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Theory article

Blowup for regular solutions and C^1 solutions of the two-phase model in $\mathbb{R}^N$ with a free boundary

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Abstract: In this paper, under the assumption of an initial bounded region $\Omega(0)$, we establish the blowup phenomenon of the regular solutions and C^1 solutions to the two-phase model in $\mathbb{R}^N$. If the total energy E and the total mass $M > 0$ satisfy

$$\max_{\vec{x}_0\in\partial\Omega(0)} \sum_{i=1}^{N} u_i^2(0,\vec{x}_0) < \frac{\min\{2, N(\Gamma-1), N(\gamma-1)\}E}{M},$$

where $E = \int_{\Omega(0)} \left(\frac{1}{2}n\left|\vec{u}\right|^2 + \frac{1}{2}\rho\left|\vec{u}\right|^2 + \frac{1}{\Gamma-1}n^\Gamma + \frac{1}{\gamma-1}\rho^\gamma\right)dV$ and $M = \int_{\Omega(0)}(n+\rho)dV > 0$, then the blowup of the solutions to the two-phase model will be formed in finite time in $\mathbb{R}^N$. Furthermore, under the assumptions that the radially symmetric initial data and initial density contain vacuum states, the blowup of the smooth solutions to the two-phase model will be formed in finite time in $\mathbb{R}^N (N \geq 2)$.

Keywords: two-phase model; singularity formation; free boundary problems; regular solutions; vacuum
Mathematics Subject Classification: 35B44, 35R35, 76T10

1. Introduction

The compressible inviscid liquid-gas two-phase model in $\mathbb{R}^N$ that will be considered is as follows:

$$\begin{cases} n_t + div(n\vec{u}) = 0, \\ \rho_t + div(\rho\vec{u}) = 0, \\ [(\rho + n)\vec{u}]_t + div[(\rho + n)\vec{u} \otimes \vec{u}] + \nabla P(n, \rho) = 0, \end{cases} \tag{1.1}$$

where $\vec{u} = \vec{u}(t, \vec{x})$, $P = P(n, \rho) = n^\Gamma + \rho^\gamma$ are the velocity and pressure while $n = n(t, \vec{x}) \geq 0$, $\rho = \rho(t, \vec{x}) \geq 0$ are the densities of two phases, where $\gamma, \Gamma > 1$.

Euler equations have been used as one of the basic models for studying fluids, plasmas, atmospheric dynamics, and condensed matter in [2, 5–7, 11, 30], and the blowup analysis for these equations is carried out in [3, 4, 12, 17, 21, 25, 26, 28, 34, 36]. In 1985, Sideris [25] constructed the functional

$$F(0) = \int_{\mathbb{R}^3} \vec{x} \cdot \rho\vec{u} d\vec{x} \tag{1.2}$$

to prove that the C^1 solutions of the three-dimensional compressible Euler equations will blow up in a finite time when the initial functional $F(0)$ is sufficiently large. Yuen [35] used the energy method to consider the blowup results of the C^1 solutions and the weakened regular solutions of the Euler equations in $\mathbb{R}^N$. In [18], the authors rewrote the system in the form of a quasilinear wave equation about the density ρ to study the blowup of solutions to Euler equations. Liu, Wang and Yuen studied the blowup results of solutions to the compressible Euler equations with time-dependent damping with vacuum and C^1 solutions of the irrotational compressible Euler equations with time-dependent damping in [19]. In [24], the author studied the three-dimensional Euler equations with a free boundary subjected to tension.

In the two-phase fluid, because of its wide application in aerospace, micro-technology, chemical engineering and other fields, it has aroused many researchers' interest. Zuber studied the two-phase model firstly in [39]. The motion of liquid and gas mixture is studied by the two-phase model in [16]. In [23], the authors considered a hyperbolic two-phase model. For existence, asymptotic and uniqueness of global weak solutions to the two-phase flow model with vacuum, Yao, Zhang and Zhu used the line method and a priori estimate to obtain relevant results in [31–33], and there are also studies of these issues in [1, 8, 13–15, 27]. In [37], under the assumption of H^2-norm of the initial perturbation with a constant state is sufficiently small and L^1-norm is bounded, Zhang and Zhu studied the global existence of Cauchy problem to viscous liquid-gas two-phase flow in three dimensions. Furthermore, for the Cauchy problem of 3D inviscid liquid-gas two-phase flow, Zhang considered the optimal $L^P - L^2$ $(1 \leq p < \frac{6}{5})$ time decay rates of the solutions with the damping on the qualitative behaviors in [38]. Wen and Zhu [29] considered global existence of weak solutions to two-fluid about the Dirichlet problem in one dimension. In [10], Dong et al. considered the energy integration method to prove the singularity of the smooth solutions to the Cauchy problem for the viscous two-phase model in arbitrary dimensions. Furthermore, Dong and Yuen provided the blowup phenomena of self-similar solutions for the inviscid liquis-gas two-phase flow [9]. By introducing the definition of regular solutions, Makino and Perthame studied the blowup phenomena of radical symmetric solutions to the Euler-Poisson equations with compact support and with a repulsive force and an attractive force in [20, 22].

2. Materials and methods

In this paper, we study the blowup results of regular solutions and C^1 solutions of the two-phase model in $\mathbb{R}^N$ with a free boundary. Our method depends on the energy integration method and a quasi-linear wave equation about ρ, the singularity of the two-phase model will form in finite time.

3. Results

3.1. Main theorems

In this section, we give the blowup results of regular solutions and C^1 solutions of the two-phase model in $\mathbb{R}^N$ with a free boundary. Firstly, we introduce the definition of regular solutions to the two-phase (1.1).

Definiton 3.1. *(Weakened regular solution) A solution $(n, \rho, \vec{u})$ to the system (1.1) which is in $\mathbb{R}^N$ and in the non-vacuum region is regular for $1 < \Gamma < 2$ or $1 < \gamma < 2$, if*

$$\left(n^{\frac{\Gamma-1}{\eta}}, \rho^{\frac{\gamma-1}{\eta}}, \vec{u}\right) \in C^1, \tag{3.1}$$

with a fixed constant $\eta > 1$.

In the following, we state the blowup results of the two-phase model with a free boundary in $\mathbb{R}^N$.

Theorem 3.1. *Assume that the fluid enters a bounded open region $\Omega(t) \subsetneq \mathbb{R}^N$, with the contacting vacuum boundary $\partial\Omega(t)$. Suppose that the $(n, \rho, \vec{u})$ is a weakened regular solution on $[0, T) \times \Omega(t)$ of the two-phase model (1.1) in $\mathbb{R}^N$. If*

$$\max_{\vec{x_0} \in \partial\Omega(0)} \sum_{i=1}^{N} u_i^2(0, \vec{x_0}) < \frac{\min\{2, N(\Gamma - 1), N(\gamma - 1)\}E}{M}, \tag{3.2}$$

where the total energy $E = \int_{\Omega(0)} \left(\frac{1}{2} n \left|\vec{u}\right|^2 + \frac{1}{2}\rho \left|\vec{u}\right|^2 + \frac{1}{\Gamma-1} n^\Gamma + \frac{1}{\gamma-1}\rho^\gamma\right) dV$ and the total mass $M = \int_{\Omega(0)} (n + \rho) dV > 0$, then the solutions of the two-phase model will blow up in finite time T.

