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*Research article*

# Stationary Kirchhoff equations and systems with reaction terms

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**Abstract:** In this paper, the operator approach based on the fixed point principles of Banach and Schaefer is used to establish the existence of solutions to stationary Kirchhoff equations with reaction terms. Next, for a coupled system of Kirchhoff equations, it is proved that under suitable assumptions, there exists a unique solution which is a Nash equilibrium with respect to the energy functionals associated to the equations of the system. Both global Nash equilibrium, in the whole space, and local Nash equilibrium, in balls are established. The solution is obtained by using an iterative process based on Ekeland's variational principle and whose development simulates a noncooperative game.

**Keywords:** Kirchhoff equation; fixed point principle; Nash equilibrium; Ekeland variational principle
**Mathematics Subject Classification:** 35J65, 47J25

## 1. Introduction

The famous Kirchhoff equation [1]

$$u_{tt} - \left(a + b \int_{\Omega} u'^2 dx\right) u'' = h(t, x)$$

$(a, b > 0)$ is an extension of the classical D'Alembert's wave equation for vibrations of elastic strings, which takes into account the changes in mass density and/or tension force of the string produced by transverse vibrations. In higher dimensions, the equation reads as follows

$$u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = h(t, x).$$

One can also consider the parabolic type equation

$$u_t - \left(a + b \int_\Omega |\nabla u|^2 \, dx\right) \Delta u = h(t, x)$$

which models diffusion processes with a diffusion coefficient globally dependent on gradient.

Several authors (see, e.g., [2–8]) have considered a more general Kirchhoff type equation, by replacing the integral factor $a + b\,|\nabla u|_{L^2}^2$ with an expression of the form $\eta(|\nabla u|_{L^2})$, where $\eta$ is an increasing and nonnegative function.

Kirchhoff type problems are referred to be nonlocal due to the presence of the integral over the entire $\Omega$, and due to this specificity, some difficulties arise in their investigation.

The study of such equations and systems have been made using variational and topological methods, as well as upper and lower solution techniques (see, e.g., [9–20] and the references therein).

In this paper, we first study the Dirichlet problem for a stationary integro-differential equation of Kirchhoff type with a reaction external force term, on a bounded domain $\Omega \subset \mathbb{R}^n$,

$$\begin{cases} -\left(a + b \int_\Omega |\nabla u|^2 \, dx\right) \Delta u = f + g(x, u, \nabla u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases}$$

and next we focus on the Dirichlet problem for a coupled system of Kirchhoff equations

$$\begin{cases} -\left(a + b\,|u|_{H_0^1}^2\right) \Delta u = f_1 + g_1(x, u, v) \\ -\left(a + b\,|v|_{H_0^1}^2\right) \Delta v = f_2 + g_2(x, u, v) \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

for which the solution is a Nash type equilibrium.

To our knowledge, Nash equilibria of system (1.1) have not been considered so far, and our objective is to provide sufficient conditions for such solutions to exist. To this aim we use the approach initiated in [21] (see also [22–27]). The idea is to put system (1.1) in an operator form, as a fixed point system,

$$\begin{cases} N_1(u, v) = u \\ N_2(u, v) = v\,, \end{cases} \tag{1.2}$$

where the operators $N_1$ and $N_2$ admit a variational structure, i.e., there exist (energy) functionals $E_1(u, v)$ and $E_2(u, v)$ such that system (1.2) is equivalent with

$$\begin{cases} E_{11}(u, v) = 0 \\ E_{22}(u, v) = 0\,. \end{cases} \tag{1.3}$$

where $E_{11}$ is the partial Fréchet derivative of $E_1$ with respect to the first variable and $E_{22}$ is the partial Fréchet derivative of $E_2$ with respect to the second variable. A solution $(u^*, v^*)$ of (1.1) is a *Nash equilibrium* if

$$\begin{cases} E_1(u^*, v^*) = \inf E_1(\cdot, v^*) \\ E_2(u^*, v^*) = \inf E_2(u^*, \cdot)\,. \end{cases}$$

The notion of a Nash equilibrium originated in game theory and economics, where a number of players or traders with their own costing criteria are in competition and each aims to optimize its cost in relation to the others. When no one can further improve his criterion, it means that the system has reached a Nash equilibrium state. Such kind of situations also hold for systems modeling real processes from physics, biology etc., when stationary states are Nash equilibria for the associated energy functionals.

Non-cooperative games in which the players move alternately suggest an iterative method based on Ekeland's variational principle for finding and approximating Nash equilibria. The convergence of the iterative process is established by using unilateral Lipschitz conditions on the reaction terms and working techniques with inverse-positive matrices.

The outline of this paper is as follows: Section 3 provides a comprehensive picture of the theoretical aspects of the Kirchhoff solution operator for the Dirichlet problem. Section 4 is dedicated to the Dirichlet problem for the stationary Kirchhoff equation with a reaction force term; the existence of solutions is established via Banach contraction principle and Schaefer's fixed point theorem. Finally in Section 5 there are provided sufficient conditions for a system of two Kirchhoff equations to admit a Nash equilibrium.

## 2. Preliminaries

In this section we collect a number of notions and results that will be used in the following.

First we recall the weak form of Ekeland's variational principle (see, e.g., [28, Corollary 8.1]).

**Theorem 1** (Ekeland)**.** *Let $(X, d)$ be a complete metric space and $E : X \to \mathbb{R}$ a lower semicontinuous functional bounded from below. For each $\varepsilon > 0$, there is an element $x \in X$ such that the following two properties hold:*

$$\begin{aligned} E(x) &\leq \inf_{y \in X} E(y) + \varepsilon, \\ E(x) &\leq E(y) + \varepsilon d(x, y) \quad \text{for all } y \in X. \end{aligned}$$

Next we recall Perov's fixed point theorem (see, e.g., [28, pp 151–154]) for mappings defined on the Cartesian product of two metric spaces.

**Theorem 2** (Perov)**.** *Let $(X_i, d_i)$, $i = 1, 2$ be complete metric spaces and $N_i : X_1 \times X_2 \to X_i$ be two mappings for which there exists a square matrix $M$ of size two with nonnegative entries and the spectral radius $\rho(M) < 1$ such that the following vector inequality*

$$\begin{pmatrix} d_1(N_1(x, y), N_1(u, v)) \\ d_2(N_2(x, y), N_2(u, v)) \end{pmatrix} \leq M \begin{pmatrix} d_1(x, y) \\ d_2(u, v) \end{pmatrix}$$

*holds for all $(x, y), (u, v) \in X_1 \times X_2$. Then there exists a unique point $(x^*, y^*) \in X_1 \times X_2$ with $x^* = N_1(x^*, y^*)$ and $y^* = N_2(x^*, y^*)$. Moreover, the point $(x^*, y^*)$ can be obtained by the method of successive approximations starting from any initial point $(x_0, y_0)$, and*

$$\begin{pmatrix} d_1\left(N_1^k(x_0, y_0), x^*\right) \\ d_2\left(N_2^k(x_0, y_0), y^*\right) \end{pmatrix} \leq M^k (I - M)^{-1} \begin{pmatrix} d_1(x_0, N_1(x_0, y_0)) \\ d_2(y_0, N_2(x_0, y_0)) \end{pmatrix}$$

*for every $k \in \mathbb{N}$.*

Here $I$ stands for the unit matrix of size two. Note that the property of a square matrix $M$ with nonnegative entries of having the spectral radius $\rho(M)$ less than 1 is equivalent to each one of the properties: (a) $M^k$ tends to the zero matrix as $k \to +\infty$; (b) The matrix $I - M$ is nonsingular and the entries of its inverse $(I-M)^{-1}$ are nonnegative.

For our Kirchhoff system (1.1) both fixed point and critical point formulations ((1.2) and (1.3)) being available, both Perov approach and Ekeland variational approach can be used. The first approach offers the approximation procedure for the solution given by the method of successive approximations, while by the second approach, an approximation procedure more appropriate to the concept of Nash equilibrium can be established.

