(x_1)}dx\end{aligned}$$

$$= \Pi_{N_1} \int_\Omega E_j^{(0)} h_j^{(1)} e^{-\psi(x_1)} dx, \tag{2.19}$$

and

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (w_j^{(1)})^2 e^{-\psi(x_1)} dx \quad + \quad & \gamma\lambda^2 \sum_{i=1}^{3} \int_\Omega (\partial_{x_i} w_j^{(1)})^2 e^{-\psi(x_1)} dx \\
& + \frac{\gamma\lambda^2}{2} \int_\Omega \Big(\psi''(x_1) - (\psi'(x_1))^2\Big)(w_j^{(1)})^2 e^{-\psi(x_1)} dx \\
& + \lambda\Pi_{N_1} \sum_{i=1}^{3} \int_\Omega \Big(u_i^{(0)} \partial_{x_i} w_j^{(1)}\Big) w_j^{(1)} e^{-\psi(x_1)} dx \\
& + \Pi_{N_1} \sum_{i=1}^{3} \int_\Omega \Big(h_i^{(1)} \partial_{x_i} d_j^{(0)}\Big) w_j^{(1)} e^{-\psi(x_1)} dx \\
& + \frac{\gamma}{\varepsilon^2}\Pi_{N_1} \int_\Omega \Big(|d^{(0)}|^2 - 1\Big)(w_j^{(1)})^2 e^{-\psi(x_1)} dx \\
= \quad & \Pi_{N_1} \int_\Omega \overline{E}_j^{(0)} w_j^{(1)} e^{-\psi(x_1)} dx.
\end{aligned} \tag{2.20}$$

We sum up (2.19) and (2.20) from $j = 1$ to $j = 3$, hence

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\sum_{j=1}^{3}\int_\Omega \Big((h_j^{(1)})^2 + (w_j^{(1)})^2\Big) e^{-\psi(x_1)} dx + \lambda^2 \sum_{j=1}^{3}\sum_{i=1}^{3} \int_\Omega \Big(\nu(\partial_{x_i} h_j^{(1)})^2 + \gamma(\partial_{x_i} w_j^{(1)})^2\Big) e^{-\psi(x_1)} dx \\
& + \frac{\lambda^2}{2} \sum_{j=1}^{3} \int_\Omega \Big(\psi''(x_1) - (\psi'(x_1))^2\Big)\Big(\nu(h_j^{(1)})^2 + \gamma(w_j^{(1)})^2\Big) e^{-\psi(x_1)} dx \\
& + \frac{\gamma}{\varepsilon^2}\Pi_{N_1} \int_\Omega \Big(|d^{(0)}|^2 - 1\Big)(w_j^{(1)})^2 e^{-\psi(x_1)} dx \\
& + \lambda\Pi_{N_1} \sum_{j=1}^{3}\sum_{i=1}^{3} \int_\Omega \Big(u_i^{(0)} \partial_{x_i} h_j^{(1)}\Big) h_j^{(1)} e^{-\psi(x_1)} dx + \Pi_{N_1} \sum_{j=1}^{3}\sum_{i=1}^{3} \int_\Omega \Big(h_i^{(1)} \partial_{x_i} u_j^{(0)}\Big) h_j^{(1)} e^{-\psi(x_1)} dx \\
& + \lambda\Pi_{N_1} \sum_{j=1}^{3} \int_\Omega \partial_{x_j}\Big((\mathcal{D}_{u^{(0)}} P)\mathbf{h}^{(1)} + (\mathcal{D}_{d^{(0)}} P)\mathbf{w}^{(1)}\Big) h_j^{(1)} e^{-\psi(x_1)} dx \\
& + \kappa\lambda\Pi_{N_1} \sum_{j=1}^{3}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3} \int_\Omega \Big((\partial^2_{x_{k_1}} d_{k_2}^{(0)})(\partial_{x_j} w_{k_2}^{(1)}) + \lambda(\partial_{x_{k_1}} d_{k_2}^{(0)})(\partial_{x_{k_1}}\partial_{x_j} w_{k_2}^{(1)}) \qquad (2.21) \\
& + (\partial^2_{x_{k_1}} w_{k_2}^{(1)})(\partial_{x_j} d_{k_2}^{(0)}) + \lambda(\partial_{x_{k_1}} w_{k_2}^{(1)})(\partial_{x_{k_1}}\partial_{x_j} d_{k_2}^{(0)})\Big) h_j^{(1)} e^{-\psi(x_1)} dx \\
& + \lambda\Pi_{N_1} \sum_{j=1}^{3}\sum_{i=1}^{3} \int_\Omega \Big(u_i^{(0)} \partial_{x_i} w_j^{(1)}\Big) w_j^{(1)} e^{-\psi(x_1)} dx + \Pi_{N_1} \sum_{j=1}^{3}\sum_{i=1}^{3} \int_\Omega \Big(h_i^{(1)} \partial_{x_i} d_j^{(0)}\Big) w_j^{(1)} e^{-\psi(x_1)} dx \\
= \ & \Pi_{N_1} \sum_{j=1}^{3} \int_\Omega \Big(E_j^{(0)} h_j^{(1)} + \overline{E}_j^{(0)} w_j^{(1)}\Big) e^{-\psi(x_1)} dx.
\end{aligned} \tag{2.22}$$

On the one hand, note that we have chosen the initial approximation function $(u^{(0)}, d^{(0)})$ satisfying (2.4)–(2.8). We integrate by parts and we get

$$\begin{aligned}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(u_i^{(0)}\partial_{x_i}h_j^{(1)}\Big)h_j^{(1)}e^{-\psi(x_1)}dx = &-\frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\partial_{x_i}u_i^{(0)}(h_j^{(1)})^2e^{-\psi(x_1)}dx\\ &+\frac{1}{2}\sum_{j=1}^{3}\int_{\Omega}\psi'(x_1)u_1^{(0)}(h_j^{(1)})^2e^{-\psi(x_1)}dx.\end{aligned}\tag{2.23}$$

Since the initial approximation function $u^{(0)}$ satisfies $\nabla\cdot u^{(0)}$, inequality (2.23) reduces to

$$\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(u_i^{(0)}\partial_{x_i}h_j^{(1)}\Big)h_j^{(1)}e^{-\psi(x_1)}dx = \frac{1}{2}\sum_{j=1}^{3}\int_{\Omega}\psi'(x_1)u_1^{(0)}(h_j^{(1)})^2e^{-\psi(x_1)}dx.\tag{2.24}$$

By direct computation we find

$$\begin{aligned}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}h_i^{(1)}\partial_{x_i}u_j^{(0)}h_j^{(1)}e^{-\psi(x_1)}dx = &\sum_{j=1}^{3}\int_{\Omega}\partial_{x_j}u_j^{(0)}(h_j^{(1)})^2e^{-\psi(x_1)}dx\\ &+\sum_{j=1}^{3}\sum_{i\neq j}\int_{\Omega}h_i^{(1)}\partial_{x_i}u_j^{(0)}h_j^{(1)}e^{-\psi(x_1)}dx.\end{aligned}\tag{2.25}$$

Next, noticing the incompressible condition

$$\nabla\cdot\mathbf{h}^{(1)} = 0,$$

we obtain

$$\begin{aligned}&\sum_{j=1}^{3}\int_{\Omega}\partial_{x_j}\Big((\mathcal{D}_{u^{(0)}}P)\mathbf{h}^{(1)} + (\mathcal{D}_{d^{(0)}}P)\mathbf{w}^{(1)}\Big)h_j^{(1)}e^{-\psi(x_1)}dx\\ &= -\sum_{j=1}^{3}\int_{\Omega}\Big((\mathcal{D}_{u^{(0)}}P)\mathbf{h}^{(1)} + (\mathcal{D}_{d^{(0)}}P)\mathbf{w}^{(1)}\Big)\partial_{x_j}h_j^{(1)}e^{-\psi(x_1)}dx\\ &\quad+\int_{\Omega}\psi'(x_1)\Big((\mathcal{D}_{u^{(0)}}P)\mathbf{h}^{(1)} + (\mathcal{D}_{d^{(0)}}P)\mathbf{w}^{(1)}\Big)h_1^{(1)}e^{-\psi(x_1)}dx\\ &= \int_{\Omega}\psi'(x_1)\Big((\mathcal{D}_{u^{(0)}}P)\mathbf{h}^{(1)} + (\mathcal{D}_{d^{(0)}}P)\mathbf{w}^{(1)}\Big)h_1^{(1)}e^{-\psi(x_1)}dx,\end{aligned}\tag{2.26}$$

then, we integrate by part to derive

$$
\begin{aligned}
&\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega \psi'(x_1)\triangle^{-1}\Big(\lambda\partial_{x_{k_1}}d_{k_2}^{(0)}\partial_{x_{k_1}}\partial_{x_1}w_{k_2}^{(1)}+\partial_{x_{k_1}}^2 w_{k_2}^{(1)}\partial_{x_1}d_{k_2}^{(0)}\Big)h_1^{(1)}e^{-\psi(x_1)}dx\\
&=-2\sum_{k_2=1}^{3}\int_\Omega\Big(\psi''(x_1)\triangle^{-1}(\partial_{x_1}d_{k_2}^{(0)})(\partial_{x_1}w_{k_2}^{(1)})+\psi'(x_1)\triangle^{-1}(\partial_{x_1}d_{k_2}^{(0)})(\partial_{x_1}w_{k_2}^{(1)})\Big)h_1^{(1)}e^{-\psi(x_1)}dx\\
&\quad-\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega\psi'(x_1)\triangle^{-1}\Big((\partial_{x_{k_1}}^2 d_{k_2}^{(0)})(\partial_{x_1}w_{k_2}^{(1)})h_1^{(1)}+(\partial_{x_{k_1}}d_{k_2}^{(0)})(\partial_{x_1}w_{k_2}^{(1)})(\partial_{x_{k_1}}h_1^{(1)})\Big)e^{-\psi(x_1)}dx\\
&\quad-\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega\psi'(x_1)\triangle^{-1}\Big((\partial_{x_{k_1}}\partial_{x_1}d_{k_2}^{(0)})(\partial_{x_{k_1}}w_{k_2}^{(1)})h_1^{(1)}+(\partial_{x_1}d_{k_2}^{(0)})(\partial_{x_{k_1}}w_{k_2}^{(1)})(\partial_{x_{k_1}}h_1^{(1)})\Big)e^{-\psi(x_1)}dx,
\end{aligned}
\tag{2.27}
$$

furthermore, from (2.11), using the standard Calderon-Zygmund theory and Young's inequality, it holds

$$
\begin{aligned}
&\Big|\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega \psi'(x_1)\triangle^{-1}\Big(\lambda\partial_{x_{k_1}}d_{k_2}^{(0)}\partial_{x_{k_1}}\partial_{x_1}w_{k_2}^{(1)}+\partial_{x_{k_1}}^2 w_{k_2}^{(1)}\partial_{x_1}d_{k_2}^{(0)}\Big)h_1^{(1)}e^{-\psi(x_1)}dx\Big|\\
\lesssim\ &\frac{1}{2}\sum_{k_2=1}^{3}\int_\Omega\Big(|\psi''(x_1)||\partial_{x_1}d_{k_2}^{(0)}|+|\psi'(x_1)||\partial_{x_1}d_{k_2}^{(0)}|\\
&+|\psi'(x_1)|\sum_{k_1=1}^{3}(|\partial_{x_{k_1}}^2 d_{k_2}^{(0)}|+|\partial_{x_{k_1}}\partial_{x_1}d_{k_2}^{(0)}|)\Big)(h_1^{(1)})^2e^{-\psi(x_1)}dx\\
&+\frac{1}{2}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega|\psi'(x_1)|\Big(|\partial_{x_{k_1}}d_{k_2}^{(0)}|+|\partial_{x_1}d_{k_2}^{(0)}|\Big)(\partial_{x_{k_1}}h_1^{(1)})^2e^{-\psi(x_1)}dx\\
&+\frac{1}{2}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega\Big[|\psi'(x_1)|\Big(|\partial_{x_1}d_{k_2}^{(0)}|+|\partial_{x_{k_1}}^2 d_{k_2}^{(0)}|+|\partial_{x_{k_1}}d_{k_2}^{(0)}|\Big)+|\psi''(x_1)||\partial_{x_1}d_{k_2}^{(0)}|\Big](\partial_{x_1}w_{k_2}^{(1)})^2e^{-\psi(x_1)}dx\\
&+\frac{1}{2}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega|\psi'(x_1)|\Big(|\partial_{x_{k_1}}\partial_{x_1}d_{k_2}^{(0)}|+|\partial_{x_1}d_{k_2}^{(0)}|\Big)(\partial_{x_{k_1}}w_{k_2}^{(1)})^2e^{-\psi(x_1)}dx,
\end{aligned}
\tag{2.28}
$$