Furthermore, we consider the two-phase model (1.1) with the pressure $P = P(n, \rho) = n^\gamma + \rho^\gamma$. In this paper, we will rewrite the two-phase model (1.1) by forming a quasi-linear wave equation about the density ρ and studying the blowup results for solutions of the two-phase model in $\mathbb{R}^N (N \geq 2)$ with the initial data

$$t = 0 : n = n_0(\vec{x}), \;\; \rho = \rho_0(\vec{x}), \;\; \vec{u} = \vec{u}_0(\vec{x}). \tag{3.3}$$

Introducing radial symmetry, the initial data become

$$n_0(\vec{x}) = n_0(r), \quad \rho_0(\vec{x}) = \rho_0(r), \quad \vec{u}_0(\vec{x}) = \frac{\vec{x}}{r} V_0(r), \tag{3.4}$$

where $r = (\sum\limits_{i=1}^{N} x_i^2)^{\frac{1}{2}}$.

In the following, we give the blowup results of the two-phase model in $\mathbb{R}^N (N \geq 3)$.

Theorem 3.2. *Assume that the initial density $n_0 \geq 0$ and $\rho_0 \geq 0$ in (3.3), and $(n_0, \rho_0, \vec{u_0}) \in H^3(\mathbb{R}^N)$. Consider the solutions $(n, \rho, \vec{u})$ of the two-phase model (1.1) in $\mathbb{R}^N (N \geq 3)$. If the initial conditions*

$$n_0(0) = 0, \quad \rho_0(0) = 0, \tag{3.5}$$

$$\int_{\mathbb{R}^N} (n_0+\rho_0)(r)dx > 0, \tag{3.6}$$

and

$$-\int_{\mathbb{R}^N} \frac{(1+r)(n_0+\rho_0)V_0}{r^2e^r}dx \geq \sqrt{\frac{1}{2^{\gamma-1}(1+\gamma)}\left[\frac{\Gamma(1+\frac{N}{2})}{(N-2)!\pi^{\frac{N}{2}}}\right]^{\gamma-1}}\left(\int_{\mathbb{R}^N}\frac{(n_0+\rho_0)}{re^r}dx\right)^{\frac{\gamma+1}{2}} \tag{3.7}$$

are satisfied, where $\frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}$ *is the volume of the unit ball in* $\mathbb{R}^N$*, then the solutions will blow up on or before the finite time* $\frac{2H(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_0}$*, where* $H(0)=\int_{\mathbb{R}^N}\frac{(n_0+\rho_0)}{re^r}dx$ *and* $C_0=\sqrt{\frac{1}{2^{\gamma-1}(1+\gamma)}\left[\frac{\Gamma(1+\frac{N}{2})}{(N-2)!\pi^{\frac{N}{2}}}\right]^{\gamma-1}}$.

Then, under the appropriate assumptions, we will give the blowup results of the two-phase model in the two dimensional cases.

Theorem 3.3. *Assume that the initial density* $n_0 \geq 0, \rho_0 \geq 0$ *in (3.3), and* $(n_0,\rho_0,\vec{u_0}) \in H^3(\mathbb{R}^2)$*. Let* $K_0(r)$ *be the modified Bessel function*

$$K_0(r) = \int_0^\infty e^{-rcosht}dt. \tag{3.8}$$

Consider the solution $(n,\rho,\vec{u})$ *of the two-phase model (1.1) in* $\mathbb{R}^2$*. If the initial conditions*

$$n_0(0)=0, \quad \rho_0(0)=0, \tag{3.9}$$

$$\int_{\mathbb{R}^2}(n_0+\rho_0)(r)dx > 0, \tag{3.10}$$

and

$$\int_{\mathbb{R}^2}(n_0+\rho_0)V_0K_0'(r)dx \geq \sqrt{\frac{1}{2^{\gamma-1}(1+\gamma)}}\frac{\left(\int_{\mathbb{R}^2}(n_0+\rho_0)(r)K_0(r)dx\right)^{\frac{\gamma+1}{2}}}{\left(\int_{\mathbb{R}^2}K_0(r)dx\right)^{\frac{\gamma-1}{2}}} \tag{3.11}$$

are satisfied, then the solutions will blow up on or before the finite time $\frac{2G(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_1}$*, where* $G(0)=\int_{\mathbb{R}^2}(n_0+\rho_0)K_0(r)dx$ *and* $C_1=\sqrt{\frac{1}{2^{\gamma-1}(\gamma+1)}}\left(\int_{\mathbb{R}^2}K_0(r)dx\right)^{-\frac{\gamma-1}{2}}$.

Remark 3.1. *For the two-phase model in* $\mathbb{R}^N$ *with pressure* $P=P(n,\rho)=(n+\rho)^\gamma$*, we can also obtain the same blowup results.*

3.2. Blowup for the two-phase model with a free boundary

In this section, we will show the proof of the blowup result of solutions to the two-phase model in $\mathbb{R}^N$. Firstly, we give some lemmas for the conserved energy and the properties of the local second inertia function to the two-phase model.

Lemma 3.1. *Assume the density n and ρ have compact support in the region* $\Omega(t)$ *for non-trivial* C^1 *solutions of the two-phase model (1.1) in* $\mathbb{R}^N$*, where* $\Omega(t)$ *is a moving region, and the region* $\Omega(0)$ *is bounded by the contacting vacuum boundary, then*

$$M(t)=M(0), \tag{3.12}$$

and

$$0 < E = E(t) = \int_{\Omega(t)} \left(\frac{1}{2} n \left| \vec{u} \right|^2 + \frac{1}{2} \rho \left| \vec{u} \right|^2 + \frac{1}{\Gamma - 1} n^{\Gamma} + \frac{1}{\gamma - 1} \rho^{\gamma} \right) dV < \infty, \tag{3.13}$$

that is, the total energy energy $E(t)$ is conserved, where $M(t) = \int_{\Omega(t)} (n+\rho) dV$ represents the total mass.

Proof. By $(1.1)_1$ and $(1.1)_2$, we obtain

$$\frac{d}{dt} M(t) = \int_{\Omega(t)} (n_t + \rho_t) dV = -\int_{\Omega(t)} div(n\vec{u}) + div(\rho\vec{u}) dV = 0. \tag{3.14}$$

Therefore, $M(t) = M(0)$.