We conclude this preliminary section by some notations and results related to Laplacian. For details we refer the reader to the book [29]. We consider the well-known Sobolev space $H_0^1(\Omega)$ whose scalar product and norm are

$$(u,v)_{H_0^1} = \int_\Omega \nabla u \cdot \nabla v dx, \quad |u|_{H_0^1} = |\nabla u|_{L^2} = \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}.$$

The notation $H^{-1}(\Omega)$ stands for the dual of $H_0^1(\Omega)$ and for any $f \in H^{-1}(\Omega)$, $u \in H_0^1(\Omega)$, by $(f,u)$ we mean the value at $u$ of the continuous linear functional $f$. One has the continuous embeddings $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ and the Poincaré inequalities

$$\begin{aligned} |u|_{L^2} &\leq \frac{1}{\sqrt{\lambda_1}} |u|_{H_0^1} \quad \left(u \in H_0^1(\Omega)\right), \\ |u|_{H^{-1}} &\leq \frac{1}{\sqrt{\lambda_1}} |u|_{L^2} \quad \left(u \in L^2(\Omega)\right), \end{aligned}$$

where $\lambda_1$ is the first eigenvalue of the Dirichlet problem for the operator $-\Delta$. We use the notation $(-\Delta)^{-1}$ for the inverse of the Laplacian with respect to the homogeneous Dirichlet boundary condition. More exactly, $(-\Delta)^{-1} f = u$, where $u$ is the unique function in $H_0^1(\Omega)$ satisfying $(u,v)_{H_0^1} = (f,v)$ for all $v \in H_0^1(\Omega)$, i.e., $u$ is the weak solution of the Dirichlet problem $-\Delta u = f$ in $\Omega$; $u = 0$ on $\partial\Omega$. Recall that $(-\Delta)^{-1}$ is an isometry between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

## 3. Stationary Kirchhoff equations

### *3.1. The Kirchhoff solution operator*

First we focus on the stationary equation

$$-\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = h$$

under the Dirichlet condition $u = 0$ on $\partial\Omega$.

**Theorem 3.** *(The solution operator) For each $h \in H^{-1}(\Omega)$, the Dirichlet problem has a unique weak solution $u \in H_0^1(\Omega)$, i.e.,*

$$\left(a + b\int_\Omega |\nabla u|^2 dx\right)(u,v)_{H_0^1} = (h,v), \quad v \in H_0^1(\Omega), \tag{3.1}$$

*and the solution operator* $S : H^{-1}(\Omega) \to H_0^1(\Omega)$, $h \longmapsto u$, *is continuous and*

$$|S(h)|_{H_0^1} \leq \frac{1}{a} |h|_{H^{-1}}. \tag{3.2}$$

*Proof.* (a) Existence: Let $h \in H^{-1}(\Omega)$ be fixed and consider the operator $S_h : H_0^1(\Omega) \to H_0^1(\Omega)$ defined by

$$S_h(v) = \frac{1}{a + b|v|_{H_0^1}^2} (-\Delta)^{-1} h.$$

Clearly, $S_h$ is completely continuous. In addition,

$$|S_h(v)|_{H_0^1} \leq \frac{1}{a} |h|_{H^{-1}}, \quad v \in H_0^1(\Omega). \tag{3.3}$$

Hence, if we denote $B = \left\{ v \in H_0^1(\Omega) : |v|_{H_0^1} \leq \frac{1}{a} |h|_{H^{-1}} \right\}$, then $S_h(B) \subset B$ and according to Schauder's fixed point theorem, there exists at least one $u$ such that $S_h(u) = u$. Clearly $u$ is a solution of the Dirichlet problem.

(b) Uniqueness: Assume that $u_1, u_2$ are two solutions of (3.1). Then

$$\begin{aligned} \left(a + b|u_1|_{H_0^1}^2\right)|u_1|_{H_0^1}^2 &= (h, u_1), \\ \left(a + b|u_2|_{H_0^1}^2\right)(u_1, u_2)_{H_0^1} &= (h, u_1). \end{aligned}$$

It follows that

$$\begin{aligned} \left(a + b|u_1|_{H_0^1}^2\right)|u_1|_{H_0^1}^2 &= \left(a + b|u_2|_{H_0^1}^2\right)(u_1, u_2)_{H_0^1} \\ &\leq \left(a + b|u_2|_{H_0^1}^2\right)|u_1|_{H_0^1} |u_2|_{H_0^1}. \end{aligned}$$

Hence

$$\left(a + b|u_1|_{H_0^1}^2\right)|u_1|_{H_0^1} \leq \left(a + b|u_2|_{H_0^1}^2\right)|u_2|_{H_0^1}.$$

The function $\left(a + bx^2\right)x$ being strictly increasing on $\mathbb{R}_+$, we have that $|u_1|_{H_0^1} \leq |u_2|_{H_0^1}$. By symmetry the converse inequality also holds. Thus $|u_1|_{H_0^1} = |u_2|_{H_0^1}$. Now the uniqueness of solution for the Dirichlet problem related to $-\Delta$ yields $u_1 = u_2$.

(c) Continuity: Let $h_k \to h$ in $H^{-1}(\Omega)$ and let $u_k := S(h_k)$. Using (3.3) we have that the sequence $(u_k)$ is bounded. Hence, passing if necessary to a subsequence, we may assume that the sequence of real numbers $(|u_k|)$ is convergent. We now prove that the sequence $(u_k)$ is Cauchy. From

$$-\Delta u_k = \frac{1}{a + b|u_k|_{H_0^1}^2} h_k,$$

we have

$$-\Delta\left(u_k - u_p\right) = \frac{1}{a + b|u_k|_{H_0^1}^2} h_k - \frac{1}{a + b\left|u_p\right|_{H_0^1}^2} h_p$$

in the weak sense. Consequently

$$\begin{aligned}\left|u_k-u_p\right|^2_{H_0^1} &= \left(\frac{1}{a+b\left|u_k\right|^2_{H_0^1}}h_k-\frac{1}{a+b\left|u_p\right|^2_{H_0^1}}h_p,\ u_k-u_p\right)\\ &= \frac{1}{a+b\left|u_k\right|^2_{H_0^1}}\left(h_k-h_p,\ u_k-u_p\right)\\ &\quad+\left(\frac{1}{a+b\left|u_k\right|^2_{H_0^1}}-\frac{1}{a+b\left|u_p\right|^2_{H_0^1}}\right)\left(h_p,u_k-u_p\right).\end{aligned}$$

Furthermore

$$\left|u_k-u_p\right|^2_{H_0^1}\le\frac{1}{a}\left|h_k-h_p\right|_{H^{-1}}\left|u_k-u_p\right|_{H_0^1}+\frac{b}{a^2}\left|\left|u_k\right|^2_{H_0^1}-\left|u_p\right|^2_{H_0^1}\right|\left|h_p\right|_{H^{-1}}\left|u_k-u_p\right|_{H_0^1},$$

whence

$$\left|u_k-u_p\right|_{H_0^1}\le\frac{1}{a}\left|h_k-h_p\right|_{H^{-1}}+\frac{b}{a^2}\left|\left|u_k\right|^2_{H_0^1}-\left|u_p\right|^2_{H_0^1}\right|\left|h_p\right|_{H^{-1}}.$$

Since $\left|h_p\right|_{H^{-1}}$ is bounded, $(h_k)$ and $\left(\left|u_k\right|^2_{H_0^1}\right)$ are convergent, one immediately obtain that $(u_k)$ is Cauchy. Hence there is $u$ with $u_k\to u$ and passing to the limit we see that $u=S(h)$. Finally the uniqueness of the solution implies that the whole sequence $(u_k)$ converges to $S(h)$, that is $S(h_k)\to S(h)$. □

**Theorem 4.** *(Monotonicity) If* $0\le h_1\le h_2$, *then* $\left|S(h_1)\right|_{H_0^1}\le\left|S(h_2)\right|_{H_0^1}$.