thus, it holds

$$
\begin{aligned}
&\Big|\int_\Omega \psi'(x_1)\Big((\mathcal{D}_{u^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{d^{(0)}}P)\mathbf{w}^{(1)}\Big)h_1^{(1)}e^{-\psi(x_1)}dx\Big|\\
&\le \lambda\Big|\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega \psi'(x_1)\triangle^{-1}\Big(\partial_{x_{k_2}}h_{k_1}^{(1)}\partial_{x_{k_1}}u_{k_2}^{(0)}+\partial_{x_{k_2}}u_{k_1}^{(0)}\partial_{x_{k_1}}h_{k_2}^{(1)}\Big)h_1^{(1)}e^{-\psi(x_1)}dx\Big|\\
&\quad+\kappa\lambda\Big|\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega \psi'(x_1)\triangle^{-1}\Big(\partial_{x_{k_1}}^2 d_{k_2}^{(0)}\partial_{x_1}w_{k_2}^{(1)}+\lambda\partial_{x_{k_1}}w_{k_2}^{(1)}\partial_{x_{k_1}}\partial_{x_1}d_{k_2}^{(0)}\Big)h_1^{(1)}e^{-\psi(x_1)}dx\Big|\\
&\quad+\kappa\lambda\Big|\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega \psi'(x_1)\triangle^{-1}\Big(\lambda\partial_{x_{k_1}}d_{k_2}^{(0)}\partial_{x_{k_1}}\partial_{x_1}w_{k_2}^{(1)}+\partial_{x_{k_1}}^2 w_{k_2}^{(1)}\partial_{x_1}d_{k_2}^{(0)}\Big)h_1^{(1)}e^{-\psi(x_1)}dx\Big| \qquad (2.29)\\
&\lesssim \frac{1}{2}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega a_{k_1k_2}(t,x)(h_1^{(1)})^2e^{-\psi(x_1)}dx+\frac{1}{2}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega b_{k_1k_2}(t,x)(\partial_{x_{k_1}}h_1^{(1)})^2e^{-\psi(x_1)}dx\\
&\quad+\frac{1}{2}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega |\psi'(x_1)|\Big(|\partial_{x_{k_1}}u_{k_2}^{(0)}|(\partial_{x_{k_2}}h_{k_1}^{(1)})^2+|\partial_{x_{k_2}}u_{k_1}^{(0)}|(\partial_{x_{k_1}}h_{k_2}^{(1)})^2\Big)e^{-\psi(x_1)}dx,\\
&+\frac{1}{2}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega c_{k_1k_2}(t,x)(\partial_{x_1}w_{k_2}^{(1)})^2e^{-\psi(x_1)}dx+\frac{1}{2}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega e_{k_1k_2}(t,x)(\partial_{x_{k_1}}w_{k_2}^{(1)})^2e^{-\psi(x_1)}dx,
\end{aligned}
$$

where

$$
\begin{aligned}
a_{k_1k_2}(t,x) &:= |\psi'(x_1)|\Big(|\partial_{x_{k_1}}u_{k_2}^{(0)}|+|\partial_{x_{k_2}}u_{k_1}^{(0)}|+|\partial_{x_{k_1}}^2 d_{k_2}^{(0)}|+|\partial_{x_{k_1}}\partial_{x_1}d_{k_2}^{(0)}|+|\partial_{x_1}d_{k_2}^{(0)}|\Big)+|\psi''(x_1)||\partial_{x_1}d_{k_2}^{(0)}|,\\
b_{k_1k_2}(t,x) &:= |\psi'(x_1)|\Big(|\partial_{x_{k_1}}d_{k_2}^{(0)}|+|\partial_{x_1}d_{k_2}^{(0)}|\Big),\\
c_{k_1k_2}(t,x) &:= |\psi'(x_1)|\Big(|\partial_{x_1}d_{k_2}^{(0)}|+|\partial_{x_{k_1}}^2 d_{k_2}^{(0)}|+|\partial_{x_{k_1}}d_{k_2}^{(0)}|\Big)+|\psi''(x_1)||\partial_{x_1}d_{k_2}^{(0)}|,\\
e_{k_1k_2}(t,x) &:= |\psi'(x_1)|\Big(|\partial_{x_{k_1}}\partial_{x_1}d_{k_2}^{(0)}|+|\partial_{x_1}d_{k_2}^{(0)}|\Big).
\end{aligned}
$$

Note that $\psi'(x_1)$ is bounded in Ω. It follows from inquality (2.29) that

$$
\begin{aligned}
&\Big|\int_\Omega \psi'(x_1)\Big((\mathcal{D}_{u^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{d^{(0)}}P)\mathbf{w}^{(1)}\Big)h_1^{(1)}e^{-\psi(x_1)}dx\Big|\\
\lesssim\ &\delta\int_\Omega (h_1^{(1)})^2e^{-\psi(x_1)}dx+\delta\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega(\partial_{x_j}h_i^{(1)})^2e^{-\psi(x_1)}dx+\delta\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega(\partial_{x_j}w_{x_i}^{(1)})^2e^{-\psi(x_1)}dx. \qquad (2.30)
\end{aligned}
$$

On the other hand, we integrate by part to derive

$$
\begin{aligned}
&\sum_{j=1}^{3}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_{\Omega}\Big(\lambda(\partial_{x_{k_1}}d_{k_2}^{(0)})(\partial_{x_{k_1}}\partial_{x_j}w_{k_2}^{(1)})+(\partial_{x_{k_1}}^2w_{k_2}^{(1)})(\partial_{x_j}d_{k_2}^{(0)})\Big)h_j^{(1)}e^{-\psi(x_1)}dx\\
&=-\lambda\sum_{j=1}^{3}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_{\Omega}\Big((\partial_{x_{k_1}}^2d_{k_2}^{(0)})(\partial_{x_j}w_{k_2}^{(1)})h_j^{(1)}+(\partial_{x_{k_1}}d_{k_2}^{(0)})(\partial_{x_j}w_{k_2}^{(1)})(\partial_{x_{k_1}}h_j^{(1)})\Big)e^{-\psi(x_1)}dx\\
&\quad+\lambda\sum_{j=1}^{3}\sum_{k_2=1}^{3}\int_{\Omega}\psi'(x_1)(\partial_{x_1}d_{k_2}^{(0)})(\partial_{x_j}w_{k_2}^{(1)})h_je^{-\psi(x_1)}dx\\
&\quad-\sum_{j=1}^{3}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_{\Omega}\Big((\partial_{x_{k_1}}w_{k_2}^{(1)})(\partial_{x_{k_1}}\partial_{x_j}d_{k_2}^{(0)})h_j+(\partial_{x_{k_1}}w_{k_2}^{(1)})(\partial_{x_j}d_{k_2}^{(0)})(\partial_{x_{k_1}}h_j)\Big)e^{-\psi(x_1)}dx\\
&\quad+\sum_{j=1}^{3}\sum_{k_2=1}^{3}\int_{\Omega}\psi'(x_1)(\partial_{x_j}d_{k_2}^{(0)})(\partial_{x_1}w_{k_2}^{(1)})h_je^{-\psi(x_1)}dx,
\end{aligned}
$$

from which, by Young's inequality, we obtain

$$
\begin{aligned}
&\kappa\lambda\Pi_{N_1}\Big|\sum_{j=1}^{3}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_{\Omega}\Big((\partial_{x_{k_1}}^2d_{k_2}^{(0)})(\partial_{x_j}w_{k_2}^{(1)})+\lambda(\partial_{x_{k_1}}d_{k_2}^{(0)})(\partial_{x_{k_1}}\partial_{x_j}w_{k_2}^{(1)})+(\partial_{x_{k_1}}^2w_{k_2}^{(1)})(\partial_{x_j}d_{k_2}^{(0)})\\
&\quad+\lambda(\partial_{x_{k_1}}w_{k_2}^{(1)})(\partial_{x_{k_1}}\partial_{x_j}d_{k_2}^{(0)})\Big)h_j^{(1)}e^{-\psi(x_1)}dx\Big|\\
&\lesssim\kappa\lambda\delta\Pi_{N_1}\sum_{j=1}^{3}\int_{\Omega}(h_j^{(1)})^2e^{-\psi(x_1)}dx+\kappa\lambda\delta\Pi_{N_1}\sum_{j=1}^{3}\sum_{k_1=1}^{3}\int_{\Omega}(\partial_{x_{k_1}}h_j)^2e^{-\psi(x_1)}dx\\
&\quad+\kappa\lambda\delta\Pi_{N_1}\sum_{j=1}^{3}\sum_{k_2=1}^{3}\int_{\Omega}(\partial_{x_j}w_{k_2}^{(1)})^2e^{-\psi(x_1)}dx+\kappa\lambda\delta\Pi_{N_1}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_{\Omega}(\partial_{x_{k_1}}w_{k_2}^{(1)})^2e^{-\psi(x_1)}dx,
\end{aligned}
\tag{2.31}
$$

and

$$
\begin{aligned}
\sum_{j=1}^{3}\sum_{i\neq j}\int_{\Omega}h_i^{(1)}\partial_{x_i}u_j^{(0)}h_j^{(1)}e^{-\psi(x_1)}dx\ &\le\ \frac{1}{2}\int_{\Omega}\Big|\partial_{x_2}u_1^{(0)}+\partial_{x_1}u_2^{(0)}+\partial_{x_3}u_1^{(0)}+\partial_{x_1}u_3^{(0)}\Big|(h_1^{(1)})^2e^{-\psi(x_1)}dx\\
&\quad+\frac{1}{2}\int_{\Omega}\Big|\partial_{x_2}u_1^{(0)}+\partial_{x_1}u_2^{(0)}+\partial_{x_3}u_2^{(0)}+\partial_{x_2}u_3^{(0)}\Big|(h_2^{(1)})^2e^{-\psi(x_1)}dx\\
&\quad+\frac{1}{2}\int_{\Omega}\Big|\partial_{x_2}u_3^{(0)}+\partial_{x_3}u_2^{(0)}+\partial_{x_3}u_1^{(0)}+\partial_{x_1}u_3^{(0)}\Big|(h_3^{(1)})^2e^{-\psi(x_1)}dx\\
&\lesssim\ \delta\sum_{j=1}^{3}\int_{\Omega}(h_j^{(1)})^2e^{-\psi(x_1)}dx,
\end{aligned}
\tag{2.32}
$$

and

$$
\begin{aligned}
&\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(u_i^{(0)}\partial_{x_i}w_j^{(1)}\Big)w_j^{(1)}e^{-\psi(x_1)}dx+\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(h_i^{(1)}\partial_{x_i}d_j^{(0)}\Big)w_j^{(1)}e^{-\psi(x_1)}dx\\
&\lesssim\lambda\delta\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big((h_j^{(1)})^2+(\partial_{x_i}w_j^{(1)})^2+(w_j^{(1)})^2\Big)e^{-\psi(x_1)}dx,
\end{aligned}
\tag{2.33}
$$

and by $\psi''(x_1) > \frac{1}{4}$, it holds

$$\begin{aligned}
&\sum_{j=1}^{3}\int_{\Omega} E_j^{(0)} h_j^{(1)} e^{-\psi(x_1)}dx \leq \frac{1}{2}\sum_{j=1}^{3}\int_{\Omega}\Big((E_j^{(0)})^2 + |\psi''(x_1)|(h_j^{(1)})^2\Big)e^{-\psi(x_1)}dx,\\
&\sum_{j=1}^{3}\int_{\Omega} \overline{E}_j^{(0)} w_j^{(1)} e^{-\psi(x_1)}dx \leq \frac{1}{2}\sum_{j=1}^{3}\int_{\Omega}\Big((\overline{E}_j^{(0)})^2 + |\psi''(x_1)|(w_j^{(1)})^2\Big)e^{-\psi(x_1)}dx.
\end{aligned} \tag{2.34}$$

If the domain is the unbounded domain $\mathbb{R}^+ \times \mathbb{R}^2$, we choose the parameter $\lambda > 1$. If the domain is a bounded domain Ω, we choose the parameter $\lambda = 1$ and $\frac{\gamma}{\varepsilon^2} < 1$. Next, for a suitable constant $\lambda > 1$, we substitute (2.23) and (2.34) into (2.21) to get

$$\begin{aligned}
&\frac{d}{dt}\sum_{j=1}^{3}\int_{\Omega}\Big((h_j^{(1)})^2 + (w_j^{(1)})^2\Big)e^{-\psi(x_1)}dx\\
&+\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big((\nu\lambda-\varepsilon)(\partial_{x_i}h_j^{(1)})^2 + (\gamma\lambda-\varepsilon)(\partial_{x_i}w_j^{(1)})^2\Big)e^{-\psi(x_1)}dx\\
&+C_\lambda\nu\sum_{j=1}^{3}\int_{\Omega}\Big(\psi''(x_1) - (\psi'(x_1))^2 - |\psi'(x)| - \delta\Big)(h_j^{(1)})^2 e^{-\psi(x_1)}dx\\
&+\sum_{j=1}^{3}\int_{\Omega}\Big(\gamma\lambda(\psi''(x_1) - (\psi'(x_1))^2 - |\psi'(x)| - \delta) - \frac{\gamma}{\varepsilon^2}(1-\delta^2)\Big)(w_j^{(1)})^2 e^{-\psi(x_1)}dx\\
\lesssim\ &\Pi_{N_1}\sum_{j=1}^{3}\int_{\Omega}\Big((E_j^{(0)})^2 + (\overline{E}_j^{(0)})^2\Big)e^{-\psi(x_1)}dx,
\end{aligned} \tag{2.35}$$

where C_λ denotes a positive constant depending on λ.