Multiplying $\vec{u}$ on both sides of the Eq $(1.1)_3$ and integrating over $\Omega(t)$, we obtain

$$\begin{aligned} \int_{\Omega(t)} (n\vec{u})_t \cdot \vec{u} dV + \int_{\Omega(t)} div(n\vec{u} \otimes \vec{u}) \cdot \vec{u} dx &+ \int_{\Omega(t)} (\rho\vec{u})_t \cdot \vec{u} dV + \int_{\Omega(t)} div(\rho\vec{u} \otimes \vec{u}) \cdot \vec{u} dV \\ &+ \int_{\Omega(t)} \nabla P(n, \rho) \cdot \vec{u} dV = 0. \end{aligned} \tag{3.15}$$

On the one hand, from $(1.1)_1$, we have

$$-\frac{1}{2} n_t \left| \vec{u} \right|^2 - \frac{1}{2} \left| \vec{u} \right|^2 div(n\vec{u}) = 0. \tag{3.16}$$

Due to

$$\left(\frac{1}{2} n \left| \vec{u} \right|^2 \right)_t = (n\vec{u})_t \cdot \vec{u} - \frac{1}{2} n_t \left| \vec{u} \right|^2, \tag{3.17}$$

we obtain

$$\begin{aligned} & \int_{\Omega(t)} (n\vec{u})_t \cdot \vec{u} dx + \int_{\Omega(t)} div(n\vec{u} \otimes \vec{u}) \cdot \vec{u} dV \\ = & \int_{\Omega(t)} \left(\frac{1}{2} n \left| \vec{u} \right|^2 \right)_t dV + \int_{\Omega(t)} \frac{1}{2} n_t \left| \vec{u} \right|^2 dV + \int_{\Omega(t)} \vec{u} \cdot \left[\nabla \cdot (n\vec{u} \otimes \vec{u}) \right] dV \\ = & \int_{\Omega(t)} \left(\frac{1}{2} n \left| \vec{u} \right|^2 \right)_t dV - \int_{\Omega(t)} \frac{1}{2} \left| \vec{u} \right|^2 \nabla \cdot (n\vec{u}) dV + \int_{\Omega(t)} \vec{u} \cdot \left[\nabla \cdot (n\vec{u} \otimes \vec{u}) \right] dV. \end{aligned} \tag{3.18}$$

We note that

$$\vec{u} \cdot \left[\nabla \cdot (n\vec{u} \otimes \vec{u}) \right] = \sum_{i,\, j=1}^{N} u_i \left[\partial_j \left(\rho u_j \right) u_i + n u_j \partial_j u_i \right] = \left| \vec{u} \right|^2 \sum_{i=1}^{N} \partial_i \left(\rho u_i \right) + \sum_{i,\, j=1}^{N} n u_i u_j \partial_j u_i. \tag{3.19}$$

Thus, we get

$$\begin{aligned} & \int_{\Omega(t)} \left[-\frac{1}{2} \left| \vec{u} \right|^2 \nabla \cdot (n\vec{u}) + \vec{u} \cdot \left[\nabla \cdot (n\vec{u} \otimes \vec{u}) \right] \right] dV \\ = & \int_{\Omega(t)} \left[-\frac{1}{2} \left| \vec{u} \right|^2 \sum_{i=1}^{N} \partial_i \left(n u_i \right) + \left| \vec{u} \right|^2 \sum_{i=1}^{N} \partial_i \left(n u_i \right) \sum_{i,\, j=1}^{N} n u_i u_j \partial_j u_i \right] dV \end{aligned}$$

$$= \int_{\Omega(t)} \left[\frac{1}{2} \left|\vec{u}\right|^2 \sum_{i=1}^{N} \partial_i (nu_i) + \sum_{i,\, j=1}^{N} nu_i u_j \partial_j u_i \right] dV$$
$$= \int_{\Omega(t)} \left[-\sum_{i=1}^{N} \vec{u} \cdot \partial_i \vec{u} nu_i + \sum_{i,\, j=1}^{N} nu_i u_j \partial_j u_i \right] dV$$
$$= \int_{\Omega(t)} \left[-\sum_{i,\, j=1}^{N} nu_i u_j \partial_i u_j + \sum_{i,\, j=1}^{N} nu_i u_j \partial_j u_i \right] dV$$
$$= 0. \tag{3.20}$$

Therefore, we obtain

$$\int_{\Omega(t)} (n\vec{u})_t \cdot \vec{u} dV + \int_{\Omega(t)} div(n\vec{u} \otimes \vec{u}) \cdot \vec{u} dV = \int_{\Omega(t)} \left(\frac{1}{2} n \left|\vec{u}\right|^2 \right)_t dV. \tag{3.21}$$

Similarly, we get

$$\int_{\Omega(t)} (\rho\vec{u})_t \cdot \vec{u} dV + \int_{\Omega(t)} div(\rho\vec{u} \otimes \vec{u}) \cdot \vec{u} dV = \int_{\Omega(t)} \left(\frac{1}{2} \rho \left|\vec{u}\right|^2 \right)_t dV. \tag{3.22}$$

On the other hand,

$$\begin{aligned} P_t &= \Gamma\, n^{\Gamma-1} \partial_t n + \gamma \rho^{\gamma-1} \partial_t \rho \\ &= \Gamma\, n^{\Gamma-1} \left[-\sum_{i=1}^{N} \partial_i (nu_i) \right] + \gamma \rho^{\gamma-1} \left[-\sum_{i=1}^{N} \partial_i (\rho u_i) \right] \\ &= -\sum_{i=1}^{N} \partial_i (n^\Gamma) u_i - \sum_{i=1}^{N} \Gamma n^\Gamma \partial_i u_i - \sum_{i=1}^{N} \partial_i (\rho^\gamma) u_i - \sum_{i=1}^{N} \gamma \rho^\gamma \partial_i u_i. \end{aligned} \tag{3.23}$$

Integrating over the region $\Omega(t)$ and applying the integration by parts, we obtain

$$\begin{aligned} \int_{\Omega(t)} P_t dV &= -\int_{\Omega(t)} \sum_{i=1}^{N} \partial_i (n^\Gamma) u_i dV - \int_{\Omega(t)} \sum_{i=1}^{N} \Gamma n^\Gamma \partial_i u_i dV - \int_{\Omega(t)} \sum_{i=1}^{N} \partial_i (\rho^\gamma) u_i dV - \int_{\Omega(t)} \sum_{i=1}^{N} \gamma \rho^\gamma \partial_i u_i dV \\ &= -\int_{\Omega(t)} \sum_{i=1}^{N} \partial_i (n^\Gamma) u_i dV + \int_{\Omega(t)} \sum_{i=1}^{N} \Gamma \partial_i (n^\Gamma) u_i dV - \int_{\Omega(t)} \sum_{i=1}^{N} \partial_i (\rho^\gamma) u_i dV - \int_{\Omega(t)} \sum_{i=1}^{N} \gamma \partial_i (\rho^\gamma) u_i dV \\ &= \int_{\Omega(t)} \sum_{i=1}^{N} (\Gamma - 1) \partial_i (n^\Gamma) u_i dV + \int_{\Omega(t)} \sum_{i=1}^{N} (\gamma - 1) \partial_i (\rho^\gamma) u_i dV. \end{aligned} \tag{3.24}$$

Therefore, we have

$$\int_{\Omega(t)} \vec{u} \cdot \nabla P(n, \rho) dV = \frac{1}{\Gamma - 1} \int_{\Omega(t)} \partial_t (n^\Gamma) dV + \frac{1}{\gamma - 1} \int_{\Omega(t)} \partial_t (\rho^\gamma) dV. \tag{3.25}$$