*Proof.* Denote $u:=S(h_1)$ and $v=S(h_2)$. Since $h_1,h_2\ge0$, one has $u,v\ge0$. Then

$$\left(1+|u|^2_{H_0^1}\right)|u|^2_{H_0^1}=(h_1,u)\le(h_2,u)=\left(1+|v|^2_{H_0^1}\right)(u,v)\le\left(1+|v|^2_{H_0^1}\right)|u|_{H_0^1}|v|_{H_0^1}$$

which gives

$$\left(1+|u|^2_{H_0^1}\right)|u|_{H_0^1}\le\left(1+|v|^2_{H_0^1}\right)|v|_{H_0^1},$$

whence $|u|_{H_0^1}\le|v|_{H_0^1}$. □

**Theorem 5.** *(The energy functional) A function* $u\in H_0^1(\Omega)$ *is the weak solution of the Dirichlet problem if and only if it is a critical point of the* $C^1$ *functional* $E:H_0^1(\Omega)\to\mathbb{R}$,

$$E(v)=\frac{1}{4}\left(2a+b|v|^2_{H_0^1}\right)|v|^2_{H_0^1}-(h,v).\tag{3.4}$$

*Proof.* One has

$$\begin{aligned}|v+\lambda w|^2_{H_0^1}-|v|^2_{H_0^1} &= 2\lambda(v,w)_{H_0^1}+\lambda^2|w|^2_{H_0^1},\\ |v+\lambda w|^4_{H_0^1}-|v|^4_{H_0^1} &= \left(|v+\lambda w|^2_{H_0^1}-|v|^2_{H_0^1}\right)\left(|v+\lambda w|^2_{H_0^1}+|v|^2_{H_0^1}\right)\\ &= \left(2\lambda(v,w)_{H_0^1}+\lambda^2|w|^2_{H_0^1}\right)\left(|v+\lambda w|^2_{H_0^1}+|v|^2_{H_0^1}\right).\end{aligned}$$

Consequently

$$\lim_{\lambda\to 0}\frac{E(v+\lambda w)-E(v)}{\lambda}=\left(a+b\,|v|^2_{H_0^1}\right)(v,w)_{H_0^1}-(h,w)\,.$$

Hence

$$\left(E'(v)\,,w\right)=\left(a+b\,|v|^2_{H_0^1}\right)(v,w)_{H_0^1}-(h,w)\,. \tag{3.5}$$

□

**Theorem 6.** *Function $u\in H_0^1(\Omega)$ solves the Dirichlet problem if and only if it minimizes the energy functional (3.4).*

*Proof.* If $u$ is a minimum point of $E$, then $E'(u)=0$ and according to (3.5) it solves the problem. Assume now that $u$ is a solution. Then for every $v$, by direct computation, we have

$$\begin{aligned}
E(u+v) &= E(u)+\left(a+b\,|u|^2_{H_0^1}\right)(u,v)_{H_0^1}-(h,v)\\
&\quad+\frac{a}{2}|v|^2+\frac{b}{4}\left(|v|^4+2|u|^2|v|^2+4(u,v)^2+4|v|^2(u,v)^2\right)\\
&= E(u)+\frac{a}{2}|v|^2+\frac{b}{4}\left(|v|^4+2|u|^2|v|^2+4(u,v)^2+4|v|^2(u,v)^2\right)\\
&\geq E(u)+\frac{a}{2}|v|^2+\frac{b}{4}\left(\left(|v|^2+2(u,v)\right)^2+2|u|^2|v|^2\right)>0
\end{aligned}$$

for every $v\neq 0$. Hence $u$ is the unique minimum point of $E$. □

## 4. Kirchhoff equations with reaction terms

Consider the Dirichlet problem

$$\begin{cases} -\left(a+b\int_\Omega|\nabla u|^2\,dx\right)\Delta u=f+g(x,u,\nabla u) & \text{in }\Omega\\ u=0 \quad \text{on }\partial\Omega\,. \end{cases} \tag{4.1}$$

Here $\Omega\subset\mathbb{R}^n$ is open bounded, $f\in H^{-1}(\Omega)$, $g:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ satisfies the Carathéodory conditions and $g(\cdot,0,0)=0$.

We look for weak solutions to (4.1), namely $u\in H_0^1(\Omega)$ with $g(\cdot,u,\nabla u)\in H^{-1}(\Omega)$ and

$$\left(a+b\int_\Omega|\nabla u|^2\,dx\right)(u,v)_{H_0^1}=\left(f+g(\cdot,u,\nabla u)\,,\ v\right)\quad\text{for all }v\in H_0^1(\Omega)\,.$$

A function $u\in H_0^1(\Omega)$ is a weak solution of (4.1) if

$$u=\frac{1}{a+b\int_\Omega|\nabla u|^2\,dx}(-\Delta)^{-1}\left(f+g(\cdot,u,\nabla u)\right),$$

that is $u$ is a fixed point of the operator

$$A(u)=S\left(f+g(\cdot,u,\nabla u)\right).$$

*4.1. Existence and uniqueness of solution*

We apply Banach contraction principle. Assume the Lipschitz condition

**(HL)**

$$|g(x,u,v)-g(x,\overline{u},\overline{v})|\leq L_1|u-\overline{u}|+L_2|v-\overline{v}|$$

for all $u,\overline{u}\in\mathbb{R}$, $v,\overline{v}\in\mathbb{R}^n$ and a.e. $x\in\Omega$, where

$$\theta:=\frac{1}{a}\left(\frac{L_1}{\lambda_1}+\frac{L_2}{\sqrt{\lambda_1}}\right)<1. \tag{4.2}$$

**Step 1: Invariance of a ball.**

We prove that if $L_1,\ L_2$ are small, then for any large enough number $R$, one has $|A(u)|_{H_0^1}\leq R$ for all $u\in H_0^1(\Omega)$ with $|u|_{H_0^1}\leq R$. According with (3.2), using (HL) and Poincaré's inequality, one has

$$\begin{aligned}|A(u)|_{H_0^1} &= |S(f+g(\cdot,u,\nabla u))|_{H_0^1}\leq\frac{1}{a}|f+g(\cdot,u,\nabla u)|_{H^{-1}}\\ &\leq \frac{1}{a}\left(|f|_{H^{-1}}+|g(\cdot,u,\nabla u)|_{H^{-1}}\right)\leq\frac{1}{a}\left(|f|_{H^{-1}}+\frac{1}{\sqrt{\lambda_1}}|g(\cdot,u,\nabla u)|_{L^2}\right)\\ &\leq \frac{1}{a}\left(|f|_{H^{-1}}+\frac{1}{\sqrt{\lambda_1}}\left(L_1|u|_{L^2}+L_2|\nabla u|_{L^2}\right)\right)\\ &\leq \frac{1}{a}|f|_{H^{-1}}+\frac{1}{a}\left(\frac{L_1}{\lambda_1}+\frac{L_2}{\sqrt{\lambda_1}}\right)|u|_{H_0^1}.\end{aligned}$$

Hence in virtue of (4.2), the invariance condition holds for any number $R\geq|f|_{H^{-1}}/(a(1-\theta))$.

**Step 2: Contraction condition.**

Fix any number $R$ as guaranteed at the previous step. Let $u,v\in H_0^1(\Omega)$ with $|u|_{H_0^1},\ |v|_{H_0^1}\leq R$ be arbitrary and let $w=S(f+g(\cdot,u,\nabla u))$ and $z=S(f+g(\cdot,v,\nabla v))$. Assume without loss of generality that $|w|_{H_0^1}\geq|z|_{H_0^1}$. Then

$$\begin{aligned}\left(a+b|w|_{H_0^1}^2\right)|w|_{H_0^1}^2 &= (f+g(.,u,\nabla u),w),\\ \left(a+b|z|_{H_0^1}^2\right)(w,z)_{H_0^1} &= (f+g(.,v,\nabla v),w),\end{aligned}$$

whence

$$\left(a+b|w|_{H_0^1}^2\right)|w|_{H_0^1}^2-\left(a+b|z|_{H_0^1}^2\right)(w,z)_{H_0^1}=(g(.,u,\nabla u)-g(.,v,\nabla v),\ w).$$

For the left side, one has

$$\left(a+b|w|_{H_0^1}^2\right)|w|_{H_0^1}^2-\left(a+b|z|_{H_0^1}^2\right)(w,z)_{H_0^1}\geq\left(a+b|w|_{H_0^1}^2\right)|w|_{H_0^1}^2-\left(a+b|z|_{H_0^1}^2\right)|w|_{H_0^1}|z|_{H_0^1}$$

and for the right side

$$\begin{aligned}(g(.,u,\nabla u)-g(.,v,\nabla v),\ w) &\leq |g(.,u,\nabla u)-g(.,v,\nabla v)|_{L^2}|w|_{L^2}\\ &\leq \left(\frac{L_1}{\lambda_1}+\frac{L_2}{\sqrt{\lambda_1}}\right)|u-v|_{H_0^1}|w|_{H_0^1}.\end{aligned}$$