Since $\psi(x_1)$ satisfies the ODE inequality (2.17), for suitable positive constants λ and δ, there exist two positive constants $C_{\nu,\gamma,\lambda}$ and $C_{\nu,\gamma,\lambda,\varepsilon,\delta}$ such that

$$\begin{aligned}
&\psi''(x_1) - (\psi'(x_1))^2 - |\psi'(x)| - \delta > C_{\nu,\gamma,\lambda} > 0,\\
&\gamma\lambda(\psi''(x_1) - (\psi'(x_1))^2 - |\psi'(x)| - \delta) - \frac{\gamma}{\varepsilon^2}(1-\delta^2) > C_{\nu,\gamma,\lambda,\varepsilon,\delta} > 0,
\end{aligned}$$

thus, integrating (2.35) over $(0, t)$ we find

$$\begin{aligned}
\sum_{j=1}^{3}\int_{\Omega}\Big((h_j^{(1)})^2 + (w_j^{(1)})^2\Big)e^{-\psi(x_1)}dx \quad &+ \quad C_{\nu,\gamma,\lambda}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_{\Omega}\Big((\partial_{x_i}h_j^{(1)})^2 + (\partial_{x_i}w_j^{(1)})^2\Big)e^{-\psi(x_1)}dxdt\\
&+C_{\nu,\gamma,\lambda,\varepsilon,\delta}\Pi_{N_1}\sum_{j=1}^{3}\int_0^t\int_{\Omega}\Big((h_j^{(1)})^2 + (w_j^{(1)})^2\Big)e^{-\psi(x_1)}dxdt\\
&\lesssim \quad \Pi_{N_1}\sum_{j=1}^{3}\int_0^t\int_{\Omega}\Big((E_j^{(0)})^2 + (\overline{E}_j^{(0)})^2\Big)e^{-\psi(x_1)}dxdt,
\end{aligned} \tag{2.36}$$

which gives the following inequality

$$\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_\Omega\Big((\partial_{x_i}h_j^{(1)})^2+(\partial_{x_i}w_j^{(1)})^2\Big)e^{-\psi(x_1)}dxdt\lesssim\Pi_{N_1}\sum_{j=1}^{3}\int_0^t\int_\Omega\Big((E_j^{(0)})^2+(\overline{E}_j^{(0)})^2\Big)e^{-\psi(x_1)}dxdt.$$

Therefore, we apply Gronwall's inequality to inequality (2.36) to obtain

$$\begin{aligned}&\sum_{j=1}^{3}\int_\Omega\Big((h_j^{(1)})^2+(w_j^{(1)})^2\Big)e^{-\psi(x_1)}dx\\&\lesssim e^{-C_{\nu,\gamma,\lambda,\varepsilon,\delta}t}\sum_{j=1}^{3}\Big(\int_\Omega\Big((h_{0j}^{(1)})^2+(w_{0j}^{(1)})^2\Big)e^{-\psi(x_1)}dx+\Pi_{N_1}\int_0^t\int_\Omega\Big((E_j^{(0)})^2+(\overline{E}_j^{(0)})^2\Big)e^{-\psi(x_1)}dxdt\Big),\end{aligned}$$

which yields

$$\begin{aligned}&\sum_{j=1}^{3}\int_0^t\int_\Omega\Big((h_j^{(1)})^2+(w_j^{(1)})^2\Big)e^{-\psi(x_1)}dxdt\\&\lesssim\sum_{j=1}^{3}\Big(\int_\Omega\Big((h_{0j}^{(1)})^2+(w_{0j}^{(1)})^2\Big)e^{-\psi(x_1)}dx+\Pi_{N_1}\int_0^t\int_\Omega\Big((E_j^{(0)})^2+(\overline{E}_j^{(0)})^2\Big)e^{-\psi(x_1)}dxdt\Big).\end{aligned}\tag{2.37}$$

Furthermore, since $e^{-\psi(x_1)}$ is a bounded smooth function in Ω, it follows that

$$\begin{aligned}&\sum_{j=1}^{3}\int_\Omega\Big((h_j^{(1)})^2+(w_j^{(1)})^2\Big)dx\\&\lesssim e^{-C_{\nu,\gamma,\lambda,\varepsilon,\delta}t}\sum_{j=1}^{3}\Big(\int_\Omega\Big((h_{0j}^{(1)})^2+(w_{0j}^{(1)})^2\Big)dx+\Pi_{N_1}\int_0^t\int_\Omega\Big((E_j^{(0)})^2+(\overline{E}_j^{(0)})^2\Big)dxdt\Big).\end{aligned}$$

The proof is now complete. □

Next, we derive the higher order derivatives estimates. For a fixed constant $s\geq 1$, we apply $D_i^s:=\partial_{x_i}^s$ $(\forall i=1,2,3)$ to both sides of (2.14). Therefore

$$\begin{aligned}&\partial_tD_i^sh_j^{(1)}-\nu\lambda^2\Delta D_i^sh_j^{(1)}+\lambda\Pi_{N_1}\sum_{i=1}^{3}u_i^{(0)}\partial_{x_i}D_i^sh_j^{(1)}+\Pi_{N_1}\sum_{i=1}^{3}D_i^sh_i^{(1)}\partial_{x_i}u_j^{(0)}\\&+\lambda\Pi_{N_1}\partial_{x_j}D_i^s\Big((\mathcal{D}_{u^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{d^{(0)}}P)\mathbf{w}^{(1)}\Big)\\&+\kappa\lambda\Pi_{N_1}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\Big((\partial_{x_{k_1}}^2d_{k_2}^{(0)})(D_i^s\partial_{x_j}w_{k_2}^{(1)})+\lambda(\partial_{x_{k_1}}d_{k_2}^{(0)})(D_i^s\partial_{x_{k_1}}\partial_{x_j}w_{k_2}^{(1)})\\&+(D_i^s\partial_{x_{k_1}}^2w_{k_2}^{(1)})(\partial_{x_j}d_{k_2}^{(0)})+\lambda(D_i^s\partial_{x_{k_1}}w_{k_2}^{(1)})(\partial_{x_{k_1}}\partial_{x_j}d_{k_2}^{(0)})\Big)=F_j,\quad for\ j=1,2,3,\end{aligned}\tag{2.38}$$

coupled with

$$\begin{aligned}&\partial_tD_i^sw_j^{(1)}-\gamma\lambda^2\Delta D_i^sw_j^{(1)}+\lambda\Pi_{N_1}\sum_{i=1}^{3}u_i^{(0)}(D_i^s\partial_{x_i}w_j^{(1)})+\Pi_{N_1}\sum_{i=1}^{3}(D_i^sh_i^{(1)})\partial_{x_i}d_j^{(0)}\\&+\frac{\gamma}{\varepsilon^2}D_i^s\Big(|d^{(0)}|^2w_j^{(1)}-w_j^{(1)}\Big)=\overline{F}_j,\quad for\ j=1,2,3,\end{aligned}\tag{2.39}$$

with the boundary condition

$$D_i^l h_j^{(1)}(t,\lambda x)|_{x\in\partial\Omega} = 0, \quad D_i^l w_j^{(1)}(t,\lambda x)|_{x\in\partial\Omega} = 0, \tag{2.40}$$

where the constant $1 \le l \le s$, and

$$\begin{aligned}
F_j := {} & \Pi_{N_1} D_i^s E_j^{(0)} - \lambda \Pi_{N_1} \sum_{s_1+s_2=s,\ 0\le s_2\le s-1} \sum_{i=1}^{3} (D_i^{s_1} u_i^{(0)})(\partial_{x_i} D_i^{s_2} h_j^{(1)}) \\
& - \Pi_{N_1} \sum_{s_1+s_2=s,\ 0\le s_2\le s-1} \sum_{i=1}^{3} (D_i^{s_2} h_i^{(1)})(D_i^{s_1}\partial_{x_i} u_j^{(0)}) \\
& - \kappa\lambda \Pi_{N_1} \sum_{s_1+s_2=s,\ 0\le s_2\le s-1} \sum_{k_1=1}^{3}\sum_{k_2=1}^{3} \Big((D_i^{s_1}\partial_{x_{k_1}}^2 d_{k_2}^{(0)})(D_i^{s_2}\partial_{x_j} w_{k_2}^{(1)}) + \lambda (D_i^{s_1}\partial_{x_{k_1}} d_{k_2}^{(0)})(D_i^{s_2}\partial_{x_{k_1}}\partial_{x_j} w_{k_2}^{(1)}) \\
& + (D_i^{s_1}\partial_{x_{k_1}}^2 w_{k_2}^{(1)})(D_i^{s_2}\partial_{x_j} d_{k_2}^{(0)}) + \lambda (D_i^{s_1}\partial_{x_{k_1}} w_{k_2}^{(1)})(D_i^{s_2}\partial_{x_{k_1}}\partial_{x_j} d_{k_2}^{(0)})\Big), \\
\overline{F}_j := {} & \Pi_{N_1} D_i^s \overline{E}_j^{(0)} - \lambda \Pi_{N_1} \sum_{s_1+s_2=s,\ 0\le s_2\le s-1} \sum_{i=1}^{3} (D_i^{s_1} u_i^{(0)})(\partial_{x_i} D_i^{s_2} w_j^{(1)}) \\
& - \Pi_{N_1} \sum_{s_1+s_2=s,\ 0\le s_2\le s-1} \sum_{i=1}^{3} (D_i^{s_2} h_i^{(1)})(D_i^{s_1}\partial_{x_i} d_j^{(0)}).
\end{aligned}$$

Next, we derive higher derivative estimate of solution for (2.14).

Lemma 2.3. *If the domain is the unbounded domain $\mathbb{R}^+ \times \mathbb{R}^2$, let the parameter $\lambda > 1$. If the domain is a bounded domain Ω, let the parameter $\lambda = 1$ and $\frac{\gamma}{\varepsilon^2} < 1$. Assume that the initial approximation function $(u^{(0)}, d^{(0)})$ satisfies conditions (2.4)–(2.8). Then the solution $(\boldsymbol{h}^{(1)}(t,\lambda x), \boldsymbol{w}^{(1)}(t,\lambda x))$ of the linear system (2.14) and (2.15) satisfies*

$$\begin{aligned}
\sum_{i=1}^{3}\sum_{j=1}^{3} \int_\Omega \Big((D_i^s h_j^{(1)})^2 + (D_i^s w_j^{(1)})^2\Big) dx \lesssim e^{-C_{\mu,\gamma,\lambda,\varepsilon,\delta} t} \Pi_{N_1} \sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{l_0=0}^{s} \Big[\int_\Omega \Big((D_i^{l_0} h_{0j}^{(1)})^2 + (D_i^{l_0} w_{0j}^{(1)})^2\Big) dx \\
+ \int_0^\infty \int_\Omega \Big((D_i^{l_0} E_j^{(0)})^2 + (D_i^{l_0} \overline{E}_j^{(0)})^2\Big) dx dt\Big],
\end{aligned} \tag{2.41}$$

where $C_{\nu,\gamma,\lambda,\varepsilon,\delta}$ denotes a positive constant depending on $\nu, \gamma, \lambda, \varepsilon, \delta$.

