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Research article

Double controlled M-metric spaces and some fixed point results

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Abstract: In this article, we introduce the idea of double controlled M-metric space by employing two control functions $a(u, w)$ and $\beta(w, v)$ on the right-hand side of the triangle inequality of M-metric space. We provide some examples of double controlled M-metric spaces. We also provide some fixed point results under new type of contractions in the setting of double controlled M-metric spaces. Moreover, we give an example to highlight the importance of one of our main results.

Keywords: fixed point; Fredholm integral equation; double controlled M-metric space
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

Fixed point results is a well-known and established concept in mathematical analysis and also has a firm utilization in many mathematical fields. Fixed point results are also used to solve differential equations, integral equations [3–6, 12, 14, 15].

In 1994, Matthews [7] developed the concept of partial metric space. In 2014, m-metric space was presented by Asadi et al. [2] which is the extended form of a partial metric space. In 2016, Mlaiki et al. [10] developed the notion of an m_b-metric space which extends m-metric space and also which is the generalize form of b-metric space (see [13]). In 2018, Mlaiki et al. [9] established the idea of extended m_b-metric spaces. In 2018, Mlaiki et al. [8] presented the concept of controlled metric space. In 2018, Abdeljawad et al. [1] developed the idea of double controlled metric space by using two control functions. More details can be found in [16].

In this article, we establish a new double controlled M-metric space, by employing two control functions $\alpha, \beta : E \times E \to [1, \infty)$ with the following new double controlled M-metric type triangle inequality:

$$(M(\omega, \nu) - M_{\omega,\nu}) \leq \alpha(\omega, \mu)\left(M(\omega, \mu) - M_{\omega,\mu}\right) + \beta(\mu, \nu)\left(M(\mu, \nu) - M_{\mu,\nu}\right).$$

Further we provide some important examples to validate our work. At last, we present an application related to our main results for Fredholm type integral equation.

2. Preliminaries

In this part, we discuss some important definitions, which would be useful in understanding this work.

Definition 2.1. [7] Let $E \neq \phi$. A function $g : E \times E \to [0, \infty)$ is said to be a partial metric if the following conditions hold, for all $u, v, w \in E$,

1) $u = v$ if and only if $g(u, u) = g(u, v) = g(v, v)$;
2) $g(u, v) = g(v, u)$;
3) $g(u, u) \leq g(u, v)$;
4) $g(u, v) \leq g(u, w) + g(w, v) - g(w, w)$.

Example 2.2. [7] Let $E = [0, \infty)$ with $g(u, v) = max\{u, v\}$. Then (E, g) is a partial metric space.

We will use the following notations given in [7].

1) $m_{u,v} = m_{b_{u,v}} = \min\{m(u, u), m(v, v)\}$;
2) $M_{u,v} = M_{b_{u,v}} = \max\{m(u, u), m(v, v)\}$.

Definition 2.3. [2] Let $E \neq \phi$. A function $m : E \times E \to [0, \infty)$ is said to be an m-metric if the following conditions hold, for all $u, v, w \in E$,

1) $u = v \Leftrightarrow m(u, u) = m(u, v) = m(v, v)$;
2) $m_{u,v} \leq m(u, v)$;
3) $m(u, v) = m(v, u)$;
4) $(m(u, v) - m_{u,v}) \leq (m(u, w) - m_{u,w}) + (m(w, v) - m_{w,v})$.

Example 2.4. [2] Let $E = [0, \infty)$. Then $m(u, v) = u + v$ on E is an m-metric.

Definition 2.5. [11] Let $E \neq \phi$. A function $m_b : E \times E \to [0, \infty)$ is said be an m_b-metric with coefficient $s \geq 1$ if the following conditions hold, for all $u, v, w \in E$,

1) $u = v$ if and only if $m_b(u, u) = m_b(u, v) = m_b(v, v)$;
2) $m_{b\,u,v} \leq m_b(u, v)$;
3) $m_b(u, v) = m_b(v, u)$;
4) $(m_b(u, v) - m_{b\,u,v}) \leq s\left[(m_b(u, w) - m_{b\,u,w}) + (m_b(w, v) - m_{b\,w,v})\right] - m_b(w, w)$.

Definition 2.6. [9] Let $E \neq \phi$ and $\alpha : E^2 \to [1, \infty)$ be a function. A function $m_b : E^2 \to \mathbb{R}^+$ is said be an extended m_b-metric if the following conditions hold, for all $u, v, w \in E$,

1) $u = v$ if and only if $m_b(u, u) = m_b(u, v) = m_b(v, v)$;
2) $m_{b\,u,v} \leq m_b(u, v)$;
3) $m_b(u, v) = m_b(v, u)$;
4) $(m_b(u, v) - m_{bu,v}) \leq \alpha(u, v)\left[(m_b(u, w) - m_{u,w}) + (m_b(w, v) - m_{bw,v})\right]$.

The pair (E, m_b) is said to be an extended m_b-metric space.

Example 2.7. [9] Let $E = C([a, b], \mathbb{R})$ be the set of all continuous real valued functions on $[a, b]$. We define the functions

$$m_b : E^2 \to [0, +\infty) \text{ and } \alpha : E^2 \to [1, +\infty)$$

by

$$m_b(u(t), v(t)) = sup_{t\in[a,b]} |u(t) - v(t)|^2 \text{ and } \alpha(u(t), v(t)) = |u(t) + v(t)| + 2.$$

Then (E, m_b) is an extended m_b-metric space with the function α.