Hence

$$\left(a+b\,|w|^2_{H^1_0}\right)|w|^2_{H^1_0}-\left(a+b\,|z|^2_{H^1_0}\right)|w|_{H^1_0}\,|z|_{H^1_0}\leq\left(\frac{L_1}{\lambda_1}+\frac{L_2}{\sqrt{\lambda_1}}\right)|u-v|_{H^1_0}\,|w|_{H^1_0}$$

and since $|w|_{H^1_0}\geq|z|_{H^1_0}$,

$$0\leq a\left(|w|_{H^1_0}-|z|_{H^1_0}\right)\leq\left(a+b\,|w|^2_{H^1_0}\right)|w|_{H^1_0}-\left(a+b\,|z|^2_{H^1_0}\right)|z|_{H^1_0}\leq\left(\frac{L_1}{\lambda_1}+\frac{L_2}{\sqrt{\lambda_1}}\right)|u-v|_{H^1_0}\,.$$

Consequently

$$0\leq|w|_{H^1_0}-|z|_{H^1_0}\leq\theta\,|u-v|_{H^1_0}\,.$$

On the other hand, from

$$\begin{aligned}\left(a+b\,|w|^2_{H^1_0}\right)(w,w-z)_{H^1_0}&=\left(f+g\,(.,u,\nabla u)\,,\;w-z\right),\\ \left(a+b\,|z|^2_{H^1_0}\right)(z,w-z)_{H^1_0}&=\left(f+g\,(.,v,\nabla v)\,,\;w-z\right),\end{aligned}$$

we deduce that

$$\begin{aligned}|w-z|^2_{H^1_0}&=\left(\frac{f+g\,(.,u,\nabla u)}{a+b\,|w|^2_{H^1_0}}-\frac{f+g\,(.,v,\nabla v)}{a+b\,|z|^2_{H^1_0}},\;w-z\right)\\ &=\frac{1}{a+b\,|w|^2_{H^1_0}}\left(g\,(.,u,\nabla u)-g\,(.,v,\nabla v)\,,\;w-z\right)\\ &\quad+\left(\frac{1}{a+b\,|w|^2_{H^1_0}}-\frac{1}{a+b\,|z|^2_{H^1_0}}\right)\left(f+g\,(.,v,\nabla v)\,,\;w-z\right).\end{aligned}$$

We have

$$\left(g\,(.,u,\nabla u)-g\,(.,v,\nabla v)\,,\;w-z\right)\leq\left(\frac{L_1}{\lambda_1}+\frac{L_2}{\sqrt{\lambda_1}}\right)|u-v|_{H^1_0}\,|w-z|_{H^1_0}$$

and

$$\begin{aligned}&\left(\frac{1}{a+b\,|w|^2_{H^1_0}}-\frac{1}{a+b\,|z|^2_{H^1_0}}\right)\left(f+g\,(.,v,\nabla v)\,,\;w-z\right)\\ \leq\;&b\left(|w|_{H^1_0}-|z|_{H^1_0}\right)\frac{|w|_{H^1_0}+|z|_{H^1_0}}{\left(a+b\,|w|^2_{H^1_0}\right)\left(a+b\,|z|^2_{H^1_0}\right)}\,|f+g\,(.,v,\nabla v)|_{H^{-1}}\,|w-z|_{H^1_0}\,.\end{aligned}$$

Since

$$\frac{|w|_{H^1_0}+|z|_{H^1_0}}{\left(a+b\,|w|^2_{H^1_0}\right)\left(a+b\,|z|^2_{H^1_0}\right)}\leq\frac{3\sqrt{3}}{8}\frac{1}{a\sqrt{ab}}$$

and

$$|g\,(.,v,\nabla v)|_{H^{-1}}\leq a\theta R,$$

we obtain

$$|w-z|_{H_0^1} \le \theta\left(1+\theta\frac{3\sqrt{3}}{8a}\sqrt{\frac{b}{a}}\left(|f|_{H^{-1}}+a\theta R\right)\right)|u-v|_{H_0^1}\,.$$

Hence if

**(HC)**

$$\theta\left(1+\theta\frac{3\sqrt{3}}{8a}\sqrt{\frac{b}{a}}\left(|f|_{H^{-1}}+a\theta R\right)\right)<1,$$

then the operator $A$ is a contraction on the ball of $H_0^1(\Omega)$ centered at the origin and of radius $R$. Notice that condition (HC) is fulfilled for example if $\theta<1$ (invariance condition for the ball of radius $R$) and $b$ is small enough.

Thus Banach's contraction principle applied to operator $A$ in the ball of radius $R$ yields the following existence and uniqueness result.

**Theorem 7.** *Assume that conditions* (HL) *and* (HC) *hold. Then problem* (4.1) *has a unique solution u such that*

$$|u|_{H_0^1} \le |f|_{H^{-1}}\,/\,(a\,(1-\theta))\,.$$

**Example 8.** Consider the Dirichlet problem,

$$\begin{cases} -\left(4+\int_{\mathcal{B}}|\nabla u|^2dx\right)\Delta u = \frac{2}{|x|}+\lambda_1\,u+\sqrt{\lambda_1}\,\sin|\nabla u| \quad \text{on } \mathcal{B} \\ u|_{\partial\mathcal{B}}=0\,, \end{cases} \tag{4.3}$$

where $\Omega=\mathcal{B}$ and $\mathcal{B}$ is the open ball centered at the origin of $\mathbb{R}^n$ and of radius $\rho$ whose measure equals 1. Here

$$a=4,\ b=1,\ f(x)=\frac{2}{|x|}\ \text{ and }\ g(x,u,v)=\lambda_1\,u+\sqrt{\lambda_1}\,\sin|v|\,,$$

for $u\in\mathbb{R}$ and $v\in\mathbb{R}^n$. Note that $f\in H^{-1}(\mathcal{B})$ with $|f|_{H^{-1}}=1$. Indeed, the function $u_0(x)=|x|-1$ is the weak solution of Dirichlet problem $-\Delta u=f$ in $\mathcal{B}$, $u|_{\partial\mathcal{B}}=0$ and consequently

$$|f|_{H^{-1}}=|u_0|_{H_0^1}=|\nabla u_0|_{L^2}=\left|\frac{x}{|x|}\right|_{L^2}=1.$$

Clearly, $g$ is a Carathéodory function, $g(\cdot,0,0)=0$ and satisfies condition (HL) with $L_1=\lambda_1$ and $L_2=\sqrt{\lambda_1}$ and $\theta=2/a=1/2$.

For $R=|f|_{H^{-1}}\,/\,(a\,(1-\theta))=1\,/\,2$, the condition (HC) is fulfilled, since

$$\theta\left(1+\theta\frac{3\sqrt{3}}{8a}\sqrt{\frac{b}{a}}\left(|f|_{H^{-1}}+a\theta R\right)\right)=\frac{1}{2}\left(1+\frac{3\sqrt{3}}{64}\right)<1.$$

Therefore, the problem (4.3) has a unique solution $u\in H_0^1(\mathcal{B})$ with $|u|_{H_0^1}\le 1/2$.

## 4.2. *Existence via Schaefer's fixed point theorem*

**Step 1: Complete continuity of the operator** $A : H_0^1(\Omega) \to H_0^1(\Omega)$.

Recall that $(-\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ is an isometry between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. This implies that the operator $A$ is completely continuous if the operator

$$u \mapsto B(u) := g(\cdot, u, \nabla u)$$

is well-defined and completely continuous from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$.

Assume that $n \geq 3$. Then the embedding $H_0^1(\Omega) \subset L^p(\Omega)$ is continuous for $1 \leq p \leq 2^* = 2n/(n-2)$, and compact for $1 \leq p < 2^*$, and consequently the embedding $L^q(\Omega) \subset H^{-1}(\Omega)$ holds and is compact for $q > (2^*)' = 2n/(n+2)$.