Thus, by (3.15), we obtain

$$\frac{d}{dt} E(t) = \frac{d}{dt} \left(\int_{\Omega(t)} \frac{1}{2} n \left|\vec{u}\right|^2 + \frac{1}{2} \rho \left|\vec{u}\right|^2 + \frac{1}{\Gamma - 1} n^\Gamma + \frac{1}{\gamma - 1} \rho^\gamma dV \right)$$

$$
\begin{aligned}
&= \int_{\Omega(t)} (n\vec{u})_t \cdot \vec{u} dV + \int_{\Omega(t)} div(n\vec{u}\otimes\vec{u})\cdot\vec{u}dx + \int_{\Omega(t)} (\rho\vec{u})_t \cdot \vec{u}dV \\
&+ \int_{\Omega(t)} div(\rho\vec{u}\otimes\vec{u})\cdot\vec{u}dV + \int_{\Omega(t)} \nabla P(n,\rho)\cdot\vec{u}dV = 0.
\end{aligned} \tag{3.26}
$$

Therefore, for non-trivial C^1 solutions, we obtain

$$
E(t) = E(0) < \infty. \tag{3.27}
$$

The proof is complete. □

Next, we show the second derivative of the local second inertia function for the two-phase model (1.1) in $\mathbb{R}^N$ for the solutions on $[0,T)\times\Omega(t)$.

Before stating the following lemma, we first give some physical quantities as follows:

$$
F(t) = \int_{\Omega(t)} n\vec{u}\cdot x dV + \int_{\Omega(t)} \rho\vec{u}\cdot x dV = F_n(t) + F_\rho(t), \tag{3.28}
$$

$$
H(t) = \frac{1}{2}\int_{\Omega(t)} n\left|\vec{x}\right|^2 dV + \frac{1}{2}\int_{\Omega(t)} \rho\left|\vec{x}\right|^2 dV = H_n(t) + H_\rho(t), \tag{3.29}
$$

and

$$
\begin{aligned}
E(t) &= \int_{\Omega(t)} \frac{1}{2} n\left|\vec{u}\right|^2 dV + \int_{\Omega(t)} \frac{1}{2}\rho\left|\vec{u}\right|^2 dV + \frac{1}{\Gamma-1}\int_{\Omega(t)} n^\Gamma dV + \frac{1}{\gamma-1}\int_{\Omega(t)} \rho^\gamma dV \\
&= E_{kn}(t) + E_{k\rho}(t) + E_{in}(t) + E_{i\rho}(t),
\end{aligned} \tag{3.30}
$$

where $F_n(t)$ and $F_\rho(t)$, $H_n(t)$ and $H_\rho(t)$, $E_{kn}(t)$ and $E_{k\rho}(t)$, $E_{in}(t)$ and $E_{i\rho}(t)$ represent the momentum weight, the local second inertia, the kinetic energy and the internal energy for the two-phase fluid, respectively.

Lemma 3.2. *For the two-phase model (1.1) in $\mathbb{R}^N$, we have*

$$
H'(t) = F(t), \tag{3.31}
$$

and

$$
H''(t) = 2E_{kn}(t) + 2E_{k\rho}(t) + N(\Gamma-1)E_{in}(t) + N(\gamma-1)E_{i\rho}(t). \tag{3.32}
$$

Proof. Differentiating $H(t)$ with respect to t, we obtain

$$
H'(t) = \frac{1}{2}\int_{\Omega(t)} n_t\left|\vec{x}\right|^2 dV + \frac{1}{2}\int_{\Omega(t)} \rho_t\left|\vec{x}\right|^2 dV = H_n'(t) + H_\rho'(t). \tag{3.33}
$$

Applying the integration by parts to (3.33) and using $(1.1)_1$, we get

$$
H_n'(t) = -\frac{1}{2}\int_{\Omega(t)} div(n\vec{u})\left|\vec{x}\right|^2 dV = -\frac{1}{2}\int_{\Omega(t)} \sum_{i,j}^{N} \partial_i(nu_i)x_j^2 dV
$$

$$= \frac{1}{2}\int_{\Omega(t)}\sum_{i,j}^{N} 2nu_i x_j dV = \int_{\Omega(t)}\sum_{i}^{N} nu_i x_i dV$$
$$= \int_{\Omega(t)} \vec{x}\cdot n\vec{u} dV = F_n(t). \tag{3.34}$$

Similarly, we have

$$H_\rho^{'}(t) = F_\rho(t). \tag{3.35}$$

Therefore, we obtain

$$H^{'}(t) = F(t). \tag{3.36}$$

Next, we calculate $H^{''}(t)$ as follows:

$$\begin{aligned} H^{''}(t) &= F^{'}(t) = \int_{\Omega(t)} (n\vec{u})_t \cdot \vec{x} dV + \int_{\Omega(t)} (\rho\vec{u})_t \cdot \vec{x} dV \\ &= -\int_{\Omega(t)} div(n\vec{u}\otimes\vec{u})\cdot\vec{x}dV - \int_{\Omega(t)} div(\rho\vec{u}\otimes\vec{u})\cdot\vec{x}dV - \int_{\Omega(t)} \nabla P(n,\rho)\cdot\vec{x}dV. \end{aligned} \tag{3.37}$$

By using the integration by parts, we get

$$\begin{aligned} \int_{\Omega(t)} div(n\vec{u}\otimes\vec{u})\cdot\vec{x}dV &= \int_{\Omega(t)}\sum_{i,j}^{N} \partial_i(nu_iu_j)x_j dV = -\int_{\Omega(t)}\sum_{i,j}^{N} nu_iu_j\partial_i x_j dV \\ &= -\int_{\Omega(t)} n\vec{u}\cdot\vec{u}dV = -\int_{\Omega(t)} n\left|\vec{u}\right|^2 dV. \end{aligned} \tag{3.38}$$

Similarly, we have

$$\int_{\Omega(t)} div(\rho\vec{u}\otimes\vec{u})\cdot\vec{x}dV = -\int_{\Omega(t)} \rho\left|\vec{u}\right|^2 dV, \tag{3.39}$$

and

$$\int_{\Omega(t)} \nabla P(n,\rho)\cdot\vec{x}dV = -N\int_{\Omega(t)} n^\Gamma dV - N\int_{\Omega(t)} \rho^\gamma dV. \tag{3.40}$$

From (3.38) to (3.40) and Gauss formula, we obtain

$$\begin{aligned} H^{''}(t) &= F^{'}(t) = \int_{\Omega(t)} n\left|\vec{u}\right|^2 dV + \int_{\Omega(t)} \rho\left|\vec{u}\right|^2 dV + N\int_{\Omega(t)} n^\Gamma dV + N\int_{\Omega(t)} \rho^\gamma dV \\ &= 2E_{kn}(t) + 2E_{k\rho}(t) + N(\Gamma-1)E_{in}(t) + N(\gamma-1)E_{i\rho}(t). \end{aligned} \tag{3.41}$$

The proof is complete. □

Using the above lemmas, we will give the proof of the blowup results of the solutions to the two-phase model with a free boundary in $\mathbb{R}^N$.