3. Main results

Definition 3.1. Let $E \neq \phi$ and $\alpha, \beta : E \times E \to [1, \infty)$ be functions. A mapping $M : E \times E \to [0, \infty)$ is said be a double controlled M-metric if the following conditions hold, for all $u, v, w \in E$,

(1) $u = v$ if and only if $M(u, u) = M(u, v) = M(v, v)$;
(2) $M_{u,v} \leq M(u, v)$;
(3) $M(u, v) = M(v, u)$;
(4) $(M(u, v) - M_{u,v}) \leq \alpha(u, w)(M(u, w) - M_{u,w}) + \beta(w, v)(M(w, v) - M_{w,v})$.

Example 3.2. Let $E = [0, 1]$ and for distinct u, v and $w \in E$ we take

$$\begin{aligned} M(u, v) &= 18 = M(v, u), \ M(1, 1) = 1, \\ M(u, w) &= 6 = M(w, u), \ M(2, 2) = 8, \\ M(w, v) &= 7 = M(v, w), \ M(3, 3) = 3. \end{aligned}$$

Define $\alpha, \beta : E \times E \to [1, \infty)$ by

$$\alpha(u, v) = \begin{cases} 2u + 3, & \text{if } u > v \\ 3v^2 + 1, & \text{otherwise} \end{cases}$$

and

$$\beta(u, v) = \begin{cases} 8u + 4, & \text{if } u > v \\ 2v^2 + 1, & \text{otherwise.} \end{cases}$$

Clearly, (1)–(3) are satisfied. Now for (4)

$$\begin{aligned} M(u, v) - M_{u,v} &\leq \alpha(u, w)(M(u, w) - M_{u,w}) + \beta(w, v)(M(w, v) - M_{w,v}), \\ 18 - 8 &\leq 7(3) + 8(3), \\ 10 &\nleq 4 + 2 = 6. \end{aligned}$$

Clearly M is not an m-metric space, but if we take

$$\alpha(u, w) = 4, \beta(w, v) = 6,$$

then

$$10 \leq (4)(4) + (6)(2) = 28.$$

Hence the space is not an m-metric space, but it is a double controlled M-metric space

Example 3.3. Let $E = \{1,\ 2,\ 3\}$ and for distinct u, v and $w \in E$ we take

$$\begin{aligned} M(u,\ v) &= 16 = M(v,\ u),\ M(1,\ 1) = 1, \\ M(u,\ w) &= 4 = M(w,\ u),\ M(2,\ 2) = 9, \\ M(w,\ v) &= 5 = M(v,\ w),\ M(3,\ 3) = 2. \end{aligned}$$

Define $\alpha, \beta : E \times E \to [1,\ \infty)$ by

$$\alpha(u,\ v) = \begin{cases} u + 1, & \text{if } u > v \\ v^2, & \text{otherwise} \end{cases}$$

and

$$\beta(u,\ v) = \begin{cases} 3u, & \text{if } u > v \\ 2v, & \text{otherwise.} \end{cases}$$

Clearly, (1)–(3) are satisfied. Now for (4)

$$\begin{aligned} M(u,v) - M_{u,v} &\leq \alpha(u,\ w)\left(M(u,w) - M_{u,w}\right) + \beta(w,\ v)\left(M(w,v) - M_{w,v}\right), \\ 15 &\leq 7(3) + 8(3) \end{aligned}$$

if we take

$$\alpha(u,\ w) = 7, \beta(w,\ v) = 8.$$

Also clearly M is not an m-metric space, since

$$15 \nleq 3 + 3 = 6.$$

Hence the space is not an m-metric space, but it is a double controlled M-metric space.

Remark 3.4. Observe that if $\alpha(u,\ v) = \beta(u,\ v)$, then M is an extended m_b-metric but if $\alpha(u,\ v) = \beta(u,\ v) = 1$, then M becomes an m-metric.

Example 3.5. Let $E = [0,\ \infty)$ and define

$$M(1,\ 1) = 7, M(2,\ 2) = 13, M(3,\ 3) = 9, M(1,\ 2) = M(2,\ 1) = 12$$

and for distinct u, v and $w \in E$

$$\begin{aligned} M(u,\ v) &= 19 = M(v,\ u), \\ M(u,\ w) &= 8 = M(w,\ u), \\ M(w,\ v) &= 4 = M(v,\ w). \end{aligned}$$

Also, define $\alpha, \beta : E^2 \to [1,\ \infty)$ by

$$\alpha(u,\ v) = \begin{cases} 3u, & \text{if } u,\ v > 1 \\ \min\{u,\ v\}, & \text{otherwise} \end{cases}$$

and

$$\beta(u,\ v) = \begin{cases} 10, & \text{if } u,\ v < 1 \\ \min\{u,\ v\}, & \text{otherwise.} \end{cases}$$

Then clearly E is not an m-metric and extended m_b-metric space but it is a double controlled M-metric space.

Definition 3.6. Let (E, M) be a double controlled M-metric space.