We would like to represent $B$ as a composition of three operators: $B = JNP$, where

$$\begin{aligned} P &: H_0^1(\Omega) \to L^{2^*}(\Omega) \times L^2(\Omega; \mathbb{R}^n), \quad P(u) = (u, \nabla u), \\ N &: L^{2^*}(\Omega) \times L^2(\Omega; \mathbb{R}^n) \to L^q(\Omega), \quad N(w_1, w_2) = g(\cdot, w_1, w_2), \\ J &: L^q(\Omega) \to H^{-1}(\Omega), \quad J(v) = v. \end{aligned}$$

Clearly, since the embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$ is continuous, $P$ is a bounded linear operator. Also, if $q > (2^*)'$, then $J$ is completely continuous. It remains to clarify the case of Nemytskii's operator $N$. It suffices that $N$ is well-defined, continuous and bounded (maps bounded sets into bounded sets). To this aim, recall the main result about Nemytskii's operator (see, e.g., [29, Section 9.1]). According to this result, we need a growth condition on $g$, namely

$$|g(x, w_1, w_2)| \leq c_1 |w_1|^{\frac{2^*}{q}} + c_2 |w_2|^{\frac{2}{q}} + h(x) \quad (w_1 \in \mathbb{R},\ w_2 \in \mathbb{R}^n,\ \text{a.a. } x \in \Omega)$$

where $c_1, c_2 \in \mathbb{R}_+$ are constants and $h \in L^q(\Omega)$. Notice that instead of the exponents $2^*/q$, $2/q$ one may have smaller exponents, let they be $\alpha$ and $\beta$, hence a growth condition like

$$|g(x, w_1, w_2)| \leq c_1 |w_1|^{\alpha} + c_2 |w_2|^{\beta} + h(x) \tag{4.4}$$

with $1 \leq \alpha \leq \frac{2^*}{q}$, $1 \leq \beta \leq \frac{2}{q}$. These give some conditions on $q$ :

$$q \leq \frac{2^*}{\alpha}, \quad q \leq \frac{2}{\beta}.$$

Thus we can take

$$q = \min\left\{\frac{2^*}{\alpha}, \frac{2}{\beta}\right\}.$$

Finally, the condition $q > (2^*)'$ holds if

$$\alpha < \frac{2^*}{(2^*)'}, \quad \beta < \frac{2}{(2^*)'}.$$

Note that

$$\frac{2^*}{(2^*)'} = \frac{n+2}{n-2}, \quad \frac{2}{(2^*)'} = \frac{n+2}{n}.$$

Therefore, the operator $N$ is as desired provided that $g$ satisfies the growth condition (4.4) for

$$1 \leq \alpha < \frac{n+2}{n-2}, \quad 1 \leq \beta < \frac{n+2}{n}$$

and $h \in L^2(\Omega)$

**Step 2: A priori boundedness of solutions.**

Let $u \in H_0^1(\Omega)$ be any solution of the equation $\lambda A(u) = u$ for some $\lambda \in (0,1)$. Then $u$ is a weak solution of the problem

$$\begin{cases} -\left(a + \frac{b}{\lambda^2}\int_\Omega |\nabla u|^2\, dx\right)\Delta u = \lambda f + \lambda g(x,u,\nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence

$$\left(a + \frac{b}{\lambda^2}\int_\Omega |\nabla u|^2\, dx\right)(u,v)_{H_0^1} = (\lambda f + \lambda g(\cdot,u,\nabla u),\ v), \quad v \in H_0^1(\Omega).$$

Letting $v = u$ gives

$$\left(a + \frac{b}{\lambda^2}|u|^2_{H_0^1}\right)|u|^2_{H_0^1} = \lambda(f,u) + \lambda(g(\cdot,u,\nabla u),\ u).$$

Since $g(\cdot,u,\nabla u) \in L^q(\Omega)$, one has $(g(\cdot,u,\nabla u),\ u) = \int_\Omega ug(x,u,\nabla u)$. Assume that $g$ satisfies the sign condition

$$ug(x,u,v) \leq 0 \quad \text{for all } u \in \mathbb{R},\ v \in \mathbb{R}^n,\ \text{a.a. } x \in \Omega. \tag{4.5}$$

Then $(g(\cdot,u,\nabla u),\ u) \leq 0$ and so

$$\left(a + \frac{b}{\lambda^2}|u|^2_{H_0^1}\right)|u|^2_{H_0^1} \leq \lambda(f,u) \leq |f|_{H^{-1}}|u|_{H_0^1}.$$

Thus

$$a|u|_{H_0^1} \leq \left(a + \frac{b}{\lambda^2}|u|^2_{H_0^1}\right)|u|_{H_0^1} \leq |f|_{H^{-1}},$$

that is the solutions are bounded independently of $\lambda$, namely $|u|_{H_0^1} \leq |f|_{H^{-1}}/a$.

Therefore, based on Schaefer's fixed point theorem, we have the following existence result.

**Theorem 9.** *Assume that $g$ satisfies the growth condition* (4.4) *for some numbers* $1 \leq \alpha < (n+2)/(n-2)$, $1 \leq \beta < (n+2)/n$ *and function* $h \in L^2(\Omega)$. *Also assume that $g$ has the sign property* (4.5). *Then problem* (4.1) *has at least one weak solution $u \in H_0^1(\Omega)$ with $|u|_{H_0^1} \leq |f|_{H^{-1}}/a$.*

**Example 10.** Consider the Dirichlet problem,

$$\begin{cases} -\left(1 + \int_{\mathcal{B}} |\nabla u|^2 dx\right)\Delta u = \frac{2}{|x|} - \frac{u^3}{u^2+1} - \frac{u}{u^2+1}|\nabla u| & \text{on } \mathcal{B} \\ \quad u|_{\partial\mathcal{B}} = 0, \end{cases} \tag{4.6}$$

where $\mathcal{B}$ is as in Example 8. We apply Theorem 9. Here

$$f(x) = \frac{2}{|x|}, \quad g(\cdot,u,v) = -\frac{u^3}{u^2+1} - \frac{u}{u^2+1}|v|$$

for $u \in \mathbb{R}$ and $v \in \mathbb{R}^n$. Similarly to Example 8, one has $f \in H^{-1}(\Omega)$ and $|f|_{H^{-1}} = 1$. Moreover, $g$ satisfies the growth condition (4.4) with $\alpha = \beta = 1$ and the sign condition (4.5) since

$$|g(x,u,v)| \le |u| + \frac{1}{2}|v|$$

and

$$u\,g(x,u,v) = -\frac{u^4}{u^2+1} - \frac{u^2}{u^2+1}\,|\,v\,| \le 0,$$

for all $u \in \mathbb{R}$ and $v \in \mathbb{R}^n$. Consequently, problem (4.6) has at least one weak solution in $H_0^1(\mathcal{B})$ with $|u|_{H_0^1} \le 1$.

## 5. Nash equilibrium for Kirchhoff systems

In this section our focus is on system (1.1), where we look for a solution which is a Nash equilibrium.

### 5.1. Global Nash equilibrium

We start by an existence and uniqueness result in the whole space $H_0^1(\Omega) \times H_0^1(\Omega)$.

Each equation of system (1.1) has a variational structure given respectively by the energy functionals $E_1, E_2 \colon H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$,

$$\begin{aligned} E_1(u,v) &= \frac{1}{4}\left(2a + b\,|u|_{H_0^1}^2\right)|u|_{H_0^1}^2 - (f_1,u) - \int_\Omega G_1(x,u(x),v(x))\,dx, \\ E_2(u,v) &= \frac{1}{4}\left(2a + b\,|v|_{H_0^1}^2\right)|v|_{H_0^1}^2 - (f_2,v) - \int_\Omega G_2(x,u(x),v(x))\,dx, \end{aligned}$$

where $G_1(x,u,v) = \int_0^u g_1(x,s,v)\,ds$ and $G_2(x,u,v) = \int_0^v g_2(x,u,s)\,ds$. Using (3.5), we easily see that

$$\begin{aligned} E_{11}(u,v) &= \left(a + b\,|u|_{H_0^1}^2\right)u - (-\Delta)^{-1}(f_1 + g_1(\cdot,u,v)), \\ E_{22}(u,v) &= \left(a + b\,|v|_{H_0^1}^2\right)v - (-\Delta)^{-1}(f_2 + g_2(\cdot,u,v)), \end{aligned}$$

for every $u, v \in H_0^1(\Omega)$.

Before stating the main result of this section we introduce the following notion: A function $H : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be of *coercive-type* if the functional $\phi : H_0^1(\Omega) \to \mathbb{R}$,

$$\phi(v) = \frac{1}{4}\left(2a + b\,|v|_{H_0^1}^2\right)|v|_{H_0^1}^2 - (f_2,v) - \int_\Omega H(x,v)\,dx \tag{5.1}$$

is coercive, i.e., $\phi(v) \to +\infty$ as $|v|_{H_0^1} \to +\infty$.