Proof of Theorem 3.1. Introducing

$$\varphi = n^{\frac{\Gamma-1}{\eta}}, \omega = \rho^{\frac{\gamma-1}{\eta}}, \tag{3.42}$$

for $n(t,x)\neq 0$ and $\rho(t,x)\neq 0$, we transform the momentum equations (1.1) into

$$(\varphi^{\frac{\eta}{\Gamma-1}}+\omega^{\frac{\eta}{\gamma-1}})\left(\frac{\partial u_i}{\partial t}+\sum_{l=1}^{N}u_l\frac{\partial u_i}{\partial x_l}\right)+\frac{\partial}{\partial x_i}(\varphi^{\frac{\eta\Gamma}{\Gamma-1}}+\omega^{\frac{\eta\gamma}{\gamma-1}})=0. \tag{3.43}$$

Simplifying the above equation, we have

$$(\varphi^{\frac{\eta}{\Gamma-1}}+\omega^{\frac{\eta}{\gamma-1}})\left(\frac{\partial u_i}{\partial t}+\sum_{l=1}^{N}u_l\frac{\partial u_i}{\partial x_l}\right)+\frac{\eta\Gamma}{\Gamma-1}\varphi^{\frac{\eta\Gamma}{\Gamma-1}-1}\frac{\partial}{\partial x_i}\varphi+\frac{\eta\gamma}{\gamma-1}\omega^{\frac{\eta\gamma}{\gamma-1}-1}\frac{\partial}{\partial x_i}\omega=0, \tag{3.44}$$

where $\eta>1$ is an arbitrary constant, $(\eta-1)\Gamma+1-\eta>0$ and $(\eta-1)\gamma+1-\eta>0$.

We study the solutions near the contacting vacuum boundary point $\vec{x}_0(t)$ in the region $\Omega(t)$. Firstly, we consider the local second inertial function

$$H(t)=\frac{1}{2}\int_{\Omega(t)}n\left|\vec{x}\right|^2dV+\frac{1}{2}\int_{\Omega(t)}\rho\left|\vec{x}\right|^2dV. \tag{3.45}$$

By Lemma 3.2, we obtain

$$\begin{aligned}H''(t) &= 2E_{kn}(t)+2E_{k\rho}(t)+N(\Gamma-1)E_{in}(t)+N(\gamma-1)E_{i\rho}(t)\\ &\geq \min\{2,N(\Gamma-1),N(\gamma-1)\}E(t)\\ &= C_1E,\end{aligned} \tag{3.46}$$

with the conserved total energy $E=\int_{\Omega(0)}\left(\frac{1}{2}n\left|\vec{u}\right|^2+\frac{1}{2}\rho\left|\vec{u}\right|^2+\frac{1}{\Gamma-1}n^\Gamma+\frac{1}{\gamma-1}\rho^\gamma\right)dV$ by Lemma 3.1, where $C_1=\min\{2,N(\Gamma-1),N(\gamma-1)\}$.

Therefore, we obtain

$$H(t)\geq H(0)+\dot{H}(0)t+\frac{C_1E}{2}t^2. \tag{3.47}$$

For an arbitrary point $\vec{x}_0(t)\in\partial\Omega(t)$ with the contacting vacuum, by applying Lemma 3.1 and $\rho(0,\vec{x}_0)=0$, we obtain

$$\frac{\partial u_i}{\partial t}+\sum_{l=1}^{N}u_l\frac{\partial u_i}{\partial x_l}=0. \tag{3.48}$$

We consider the governing differential dynamic system at the contacting vacuum point $\vec{x}_0(t)$:

$$\begin{cases}\frac{D^2x_{0,i}(t)}{Dt^2}=0, \quad \text{for } i=1,2,\cdots,N,\\ x_{0,i}(0,\vec{x}_0)=x_{0,i}, \quad \dot{x}_{0,i}(0,\vec{x}_0)=u_i(0,\vec{x}_0),\end{cases} \tag{3.49}$$

where $\frac{dx_{0,i}(t)}{dt}=u_i(t,\vec{x}_0(t))$.

The solutions of the above system are as follows:

$$x_{0,i}(t)=x_{0,i}+u_i(0,\vec{x}_0)t. \tag{3.50}$$

Then, we get

$$u_i(t,\vec{x}_0(t))=\frac{dx_{0,i}(t)}{dt}=u_i(0,\vec{x}_0). \tag{3.51}$$

By applying the Euclidean norm, we obtain

$$\max_{\vec{x}\in\partial\Omega(t)}\left(x_1^2+x_2^2+\cdots+x_N^2\right)^{\frac{1}{2}}\geq\max_{\vec{x}\in\Omega(t)}\left(x_1^2+x_2^2+\cdots+x_N^2\right)^{\frac{1}{2}}\geq 0. \tag{3.52}$$

Therefore, we have

$$\max_{\vec{x}_0\in\partial\Omega(t)}\left|\vec{x_0}\right|^2\geq\max_{\vec{x}\in\Omega(t)}\left|\vec{x}\right|^2. \tag{3.53}$$

Furthermore, we obtain

$$\frac{1}{2}\max_{\vec{x}_0\in\partial\Omega(t)}\sum_{i=1}^{N}x_{0,i}^2(t)\int_{\Omega(t)}ndV\geq\frac{1}{2}\int_{\Omega(t)}n\left|\vec{x}\right|^2dV. \tag{3.54}$$

Similarly, we obtain

$$\frac{1}{2}\max_{\vec{x}_0\in\partial\Omega(t)}\sum_{i=1}^{N}x_{0,i}^2(t)\int_{\Omega(t)}\rho dV\geq\frac{1}{2}\int_{\Omega(t)}\rho\left|\vec{x}\right|^2dV. \tag{3.55}$$

Therefore,

$$\begin{aligned}\frac{1}{2}\max_{\vec{x}_0\in\partial\Omega(t)}\sum_{i=1}^{N}x_{0,i}^2(t)\left(\int_{\Omega(t)}ndV+\int_{\Omega(t)}\rho dV\right) &\geq \frac{1}{2}\left(\int_{\Omega(t)}n\left|\vec{x}\right|^2dV+\int_{\Omega(t)}\rho\left|\vec{x}\right|^2dV\right)\\ &= H(t)\geq H(0)+\dot{H}(0)t+\frac{C_1E}{2}t^2,\end{aligned} \tag{3.56}$$

that is

$$\max_{\vec{x}_0\in\partial\Omega(t)}\sum_{i=1}^{N}x_{0,i}^2(t)\geq\frac{2}{M}\left(H(0)+\dot{H}(0)t+\frac{C_1E}{2}t^2\right), \tag{3.57}$$

with the conserved total mass $M=\int_{\Omega(0)}(n+\rho)dV>0$ in Lemma 3.1.