(1) A sequence $\{u_n\}$ in E converges at a point u if

$$\lim_{n\to\infty}\left(M\left(u_n, u\right) - M_{u_n,u}\right) = 0.$$

(2) A sequence $\{u_n\}$ in E is said to be an M-Cauchy sequence if

$$\lim_{n,m\to\infty}\left(M\left(u_n, u_m\right) - M_{u_n,u_m}\right) \text{ and } \lim_{n\to\infty}\left(M_{u_n,u_m} - m_{u_n,u_m}\right) < \infty.$$

(3) A double controlled M-metric space is said to be M-complete if every M-Cauchy sequence $\{u_n\}$ converges to a point u, i.e.,

$$\lim_{n\to\infty}\left(M\left(u_n, u\right) - M_{u_n,u}\right) = 0 \text{ and } \lim_{n\to\infty}\left(M_{u_n,u} - m_{u_n,u}\right) = 0.$$

Theorem 3.7. *Let (E, M) be a complete double controlled M-metric space by functions $\alpha, \mu : E\times E \to [1, \infty)$. Suppose that a continuous mapping $R : E \to E$ satisfies*

$$M(Ru, Rv) \leq K\left(M(u, v) + M_{u,v}\right) \tag{3.1}$$

for all u, $v \in E$ where $K \in (0, 1)$. For $u_0 \in E$, let $u_n = R^n u_0$. Assume that

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha(u_{i+1}, u_{i+2})}{\alpha(u_i, u_{i+1})}\mu(u_{i+1}, u_m) < \frac{1}{k}. \tag{3.2}$$

In addition, for each $u \in E$, suppose that

$$\lim_{n\to\infty}\alpha(u, u_n) \text{ and } \lim_{n\to\infty}\mu(u_n, u) < \infty. \tag{3.3}$$

Then the mapping R has a unique fixed point.

Proof. Let $\{u_n = R^n u_0\}$ be a sequence in E satisfying the hypothesis of the theorem. By (3.1), we get

$$M(u_n, u_{n+1}) \leq K^n\left(M(u_o, u_1) + M_{u_o,u_1}\right)$$

for all $n \geq 0$. For $n, m \in \mathbb{Z}$ with $n \leq m$, we get

$$\begin{aligned}
M(u_n, u_m) - M_{u_n,u_m} &\leq \alpha(u_n, u_{n+1})\left(M(u_n, u_{n+1}) - M_{u_n,u_{n+1}}\right) + \mu(u_{n+1}, u_m)\left(M(u_{n+1}, u_m) - M_{u_{n+1},u_m}\right)\\
&\leq \alpha(u_n, u_{n+1})\left(M(u_n, u_{n+1}) - M_{u_n,u_{n+1}}\right)\\
&+ \mu(u_{n+1}, u_m)\left[\begin{array}{c}\alpha(u_{n+1}, u_{n+2})\left(M(u_{n+1}, u_{n+2}) - M_{u_{n+1},u_{n+2}}\right)\\ +\mu(u_{n+2}, u_m)\left(M(u_{n+2}, u_m) - M_{u_{n+2},u_m}\right)\end{array}\right]\\
&\leq \alpha(u_n, u_{n+1})\left(M(u_n, u_{n+1}) - M_{u_n,u_{n+1}}\right)\\
&\quad+\mu(u_{n+1}, u_m)\alpha(u_{n+1}, u_{n+2})[(M(u_{n+1}, u_{n+2}) - M_{u_{n+1},u_{n+2}})]\\
&\quad+\mu(u_{n+1}, u_m)\mu(u_{n+2}, u_m)[M(u_{n+2}, u_m) - M_{u_{n+2},u_m}],\\
M(u_{n+2}, u_m) - M_{u_{n+2},u_m} &\leq \alpha(u_{n+2}, u_{n+3})\left(M(u_{n+2}, u_{n+3}) - M_{u_{n+2},u_{n+3}}\right)\\
&\quad+\mu(u_{n+3}, u_m)\left(M(u_{n+3}, u_m) - M_{u_{n+3},u_m}\right)
\end{aligned}$$

and so we have

$$M(u_n, u_m) - M_{u_n,u_m} \leq \alpha(u_n, u_{n+1})\left(M(u_n, u_{n+1}) - M_{u_n,u_{n+1}}\right)$$