We have the following result on the existence of a Nash equilibrium under unilateral Lipschitz (monotonicity type) conditions.

**Theorem 11.** *Assume that for $i = 1,2$, $f_i \in H^{-1}(\Omega)$, $g_i \colon \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function and $g_i(\cdot,0,0) = 0$. In addition assume that the following conditions are satisfied:*

**(h1)** *There exist constants* $a_{ij} \in \mathbb{R}_+$ $(i, j = 1, 2)$ *such that*

$$
\begin{aligned}
&a_{ii} < \lambda_1 a, \quad i = 1, 2, \\
&a_{12}a_{21} < (\lambda_1 a - a_{11})(\lambda_1 a - a_{22})
\end{aligned} \tag{5.2}
$$

*and*

$$
\begin{aligned}
(g_1(x,u,v) - g_1(x,\overline{u},\overline{v}))(u-\overline{u}) &\leq a_{11}|u-\overline{u}|^2 + a_{12}|u-\overline{u}||v-\overline{v}|, \\
(g_2(x,u,v) - g_2(x,\overline{u},\overline{v}))(v-\overline{v}) &\leq a_{21}|u-\overline{u}||v-\overline{v}| + a_{22}|v-\overline{v}|^2
\end{aligned} \tag{5.3}
$$

*for all* $u, v, \overline{u}, \overline{v} \in \mathbb{R}$ *and a.e.* $x \in \Omega$.

**(h2)** *There exist two functions* $H_1,\ H_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ *of coercive-type such that*

$$
H_1(x,v) \leq G_2(x,u,v) \leq H_2(x,v)
$$

*for all* $u, v \in \mathbb{R}$, *a.e.* $x \in \Omega$.

*Then system* (1.1) *has a unique solution which is a Nash equilibrium for the pair of functionals* $(E_1, E_2)$.

*Proof.* The proof follows the idea from [22]. For a clear comprehending, we structure our proof in six steps.

**Step 1:** The functionals $E_1(\cdot, v)$ and $E_2(u, \cdot)$ are bounded from below. First let us remark that from (5.3), for every $u, v \in \mathbb{R}$, there exist $\theta \in (0, 1)$ such that

$$
\begin{aligned}
G_1(x,u,v) &= \int_0^u g_1(x,s,v)\,ds = ug_1(x,\theta u,v) \\
&= \frac{1}{\theta} g_1(x,\theta u,v)\,\theta u \leq \frac{1}{\theta}\left(a_{11}|\theta u|^2 + a_{12}|\theta u||v|\right) \\
&= a_{11}\theta u^2 + a_{12}|u||v| \leq a_{11}u^2 + a_{12}|u||v|.
\end{aligned}
$$

Similarly

$$
G_2(x,u,v) \leq a_{21}|u||v| + a_{22}v^2.
$$

Now let $v \in H_0^1(\Omega)$ be fixed. For any $u \in H_0^1(\Omega)$, one has

$$
\begin{aligned}
E_1(u,v) =& \frac{1}{4}\left(2a + b|u|_{H_0^1}^2\right)|u|_{H_0^1}^2 - (f_1, u) - \int_\Omega G_1(x, u(x), v(x))\,dx \\
\geq& \frac{1}{4}\left(2a + b|u|_{H_0^1}^2\right)|u|_{H_0^1}^2 - |f_1|_{H^{-1}}|u|_{H_0^1} - \left(a_{11}|u|_{L^2}^2 + a_{12}|u|_{L^2}|v|_{L^2}\right) \\
\geq& \frac{1}{4}\left(2a + b|u|_{H_0^1}^2\right)|u|_{H_0^1}^2 - a_{11}\frac{1}{\lambda_1}|u|_{H_0^1}^2 - a_{12}\frac{1}{\lambda_1}|u|_{H_0^1}|v|_{H_0^1} - |f_1|_{H^{-1}}|u|_{H_0^1} \\
\geq& \frac{b}{4}|u|_{H_0^1}^4 + \left(\frac{a}{2} - \frac{a_{11}}{\lambda_1}\right)|u|_{H_0^1}^2 - \left(|f_1|_{H^{-1}} + \frac{a_{12}}{\lambda_1}|v|_{H_0^1}\right)|u|_{H_0^1},
\end{aligned}
$$

which is bounded from below since the coefficient of the term of forth degree of the quartic expression in $|u|_{H_0^1}$ is positive. Similarly the functional $E_2(u, \cdot)$ is bounded from below for each $u$.

**Step 2:** Construction of an approximation sequence $(u_k, v_k)$.

Now, similarly to [21], starting with an arbitrary $v_0$ and using Ekeland's variational principle, we recursively construct a sequence $(u_k, v_k) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$E_1(u_k, v_{k-1}) \leq \inf_{H_0^1(\Omega)} E_1(\cdot, v_{k-1}) + \frac{1}{k}, \quad E_2(u_k, v_k) \leq \inf_{H_0^1(\Omega)} E_2(u_k, \cdot) + \frac{1}{k},$$
$$|E_{11}(u_k, v_{k-1})|_{H_0^1} \leq \frac{1}{k}, \quad |E_{22}(u_k, v_k)|_{H_0^1} \leq \frac{1}{k}. \tag{5.4}$$

**Step 3:** Boundedness of the sequence $(v_k)$.

Let $\phi_1, \phi_2$ be the functionals of type (5.1) with $\phi$ replaced by $\phi_1$ and $\phi_2$, respectively. As coercive functionals they are bounded from below.

Obviously, for every $u, v$, one has

$$\phi_1(v) \geq E_2(u, v) \geq \phi_2(v).$$

The coerciveness of $\phi_2$ implies that there is $R > 0$ with

$$\phi_2(v) \geq \inf_{H_0^1(\Omega)} \phi_1 + 1, \quad |v|_{H_0^1} > R.$$

Since $\inf_{H_0^1(\Omega)} \phi_1 \geq \inf_{H_0^1(\Omega)} E_2(u, \cdot)$ for all $u$, we obtain

$$E_2(u, v) \geq \inf_{H_0^1(\Omega)} E_2(u, \cdot) + 1 \quad \text{for all } u, v \in H_0^1(\Omega), \ |v|_{H_0^1} > R. \tag{5.5}$$

Since for $k \geq 2$,

$$E_2(u_k, v_k) \leq \inf_{H_0^1(\Omega)} E_2(u_k, \cdot) + \frac{1}{k} < \inf_{H_0^1(\Omega)} E_2(u_k, \cdot) + 1,$$

in view of (5.5) we must have $|v_k|_{H_0^1} \leq R$, that is the boundedness of the sequence $(v_k)$.

**Step 4:** Convergence of the sequences $(u_k)$ and $(v_k)$.

For every $u, \overline{u}, v, \overline{v} \in H_0^1(\Omega)$, we have

$$\begin{aligned}
(E_{11}(u, v) - E_{11}(\overline{u}, \overline{v}), u - \overline{u})_{H_0^1} &= \left((a + b|u|_{H_0^1}^2)u - (a + b|\overline{u}|_{H_0^1}^2)\overline{u}, u - \overline{u}\right)_{H_0^1} \\
&\quad - (g_1(\cdot, u, v) - g_1(\cdot, \overline{u}, \overline{v}), \ u - \overline{u})_{L^2} \\
&= a|u - \overline{u}|_{H_0^1}^2 + b\left(|u|_{H_0^1}^2 u - |\overline{u}|_{H_0^1}^2 \overline{u}, \ u - \overline{u}\right)_{H_0^1} \\
&\quad - (g_1(\cdot, u, v) - g_1(\cdot, \overline{u}, \overline{v}), \ u - \overline{u})_{L^2}.
\end{aligned}$$