By (3.50), we have

$$\frac{2}{M}\left(H(0)+\dot{H}(0)t+\frac{C_1E}{2}t^2\right)\leq\max_{\vec{x}_0\in\partial\Omega(0)}\sum_{i=1}^{N}\left(x_{0,i}^2+2x_{0,i}\cdot u_i(0,\vec{x}_0)t+u_i^2(0,\vec{x}_0)t^2\right). \tag{3.58}$$

When $t\to+\infty$, we obtain

$$\lim_{t\to+\infty}\frac{\frac{2H(0)}{M}+\frac{2\dot{H}(0)}{M}t+\frac{C_1E}{M}t^2}{t^2}\leq\lim_{t\to+\infty}\frac{\max_{\vec{x}_0\in\partial\Omega(0)}\sum_{i=1}^{N}\left(x_{0,i}^2+2x_{0,i}\cdot u_i(0,\vec{x}_0)t+u_i^2(0,\vec{x}_0)t^2\right)}{t^2}. \tag{3.59}$$

Therefore, we have

$$\frac{C_1E}{M}\leq\max_{\vec{x}_0\in\partial\Omega(0)}u_i^2(0,\vec{x}_0), \tag{3.60}$$

that is

$$\frac{\min\left(2,N(\Gamma-1),N(\gamma-1)\right)E}{M}\leq\max_{\vec{x}_0\in\partial\Omega(0)}u_i^2(0,\vec{x}_0), \tag{3.61}$$

which contradicts with inequality (3.2) in Theorem 3.1. Therefore, we conclude that the solutions for the two-phase model (1.1) in $\mathbb{R}^N$ will blow up in finite time.

This completes the proof.

□

3.3. Blowup for the two-phase model with vacuum

For proofing the blowup results of solutions to the two-phase model in $\mathbb{R}^N (N \geq 2)$, we first introduce the properties of the two-phase model.

Lemma 3.3. *(Theorem 2.1 in [18]) Assume that $n_0 \geq 0$, $\rho_0 \geq 0$ and $(n_0, \rho_0, \vec{u_0}) \in H^3(\mathbb{R}^N)$, then there exists a unique solution $(n, \rho, \vec{u})$ to the two-phase model on some time interval $[0, T)$, which satisfies*

$$n, \rho \in C\left([0, T) \times \mathbb{R}^N\right), \tag{3.62}$$

and

$$\vec{u} \in C\left([0, T), H^3(\mathbb{R}^N)\right) \cap C^1\left([0, T), H^2(\mathbb{R}^N)\right) \cap C^2\left([0, T), H^1(\mathbb{R}^N)\right). \tag{3.63}$$

Lemma 3.4. *(Lemma 4 in [35]) From the mass equations $(1.1)_1$ and $(1.1)_2$, we obtain*

$$n(t, \vec{x}(t)) = n(0, \vec{x}_0) \exp\left(-\int_0^t \nabla \cdot \vec{u} ds\right) \tag{3.64}$$

and

$$\rho(t, \vec{x}(t)) = \rho(0, \vec{x}_0) \exp\left(-\int_0^t \nabla \cdot \vec{u} ds\right). \tag{3.65}$$

Next, we give the lemma to show the property of the modified Bessel function $K_0(r)$.

Lemma 3.5. *(Lemma 3.1 in [18]) The modified Bessel function $K_0(r) = \int_0^\infty e^{-rcosht} dt$ satisfies*

$$\begin{cases} K_0(r) \leq \frac{3}{r}, |K_0'(r)| \leq \frac{1}{r^2}, \quad 0 < r < \frac{1}{2}, \\ K_0(r) \leq \frac{C_k}{r^k}, |K_0'(r)| \leq \frac{C_k}{r^k}, \quad r > 1, \end{cases} \tag{3.66}$$

for some constants C_k depending only on $k > 1$.

Then, we give the proof of the blowup results of solutions to the two-phase model in $\mathbb{R}^N (N \geq 2)$.

Proof of Theorem 3.2. We consider the solution $(n, \rho, \vec{u})$ that satisfies the conditions in Theorem 3.2. From the two-phase system (1.1), we obtain

$$\begin{aligned} n_{tt} + \rho_{tt} &= -(\nabla \cdot (n\vec{u})_t + \nabla \cdot (\rho \vec{u})_t) \\ &= \Delta P + \nabla \cdot [\nabla \cdot ((n + \rho)\vec{u} \otimes \vec{u})]. \end{aligned} \tag{3.67}$$

Then, multiplying Eq (3.67) by $\frac{1}{re^r}$ and integrating over $\mathbb{R}^N$ to obtain

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^N} \frac{n + \rho}{re^r} dx = \int_{\mathbb{R}^N} \Delta P \frac{1}{re^r} dx + \int_{\mathbb{R}^N} \nabla \cdot [\nabla \cdot ((n + \rho)\vec{u} \otimes \vec{u})] \frac{1}{re^r} dx. \tag{3.68}$$

By the divergence theorem, we obtain

$$\int_{\mathbb{R}^N} \Delta P \frac{1}{re^r} dx = \int_{\mathbb{R}^N} P \Delta \frac{1}{re^r} dx - \lim_{\varepsilon \to 0} \int_{r=\varepsilon} \left(\frac{\partial P}{\partial r} \frac{1}{re^r} - \frac{\partial}{\partial r}(\frac{1}{re^r}) P\right) ds. \tag{3.69}$$

By applying the assumption (3.5) and Lemma 3.4, we obtain $n(t,\vec{0}) \equiv 0, \rho(t,\vec{0}) \equiv 0$. Using the continuity of P, we get that for any sufficiently small $\delta > 0$, there is $0 < \varepsilon < 1$, such that $P(t,\vec{x}) < \delta$ while $r = \varepsilon$, then,

$$\begin{aligned}\int_{r=\varepsilon} \frac{\partial(\frac{1}{re^r})}{\partial r} P ds &= \int_{r=\varepsilon} \frac{re^r + e^r}{r^2 e^{2r}} P ds \\ &\leq \delta \int_{r=\varepsilon} \frac{re^r + e^r}{r^2 e^{2r}} ds = \delta \int_{r=\varepsilon} \frac{(\varepsilon+1)e^\varepsilon}{\varepsilon^2 e^{2\varepsilon}} ds \\ &= \delta N\alpha(N)\varepsilon^{N-1} \frac{(\varepsilon+1)e^\varepsilon}{\varepsilon^2 e^{2\varepsilon}} = \delta N\alpha(N)\varepsilon^{N-3} \frac{(\varepsilon+1)e^\varepsilon}{e^{2\varepsilon}}, \end{aligned} \tag{3.70}$$

where $\alpha(N) = N(N-2)\frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}$ and $\frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}$ is the volume of the unit ball in $\mathbb{R}^N$.
Therefore, for $N \geq 3$, we obtain

$$\lim_{\varepsilon\to 0} \int_{r=\varepsilon} \frac{\partial(\frac{1}{re^r})}{\partial r} P ds = 0. \tag{3.71}$$

Similarly, we get

$$\lim_{\varepsilon\to 0} \int_{r=\varepsilon} \frac{\partial P}{\partial r} \frac{1}{re^r} ds = 0. \tag{3.72}$$

Thus, we obtain

$$\int_{\mathbb{R}^N} \Delta P \frac{1}{re^r} dx = \int_{\mathbb{R}^N} P \frac{1}{re^r} dx = \int_{\mathbb{R}^N} \frac{n^\gamma}{re^r} dx + \int_{\mathbb{R}^N} \frac{\rho^\gamma}{re^r} dx. \tag{3.73}$$