$$
\begin{aligned}
& +\mu(u_{n+1},u_m)\alpha(u_{n+1},u_{n+2})[(M(u_{n+1},u_{n+2})-M_{u_{n+1},u_{n+2}})] \\
& +\mu(u_{n+1},u_m)\mu(u_{n+2},u_m)\left[\begin{array}{c}\alpha(u_{n+2},u_{n+3})\left(M(u_{n+2},u_{n+3})-M_{u_{n+2},u_{n+3}}\right)\\ +\mu(u_{n+3},u_m)\left(M(u_{n+3},u_m)-M_{u_{n+3},u_m}\right)\end{array}\right] \\
\leq\ & \alpha(u_n,u_{n+1})[M(u_n,u_{n+1})-M_{u_n,u_{n+1}}] \\
& +\mu(u_{n+1},u_m)\alpha(u_{n+1},u_{n+2})[(M(u_{n+1},u_{n+2})-M_{u_{n+1},u_{n+2}})] \\
& +\mu(u_{n+1},u_m)\mu(u_{n+2},u_m)\alpha(u_{n+2},u_{n+3})[(M(u_{n+2},u_{n+3})-M_{u_{n+2},u_{n+3}})] \\
& +\mu(u_{n+1},u_m)\mu(u_{n+2},u_m)\mu(u_{n+3},u_m)[(M(u_{n+3},u_m)-M_{u_{n+3},u_m})] \\
\leq\ & \\
\vdots\ & \\
\leq\ & \alpha(u_n,u_{n+1})\left(M(u_n,u_{n+1})-M_{u_n,u_{n+1}}\right) \\
& +\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu(u_j,u_m)\right)\alpha(u_i,u_{i+1})\left[(M(u_i,u_{i+1})-M_{u_i,u_{i+1}})\right] \\
& +\prod_{k=n+1}^{m-1}\mu(u_k,u_m)[(M(u_{m-1},u_m)-M_{u_{m-1},u_m})] \\
\leq\ & \alpha(u_n,u_{n+1})k^n\left(M(u_o,u_1)+M_{u_o,u_1}\right) \\
& +\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu(u_j,u_m)\right)\alpha(u_i,u_{i+1})k^i\left[(M(u_0,u_1)-M_{u_0,u_1})\right] \\
& +\left(\prod_{i=n+1}^{m-1}\mu(u_i,u_m)\right)k^{m-1}\left(M(u_o,u_1)+M_{u_o,u_1}\right) \\
\leq\ & \alpha(u_n,u_{n+1})k^n\left(M(u_o,u_1)+M_{u_o,u_1}\right) \\
& +\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu(u_j,u_m)\right)\alpha(u_i,u_{i+1})k^i\left[(M(u_0,u_1)-M_{u_0,u_1})\right] \\
& +\left(\prod_{j=n+1}^{m-1}\mu(u_i,u_m)\right)k^{m-1}\alpha(u_{m-1},u_m)\left(M(u_o,u_1)+M_{u_o,u_1}\right) \\
=\ & \alpha(u_n,u_{n+1})k^n\left(M(u_o,u_1)+M_{u_o,u_1}\right) \\
& +\sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\mu(u_j,u_m)\right)\alpha(u_i,u_{i+1})k^i\left[(M(u_0,u_1)-M_{u_0,u_1})\right] \\
\leq\ & \alpha(u_n,u_{n+1})k^n\left(M(u_o,u_1)+M_{u_o,u_1}\right) \\
& +\sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\mu(u_j,u_m)\right)\alpha(u_i,u_{i+1})k^i\left[(M(u_0,u_1)-M_{u_0,u_1})\right].
\end{aligned}
$$

Letting

$$
s_p=\sum_{i=0}^{p}\left(\prod_{j=0}^{i}\mu(u_j,u_m)\right)\alpha(u_i,u_{i+1})k^i,
$$

we have

$$(M(u_n,u_m)-M_{u_n,u_m})\leq(M(u_o,u_1)+M_{u_o,u_1})+[k^n\alpha(u_n,u_{n+1})+s_{m-1}-s_n]. \tag{3.4}$$

By (3.2), the limit of the real sequences exists and so $\{s_n\}$ is Cauchy. Indeed, the ratio test is applied to the term

$$a_i=\left(\prod_{j=0}^{i}\mu(u_j,u_m)\right)\alpha(u_i,u_{i+1}).$$

Letting n,m tend to ∞ in (3.4), we get

$$\lim_{n,m\to\infty}(M(u_n,u_m)-M_{u_n,u_m})=0,$$

which implies that $\{u_n\}$ is a Cauchy sequence and by using the completeness of M there exists $u\in E$ such that

$$M_{u,Ru}\leq M(u,Ru)$$

and

$$M(u,Ru)-M_{u,Ru}\leq\alpha(u,\psi_n)[\Big(M(u,\psi_n)-M_{u,\psi_n}\Big)]+\mu(\psi_n,Ru)[\Big(M(\psi_n,Ru)-M_{\psi_n,Ru}\Big)].$$

By the continuity of R and taking the limit, we obtain

$$\begin{aligned}M(u,Ru)-M_{u,Ru}&\leq 0,\\ M(u,Ru)&=M_{u,Ru}.\end{aligned}$$

Now let

$$\begin{aligned}K^*_{u,Ru}&=M(u,Ru)\text{ where }k^*_{u,Ru}=\max(M(u,u),M(Ru,Ru)),\\ K^*_{\psi_n,R\psi_n}&=M(\psi_n,\psi_n)\leq K^n\left(M(\psi_o,\psi_1)+M_{\psi_o,\psi_1}\right).\end{aligned}$$

Since $K\in(0,1)$, by applying limit, we get

$$K^*_{\psi_n,R\psi_n}=0.$$

Finally, since

$$M(u,Ru)=M_{u,Ru}\leq K^*_{u,Ru}$$

and

$$M(Ru,Ru)=M_{u,Ru},$$

$Ru=u$.