Proof. We use an induction argument. Let $s = 1$. By (2.38), we obtain

$$\begin{aligned}
& \partial_t D_i^1 h_j^{(1)} - \nu\lambda^2 \Delta D_i^1 h_j^{(1)} + \lambda \Pi_{N_1} \sum_{i=1}^{3} u_i^{(0)} \partial_{x_i} D_i^1 h_j^{(1)} + \Pi_{N_1} \sum_{i=1}^{3} D_i^1 h_i^{(1)} \partial_{x_i} u_j^{(0)} \\
& + \lambda \Pi_{N_1} \partial_{x_j} D_i^1 \Big((\mathcal{D}_{u^{(0)}} P)\mathbf{h}^{(1)} + (\mathcal{D}_{d^{(0)}} P)\mathbf{w}^{(1)}\Big) + \lambda \Pi_{N_1} \sum_{i=1}^{3} D_i^1 u_i^{(0)} \partial_{x_i} h_j^{(1)} \\
& + \Pi_{N_1} \sum_{i=1}^{3} h_i^{(1)} D_i^1 \partial_{x_i} u_j^{(0)} + \kappa\lambda \Pi_{N_1} \sum_{k_1=1}^{3}\sum_{k_2=1}^{3} D_i^1 \Big((\partial_{x_{k_1}}^2 d_{k_2}^{(0)})(\partial_{x_j} w_{k_2}^{(1)}) + \lambda(\partial_{x_{k_1}} d_{k_2}^{(0)})(\partial_{x_{k_1}}\partial_{x_j} w_{k_2}^{(1)}) \\
& + (\partial_{x_{k_1}}^2 w_{k_2}^{(1)})(\partial_{x_j} d_{k_2}^{(0)}) + \lambda(\partial_{x_{k_1}} w_{k_2}^{(1)})(\partial_{x_{k_1}}\partial_{x_j} d_{k_2}^{(0)})\Big) = \Pi_{N_1} D_i^1 E_j^{(0)}, \quad for\ j = 1,2,3,
\end{aligned} \tag{2.42}$$

coupled with

$$
\begin{aligned}
&\partial_t D_i^1 w_j^{(1)} - \gamma\lambda^2 \Delta D_i^1 w_j^{(1)} + \lambda \Pi_{N_1} \sum_{i=1}^{3} u_i^{(0)} (D_i^1 \partial_{x_i} w_j^{(1)}) + \Pi_{N_1} \sum_{i=1}^{3} (D_i^1 h_i^{(1)}) \partial_{x_i} d_j^{(0)} \\
&+ \lambda \Pi_{N_1} \sum_{i=1}^{3} (D_i^1 u_i^{(0)}) \partial_{x_i} w_j^{(1)} + \Pi_{N_1} \sum_{i=1}^{3} h_i^{(1)} (D_i^1 \partial_{x_i} d_j^{(0)}) \\
&+ \frac{\gamma}{\varepsilon^2} D_i^1 \Big(|d^{(0)}|^2 w_j^{(1)} - w_j^{(1)} \Big) = \Pi_{N_1} D_i^1 \overline{E}_j^{(0)}, \quad for\ j = 1, 2, 3,
\end{aligned}
\tag{2.43}
$$

with the boundary condition

$$
D_i^1 h_j^{(1)}(t, \lambda x)|_{x \in \partial\Omega} = 0, \quad D_i^1 w_j^{(1)}(t, \lambda x)|_{x \in \partial\Omega} = 0. \tag{2.44}
$$

We also choose the weighted function satisfies (2.17). Multiplying both sides of (2.42) and (2.43) by $D_i^1 h_j^{(1)} e^{-\psi(x_1)}$ and $D_i^1 w_j^{(1)} e^{-\psi(x_1)}$, respectively, then integrating over Ω by noticing (2.44), and summing up those equalities from $j = 1$ to $j = 3$, it follows that

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \sum_{j=1}^{3} \int_\Omega (D_i^1 h_j^{(1)})^2 e^{-\psi(x_1)} dx + \nu\lambda^2 \sum_{j=1}^{3} \sum_{k=1}^{3} \int_\Omega (\partial_{x_k} D_i^1 h_j^{(1)})^2 e^{-\psi(x_1)} dx \\
&+ \frac{\nu\lambda^2}{2} \sum_{j=1}^{3} \int_\Omega \Big(\psi''(x_1) - (\psi'(x_1))^2 \Big) (D_i^1 h_j^{(1)})^2 e^{-\psi(x_1)} dx \\
&+ \lambda \Pi_{N_1} \sum_{j=1}^{3} \sum_{i=1}^{3} \int_\Omega u_i^{(0)} (\partial_{x_i} D_i^1 h_j^{(1)}) (D_i^1 h_j^{(1)}) e^{-\psi(x_1)} dx \\
&+ \Pi_{N_1} \sum_{j=1}^{3} \sum_{i=1}^{3} \int_\Omega D_i^1 h_i^{(1)} \partial_{x_i} u_j^{(0)} D_i^1 h_j^{(1)} e^{-\psi(x_1)} dx \\
&+ \lambda \Pi_{N_1} \sum_{j=1}^{3} \int_\Omega \partial_{x_j} D_i^1 \Big((\mathcal{D}_{u^{(0)}} P) \mathbf{h}^{(1)} + (\mathcal{D}_{d^{(0)}} P) \mathbf{w}^{(1)} \Big) D_i^1 h_j^{(1)} e^{-\psi(x_1)} dx \\
&+ \Pi_{N_1} \sum_{j=1}^{3} \sum_{a=1}^{11} I_a = 0,
\end{aligned}
\tag{2.45}
$$

and

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \sum_{j=1}^{3} \int_\Omega (D_i^1 w_j^{(1)})^2 e^{-\psi(x_1)} dx + \gamma\lambda^2 \sum_{j=1}^{3} \sum_{k=1}^{3} \int_\Omega (\partial_{x_k} D_i^1 w_j^{(1)})^2 e^{-\psi(x_1)} dx \\
&+ \frac{\gamma\lambda^2}{2} \sum_{j=1}^{3} \int_\Omega \Big(\psi''(x_1) - (\psi'(x_1))^2 \Big) (D_i^1 w_j^{(1)})^2 e^{-\psi(x_1)} dx \\
&+ \lambda \Pi_{N_1} \sum_{j=1}^{3} \sum_{i=1}^{3} \int_\Omega \Big(u_i^{(0)} D_i^1 \partial_{x_i} w_j^{(1)} \Big) (D_i^1 w_j^{(1)}) e^{-\psi(x_1)} dx
\end{aligned}
$$

$$
\begin{aligned}
&+\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega\Big((D_i^1h_i^{(1)})\partial_{x_i}d_j^{(0)}\Big)(D_i^1w_j^{(1)})e^{-\psi(x_1)}dx\\
&+\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega\Big((D_i^1u_i^{(0)})\partial_{x_i}w_j^{(1)}\Big)(D_i^1w_j^{(1)})e^{-\psi(x_1)}dx\\
&+\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega\Big(h_i^{(1)}(D_i^1\partial_{x_i}d_j^{(0)})\Big)(D_i^1w_j^{(1)})e^{-\psi(x_1)}dx\\
&+\frac{\gamma}{\varepsilon^2}\Pi_{N_1}\sum_{j=1}^{3}\int_\Omega D_i^1\Big(|d^{(0)}|^2w_j^{(1)}-w_j^{(1)}\Big)(D_i^1w_j^{(1)})e^{-\psi(x_1)}dx\\
=\ &\Pi_{N_1}\sum_{j=1}^{3}\int_\Omega(D_i^1\overline{E}_j^{(0)})(D_i^1w_j^{(1)})e^{-\psi(x_1)}dx, \qquad (2.46)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &:= \lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega D_i^1u_i^{(0)}\partial_{x_i}h_j^{(1)}D_i^1h_j^{(1)}e^{-\psi(x_1)}dx,\\
I_2 &:= \sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega h_i^{(1)}D_i^1\partial_{x_i}u_j^{(0)}D_i^1h_j^{(1)}e^{-\psi(x_1)}dx,\\
I_3 &:= \kappa\lambda\Pi_{N_1}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega(D_i^1\partial_{x_{k_1}}^2d_{k_2}^{(0)})(\partial_{x_j}w_{k_2}^{(1)})(D_i^1h_j^{(1)})e^{-\psi(x_1)}dx,\\
I_4 &:= \kappa\lambda\Pi_{N_1}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega(\partial_{x_{k_1}}^2d_{k_2}^{(0)})(D_i^1\partial_{x_j}w_{k_2}^{(1)})(D_i^1h_j^{(1)})e^{-\psi(x_1)}dx,\\
I_5 &:= \kappa\lambda^2\Pi_{N_1}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega(D_i^1\partial_{x_{k_1}}d_{k_2}^{(0)})(\partial_{x_{k_1}}\partial_{x_j}w_{k_2}^{(1)})(D_i^1h_j^{(1)})e^{-\psi(x_1)}dx,\\
I_6 &:= \kappa\lambda^2\Pi_{N_1}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega(\partial_{x_{k_1}}d_{k_2}^{(0)})(D_i^1\partial_{x_{k_1}}\partial_{x_j}w_{k_2}^{(1)})(D_i^1h_j^{(1)})e^{-\psi(x_1)}dx,\\
I_7 &:= \kappa\lambda\Pi_{N_1}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega(D_i^1\partial_{x_{k_1}}^2w_{k_2}^{(1)})(\partial_{x_j}d_{k_2}^{(0)})(D_i^1h_j^{(1)})e^{-\psi(x_1)}dx,\\
I_8 &:= \kappa\lambda\Pi_{N_1}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega(\partial_{x_{k_1}}^2w_{k_2}^{(1)})(D_i^1\partial_{x_j}d_{k_2}^{(0)})(D_i^1h_j^{(1)})e^{-\psi(x_1)}dx,\\
I_9 &:= \kappa\lambda^2\Pi_{N_1}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega(D_i^1\partial_{x_{k_1}}w_{k_2}^{(1)})(\partial_{x_{k_1}}\partial_{x_j}d_{k_2}^{(0)})(D_i^1h_j^{(1)})e^{-\psi(x_1)}dx,\\
I_{10} &:= \kappa\lambda^2\Pi_{N_1}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega(\partial_{x_{k_1}}w_{k_2}^{(1)})(D_i^1\partial_{x_{k_1}}\partial_{x_j}d_{k_2}^{(0)})(D_i^1h_j^{(1)})e^{-\psi(x_1)}dx,
\end{aligned}
$$

$$I_{11} \;:=\; \Pi_{N_1} \sum_{j=1}^{3} \int_\Omega \Big((D_i^1 E_j^{(0)})(D_i^1 h_j^{(1)}) + (D_i^1 \overline{E}_j^{(0)})(D_i^1 w_j^{(1)}) \Big) e^{-\psi(x_1)} dx.$$

We now estimate each term in (2.45) and (2.46). On the one hand, since we have chosen the initial approximation function $(u^{(0)}, d^{(0)})$ satisfying (2.4)–(2.8), we derive

$$\sum_{j=1}^{3}\sum_{i=1}^{3} \int_\Omega u_i^{(0)} (\partial_{x_i} D_i^1 h_j^{(1)})(D_i^1 h_j^{(1)}) e^{-\psi(x_1)} dx = \frac{1}{2} \sum_{j=1}^{3} \int_\Omega \psi'(x_1) u_1^{(0)} (D_i^1 h_j^{(1)})^2 e^{-\psi(x_1)} dx, \tag{2.47}$$

$$\begin{aligned} \sum_{j=1}^{3}\sum_{i=1}^{3} \int_\Omega D_i^1 h_i^{(1)} \partial_{x_i} u_j^{(0)} D_i^1 h_j^{(1)} e^{-\psi(x_1)} dx &= \sum_{j=1}^{3} \int_\Omega \partial_{x_j} u_j^{(0)} (D_j^1 h_j^{(1)})^2 e^{-\psi(x_1)} dx \\ &\quad + \sum_{j=1}^{3} \sum_{i \neq j} \int_\Omega \partial_{x_i} u_j^{(0)} (D_i^1 h_i^{(1)})(D_i^1 h_j^{(1)}) e^{-\psi(x_1)} dx. \end{aligned} \tag{2.48}$$