Using Hölder inequality, we get

$$(\int_{\mathbb{R}^N} \frac{n}{re^r} dx)^\gamma \leq (\int_{\mathbb{R}^N} \frac{n^\gamma}{re^r} dx)(\int_{\mathbb{R}^N} \frac{1}{re^r} dx)^{\gamma-1}, \tag{3.74}$$

and

$$(\int_{\mathbb{R}^N} \frac{\rho}{re^r} dx)^\gamma \leq (\int_{\mathbb{R}^N} \frac{\rho^\gamma}{re^r} dx)(\int_{\mathbb{R}^N} \frac{1}{re^r} dx)^{\gamma-1}. \tag{3.75}$$

Therefore, (3.73) becomes

$$\begin{aligned}\int_{\mathbb{R}^N} \Delta P \frac{1}{re^r} dx &\geq \frac{1}{(\int_{\mathbb{R}^N} \frac{1}{re^r} dx)^{\gamma-1}} \left[(\int_{\mathbb{R}^N} \frac{n}{re^r} dx)^\gamma + (\int_{\mathbb{R}^N} \frac{\rho}{re^r} dx)^\gamma \right] \\ &\geq \frac{1}{2^{\gamma-1}(\int_{\mathbb{R}^N} \frac{1}{re^r} dx)^{\gamma-1}} (\int_{\mathbb{R}^N} \frac{n+\rho}{re^r} dx)^\gamma \\ &= \frac{1}{2^{\gamma-1}} \left[\frac{\Gamma(1+\frac{N}{2})}{(N-2)!\pi^{\frac{N}{2}}} \right]^{\gamma-1} (\int_{\mathbb{R}^N} \frac{n+\rho}{re^r} dx)^\gamma. \end{aligned} \tag{3.76}$$

Applying the integration by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla \cdot [\nabla \cdot ((n+\rho)\vec{u} \otimes \vec{u})] \frac{1}{re^r} dx &= \int_{\mathbb{R}^N} \frac{1}{re^r} \sum_{i,j=1}^{n} \frac{\partial^2 \left((n+\rho)u_i u_j\right)}{\partial x_i \partial x_j} dx = \int_{\mathbb{R}^N} (n+\rho)V^2 (\frac{1}{re^r})'' dx \\ &= \int_{\mathbb{R}^N} (n+\rho)V^2 [\frac{1}{r} + \frac{2}{r^2} + \frac{2}{r^3}] e^{-r} dx > 0. \end{aligned} \tag{3.77}$$

Therefore, we obtain

$$\frac{d^2}{dt^2}H(t) \geq \frac{1}{2^{\gamma-1}}\left[\frac{\Gamma(1+\frac{N}{2})}{(N-2)!\pi^{\frac{N}{2}}}\right]^{\gamma-1} H^{\gamma}(t), \tag{3.78}$$

where $H(t) = \int_{\mathbb{R}^N} \frac{(n+\rho)(t,r)}{re^r}dx$. Using integration by parts and (3.7), we obtain

$$\begin{aligned} H^{'}(0) &= \frac{d}{dt}\int_{\mathbb{R}^N} \frac{(n+\rho)}{re^r}dx|_{t=0} = -\int_{\mathbb{R}^N} \frac{\nabla\cdot[(n+\rho)\vec{u}]}{re^r}dx|_{t=0} \\ &= -\int_{\mathbb{R}^N}\sum_{i=1}^{N}(n+\rho)\frac{x_i}{r}V\frac{1+r}{r^2e^r}\frac{x_i}{r}dx|_{t=0} = -\int_{\mathbb{R}^N}\frac{(1+r)(n_0+\rho_0)V_0}{r^2e^r}dx > 0. \end{aligned} \tag{3.79}$$

Therefore,

$$H^{'}(t) = H^{'}(0) + \int_0^t \frac{d^2}{ds^2}H(s)ds \geq H^{'}(0) > 0. \tag{3.80}$$

Multiplying the both sides of (3.78) by $2H^{'}(t)$, we obtain

$$\left(H^{'}(t)^2\right)^{'} \geq \frac{1}{2^{\gamma-1}(1+\gamma)}\left[\frac{\Gamma(1+\frac{N}{2})}{(N-2)!\pi^{\frac{N}{2}}}\right]^{\gamma-1}\left(H^{\gamma+1}(t)\right)^{'}. \tag{3.81}$$

Integrate over $[0,t]$, we obtain

$$\begin{aligned} H^{'}(t)^2 &\geq \frac{1}{2^{\gamma-1}(1+\gamma)}\left[\frac{\Gamma(1+\frac{N}{2})}{(N-2)!\pi^{\frac{N}{2}}}\right]^{\gamma-1} H^{\gamma+1}(t) + \left(\frac{d}{dt}\int_{\mathbb{R}^N}\frac{(n+\rho)}{re^r}dx\right)^2|_{t=0} \\ &- \frac{1}{2^{\gamma-1}(1+\gamma)}\left[\frac{\Gamma(1+\frac{N}{2})}{(N-2)!\pi^{\frac{N}{2}}}\right]^{\gamma-1} H^{\gamma+1}(0). \end{aligned} \tag{3.82}$$

Applying the assumption (3.7), we obtain

$$\begin{aligned} H^{'}(t)^2 &\geq C_0^2H^{\gamma+1}(t) + \left(\frac{d}{dt}\int_{\mathbb{R}^N}\frac{(n+\rho)}{re^r}dx\right)^2|_{t=0} - C_0^2\left(\int_{\mathbb{R}^N}\frac{(n_0+\rho_0)}{re^r}dx\right)^{\gamma+1} \\ &\geq C_0^2H^{\gamma+1}(t), \end{aligned} \tag{3.83}$$

where $C_0 = \sqrt{\frac{1}{2^{\gamma-1}(1+\gamma)}\left[\frac{\Gamma(1+\frac{N}{2})}{(N-2)!\pi^{\frac{N}{2}}}\right]^{\gamma-1}}$.

Therefore,

$$H^{'}(t) \geq C_0H^{\frac{\gamma+1}{2}}(t). \tag{3.84}$$

By integrating over $[0,t]$, we obtain

$$\int_0^t \frac{dH(s)}{H^{\frac{\gamma+1}{2}}(s)} \geq C_0t. \tag{3.85}$$

Therefore,

$$-\frac{2}{\gamma-1}\left(H^{-\frac{\gamma-1}{2}}(t) - H^{-\frac{\gamma-1}{2}}(0)\right) \geq C_0t, \tag{3.86}$$

which means that

$$H(t) \geq \left(H^{-\frac{\gamma-1}{2}}(0) - \frac{\gamma-1}{2}C_0 t\right)^{-\frac{2}{\gamma-1}}. \tag{3.87}$$

From (3.6), we have

$$\int_{\mathbb{R}^N} \frac{n_0+\rho_0}{re^r}dx > 0. \tag{3.88}$$

Applying the mass conservation, we obtain

$$H(t) = \int_{\mathbb{R}^N} \frac{n+\rho}{re^r}dx \leq \int_{\mathbb{B}_{r_0}} \frac{n+\rho}{re^r}dx + \frac{1}{r_0}\int_{\mathbb{R}^N}(n_0+\rho_0)dx, \tag{3.89}$$

where $\mathbb{B}_{r_0}$ is the N-dimensional ball centered at the origin with any given radius r_0.