Since

$$\begin{aligned}
\left(|u|_{H_0^1}^2 u - |\overline{u}|_{H_0^1}^2 \overline{u}, \ u - \overline{u}\right)_{H_0^1} &= |u|_{H_0^1}^4 + |\overline{u}|_{H_0^1}^4 - \left(|u|_{H_0^1}^2 + |\overline{u}|_{H_0^1}^2\right)(u, \overline{u})_{H_0^1} \\
&\geq |u|_{H_0^1}^4 + |\overline{u}|_{H_0^1}^4 - \left(|u|_{H_0^1}^2 + |\overline{u}|_{H_0^1}^2\right)|u|_{H_0^1} |\overline{u}|_{H_0^1} \\
&= \left(|u|_{H_0^1}^2 + |\overline{u}|_{H_0^1}^2 + |u|_{H_0^1} |\overline{u}|_{H_0^1}\right)\left(|u|_{H_0^1} - |\overline{u}|_{H_0^1}\right)^2 \geq 0
\end{aligned}$$

we obtain

$$\begin{aligned}(E_{11}(u,v)-E_{11}(\overline{u},\overline{v}),u-\overline{u})_{H_0^1} &\geq a|u-\overline{u}|^2_{H_0^1}-(g_1(\cdot,u,v)-g_1(\cdot,\overline{u},\overline{v}),\ u-\overline{u})_{L^2}\\ &\geq a|u-\overline{u}|^2_{H_0^1}-a_{11}|u-\overline{u}|^2_{L^2}-a_{12}|u-\overline{u}|_{L^2}|v-\overline{v}|_{L^2}\\ &\geq \left(a-\frac{a_{11}}{\lambda_1}\right)|u-\overline{u}|^2_{H_0^1}-\frac{a_{12}}{\lambda_1}|u-\overline{u}|_{H_0^1}|v-\overline{v}|_{H_0^1}. \end{aligned} \tag{5.6}$$

Similarly

$$(E_{22}(u,v)-E_{22}(\overline{u},\overline{v}),v-\overline{v})_{H_0^1}\geq\left(a-\frac{a_{22}}{\lambda_1}\right)|v-\overline{v}|^2_{H_0^1}-\frac{a_{21}}{\lambda_1}|u-\overline{u}|_{H_0^1}|v-\overline{v}|_{H_0^1}. \tag{5.7}$$

On the other hand, from (5.4) we obtain

$$\Big(E_{11}(u_{k+p},v_{k+p-1})-E_{11}(u_k,v_{k-1}),u_{k+p}-u_k\Big)_{H_0^1}\leq\left(\frac{1}{k+p}+\frac{1}{k}\right)|u_{k+p}-u_k|_{H_0^1},$$

$$\Big(E_{22}(u_{k+p},v_{k+p})-E_{11}(u_k,v_k),v_{k+p}-v_k\Big)_{H_0^1}\leq\left(\frac{1}{k+p}+\frac{1}{k}\right)|v_{k+p}-v_k|_{H_0^1}.$$

Consequently, if we denote $m_{ii}=a-\frac{a_{ii}}{\lambda_1}$ $(i=1,2)$, $m_{12}=\frac{a_{12}}{\lambda_1}$ and $m_{21}=\frac{a_{21}}{\lambda_1}$, then

$$m_{11}|u_{k+p}-u_k|_{H_0^1}-m_{12}|v_{k+p-1}-v_{k-1}|_{H_0^1}\leq\frac{2}{k},\ -m_{21}|u_{k+p}-u_k|_{H_0^1}+m_{22}|v_{k+p}-v_k|_{H_0^1}\leq\frac{2}{k}. \tag{5.8}$$

Under the notations $x_{k,p}:=|u_{k+p}-u_k|_{H_0^1}$ and $y_{k,p}=|v_{k+p}-v_k|_{H_0^1}$, relations (5.8) can be put under the matrix form

$$M'\begin{bmatrix}x_{k,p}\\ y_{k,p}\end{bmatrix}\leq\frac{2}{k}\begin{bmatrix}1\\ 1\end{bmatrix}-(M-M')\begin{bmatrix}x_{k-1,p}\\ y_{k-1,p}\end{bmatrix},$$

where

$$M=\begin{bmatrix}m_{11} & -m_{12}\\ -m_{21} & m_{22}\end{bmatrix},\quad M'=\begin{bmatrix}m_{11} & 0\\ -m_{21} & m_{22}\end{bmatrix}.$$

Since $M'$ is invertible and its inverse

$$M'^{-1}=\begin{bmatrix}\frac{1}{m_{11}} & 0\\ \frac{m_{21}}{m_{11}m_{22}} & \frac{1}{m_{22}}\end{bmatrix}$$

is nonnegative, we obtain

$$\begin{bmatrix}x_{k,p}\\ y_{k,p}\end{bmatrix}\leq M'^{-1}\frac{2}{k}\begin{bmatrix}1\\ 1\end{bmatrix}-M'^{-1}\begin{bmatrix}0 & -m_{12}\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_{k-1,p}\\ y_{k-1,p}\end{bmatrix}=\begin{bmatrix}\frac{2}{k}\frac{1}{m_{11}}\\ \frac{2}{k}\left(\frac{m_{21}}{m_{11}m_{22}}+\frac{1}{m_{22}}\right)\end{bmatrix}+\begin{bmatrix}\frac{m_{12}}{m_{11}}y_{k-1,p}\\ \frac{m_{21}m_{12}}{m_{11}m_{22}}y_{n-1,p}\end{bmatrix}$$

and therefore

$$x_{k,p}\leq\frac{2}{km_{11}}+\frac{m_{12}}{m_{11}}y_{k-1,p}, \tag{5.9}$$

$$y_{k,p}\leq\frac{2}{k}\left(\frac{m_{12}}{m_{11}m_{22}}+\frac{1}{m_{22}}\right)+\frac{m_{12}m_{21}}{m_{11}m_{22}}y_{k-1,p}$$

From (5.2) one has $\alpha := \frac{m_{12}m_{21}}{m_{11}m_{22}} < 1$ and hence

$$y_{k,p} \leq \alpha\, y_{k-1,p} + \frac{2}{k}\left(\frac{m_{12}}{m_{11}m_{22}} + \frac{1}{m_{22}}\right).$$

Now we use the following lemma provided in [21].

**Lemma 12.** *Let $(y_{k,p}), (z_{k,p})$ be two sequences of real numbers depending on a parameter $p$, such that*

$$\left(y_{k,p}\right) \quad \text{is bounded uniformly with respect to } p$$

*and*

$$0 \leq y_{k,p} \leq \alpha\, y_{k-1,p} + z_{k,p} \quad \text{for some } \alpha \in (0,1).$$

*If $z_{k,p} \to 0$ as $k \to +\infty$ uniformly with respect to $p$, then $y_{k,p} \to 0$ as $k \to +\infty$ uniformly with respect to $p$.*

According to this result, since $(v_k)$ is bounded and then $\left(y_{k,p}\right)$ is bounded uniformly with respect to $p$, we conclude that $y_{k,p} \to 0$ as $k \to +\infty$ uniformly with respect to $p$. It follows that $(v_k)$ is a Cauchy sequence. Next, the inequality (5.9) implies that $(u_k)$ is also a Cauchy sequence. Denote by $u^*, v^*$ their limits.

**Step 5:** Transition to the limit.

If we pass to the limit in (5.4) we obtain

$$E_1(u^*, v^*) = \inf_{H_0^1(\Omega)} E_1(\cdot, v^*), \;\; E_2(u^*, v^*) = \inf_{H_0^1(\Omega)} E_2(u^*, \cdot), \; E_{11}(u^*, v^*) = E_{22}(u^*, v^*) = 0,$$

i.e., $(u^*, v^*)$ is a solution of (1.1) and also is a Nash equilibrium for the pair of functional $(E_1, E_2)$.