Now suppose R has two fixed points, i.e., $Ra=a$ and $Rb=b$. Then

$$M(a,b)=M(Ra,Rb)\leq K[(M(a,b)+M_{a,b})].$$

Since $M(a,b)=M_{a,b}$, we obtain

$$\begin{aligned}M(a,b)&\leq K2M(a,b),\\ M(a,b)(1-2M)&\leq 0,\\ M(a,b)&=0,\end{aligned}$$

which implies that $a=b$. □

Example 3.8. Let $E = [0, \infty)$. Consider the double controlled M-metric type defined by $M(u, v) = u+v$ for all $u, v \in E$ and the functions α, μ given by

$$\alpha(u, v) = \begin{cases} 8 \text{ if } u, v \geq 1 \\ 8(u+2) \text{ otherwise} \end{cases}$$

and

$$\mu(u, v) = \begin{cases} 7 \text{ if } u, v \geq 1 \\ 7(v+2) \text{ otherwise.} \end{cases}$$

Letting $Ru = 1$ for all $u \in E$ and $u_0 = 1$ and $k = \frac{1}{4}$, we have

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\alpha(u_{i+1}, u_{i+2})}{\alpha(u_i, u_{i+1})} \mu(u_{i+1}, u_m) = 1 < 4 = \frac{1}{k},$$

that is, (3.2) holds. In addition to that for every $u \in [0, \infty)$ we have

$$\lim_{n\to\infty} \alpha(u, u_n) = \max(1, u) < \infty \text{ and } \lim_{n\to\infty} \mu(u_n, u) = \max(u, 1) < \infty,$$

that is, (3.3) holds. All the conditions of the above theorem hold and $u = 1$ is the unique fixed point.

Theorem 3.9. *Let (E, M) be a complete double controlled M-metric type and $R : E \to E$ be a continues mapping. Suppose that there exist $\breve{o}, \check{c} \in [0, \infty)$ with*

$$\lim_{n\to\infty} \frac{\breve{o}\alpha(u_n, u_{n-1})}{1 - \check{c}\mu(u_n, u_{n+1})} < 1, \tag{3.5}$$

where $u_n = R^n u_0$ and for any $u \in E$

$$\breve{o}\alpha(u, Ru) + \check{c}\mu(u, Ru) < 1.$$

If

$$M(Ru, Rv) \leq \breve{o}\alpha(u, Ru)M(u, Ru) + \check{c}\mu(v, Rv)M(v, Rv)$$

then R has a unique fixed point.

Proof. Let $u_o \in E$ and define $\{u_n\}$ as follow: $u_1 = Ru_0$, $u_2 = Ru_1 = R^2u_0$ $u_n = R^n u_0$. We first prove that

$$M(u_n, u_{n+1}) \leq \breve{o}^n \prod_{i=o}^{n} \left[\frac{\alpha(u_i, u_{i-1})}{1 - \check{c}\mu(u_i, u_{i+1})} \right] M(u_0, u_1).$$

To prove this, let $n \in \mathbb{N}$. Then

$$\begin{aligned} M(u_n, u_{n+1}) &= M(Ru_{n-1}, u_n) \\ &\leq \breve{o}\alpha(u_{n-1}, Ru_{n-1})M(u_{n-1}, Ru_{n-1}) + \check{c}\mu(u_n, Ru_n)M(u_n, Ru_n) \\ &\leq \breve{o}\alpha(u_{n-1}, u_n)M(u_{n-1}, u_n) + \check{c}\mu(u_n, u_{n+1})M(u_n, u_{n+1}) \end{aligned}$$

and so

$$M(u_n, u_{n+1}) - \check{c}\mu(u_n, u_{n+1})M(u_n, u_{n+1}) \leq \breve{o}\alpha(u_{n-1}, u_n)M(u_{n-1}, u_n),$$

$$M(u_n,u_{n+1})\,(1-\check{c}\mu(u_n,u_{n+1})) \quad\leq\quad \check{o}\alpha(u_{n-1},u_n)M(u_{n-1},u_n).$$

Hence

$$\begin{aligned} M(u_n,u_{n+1}) &\leq \frac{\check{o}\alpha(u_{n-1},u_n)M(u_{n-1},u_n)}{(1-\check{c}\mu(u_n,u_{n+1}))} = \frac{\check{o}\alpha(u_{n-1},u_n)}{(1-\check{c}\mu(u_n,u_{n+1}))}M(Ru_{n-1},Ru_{n-2}) \\ &\leq \frac{\check{o}^2\alpha(u_n,u_{n-1})\alpha(u_{n-1},u_{n-2})}{[(1-\check{c}\mu(u_n,u_{n+1}))\,(1-\check{c}\mu(u_{n-1},u_n))]}M(u_{n-1},u_n) \\ &\leq \cdots \leq \check{o}^n\prod_{i=1}^{n}\left[\frac{\alpha(u_i,u_{i-1})}{1-\check{c}\mu(u_i,u_{i+1})}\right]M(u_0,u_1). \end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{\check{o}\alpha(u_n,u_{n-1})}{1-\check{c}\mu(u_n,u_{n+1})} < 1,$$

by the ratio test,

$$\sum_{n=1}^{\infty}\check{o}^n\prod_{i=1}^{n}\left[\frac{\alpha(u_i,u_{i-1})}{1-\check{c}\mu(u_i,x_{i+1})}\right]$$

converges. This implies that $M(u_n,u_{n+1})$ converges to 0. Further for $n,m\in\mathbb{N}$

$$\begin{aligned} M(u_n,u_m) &= M(Ru_{n-1},Ru_{m-1}) \leq \check{o}\alpha(u_{n-1},Ru_{n-1})M(u_{n-1},Ru_{n-1}) \\ &\quad +\check{c}\mu(u_{m-1},Ru_{m-1})M(u_{m-1},Ru_{m-1}) \\ &\leq \check{o}\alpha(u_{n-1},u_n)M(u_{n-1},u_n)+\check{c}\mu(u_{m-1},u_m)M(u_{m-1},u_m). \end{aligned}$$