By the incompressibility condition $\nabla \cdot \mathbf{h}^{(1)} = 0$ and integration by parts we find

$$\begin{aligned} &\sum_{j=1}^{3} \int_\Omega \partial_{x_j} D_i^1 \Big((\mathcal{D}_{u^{(0)}} P)\mathbf{h}^{(1)} + (\mathcal{D}_{d^{(0)}} P)\mathbf{w}^{(1)} \Big) D_i^1 h_j^{(1)} e^{-\psi(x_1)} dx \\ &= \int_\Omega \psi'(x_1) D_i^1 \Big((\mathcal{D}_{u^{(0)}} P)\mathbf{h}^{(1)} + (\mathcal{D}_{d^{(0)}} P)\mathbf{w}^{(1)} \Big) (D_i^1 h_1^{(1)}) e^{-\psi(x_1)} dx, \end{aligned} \tag{2.49}$$

from which, similar to (2.30), we use the standard Calderon-Zygmund theory and Young's inequality to derive

$$\begin{aligned} &\Big| \int_\Omega \psi'(x_1) D_i^1 \Big((\mathcal{D}_{u^{(0)}} P)\mathbf{h}^{(1)} + (\mathcal{D}_{d^{(0)}} P)\mathbf{w}^{(1)} \Big) (D_i^1 h_1^{(1)}) e^{-\psi(x_1)} dx \Big| \\ \lesssim\;& \delta \int_\Omega (D_i^1 h_1^{(1)})^2 e^{-\psi(x_1)} dx + \delta \sum_{j=1}^{3}\sum_{i=1}^{3} \int_\Omega (D_i^1 \partial_{x_j} h_i^{(1)})^2 e^{-\psi(x_1)} dx \\ &+ \delta \sum_{j=1}^{3}\sum_{i=1}^{3} \int_\Omega (D_i^1 \partial_{x_j} w_{x_i}^{(1)})^2 e^{-\psi(x_1)} dx. \end{aligned} \tag{2.50}$$

and

$$\begin{aligned} &\sum_{j=1}^{3} \sum_{i \neq j} \int_\Omega \partial_{x_i} u_j^{(0)} (D_i^1 h_i^{(1)})(D_i^1 h_j^{(1)}) e^{-\psi(x_1)} dx \\ \le\;& \frac{1}{2} \int_\Omega \Big| \partial_{x_2} u_1^{(0)} + \partial_{x_1} u_2^{(0)} + \partial_{x_3} u_1^{(0)} + \partial_{x_1} u_3^{(0)} \Big| (D_i^1 h_1^{(1)})^2 e^{-\psi(x_1)} dx \\ &+ \frac{1}{2} \int_\Omega \Big| \partial_{x_2} u_1^{(0)} + \partial_{x_1} u_2^{(0)} + \partial_{x_3} u_2^{(0)} + \partial_{x_2} u_3^{(0)} \Big| (D_i^1 h_2^{(1)})^2 e^{-\psi(x_1)} dx \\ &+ \frac{1}{2} \int_\Omega \Big| \partial_{x_2} u_3^{(0)} + \partial_{x_3} u_2^{(0)} + \partial_{x_3} u_1^{(0)} + \partial_{x_1} u_3^{(0)} \Big| (D_i^1 h_3^{(1)})^2 e^{-\psi(x_1)} dx, \end{aligned}$$

and

$$\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega\Big(u_i^{(0)}D_i^1\partial_{x_i}w_j^{(1)}\Big)(D_i^1w_j^{(1)})e^{-\psi(x_1)}dx\lesssim\delta\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega\Big((D_i^1\partial_{x_i}w_j^{(1)})^2+(D_i^1w_j^{(1)})^2\Big)dx,$$

$$\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega\Big((D_i^1h_i^{(1)})\partial_{x_i}d_j^{(0)}\Big)(D_i^1w_j^{(1)})e^{-\psi(x_1)}dx\lesssim\delta\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega\Big((D_i^1h_i^{(1)})^2+(D_i^1w_j^{(1)})^2\Big)dx$$

$$\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega\Big((D_i^1u_i^{(0)})\partial_{x_i}w_j^{(1)}\Big)(D_i^1w_j^{(1)})e^{-\psi(x_1)}dx\lesssim\delta\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega\Big((\partial_{x_i}w_j^{(1)})^2+(D_i^1w_j^{(1)})^2\Big)dx$$

$$\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega\Big(h_i^{(1)}(D_i^1\partial_{x_i}d_j^{(0)})\Big)(D_i^1w_j^{(1)})e^{-\psi(x_1)}dx\lesssim\delta\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega\Big((h_j^{(1)})^2+(D_i^1w_j^{(1)})^2\Big)dx$$

$$\frac{\gamma}{\varepsilon^2}\Pi_{N_1}\sum_{j=1}^{3}\int_\Omega D_i^1\Big(|d^{(0)}|^2w_j^{(1)}-w_j^{(1)}\Big)(D_i^1w_j^{(1)})e^{-\psi(x_1)}\lesssim\frac{\gamma}{\varepsilon^2}(1-\delta^2)\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega\Big((D_i^1w_j^{(1)})^2+(w_j^{(1)})^2\Big)dx.$$

On the other hand, by Young's inequality, it holds

$$\begin{aligned}
I_1&=\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega D_i^1u_i^{(0)}(D_i^1h_j^{(1)})^2e^{-\psi(x_1)}dx,\\
I_2&\lesssim\delta\sum_{j=1}^{3}\int_\Omega\Big((h_j^{(1)})^2+(D_i^1h_j^{(1)})^2\Big)e^{-\psi(x_1)}dx,\\
I_3&\lesssim\delta\sum_{k_2=1}^{3}\int_\Omega\Big((\partial_{x_j}w_{k_2}^{(1)})^2+(D_i^1h_j^{(1)})^2\Big)e^{-\psi(x_1)}dx,\\
I_4&\lesssim\delta\sum_{k_2=1}^{3}\int_\Omega\Big((D_i^1\partial_{x_j}w_{k_2}^{(1)})^2+(D_i^1h_j^{(1)})^2\Big)e^{-\psi(x_1)}dx,\\
I_5&\lesssim\delta\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega\Big((\partial_{x_{k_1}}\partial_{x_j}w_{k_2}^{(1)})^2+(D_i^1h_j^{(1)})^2\Big)e^{-\psi(x_1)}dx,\\
I_8&\lesssim\delta\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega\Big((\partial_{x_{k_1}}^2w_{k_2}^{(1)})^2+(D_i^1h_j^{(1)})^2\Big)e^{-\psi(x_1)}dx,\\
I_9&\lesssim\delta\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega\Big((D_i^1\partial_{x_{k_1}}w_{k_2}^{(1)})^2+(D_i^1h_j^{(1)})^2\Big)e^{-\psi(x_1)}dx,\\
I_{10}&\lesssim\delta\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega\Big((\partial_{x_{k_1}}w_{k_2}^{(1)})^2+(D_i^1h_j^{(1)})^2\Big)e^{-\psi(x_1)}dx,\\
I_{11}&\le\frac{1}{2}\sum_{j=1}^{3}\int_\Omega\Big((D_i^1E_j^{(0)})^2+(D_i^1\overline{E}_j^{(0)})^2+|\psi''(x_1)|(D_i^1h_j^{(1)})^2+|\psi''(x_1)|(D_i^1w_j^{(1)})^2\Big)e^{-\psi(x_1)}dx,
\end{aligned}\tag{2.51}$$

and we integrate by parts to estimate

$$\begin{aligned} I_6 &\lesssim \delta \sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega \Big((\partial_{x_{k_1}}\partial_{x_j} w_{k_2}^{(1)})^2 + (D_i^1 h_j^{(1)})^2 + (D_i^2 h_j^{(1)})^2\Big) e^{-\psi(x_1)}dx,\\ I_7 &\lesssim \delta \sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\int_\Omega \Big((\partial_{x_{k_1}}^2 w_{k_2}^{(1)})^2 + (D_i^1 h_j^{(1)})^2 + (D_i^2 h_j^{(1)})^2\Big) e^{-\psi(x_1)}dx. \end{aligned} \tag{2.52}$$

Thus, summing up (2.45) and (2.46) from $i = 1$ to $i = 3$, we use (2.47)–(2.52) to derive

$$\begin{aligned} &\frac{d}{dt}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega \Big((D_i^1 h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2\Big)e^{-\psi(x_1)}dx\\ &+\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3}\int_\Omega \Big((\nu\lambda-\delta)(\partial_{x_k} D_i^1 h_j^{(1)})^2 + (\gamma\lambda-\delta)(\partial_{x_k} D_i^1 w_j^{(1)})^2\Big)e^{-\psi(x_1)}dx\\ &+C_\lambda \nu \sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega \Big(\psi''(x_1) - (\psi'(x_1))^2 - |\psi'(x)| - \delta\Big)(D_i^1 h_j^{(1)})^2 e^{-\psi(x_1)}dx\\ &+\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega \Big(\gamma\lambda(\psi''(x_1) - (\psi'(x_1))^2 - |\psi'(x)| - \delta) - \frac{\gamma}{\varepsilon^2}(1-\delta^2)\Big)(D_i^1 w_j^{(1)})^2 e^{-\psi(x_1)}dx\\ \lesssim\;\; &\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega \Big((D_i^1 E_j^{(0)})^2 + (D_i^1 \overline{E}_j^{(0)})^2\Big)e^{-\psi(x_1)}dx\\ &+\delta\Pi_{N_1}\sum_{j=1}^{3}\int_\Omega \Big((h_j^{(1)})^2 + (w_j^{(1)})^2\Big)e^{-\psi(x_1)}dx. \end{aligned} \tag{2.53}$$

If the domain is the unbounded domain $\mathbb{R}^+ \times \mathbb{R}^2$, we choose the parameter $\lambda > 1$. If the domain is a bounded domain Ω, we choose the parameter $\lambda = 1$ and $\frac{\gamma}{\varepsilon^2} < 1$. Furthermore, note that $\psi(x_1)$ satisfies the ODE inequality (2.17), for suitable positive constants λ and δ, there exist two positive constants $C_{\nu,\gamma,\lambda}$ and $C_{\nu,\gamma,\lambda,\varepsilon,\delta}$ such that we integrate inequality (2.53) over $(0, t)$ to get

$$\begin{aligned} &\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_\Omega \Big((D_i^1 h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2\Big)e^{-\psi(x_1)}dxdt\\ &+C_{\nu,\gamma,\lambda}\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3}\int_0^t\int_\Omega \Big((\partial_{x_k} D_i^1 h_j^{(1)})^2 + (\partial_{x_k} D_i^1 w_j^{(1)})^2\Big)e^{-\psi(x_1)}dx\\ &+C_{\nu,\gamma,\lambda,\varepsilon,\delta}\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_\Omega \Big((D_i^1 h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2\Big)e^{-\psi(x_1)}dxdt\\ \lesssim\;\; &\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_\Omega \Big((D_i^1 E_j^{(0)})^2 + (D_i^1 \overline{E}_j^{(0)})^2\Big)e^{-\psi(x_1)}dxdt\\ &+\delta\Pi_{N_1}\sum_{j=1}^{3}\int_0^t\int_\Omega \Big((h_j^{(1)})^2 + (w_j^{(1)})^2\Big)e^{-\psi(x_1)}dxdt. \end{aligned} \tag{2.54}$$

We observe that the last term in the right-hand side of (2.54) can be controlled by using (2.37). Therefore

$$
\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3}\int_0^t\int_\Omega\Big((\partial_{x_k}D_i^1h_j^{(1)})^2+(\partial_{x_k}D_i^1w_j^{(1)})^2\Big)e^{-\psi(x_1)}dx\\
\lesssim\ &\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_\Omega\Big((D_i^1E_j^{(0)})^2+(D_i^1\overline{E}_j^{(0)})^2\Big)e^{-\psi(x_1)}dxdt\\
&+\sum_{j=1}^{3}\Big(\int_\Omega\Big((h_{0j}^{(1)})^2+(w_{0j}^{(1)})^2\Big)e^{-\psi(x_1)}dx+\Pi_{N_1}\int_0^t\int_\Omega\Big((E_j^{(0)})^2+(\overline{E}_j^{(0)})^2\Big)e^{-\psi(x_1)}dxdt\Big). \qquad (2.55)
\end{aligned}
$$

Hence, by (2.37), we apply Gronwall's inequality to (2.54) to derive

$$
\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_\Omega\Big((D_i^1h_j^{(1)})^2+(D_i^1w_j^{(1)})^2\Big)e^{-\psi(x_1)}dxdt\\
\lesssim\ &e^{-C_{\nu,\gamma,\lambda,\varepsilon,\delta}t}\Pi_{N_1}\sum_{j=1}^{3}\Big[\int_\Omega\Big((h_{0j}^{(1)})^2+(w_{0j}^{(1)})^2\Big)e^{-\psi(x_1)}dx\\
&+\sum_{i=1}^{3}\int_\Omega\Big((D_i^1h_{0j}^{(1)})^2+(D_i^1w_{0j}^{(1)})^2\Big)e^{-\psi(x_1)}dx\\
&+\int_0^{+\infty}\int_\Omega\Big((E_j^{(0)})^2+(\overline{E}_j^{(0)})^2\Big)e^{-\psi(x_1)}dxdt\\
&+\sum_{i=1}^{3}\int_0^{\infty}\int_\Omega\Big((D_i^1E_j^{(0)})^2+(D_i^1\overline{E}_j^{(0)})^2\Big)e^{-\psi(x_1)}dxdt\Big]. \qquad (2.56)
\end{aligned}
$$

Assume that the $2\le l\le s-1$ derivative case holds, that is,

$$
\begin{aligned}
&\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega\Big((D_i^lh_j^{(1)})^2+(D_i^lw_j^{(1)})^2\Big)e^{-\psi(x_1)}dx\\
&\lesssim e^{-C_{\nu,\gamma,\lambda,\varepsilon,\delta}t}\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{l_0=0}^{l}\Big(\int_\Omega\Big((D_i^{l_0}h_{0j}^{(1)})^2+(D_i^{l_0}w_{0j}^{(1)})^2\Big)e^{-\psi(x_1)}dx \qquad (2.57)\\
&\quad+\int_0^{\infty}\int_\Omega\Big((D_i^{l_0}E_j^{(0)})^2+(D_i^{l_0}\overline{E}_j^{(0)})^2\Big)e^{-\psi(x_1)}dxdt\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3}\int_0^t\int_\Omega\Big((\partial_{x_k}D_i^lh_j^{(1)})^2+(\partial_{x_k}D_i^lw_j^{(1)})^2\Big)e^{-\psi(x_1)}dxdt\\
&\lesssim\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{l_0=0}^{l-1}\Big(\int_\Omega\Big((D_i^{l_0}h_{0j}^{(1)})^2+(D_i^{l_0}w_{0j}^{(1)})^2\Big)e^{-\psi(x_1)}dx \qquad (2.58)\\
&\quad+\Pi_{N_1}\int_0^t\int_\Omega\Big((D_i^{l_0}E_j^{(0)})^2+(D_i^{l_0}\overline{E}_j^{(0)})^2\Big)e^{-\psi(x_1)}dxdt\Big).
\end{aligned}
$$