When $t \to \frac{2H(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_0}$, $H(t)$ cannot be bounded. Therefore, we obtain that $\int_{r\leq r_0}(n+\rho)(t,r)rdr$ cannot be bounded as $t \to \frac{2H(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_0}$, hence the solutions will blow up on or before the finite time $t \to \frac{2H(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_0}$.

The proof is finished. □

In the following, we will prove Theorem 3.3.

Proof of Theorem 3.3. We consider the solution $(n, \rho, \vec{u})$ satisfies the conditions in Theorem 3.3. From (3.67), we obtain

$$n_{tt} + \rho_{tt} = \Delta P + \nabla\cdot[\nabla\cdot((n+\rho)\vec{u}\otimes\vec{u})]. \tag{3.90}$$

Then, multiplying Eq (3.90) by $K_0(r)$ and taking the integration over $\mathbb{R}^2$ to obtain

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^2}(n+\rho)K_0(r)dx = \int_{\mathbb{R}^2}\Delta P K_0(r)dx + \int_{\mathbb{R}^2}\nabla\cdot[\nabla\cdot((n+\rho)\vec{u}\otimes\vec{u})]K_0(r)dx. \tag{3.91}$$

Using the same method as that we obtain from (3.69) to (3.73), we obtain from Lemma 3.5 that

$$\begin{aligned}\int_{\mathbb{R}^2}\Delta P K_0(r)dx &= \int_{\mathbb{R}^2}(n^\gamma+\rho^\gamma)K_0(r)dx \\ &\geq \frac{1}{2^{\gamma-1}\left(\int_{\mathbb{R}^2}K_0(r)dx\right)^{\gamma-1}}\left(\int_{\mathbb{R}^2}(n+\rho)K_0(r)dx\right)^{\gamma}.\end{aligned} \tag{3.92}$$

Using the integration by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}\nabla\cdot[\nabla\cdot((n+\rho)\vec{u}\otimes\vec{u})]K_0(r)dx &= \int_{\mathbb{R}^2}\sum_{i,j=1}^{2}\frac{\partial^2((n+\rho)u_iu_j)}{\partial x_i\partial x_j}K_0(r)dx \\ &= \int_{\mathbb{R}^2}(n+\rho)V^2K_0''(r)dx > 0.\end{aligned} \tag{3.93}$$

Therefore,

$$\frac{d^2}{dt^2}G(t) \geq \frac{1}{2^{\gamma-1}\left(\int_{\mathbb{R}^2}K_0(r)dx\right)^{\gamma-1}}G^\gamma(t), \tag{3.94}$$

where $G(t) = \int_{\mathbb{R}^2}(n+\rho)(t,r)K_0(r)dx$.
Using integration by parts, $(1.1)_1$, $(1.1)_2$ and (3.11), we obtain

$$\begin{aligned} G^{'}(0) &= \frac{d}{dt}\int_{\mathbb{R}^2}(n+\rho)K_0(r)dx|_{t=0} \\ &= -\int_{\mathbb{R}^2}\nabla\cdot[(n+\rho)\vec{u}]K_0(r)dx|_{t=0} = \int_{\mathbb{R}^2}\sum_{i=1}^{2}(n+\rho)u_iK_0^{'}(r)\frac{x_i}{r}dx|_{t=0} \\ &= \int_{\mathbb{R}^2}\sum_{i=1}^{2}(n+\rho)\frac{x_i}{r}VK_0^{'}(r)\frac{x_i}{r}dx|_{t=0} = \int_{\mathbb{R}^2}(n+\rho)VK_0^{'}(r)dx|_{t=0} \\ &= \int_{\mathbb{R}^2}(n_0+\rho_0)V_0K_0^{'}(r)dx > 0. \end{aligned} \tag{3.95}$$

Similarly, we obtain $G^{'}(t) \geq 0$.
Multiplying both sides of (3.94) by $2G^{'}(t)$, we obtain

$$\left(G^{'}(t)^2\right)^{'} \geq \frac{1}{2^{\gamma-1}(1+\gamma)\left(\int_{\mathbb{R}^2}K_0(r)dx\right)^{\gamma-1}}\left(G^{\gamma+1}(t)\right)^{'}. \tag{3.96}$$

Integrate over $[0,t]$ and using the assumption (3.11), we obtain

$$\begin{aligned} G^{'}(t)^2 &\geq C_1^2G^{\gamma+1}(t) + \left(\frac{d}{dt}\int_{\mathbb{R}^2}(n+\rho)K_0(r)dx\right)^2|_{t=0} - C_1^2G^{\gamma+1}(0) \\ &\geq C_1^2G^{\gamma+1}(t), \end{aligned} \tag{3.97}$$

where $C_1 = \sqrt{\frac{1}{2^{\gamma-1}(1+\gamma)}}\left(\int_{\mathbb{R}^2}K_0(r)dx\right)^{-\frac{\gamma-1}{2}}$.
Therefore,

$$G^{'}(t) \geq C_1G^{\frac{\gamma+1}{2}}(t). \tag{3.98}$$

By integrating over $[0,t]$, we obtain

$$\int_0^t \frac{dG(s)}{G^{\frac{1+\gamma}{2}}(s)} \geq C_1t. \tag{3.99}$$

Therefore,

$$-\frac{2}{\gamma-1}\left(G^{-\frac{\gamma-1}{2}}(t) - G^{-\frac{\gamma-1}{2}}(0)\right) \geq C_1t, \tag{3.100}$$

which means that

$$G(t) \geq \left(G^{-\frac{\gamma-1}{2}}(0) - \frac{\gamma-1}{2}C_1t\right)^{-\frac{2}{\gamma-1}}. \tag{3.101}$$

From (3.10), we have

$$\int_{\mathbb{R}^2}(n_0+\rho_0)K_0(r)dx > 0. \tag{3.102}$$

Applying the mass conservation, we obtain

$$G(t) = \int_{\mathbb{R}^2}(n+\rho)K_0(r)dx \leq \int_{\mathbb{B}_{r_0}}(n+\rho)K_0(r)dx + \max_{r\geq r_0}K_0(r)\int_{\mathbb{R}^2}(n_0+\rho_0)dx, \tag{3.103}$$

where $\mathbb{B}_{r_0}$ is the 2-dimensional ball centered at the origin with any given radius r_0.

When $t \to \frac{2G(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_1}$, $G(t)$ cannot be bounded. Thus, the solutions will blow up on or before the finite time $t \to \frac{2G(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_1}$. Therefore, we conclude that the solutions of the two-phase model in $\mathbb{R}^2$ will blow up. □

4. Conclusions

In this paper, we study the blowup results of solutions to the two-phase model in $\mathbb{R}^N$. Our method depends on the energy integration method and a quasi-linear wave equation about ρ, the singularity of the two-phase model will form in finite time.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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