Using the observation of the above inequality, we obtain that $M(u_n,u_m)$ converges to 0. Since

$$M_{u_n,u_m} = \min(M(u_n,u_n),M(u_m,u_m)) \leq M(u_m,u_m),$$

we conclude that

$$\lim_{n,m\to\infty}(M(u_n,u_m)-M_{u_n,u_m}) = 0.$$

Also suppose that

$$k^*_{u_n,u_m} = \max(M(u_n,u_n),M(u_m,u_m)) = M(u_n,u_n).$$

Then

$$\begin{aligned} k^*_{u_n,u_m}-M_{u_n,u_m} &\leq k^*_{u_n,u_m} = M(u_n,u_n) = M(Ru_{n-1},Ru_{n-1}) \\ &\leq \check{o}\alpha(u_{n-1},u_n)M(u_{n-1},u_n)+\check{c}\mu(u_{n-1},u_n)M(u_{n-1},u_n). \end{aligned}$$

Taking the limit $n\to\infty$, we obtain

$$\lim_{n,m\to\infty}(k^*_{u_n,u_m}-M_{u_n,u_m}) = 0.$$

Thus $\{u_n\}$ is an M-Cauchy sequence and by using the completeness of M we obtain that $\{u_n\}$ converges to some $u\in E$ and so $\{Ru_n = u_{n+1}\}$ converges to $u\in E$. Furthermore by the hypothesis of the theorem, i.e, $R : (E, M) \to (E, M)$ is continues, it is not difficult to show that $\{Ru_n\}$ converges to $R\bar{e}$. By [9, Lemma 3.3], we have

$$(M(\bar{e}, R\bar{e}) - M_{\bar{e},R\bar{e}}) = 0$$

and so

$$\begin{aligned} M(\bar{e}, R\bar{e}) &= M(R\bar{e}, R\bar{e}) \leq \breve{o}\alpha(\bar{e}, R\bar{e})M(\bar{e}, R\bar{e}) + \breve{c}\mu(\bar{e}, R\bar{e})M(\bar{e}, R\bar{e}) \\ &\leq (\breve{o}\alpha(\bar{e}, R\bar{e}) + \breve{c}\mu(\bar{e}, R\bar{e}))M(\bar{e}, R\bar{e}) \\ &\leq M(\bar{e}, R\bar{e}). \end{aligned}$$

Hence

$$M(\bar{e}, R\bar{e}) = M(R\bar{e}, R\bar{e}) = 0.$$

By using the same observation given in the above equality, we get

$$M(R\bar{e}, R\bar{e}) = M(R^2\bar{e}, R^2\bar{e}) = 0. \tag{3.6}$$

Since (E, M) is complete, it follows that

$$RR\bar{e} = R\bar{e}.$$

Thus we obtain that

$$\bar{e}' = R\bar{e}$$

is a fixed point of R.

Next we will show the uniqueness. Suppose there exists $v \in E$ such that

$$M(v, \bar{e}') = M(Rv, R\bar{e}') \leq \breve{o}\alpha(v, Rv)M(v, Rv) + \breve{c}\mu(\bar{e}', R\bar{e}')M(\bar{e}', R\bar{e}').$$

By (1.6), we obtain

$$M(v, \bar{e}') \leq \breve{o}\alpha(v, Rv)M(v, Rv) + 0 = \breve{o}\alpha(v, Rv)M(v, Rv).$$

Hence

$$\begin{aligned} M(v, \bar{e}') &\leq \breve{o}\alpha(v, Rv)M(Rv, Rv) \leq \breve{o}\alpha(v, Rv)[\breve{o}\alpha(v, Rv) + \breve{c}\mu(v, Rv)]M(v, Rv) \\ &\leq \cdots \leq \breve{o}\alpha(v, Rv)[\breve{o}\alpha(v, Rv) + \breve{c}\mu(v, Rv)]^n M(v, Rv). \end{aligned}$$

Since

$$\breve{o}\alpha(v, Rv) + \breve{c}\mu(v, Rv) = \breve{o}\alpha(Rv, R^2v) + \breve{c}\mu(Rv, R^2v) < 1,$$

$$[\breve{o}\alpha(v, Rv) + \breve{c}\mu(v, Rv)]^n \to 0$$

and so

$$M(v, \bar{e}') = M(v, v) = 0.$$

Using (3.6), we obtain

$$M(v, \bar{e}') = M(v, v) = M(\bar{e}', \bar{e}').$$

So the mapping R has a unique fixed point. □

Example 3.10. Let $E = \{1, 2, 3\}$ and α, μ be defined in Example 3.3. Let

$$M(u, v) = u + v$$

and $R : E \to E$ be defined by

$$R(u) = \frac{u}{2}.$$

Then clearly

$$M(Ru, Rv) \leq \breve{o}\alpha(u, Ru)M(u, Ru) + \check{c}\mu(v, Rv)M(v, Rv).$$

Letting $Ru = 1$ for $u_o = 1$, we have

$$\lim_{n\to\infty} \frac{\breve{o}\alpha(u_n, u_{n-1})}{1 - \check{c}\mu(u_n, u_{n+1})} < 1.$$

That is, (3.5) holds. Since

$$\lim_{n\to\infty} \alpha(u, u_n) = \lim_{n\to\infty} \mu(u_n, u) < \infty,$$

all the conditions of Theorem 3.9 are satisfied and hence $u = 1$ is a fixed point.