We now prove the sth derivative case holds. Multiplying both sides of Eqs (2.38) and (2.39) by $D_i^s h_j^{(1)} e^{-\psi(x_1)}$ and $D_i^s w_j^{(1)} e^{-\psi(x_1)}$, then integrating over Ω by using the boundary condition (2.40), and summing up those equalities from $i, j = 1$ to $i, j = 3$, with similar arguments as for getting (2.53), we can obtain

$$\begin{aligned}
&\frac{d}{dt}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega \Big((D_i^s h_j^{(1)})^2 + (D_i^s w_j^{(1)})^2\Big) e^{-\psi(x_1)} dx \\
&+\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega \Big((\nu\lambda-\delta)(\partial_{x_j} D_i^s h_j^{(1)})^2 + (\gamma\lambda-\delta)(\partial_{x_j} D_i^s w_j^{(1)})^2\Big) e^{-\psi(x_1)} dx \\
&+C_{\nu,\gamma,\lambda,\varepsilon,\delta}\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega \Big((D_i^s h_j^{(1)})^2 + (D_i^s w_j^{(1)})^2\Big) e^{-\psi(x_1)} dx \\
\lesssim\ & \Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega \Big((D_i^s E_j^{(0)})^2 + (D_i^s \overline{E}_j^{(0)})^2\Big) e^{-\psi(x_1)} dx \\
&+\varepsilon\Pi_{N_1}\sum_{k=0}^{s-1}\sum_{i=1}^{3}\int_\Omega \Big((D_i^k h_i^{(1)})^2 + (\partial_{x_i} D_i^k h_i^{(1)})^2 + (\partial_{x_i} D_i^k w_i^{(1)})^2\Big) e^{-\psi(x_1)} dx \\
&+\varepsilon\Pi_{N_1}\sum_{k=0}^{s-1}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\Big((D_i^k \partial_{x_j} w_{k_2}^{(1)})^2 + (D_i^k \partial_{x_{k_1}}^2 w_{k_2}^{(1)})^2 + (D_i^k \partial_{x_{k_1}} w_{k_2}^{(1)})^2\Big),
\end{aligned} \tag{2.59}$$

where we intergrate by parts to estimate the term $(D_i^{s_1}\partial_{x_{k_1}} d_{k_2}^{(0)})(D_i^{s_2}\partial_{x_{k_1}}\partial_{x_j} w_{k_2}^{(1)})$ in F_j. Thus, by (2.57) and (2.58), in a similar way as for getting (2.55) and (2.56), applying Gronwall's inequality to (2.59), we obtain

$$\begin{aligned}
&\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega \Big((D_i^s h_j^{(1)})^2 + (D_i^s w_j^{(1)})^2\Big) e^{-\psi(x_1)} dx \\
&\lesssim e^{-C_{\mu,\gamma,\lambda,\varepsilon,\delta}t}\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{l_0=0}^{s}\Big[\int_\Omega \Big((D_i^{l_0} h_{0j}^{(1)})^2 + (D_i^{l_0} w_{0j}^{(1)})^2\Big) e^{-\psi(x_1)} dx \\
&\quad + \int_0^\infty \int_\Omega \Big((D_i^{l_0} E_j^{(0)})^2 + (D_i^{l_0} \overline{E}_j^{(0)})^2\Big) e^{-\psi(x_1)} dxdt\Big],
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t \int_\Omega \Big((\partial_{x_j} D_i^l h_j^{(1)})^2 + (\partial_{x_j} D_i^l w_j^{(1)})^2\Big) e^{-\psi(x_1)} dxdt \\
&\lesssim \sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{l_0=0}^{s-1}\Big(\int_\Omega \Big((D_i^{l_0} h_{0j}^{(1)})^2 + (D_i^{l_0} w_{0j}^{(1)})^2\Big)^2 e^{-\psi(x_1)} dx \\
&\quad + \Pi_{N_1}\int_0^t \int_\Omega \Big((D_i^{l_0} E_j^{(0)})^2 + (D_i^{l_0} \overline{E}_j^{(0)})^2\Big) e^{-\psi(x_1)} dxdt\Big).
\end{aligned}$$

Noticing that $e^{-\psi(x_1)}$ is be a bounded function in $(0,+\infty)$, it follows from inequality (2.60) that

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}\Big((D_i^s h_j^{(1)})^2+(D_i^s w_j^{(1)})^2\Big)dx \lesssim e^{-C_{\mu,\gamma,\lambda,\varepsilon,\delta}t}\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{l_0=0}^{s}\Big[\int_{\Omega}\Big((D_i^{l_0}h_{0j}^{(1)})^2+(D_i^{l_0}w_{0j}^{(1)})^2\Big)dx$$
$$+\int_0^{\infty}\int_{\Omega}\Big((D_i^{l_0}E_j^{(0)})^2+(D_i^{l_0}\overline{E}_j^{(0)})^2\Big)dxdt\Big].$$

The proof is now complete. □

2.3. Existence of the first approximation step

Based on above *a priori* estimates, we are ready to prove the existence of the first approximation step by using semigroup theory arguments, see [36].

Proposition 2.1. *If the domain is the unbounded domain $\mathbb{R}^+\times\mathbb{R}^2$, let the parameter $\lambda>1$. If the domain is a bounded domain Ω, let the parameter $\lambda=1$ and $\frac{\gamma}{\varepsilon^2}<1$. Assume the initial approximation function $(u^{(0)},d^{(0)})$ satisfying (2.4)–(2.8). Then the linearized system*

$$\begin{cases} \mathcal{J}_1[u^0,d^{(0)}](\boldsymbol{h}^{(1)},\boldsymbol{w}^{(1)})=\Pi_{N_1}E^{(0)},\\ \mathcal{J}_2[u^0,d^{(0)}](\boldsymbol{h}^{(1)},\boldsymbol{w}^{(1)})=\Pi_{N_1}\overline{E}^{(0)},\\ \nabla\cdot\boldsymbol{h}^{(1)}=0,\\ \boldsymbol{h}^{(1)}(0,\lambda x)=0,\quad \boldsymbol{w}^{(1)}(0,\lambda x)=0, \end{cases}\tag{2.60}$$

and the boundary condition

$$\boldsymbol{h}^{(1)}(t,\lambda x)|_{x\in\partial\Omega}=0,\quad \boldsymbol{w}^{(1)}(t,\lambda x)|_{x\in\partial\Omega}=0,$$

admit a global solution

$$(\boldsymbol{h}^{(1)}(t,\lambda x),\boldsymbol{h}^{(1)}(t,\lambda x))\in\mathbb{C}((0,+\infty);H^s(\Omega)\times H^s(\Omega)).$$

Moreover, the global solution satisfies

$$\|\boldsymbol{h}^{(1)}\|_{H^s}^2+\|\boldsymbol{w}^{(1)}\|_{H^s}^2\lesssim\|\Pi_{N_1}E^{(0)}\|_{H^s}^2+\|\Pi_{N_1}\overline{E}^{(0)}\|_{H^s}^2,\quad \forall t>0.\tag{2.61}$$

Proof. The idea of proof is based on the semigroup theory. We first rewrite it as an evolution equation, then by the semigroup generator given in [36], we know the evolution operator generate a semigroup, a local existence can been shown. It combine with the dissipative energy estimate given in Lemma 2.3, the global existence of linear evolution is obtained. More precisely, let $\mathbb{P}$ be the Leray projector onto the space of divergence free functions. We apply the Leray projector to Eq (2.12). Therefore

$$\begin{aligned} &\mathbf{h}_t^{(1)}-\nu\lambda^2\mathbb{P}\Delta\mathbf{h}^{(1)}+\mathbb{N}_1(\mathbf{h}^{(1)},\mathbf{w}^{(1)})=\mathbb{P}\Pi_{N_1}E^{(0)},\\ &\mathbf{w}_t^{(1)}-\gamma\lambda^2\mathbb{P}\Delta\mathbf{w}^{(1)}+\mathbb{N}_2(\mathbf{h}^{(1)},\mathbf{w}^{(1)})=\mathbb{P}\Pi_{N_1}\overline{E}^{(0)}, \end{aligned}\tag{2.62}$$

where

$$\mathbb{N}_1(\mathbf{h}^{(1)},\mathbf{w}^{(1)}):=\mathbb{P}\Pi_{N_1}\Big[\lambda\Big(u^{(0)}\cdot\nabla\Big)\mathbf{h}^{(1)}+\Big(\mathbf{h}^{(1)}\cdot\nabla\Big)u^{(0)}$$
$$+\kappa\nabla\cdot(\nabla d^{(0)}\odot\lambda\nabla\mathbf{w}^{(1)})+\kappa\nabla\cdot(\lambda\nabla\mathbf{w}^{(1)}\odot\nabla d^{(0)})\Big].$$

and

$$\mathbb{N}_2(\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) := \mathbb{P}\Pi_{N_1}\Big[\lambda\Big(u^{(0)} \cdot \nabla\Big)\mathbf{w}^{(1)} + \Big(\mathbf{w}^{(1)} \cdot \nabla\Big)u^{(0)} \\ + \gamma\Big(|\nabla d^{(0)}|^2\mathbf{w}^{(1)} + 2\lambda(\nabla d^{(0)}, \nabla\mathbf{w}^{(1)})d^{(0)}\Big)\Big].$$

For convenience, we rewrite (2.62) as

$$\frac{d}{dt}\begin{pmatrix} \mathbf{h}^{(1)} \\ \mathbf{w}^{(1)} \end{pmatrix} + \begin{pmatrix} -\nu\lambda^2\mathbb{P}\triangle & 0 \\ 0 & -\gamma\lambda^2\mathbb{P}\triangle \end{pmatrix}\begin{pmatrix} \mathbf{h}^{(1)} \\ \mathbf{w}^{(1)} \end{pmatrix} + \begin{pmatrix} \mathbb{N}_1(\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) \\ \mathbb{N}_2(\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) \end{pmatrix} = \begin{pmatrix} \mathbb{P}\Pi_{N_1}E^{(0)} \\ \mathbb{P}\Pi_{N_1}\overline{E}^{(0)} \end{pmatrix}. \tag{2.63}$$

We notice that the term $\begin{pmatrix} \mathbb{N}_1(\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) \\ \mathbb{N}_2(\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) \end{pmatrix}$ can be seen as a bounded perturbation of the linear operator $ML := \begin{pmatrix} -\nu\lambda^2\mathbb{P}\triangle & 0 \\ 0 & -\gamma\lambda^2\mathbb{P}\triangle \end{pmatrix}$. We follow the idea of [36, 37] to show that the linear operator

$$\mathbb{Z} := ML + BP$$

can generate a strongly continuous semigroup $e^{\mathbb{Z}\tau}$ in Sobolev space $H^s(\Omega) \times H^s(\Omega)$ by the Lumer-Phillips theorem [38]. Hence linear Eq (2.60) has a solution in $\mathbb{C}((0, +\infty); H^s(\Omega) \times H^s(\Omega))$. Furthermore, it follows from Lemmas 2.2 and 2.3 that the estimate (2.61) holds. □

3. The *m*th approximation step

Let $\varepsilon \in (0, 1)$ be a fixed constant. We define

$$\mathcal{B}_\varepsilon := \{(u^{(k)}(t, \lambda x), d^{(k)}(t, \lambda x)) : \|u^{(k)}\|_{H^s} + \|d^{(k)}\|_{H^s} \lesssim \delta < 1\} \tag{3.1}$$

with the integer $2 \le k \le m - 1$ and the constant $s \ge 1$.