Theorem 3.11. *Let (E, M) be a complete double controlled M-metrc space by functions $\alpha, \mu : E\times E \to [1, \infty)$. Suppose that a continuous mapping R satisfies Bianchini type condition*

$$M(Ru, Rv) \leq \hat{h} \max\{M(u, Ru), M(v, Rv)\} \tag{3.7}$$

for all $u, v \in E$ where $0 < \hat{h} < 1$. For $u_0 \in E$, choose $u_n = R^n u_0$. Suppose that

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\alpha(u_{i+1}, u_{i+2})}{\alpha(u_i, u_{i+1})}\mu(u_{i+1}, u_m) < \frac{1}{\hat{h}}. \tag{3.8}$$

In addition, for each $u \in E$ suppose that

$$\lim_{n\to\infty} \alpha(u, u_n) \text{ and } \lim_{n\to\infty} \mu(u_n, u) < \infty.$$

Then R has a unique fixed point.

Proof. Let u_o and u_1 be points as in Theorem 3.7. If for some m, $u_m = u_{m+1} = Ru_m$, then clearly u_m is a fixed point of R.

Now suppose $u_n \neq u_{n+1}$ for all n. By (3.7), we have

$$M(u_n, u_{n+1}) = M(Ru_{n-1}, Ru_n) \leq \hat{h} \max\{M(u_n, u_{n+1}), M(u_{n-1}, u_n)\}.$$

If $\max\{M(u_n, u_{n+1}), M(u_{n-1}, u_n)\} = M(u_n, u_{n+1})$, then

$$\begin{aligned} M(u_n, u_{n+1}) &\leq \hat{h}M(u_n, u_{n+1}), \\ 1 &= \frac{M(u_n, u_{n+1})}{M(u_n, u_{n+1})} \leq \hat{h} \end{aligned}$$

which is a contradiction since $0 < \hat{h} < 1$.

If $\max\{M(u_n,u_{n+1}),M(u_{n-1},u_n)\}=M(u_{n-1},u_n)$, then

$$M(u_n,u_{n+1})\leq\hat{h}M(u_{n-1},u_n).$$

If we proceed it continually, then we come to the conclusion that for each $n\geq 0$ we obtain

$$M(u_n,u_{n+1})\leq\hat{h}^n M(u_o,u_1).$$

Taking the limit on both sides, we get

$$\lim_{n\to\infty} M(u_n,u_{n+1})=0,$$

which implies that $\{u_n\}$ is a Cauchy sequence. By the same procedure used in Theorem 3.7, for all $m\geq n$, we may get

$$\begin{aligned} M(u_n,u_{n+1}) &\leq \alpha(u_n,u_{n+1})h^n\left(M(u_o,u_1)+M_{u_o,u_1}\right)\\ &+ \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\mu(u_j,u_m)\right)\alpha(u_i,u_{i+1})h^i\left[(M(u_o,u_1)-M_{u_0,u_1})\right].\end{aligned}$$

By using the assumption used in (3.8) and the ratio test to the series derived in the above inequality as in Theorem 3.7, the sequence $\{u_n\}$ is Cauchy. By the completeness of double controlled M-metric space, there exists $u\in E$ such that

$$\lim_{n,m\to\infty}(M(u_n,u_m)-M_{u_n,u_m})=0.$$

Thus u is a fixed point of R, which follows from the same procedure as we done in Theorem 3.7. □

Example 3.12. Let $E=[0,\infty)$ and $R:E\to E$ be defined by $R(u)=\frac{u}{2+2u}$ and $M(u,v)=\frac{(u+v)^2}{2}$. Then $M(u,Rv)=M(Ru,v)=(u+v)^2$. If either one of the elements in the form of Ru or Rv, we get

$$\begin{aligned} M(Ru,Rv) &= \frac{(Ru+Rv)^2}{2},\\ M(Ru,Rv) &= \frac{(\frac{\lambda}{2+2\lambda}+\frac{\lambda}{2+2\lambda})^2}{4},\\ M(Ru,Rv) &\leq \frac{1}{2}\times\frac{(u+v)^2}{2}\leq\frac{1}{2}\times(u+v)^2,\\ M(Ru,Rv) &\leq h\max\{M(u,Rv),M(Ru,v)\},\ \text{where } h=\frac{1}{2}.\end{aligned}$$

Define $\alpha(u,v)=\mu(u,v)=1+u+v$. Now by induction it is not difficult to deduce that

$$\lambda_n=f^n(\lambda)=\frac{\lambda}{2^n+(\sum_{i=1}^n 2^k)\lambda}$$

for all $n\in N$. Thus

$$\lim_{n\to\infty}\alpha(\lambda,\lambda_n)=\lim_{n\to\infty}\alpha(\lambda_n,\lambda)=1+\lambda.$$

On the other hand,

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha(u_{i+1},u_{i+2})}{\alpha(u_i,u_{i+1})}\mu(u_{i+1},u_m)=1+\frac{\lambda}{2+2\lambda}\leq 2=\frac{1}{h}.$$

Hence all the hypothesis of Theorem 3.11 hold. Therefore, f has a unique fixed point in E.

4. Application

Let $E = C([0,1],\mathbb{R})$ and

$$u^{'}(p) = \int_0^1 G(p,q,u^{'}(p)), \text{ for } p,q \in [0,1], \tag{4.1}$$

be a Fredholm type integral equation, where $G(p,q,u^{'}(p))$ is a continuous function from $[0,1]\times[0,1] \to \mathbb{R}$. Now define

$$\begin{aligned} M &: E \times E \to R \\ (u,v) &\to \sup_{p\in[0,1]} \frac{|u^{'}(p)+v(q)|}{2}. \end{aligned}$$

Note that (E,M) is a double controlled M-metric type space as already shown in Example 3.2.

Theorem 4.1. *Suppose that for all $u, v \in E$*

(1)

$$\begin{aligned} |G(p,q,u^{'}(p))| + |G(p,q,v(p))| &\le \sup_{p\in[0,1]} (\{|u^{'}(p)|(v(p)|\})(|u^{'}(p)+v(p)|) \\ &\le k(\sup_{p\in[0,1]} (\{|u^{'}(p)|(v(p)|\})(|u^{'}(p)+v(p)|) + R_{|u(p)v(p)|}), \end{aligned}$$

where $k \in [0, \frac{1}{(1+sup_{p,q}|G(p,q,u^{'}(p))||G(p,q,v(p))|)^2}]$ and for some $p \in [0,1]$

(2)

$$G\left(p,q,\int_0^1 G(p,q,u^{'}(p))dq\right) \le G(p,q,u^{'}(p))$$

for all p,q. Then the above integral integral has a unique solution.

Proof. Let $R : E \to E$ be defined by $Ru^{'}(p) = \int_0^1 G(p,q,u^{'}(p))$. Then

$$M(R(u),R(v)) = \sup_{p\in[0,1]} \left(\frac{|Ru^{'}(p)+Rv(p)|}{2}\right).$$

So we get

$$\begin{aligned} \frac{|Ru^{'}(p)+Rv(p)|}{2} &= \frac{\left|\int_0^1 G(p,q,u^{'}(p))dq\right| + \left|\int_0^1 G(p,q,v(p))dq\right|}{2} \\ &\le \frac{\int_0^1 \left|G(p,q,u^{'}(p))\right| dq + \int_0^1 |G(p,q,v(p))|\, dq}{2} \\ &= \frac{\int_0^1 \left(\left|G(p,q,u^{'}(p))\right| + |G(p,q,v(p))|\right) dq}{2} \\ &\le \frac{\int_0^1 \sup_{p\in[0,1]}(\{|u^{'}(p)|(v(p)|\})(|u^{'}(p)+v(p)|)dq}{2} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\int_0^1 k(\sup_{p\in[0,1]}(\{|u^{'}(p)|(v(p)|\}|u(p)+v(p)|))dq}{2} \\
&\quad +k(R_{|u(p)v(p)|}) \text{ using } (1), \\
\frac{|Ru(p)+Rv(p)|}{2} &\leq k(M(u,v)+M_{u,v}), \\
M(Ru,Rv) &\leq k(M(u,v)+M_{u,v}).
\end{aligned}
$$

Furthermore, let $n \in N$ and $u \in E$. Then

$$
\begin{aligned}
R^n u(p) &= R(R^{n-1}u^{'}(p) = \int_0^1 G(p,q,R^{n-1}u^{'}(p)) \\
&= \int_0^1 G(p,q,R(R^{n-2}(u(p)) \leq \int_0^1 G(p,q,\int_0^1 G(p,q,R^{n-2}(u^{'}(p))ds \\
&\leq \int_0^1 G(p,q,R^{n-2}(u^{'}(p)))ds = R^{n-1}u^{'}(p).
\end{aligned}
$$

Thus for all $p \in [0,1]$ we find that the sequences $\{R^n u^{'}(p)\}$ is bounded below and strictly decreasing and so it converges to some l. Since $\{R^n\}_n$ is monotonic, by using Denis theorem, the sup value converges to some point l. Observe that $M(R(u),R(v)) = \sup_{p\in[0,1]}\left(\frac{|Ru(p)+Rv(p)|}{2}\right)$ converges to l. So all the hypothesis of Theorem 3.7 are satisfied and the Eq (4.1) has a unique solution. □

5. Conclusions

This paper dealt with the achievement of introducing the notion of double controlled m-metric spaces as a generalization of extended m-b-metric space and studiedg some of its properties and results. Moreover, some fixed points have been investigated for mapping satisfying different conditions in this new frame work. This idea can be applied for further investigations in studying fixed points for other structures in metric spaces.

Conflict of interest

The authors declare that they have no competing interests.

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