Assume that the m-th approximation solutions of (2.3) is denoted by $(\mathbf{h}^{(m)}(t, \lambda x), \mathbf{w}^{(m)}(t, \lambda x))$ with $m = 2, 3, \ldots$. Let

$$\mathbf{h}^{(m)}(t, \lambda x) := u^{(m)}(t, \lambda x) - u^{(m-1)}(t, \lambda x), \\ \mathbf{w}^{(m)}(t, \lambda x) := d^{(m)}(t, \lambda x) - d^{(m-1)}(t, \lambda x),$$

it holds

$$u^{(m)}(t, \lambda x) = u^{(0)}(t, x) + \mathbf{h}^{(1)}(t, \lambda x) + \sum_{i=2}^{m} \mathbf{h}^{(i)}(t, \lambda x), \\ d^{(m)}(t, \lambda x) = d^{(0)}(t, x) + \mathbf{w}^{(1)}(t, \lambda x) + \sum_{i=2}^{m} \mathbf{w}^{(i)}(t, \lambda x).$$

We linearize the nonlinear Eq (2.3) around $(u^{(m-1)}(t, \lambda x), d^{(m-1)}(t, \lambda x))$ to get the following initial value problem

$$\begin{cases} \mathcal{J}_1[u^{m-1}, d^{(m-1)}](\mathbf{h}^{(m)}, \mathbf{d}^{(m)}) = \Pi_{N_m}E^{(m-1)}, \\ \mathcal{J}_2[u^{m-1}, d^{(m-1)}](\mathbf{h}^{(m)}, \mathbf{d}^{(m)}) = \Pi_{N_m}\overline{E}^{(m-1)}, \\ \nabla \cdot \mathbf{h}^{(m)} = 0, \\ \mathbf{h}^{(m)}(0, \lambda x) = 0, \quad \mathbf{w}^{(m)}(0, \lambda x) = 0, \end{cases} \tag{3.2}$$

with the boundary conditions

$$\mathbf{h}^{(m)}(t,\lambda x)|_{x\in\partial\Omega}=0,\quad \mathbf{w}^{(m)}(t,\lambda x)|_{x\in\partial\Omega}=0 \tag{3.3}$$

where the error term is

$$\begin{aligned} E^{(m-1)} &:= \mathcal{L}_1[u^{m-1}(t,\lambda x),d^{m-1}(t,\lambda x)]=\mathcal{R}_1(\mathbf{h}^{(m)},\mathbf{w}^{(m)}),\\ \overline{E}^{(m-1)} &:= \mathcal{L}_2[u^{m-1}(t,\lambda x),d^{m-1}(t,\lambda x)]=\mathcal{R}_2(\mathbf{h}^{(m)},\mathbf{w}^{(m)}), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \mathcal{R}_1(\mathbf{h}^{(m)},\mathbf{w}^{(m)}) :=\ & \mathcal{L}_1(u^{(m-1)}+\mathbf{h}^{(m)},d^{(m-1)}+\mathbf{w}^{(m)})-\mathcal{L}_1(u^{(m-1)},d^{(m-1)})\\ &-\mathcal{J}_1[u^{(m-1)},d^{(m-1)}](\mathbf{h}^{(m)},\mathbf{d}^{(m)})\\ =\ & \Pi_{N_m}\Big(\lambda\mathbf{h}^{(m)}\cdot\nabla\mathbf{h}^{(m)}+\nabla P^{(m)}+\kappa\lambda^2\nabla\cdot(\nabla\mathbf{w}^{(m)}\odot\nabla\mathbf{w}^{(m)})\Big), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \mathcal{R}_2(\mathbf{h}^{(m)},\mathbf{w}^{(m)}) :=\ & \mathcal{L}_2(u^{(m-1)}+\mathbf{h}^{(m)},d^{(m-1)}+\mathbf{w}^{(m)})-\mathcal{L}_2(u^{(m-1)},d^{(m-1)})\\ &-\mathcal{J}_2[u^{(m-1)},d^{(m-1)}](\mathbf{h}^{(m)},\mathbf{d}^{(m)})\\ =\ & \lambda\Pi_{N_m}\Big(\mathbf{h}^{(m)}\cdot\nabla\mathbf{w}^{(m)}-\frac{\gamma}{\varepsilon^2}(|\mathbf{w}^{(m)}|^2-1)\mathbf{w}^{(m)}\Big), \end{aligned} \tag{3.6}$$

This is also the nonlinear term in the approximation problem (2.3) at $(u^{(m-1)}(t,\lambda x),d^{(m-1)}(t,\lambda x))$. Here the approximation pressure satisfies

$$\Delta P^{(m)}(t,x)=-\nabla\cdot\Big[u^{(m)}\cdot\nabla u^{(m)}+\kappa\nabla\cdot(\nabla d^{(m)}\odot\nabla d^{(m)})\Big].$$

The following result describes how to construct the m-th approximation solution.

Proposition 3.1. *If the domain is the unbounded domain $\mathbb{R}^+\times\mathbb{R}^2$, let the parameter $\lambda>1$. If the domain is a bounded domain Ω, let the parameter $\lambda=1$ and $\frac{\gamma}{\varepsilon^2}<1$. Assume the initial approximation function $(u^{(0)},d^{(0)})$ satisfying (2.4)–(2.8). $(u^{(m-1)},d^{(m-1)})\in\mathcal{B}_\varepsilon$ and $\sum_{i=1}^{m-1}\Big(\|\boldsymbol{h}^{(i)}\|^2_{H^s}+\|\boldsymbol{w}^{(i)}\|^2_{H^s}\Big)\lesssim\delta^2$. Then the linearized problem (3.2) with the boundary condition (3.3) admits a global solution*

$$(\boldsymbol{h}^{(m)}(t,\lambda x),\boldsymbol{w}^{(m)}(t,\lambda x))\in\mathbb{C}((0,+\infty);H^s(\Omega)\times H^s(\Omega)),$$

which satisfies

$$\|\boldsymbol{h}^{(m)}\|^2_{H^s}+\|\boldsymbol{w}^{(m)}\|^2_{H^s}\lesssim\|\Pi_{N_m}E^{(m-1)}\|^2_{H^s}+\|\Pi_{N_m}\overline{E}^{(m-1)}\|^2_{H^s},\quad\forall t>0, \tag{3.7}$$

where the error term verifies

$$\begin{aligned} \|E^{(m)}\|_{H^s}+\|\overline{E}^{(m)}\|_{H^s} &= \|\mathcal{R}_1(\boldsymbol{h}^{(m)},\boldsymbol{w}^{(m)})\|_{H^s}+\|\mathcal{R}_2(\boldsymbol{h}^{(m)},\boldsymbol{w}^{(m)})\|_{H^s}\\ &\lesssim N_m^3\Big(\|\boldsymbol{h}^{(m)}\|^2_{H^s}+\|\boldsymbol{w}^{(m)}\|^2_{H^s}+\|\boldsymbol{w}^{(m)}\|^4_{H^s}\Big). \end{aligned} \tag{3.8}$$

Proof. By direct computation we find

$$\begin{aligned} \partial_{x_i}u_j^{(m-1)}(t,x) &= \partial_{x_i}u_j^{(0)}(t,x)+\lambda\partial_{x_i}\mathbf{h}^{(1)}(t,\lambda x)+\lambda\sum_{i=2}^{m-1}\partial_{x_i}\mathbf{h}^{(i)}(t,\lambda x),\\ \partial_{x_i}d_j^{(m-1)}(t,x) &= \partial_{x_i}d_j^{(0)}(t,x)+\lambda\partial_{x_i}\mathbf{w}^{(1)}(t,\lambda x)+\lambda\sum_{i=2}^{m-1}\partial_{x_i}\mathbf{w}^{(i)}(t,\lambda x). \end{aligned} \tag{3.9}$$

By the assumption

$$\sum_{i=1}^{m-1}\left(\|\mathbf{h}^{(i)}\|_{H^s}^2+\|\mathbf{w}^{(i)}\|_{H^s}^2\right)\lesssim\delta^2,$$

for a fixed constant $\lambda > 1$, it follows that

$$\begin{aligned}\partial_{x_i}u_j^{(m-1)}(t,x)&\sim\partial_{x_i}u_j^{(0)}(t,x)+O(\delta^2),\\ \partial_{x_i}d_j^{(m-1)}(t,x)&\sim\partial_{x_i}d_j^{(0)}(t,x)+O(\delta^2).\end{aligned}$$

Thus, noticing that $(u^{(0)}(t,x), d^{(0)}(t,x))$ satisfies (2.4)–(2.8), by small modification of $\partial_{x_i}u_j^{(0)}(t,x)$ and $\partial_{x_i}d_j^{(0)}(t,x))$, it follows that

$$\sum_{k=0}^{s}\|\Pi_{N_m}\partial_{x_i}^k u_j^{(m-1)}(t,x)\|_{L^\infty}\lesssim\delta_0,\tag{3.10}$$

and

$$\sum_{k=0}^{s}\|\Pi_{N_m}\partial_{x_i}^k d_j^{(m-1)}(t,x)\|_{L^\infty}\lesssim\delta_0.\tag{3.11}$$

Moreover, we notice that the $(m-1)$-th approximation solution is

$$\begin{aligned}u^{(m-1)}(t,\varepsilon x)&=u^{(0)}(t,x)+\mathbf{h}^{(1)}(t,\lambda x)+\sum_{i=2}^{m-1}\mathbf{h}^{(i)}(t,\lambda x),\\ d^{(m-1)}(t,\varepsilon x)&=d^{(0)}(t,x)+\mathbf{w}^{(1)}(t,\lambda x)+\sum_{i=2}^{m-1}\mathbf{w}^{(i)}(t,\lambda x),\end{aligned}$$

and

$$\nabla\cdot\mathbf{h}^{(m-1)}=0.$$

We obtain

$$\begin{cases}\nabla\cdot u^{(m-1)}(t,x)=0,\\ u^{(m-1)}(0,x)=0,\\ \|u^{(m-1)}\|_{H^s}\lesssim\delta,\\ u^{(m-1)}(t,x)|_{x\in\partial\Omega}=0,\end{cases}$$

and

$$\begin{cases}d^{(m-1)}(0,x)=0,\\ \|d^{(m-1)}\|_{H^s}\lesssim\delta,\\ d^{(m-1)}(t,x)|_{x\in\partial\Omega}=0.\end{cases}$$

Then we will find the m-th ($m \geq 2$) approximation solution $(u^{(m)}(t,\lambda x), d^{(m)}(t,\lambda x))$, which is equivalent to find $(\mathbf{h}^{(m)}, \mathbf{w}^{(m)})$ such that

$$\begin{aligned}u^{(m)}(t,\lambda x)&=u^{(m-1)}(t,\lambda x)+\mathbf{h}^{(m)}(t,\lambda x),\\ d^{(m)}(t,\lambda x)&=d^{(m-1)}(t,\lambda x)+\mathbf{w}^{(m)}(t,\lambda x).\end{aligned}\tag{3.12}$$

Substituting (3.12) into (2.3), we have

$$\begin{aligned}\mathcal{L}_1(u^{(m)},d^{(m)})&=\mathcal{L}_1(u^{(m-1)},d^{(m-1)})+\mathcal{J}_1[u^{(m-1)},d^{(m-1)}](\mathbf{h}^{(m)},\mathbf{d}^{(m)})+\mathcal{R}_1(\mathbf{h}^{(m)},\mathbf{w}^{(m)}),\\ \mathcal{L}_2(u^{(m)},d^{(m)})&=\mathcal{L}_2(u^{(m-1)},d^{(m-1)})+\mathcal{J}_2[u^{(m-1)},d^{(m-1)}](\mathbf{h}^{(m)},\mathbf{d}^{(m)})+\mathcal{R}_2(\mathbf{h}^{(m)},\mathbf{w}^{(m)}).\end{aligned}$$

Set

$$\mathcal{J}_1[u^{(m-1)}, d^{(m-1)}](\mathbf{h}^{(m)}, \mathbf{d}^{(m)}) = -\mathcal{L}_1(u^{(m-1)}, d^{(m-1)}) = -E^{(m-1)},$$
$$\mathcal{J}_2[u^{(m-1)}, d^{(m-1)}](\mathbf{h}^{(m)}, \mathbf{d}^{(m)}) = -\mathcal{L}_2(u^{(m-1)}, d^{(m-1)}) = -\overline{E}^{(m-1)},$$

which we supplement with the boundary conditions (3.3).

Since we assume $(u^{(m-1)}(t, \lambda x), d^{(m-1)}(t, \lambda x)) \in \mathcal{B}_\varepsilon$, there is the same structure between the linear system (2.12) and the linear system of mth approximation solutions. Thus, by the same arguments as in the proof of Proposition 2.1, we can show that above problem admits a solution $(\mathbf{h}^{(m)}, \mathbf{w}^{(m)}) \in H^s(\Omega) \times H^s(\Omega)$. Here, we should use (2.2). Furthermore, similar to (2.61), we can use (3.9) and (3.10) to derive

$$\|\mathbf{h}^{(m)}\|^2_{H^s} + \|\mathbf{w}^{(m)}\|^2_{H^s} \lesssim \|E^{(m-1)}\|^2_{H^s} + \|\overline{E}^{(m-1)}\|^2_{H^s}, \quad \forall t > 0,$$

where one can see the $(m-1)$-th error term $(E^{(m-1)}, \overline{E}^{(m-1)})$ such that

$$E^{(m-1)} := \mathcal{L}_1(u^{(m-1)}, d^{(m-1)}) = \mathcal{R}_1(\mathbf{h}^{(m)}, \mathbf{w}^{(m)}),$$
$$\overline{E}^{(m-1)} := \mathcal{L}_2(u^{(m-1)}, d^{(m-1)}) = \mathcal{R}_2(\mathbf{h}^{(m)}, \mathbf{w}^{(m)}).$$

Moreover, by (3.5) and (3.6) and the standard Calderon-Zygmund theory, it follows that

$$\|E^{(m)}\|_{H^s} + \|\overline{E}^{(m)}\|_{H^s} \lesssim N_m^3\Big(\|\mathbf{h}^{(m)}\|^2_{H^s} + \|\mathbf{w}^{(m)}\|^2_{H^s} + \|\mathbf{w}^{(m)}\|^4_{H^s}\Big).$$

The proof is now complete. □

4. Convergence of the approximation scheme

Our target is to prove that the nonlinear Eq (1.1) admits global solutions

$$(u^{(\infty)}(t, \lambda x), d^{(\infty)}(t, \lambda x)).$$

This is equivalent to show that the series

$$\sum_{i=1}^{m} \mathbf{h}^{(i)}(t, \lambda x) < +\infty,$$
$$\sum_{i=1}^{m} \mathbf{w}^{(i)}(t, \lambda x) < +\infty.$$

For a fixed constant $s > 1$, let $1 < s = \bar{k} < k_0 \leq k$ and

$$k_m := \bar{k} + \frac{k - \bar{k}}{2^m}, \quad k_{+\infty} = \bar{k},$$
$$\alpha_{m+1} := k_m - k_{m+1} = \frac{k - \bar{k}}{2^{m+1}},$$

which yields

$$k_0 > k_1 > \ldots > k_m > k_{m+1} > \ldots. \tag{4.1}$$

Proposition 4.1. *If the domain is the unbounded domain $\mathbb{R}^+ \times \mathbb{R}^2$, let the parameter $\lambda > 1$. If the domain is a bounded domain Ω, let the parameter $\lambda = 1$ and $\frac{\gamma}{\varepsilon^2} < 1$. Assume the initial approximation function $(u^{(0)}, d^{(0)})$ satisfying (2.4)–(2.8). Then the three dimension nematic liquid crystal flow (1.1) with the small initial data (1.4) and the boundary condition (1.5) admit a global Sobolev solution*

$$u^{(\infty)}(t,x) = u^{(0)}(t,x) + \sum_{m=1}^{\infty} \boldsymbol{h}^{(m)}(t,\lambda x) + u_0(x) \in H^s(\Omega),$$
$$d^{(\infty)}(t,x) = d^{(0)}(t,x) + \sum_{m=1}^{\infty} \boldsymbol{w}^{(m)}(t,\lambda x) + d_0(x) \in H^s(\Omega).$$

Moreover, it holds the estimate

$$\|u^{(\infty)}\|_{H^s} + \|d^{(\infty)}\|_{H^s} \lesssim \delta.$$

Proof. The proof is based on the induction. For convenience, we first deal with the case of zero initial data, that is, $u_0(x) = \big(0,0,0\big)^T$ and $d_0(x) = \big(0,0,0\big)^T$. After that, we discuss the case of non-zero initial data. Note that $N_m = N_0^m$ with $N_0 > 1$. For all $m = 1, 2, \ldots$, we claim that there exists a sufficient small positive constant δ such that

$$\begin{aligned}
&\|\mathbf{h}^{(m)}\|_{H^{k_{m-1}}} + \|\mathbf{d}^{(m)}\|_{H^{k_{m-1}}} < \delta^{2^{m-1}},\\
&\|E^{(m)}\|_{H^{k_{m-1}}} + \|\overline{E}^{(m)}\|_{H^{k_{m-1}}} < \delta^{2^m},\\
&(u^{(m)}, d^{(m)}) \in \mathcal{B}_\delta.
\end{aligned} \tag{4.2}$$

For the case of $m = 1$, we recall the assumptions (2.4)–(2.8) on the initial approximation function $(u^{(0)}(t,x), d^{(0)}(t,x))$. By (2.61), letting $0 < \delta_0 < N_0^{-(8+k-\bar{k})}\delta^2 \ll 1$, we obtain

$$\|\mathbf{h}^{(1)}\|_{H^{k_0}} + \|\mathbf{w}^{(1)}\|_{H^{k_0}} \lesssim \|E^{(0)}\|_{H^{k_0}} + \|\overline{E}^{(0)}\|_{H^{k_0}} < \delta_0 < \delta^2.$$

Moreover, by (3.8) and the above estimate,

$$\begin{aligned}
\|E^{(1)}\|_{H^{k_0}} + \|\overline{E}^{(1)}\|_{H^{k_0}} &\lesssim \|\mathcal{R}_1(\mathbf{h}^1, \mathbf{w}^1)\|_{H^{k_0}} + \|\mathcal{R}_2(\mathbf{h}^1, \mathbf{w}^1)\|_{H^{k_0}}\\
&\lesssim N_1^2\Big(\|\mathbf{h}^{(1)}\|_{H^{k_0}}^2 + \|\mathbf{w}^{(1)}\|_{H^{k_0}}^2 + \|\mathbf{w}^{(1)}\|_{H^{k_0}}^4\Big)\\
&\lesssim N_1^2\Big(\|\mathbf{h}^{(1)}\|_{H^{k_0}} + \|\mathbf{w}^{(1)}\|_{H^{k_0}}\Big)^2\\
&\lesssim \delta^2,
\end{aligned}$$

and

$$\|u^{(1)}\|_{H^{k_0}} + \|d^{(1)}\|_{H^{k_0}} \lesssim \|u^{(0)}\|_{H^{k_0}} + \|d^{(0)}\|_{H^{k_0}} + \|\mathbf{h}^{(1)}\|_{H^{k_0}} + \|\mathbf{w}^{(1)}\|_{H^{k_0}} \lesssim \delta,$$

which means that $(u^{(1)}, d^{(1)}) \in \mathcal{B}_\delta$.

Assume that the case of $m-1$ holds, that is,

$$\begin{aligned}
&\|\mathbf{h}^{(m-1)}\|_{H^{k_m}} + \|\mathbf{w}^{(m-1)}\|_{H^{k_m}} < \delta^{2^{m-2}},\\
&\|E^{(m-1)}\|_{H^{k_m}} + \|\overline{E}^{(m-1)}\|_{H^{k_m}} < \delta^{2^{m-1}},\\
&(u^{(m-1)}, d^{(m-1)}) \in \mathcal{B}_\delta.
\end{aligned} \tag{4.3}$$

Then we prove that the case of m holds. Using (2.2), (3.7) and the second inequality of (4.3), we derive

$$\begin{aligned}\|\mathbf{h}^{(m)}\|_{H^{k_{m-1}}} + \|\mathbf{w}^{(m)}\|_{H^{k_{m-1}}} &\lesssim \|\Pi_{N_m} E^{(m-1)}\|_{H^{k_{m-1}}} + \|\Pi_{N_m} \overline{E}^{(m-1)}\|_{H^{k_{m-1}}} \\ &\lesssim N_m^{\alpha_m}\Big(\|E^{(m-1)}\|_{H^{k_m}} + \|\overline{E}^{(m-1)}\|_{H^{k_m}}\Big) \\ &< \delta^{2^{m-2}},\end{aligned} \tag{4.4}$$

which combined with (2.2), (3.8) and (4.1) yields

$$\begin{aligned}\|E^{(m)}\|_{H^{k_m}} + \|\overline{E}^{(m)}\|_{H^{k_m}} &\lesssim N_m^2\Big(\|\mathbf{h}^{(m)}\|^2_{H^{k_m}} + \|\mathbf{w}^{(m)}\|^2_{H^{k_m}} + \|\mathbf{w}^{(m)}\|^4_{H^{k_m}}\Big) \\ &\lesssim N_m^{2+\alpha_{m+1}}\Big(\|E^{(m-1)}\|_{H^{k_{m+1}}} + \|\overline{E}^{(m-1)}\|_{H^{k_{m+1}}}\Big)^2 \\ &\lesssim \dots, \\ &\lesssim \Big[N_0^{8+k-\bar{k}}\Big(\|E^{(0)}\|_{H^{k_{2m}}} + \|\overline{E}^{(0)}\|_{H^{k_{2m}}}\Big)\Big]^{2^m}.\end{aligned} \tag{4.5}$$

We choose a sufficiently small positive constant δ_0 such that

$$0 < N_0^{8+k-\bar{k}}\Big(\|E^{(0)}\|_{H^{k_{2m}}} + \|\overline{E}^{(0)}\|_{H^{k_{2m}}}\Big) < 2N_0^4\delta_0 < \delta^2.$$

Thus, by (4.5) we have

$$\|E^{(m)}\|_{H^{k_m}} + \|\overline{E}^{(m)}\|_{H^{k_m}} < \delta^{2^m},$$

and

$$\begin{aligned}0 &\le \lim_{m\to+\infty}\Big(\|E^{(m)}\|_{H^{k_m}} + \|\overline{E}^{(m)}\|_{H^{k_m}}\Big) \\ &\lesssim \Big[N_0^{8+k-\bar{k}}\Big(\|E^{(0)}\|_{H^{k_{+\infty}}} + \|\overline{E}^{(0)}\|_{H^{k_{+\infty}}}\Big)\Big]^{2^{+\infty}} \longrightarrow 0.\end{aligned}$$

So, the error term goes to 0 as $m \to \infty$, that is,

$$\lim_{m\to+\infty}(\|E^{(m)}\|_{H^{k_m}} + \|\overline{E}^{(m)}\|_{H^{k_m}}) = 0.$$

On the other hand, note that $N_m = N_0^m$, by (4.3) and (4.4). It follows that

$$\begin{aligned}\|u^{(m)}\|_{H^{k_m}} + \|d^{(m)}\|_{H^{k_m}} &\lesssim \|u^{(m-1)}\|_{H^{k_m}} + \|\mathbf{h}^{(m)}\|_{H^{k_m}} + \|d^{(m-1)}\|_{H^{k_m}} + \|\mathbf{w}^{(m)}\|_{H^{k_m}} \\ &\lesssim \delta + N_m^3\delta^{2^m} \lesssim \delta.\end{aligned}$$

This means that $(u^{(m)}, d^{(m)}) \in \mathcal{B}_\delta$. Hence we conclude that (4.2) holds.

Therefore, the three dimension nematic liquid crystal flow (1.1) with the zero initial data admit global solutions

$$u^{(\infty)}(t,x) = u^{(0)}(t,x) + \sum_{m=1}^{\infty}\mathbf{h}^{(m)}(t,\lambda x) + u_0(x) = u^{(0)}(t,x) + u_0(x) + O(\delta^2),$$

$$d^{(\infty)}(t,x) = d^{(0)}(t,x) + \sum_{m=1}^{\infty}\mathbf{w}^{(m)}(t,\lambda x) + d_0(x) = d^{(0)}(t,x) + d_0(x) + O(\delta^2).$$

Next, we discuss the case of small non-zero initial data.

We introduce the auxiliary function

$$\overline{u}(t,x) = u(t,x) - u_0(x), \quad x \in \Omega,$$
$$\overline{d}(t,x) = d(t,x) - d_0(x), \quad x \in \Omega,$$

Thus, the initial data reduces to

$$\overline{u}(0,x) = \Big(0,0,0\Big), \quad \overline{d}(0,x) = \Big(0,0,0\Big),$$

and the boundary condition (1.5) is changed into the Dirichlet boundary condition

$$\overline{u}(t,x)|_{x\in\partial\Omega} = 0, \quad \overline{d}(t,x)|_{x\in\partial\Omega} = 0,$$

and Eq (1.1) are transformed into equations of $(\overline{u}, \overline{d})$.

Thus, we can follow the above iteration scheme to construct a global Sobolev solution $(\overline{u}, \overline{d})$. Furthermore, the global Sobolev solution of Eq (1.1) with a small non-zero initial data takes the form $(\overline{u}(t,x) + u_0, \overline{d}(t,x) + d_0)$. We notice that by the auxiliary function, the new first equation contain the non-autonomous force:

$$f(u_0, d_0) := -\nu\Delta U_0 + (U_0\cdot\nabla)U_0 + \nabla P_0 - \kappa\nabla\cdot(\nabla d_0 \odot \nabla d_0),$$

and the new second equation contain the non-autonomous force:

$$g(u_0, d_0) := -\gamma\Delta d_0 + u_0\cdot\nabla d_0 - \frac{\gamma}{\varepsilon^2}(|d_0|^2 - 1)d_0.$$

Thus, we have to require the initial data $(u_0, d_0) \in H^{s+2}(\Omega)\times H^{s+2}(\Omega)$ to be small and $\nabla\cdot u_0 = 0$.

Finally, using (1.2), by the standard Calderon-Zygmund theory for the Riesz operator $\mathcal{R}$, we have $\|\mathcal{R}u\|_{\mathbb{L}^{s_0}} \leq \|u\|_{\mathbb{L}^{s_0}}$ with $1 < s_0 < \infty$, hence

$$\|P\|_{H^s} \lesssim \delta.$$

This completes the proof. □

Conflict of interest

The authors declare that they have no competing interests.

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