**Step 6:** Uniqueness.

Assume there are two different solutions of the system (1.1), denoted with $(u, v)$ and $(\overline{u}, \overline{v})$. Then

$$\begin{aligned} E_{11}(u, v) = 0\;,\; E_{22}(u, v) = 0, \\ E_{11}(\overline{u}, \overline{v}) = 0\;,\; E_{22}(\overline{u}, \overline{v}) = 0. \end{aligned}$$

On the other hand, from (5.6) and (5.7), we have

$$\begin{aligned} 0 &\geq m_{11}|u - \overline{u}|^2_{H_0^1} - m_{12}|u - \overline{u}|_{H_0^1}|v - \overline{v}|_{H_0^1}, \\ 0 &\geq m_{22}|v - \overline{v}|^2_{H_0^1} - m_{21}|u - \overline{u}|_{H_0^1}|v - \overline{v}|_{H_0^1}. \end{aligned} \tag{5.10}$$

If $u = \overline{u}$ or $v = \overline{v}$ then in each case $|u - \overline{u}| = 0$ or $|v - \overline{v}| = 0$, concluding that $u = \overline{u}$ and $v = \overline{v}$. In what follows we will work under assumption that $u \neq \overline{u}$ and $v \neq \overline{v}$. From (5.10) we obtain

$$\begin{aligned} |u - \overline{u}|_{H_0^1} &\leq \frac{m_{12}}{m_{11}}|v - \overline{v}|_{H_0^1}, \\ |v - \overline{v}|_{H_0^1} &\leq \frac{m_{21}}{m_{22}}|u - \overline{u}|_{H_0^1}, \end{aligned}$$

whence

$$|v - \overline{v}|_{H_0^1} \leq \frac{m_{12}m_{21}}{m_{11}m_{22}}|v - \overline{v}|_{H_0^1}.$$

Since from (5.2) one has $\frac{m_{12}m_{21}}{m_{11}m_{22}} < 1$, we conclude that

$$|v - \overline{v}|_{H_0^1} < v - \overline{v}|_{H_0^1},$$

which is impossible. Hence $u = \overline{u}$ and $v = \overline{v}$.

□

**Remark 13** (Classical Lipschitz conditions)**.** *Obviously the unilateral Lipschitz conditions (5.3) are satisfied if $g_1, g_2$ fulfill the classical Lipschitz conditions*

$$\begin{aligned} |g_1(x,u,v) - g_1(x,\overline{u},\overline{v})| &\leq a_{11}|u-\overline{u}| + a_{12}||v-\overline{v}|, \\ |g_2(x,u,v) - g_2(x,\overline{u},\overline{v})| &\leq a_{21}|u-\overline{u}| + a_{22}|v-\overline{v}|, \end{aligned}$$

*for all $u, v, \overline{u}, \overline{v} \in \mathbb{R}$ and a.e. $x \in \Omega$. In this particular case considered in [21] (see also [27]), the required conditions on the coefficients $a_{ij}$ make possible to derive the existence and uniqueness of the solution of system (1.2) directly from Perov's fixed point theorem. We note that unilateral Lipschitz conditions for Nash equilibria of systems have been used for the first time in paper [22].*

**Example 14.** Consider the Dirichlet problem for the system of Kirchhoff type

$$\begin{cases} -\left(1 + \int_0^1 |u'|^2\right) u'' = u - \sin v \\ -\left(1 + \int_0^1 |v'|^2\right) v'' = v + \sin u & \text{on } (0,1) \\ u(0) = v(0) = u(1) = v(1) = 0. \end{cases} \tag{5.11}$$

We apply Theorem 11, where

$$\Omega = (0,1),\ a = b = 1\ ,\ g_1(x,u,v) = u - \sin v\ ,\ g_2(x,u,v) = \sin u + v.$$

Note that condition (5.3) holds with $a_{ij} = 1$ $(i, j = 1, 2)$. The first eigenvalue of the Dirichlet problem $-u'' = \lambda u$ on $(0, 1)$, $u(0) = u(1) = 0$ is equal to $\pi^2$ (see, e.g., [28, p. 72]), whence relation (5.2) is valid since $1 < \pi^2$ and $1 < \left(\pi^2 - 1\right)^2$. In order to check (h2) we compute

$$G_2(x,u,v) = \int_0^v (s + \sin u)ds = \frac{1}{2}v^2 + v\ \sin u.$$

Consider the coercive-type functions $H_1(x,v) = \frac{1}{2}v^2 - |v|$ and $H_2(x,v) = \frac{1}{2}v^2 + |v|$. Clearly

$$H_1(x,v) \leq G_2(x,u,v) \leq H_2(x,v).$$

Therefore, the Dirichlet problem (5.11) has a unique solution $(u^*, v^*) \in H_0^1(0,1) \times H_0^1(0,1)$ which is a Nash equilibrium for the corresponding energy functionals.

### 5.2. Local Nash equilibrium

Let $R_1, R_2 > 0$ and denote by $B_{R_1}$, $B_{R_2}$ two closed balls of $H_0^1(\Omega)$, of center the origin and radius $R_1$ and $R_2$, respectively. Now, our interest is focused on an existence and uniqueness result of the system (1.1) on $B_{R_1} \times B_{R_2}$.

Here an additional ingredient is given by the Leray-Schauder boundary conditions

$$\begin{aligned} E_{11}(u,v) + \mu\, u &\neq 0 \ \text{ for all } (u,v) \in B_{R_1} \times B_{R_2} \text{ with } |u|_{H_0^1} = R_1 \text{ and all } \mu > 0, \\ E_{22}(u,v) + \gamma\, v &\neq 0 \text{ for all } (u,v) \in B_{R_1} \times B_{R_2} \text{ with } |v|_{H_0^1} = R_2 \text{ and all } \gamma > 0. \end{aligned} \tag{5.12}$$

**Theorem 15.** *Assume that for* $i = 1,2$, $f_i \in H^{-1}(\Omega)$, $g_i\colon\ \Omega \times \mathbb{R}^2 \to \mathbb{R}$ *is a Carathéodory function,* $g_i(\cdot,0,0) = 0$, *and that condition (h1) holds. In addition assume that*

**(h2')**

$$\begin{aligned} \frac{a_{11}}{\lambda_1} R_1 + \frac{a_{12}}{\lambda_1} R_2 + |f_1|_{H^{-1}} &\leq aR_1 + bR_1^3, \\ \frac{a_{21}}{\lambda_1} R_1 + \frac{a_{22}}{\lambda_1} R_2 + |f_2|_{H^{-1}} &\leq aR_2 + bR_2^3. \end{aligned}$$

*Then system* (1.1) *has in* $B_{R_1} \times B_{R_2}$ *a unique solution which is a Nash equilibrium in* $B_{R_1} \times B_{R_2}$ *for the pair of functionals* $(E_1, E_2)$.

*Proof.* **Step 1:** As at Step 1 from the proof of Theorem 11, the functionals $E_1$ and $E_2$ are bounded from below on $B_{R_1} \times B_{R_2}$.

**Step 2:** $E_1$ and $E_2$ satisfy the Leray-Schauder boundary conditions (5.12).

Assume that there exist $(u,v) \in B_{R_1} \times B_{R_2}$ with $|u|_{H_0^1} = R_1$ and $\mu > 0$ such that

$$E_{11}(u,v) + \mu\, u = 0.$$

Then

$$\left(a + b|u|_{H_0^1}^2\right)|u|_{H_0^1}^2 + \mu\,|u|_{H_0^1}^2 - \left((\Delta)^{-1}\left(f_1 + g_1(\cdot,u,v)\right)\,,\, u\right)_{H_0^1} = 0,$$

which gives

$$\begin{aligned} \left(a + bR_1^2\right)R_1^2 + \mu\, R_1^2 &= \left((\Delta)^{-1}\left(f_1 + g_1(\cdot,u,v)\right)\,,\, u\right)_{H_0^1} \\ &= \left(f_1 + g_1(\cdot,u,v)\,,\, u\right)_{L^2} \\ &\leq R_1\,|f_1|_{H^{-1}} + a_{11}|u|_{L^2}^2 + a_{12}|u|_{L^2}|v|_{L^2} \\ &\leq R_1\,|f_1|_{H^{-1}} + \frac{a_{11}}{\lambda_1}R_1^2 + \frac{a_{12}}{\lambda_1}R_1\,R_2, \end{aligned}$$

whence

$$\left(a + bR_1^2\right)R_1 + \mu\, R_1 \leq |f_1|_{H^{-1}} + \frac{a_{11}}{\lambda_1}R_1 + \frac{a_{12}}{\lambda_1}R_2,$$

which contradicts the first relation in (h2'). An analog reasoning applies for $E_2$.

**Step 3:** Construction of an approximation sequence.

As in the proof of Lemma 2.1 in [24], starting from an arbitrarily initial point $v_0 \in B_{R_2}$ and applying recursively Ekeland's variational principle, we obtain a sequence $(u_k, v_k) \in B_{R_1} \times B_{R_2}$ such that

$$E_1(u_k, v_{k-1}) \le \inf_{B_{R_1}} E_1(\cdot, v_{k-1}) + \frac{1}{k}, \quad E_2(u_k, v_k) \le \inf_{B_{R_2}} E_2(u_k, \cdot) + \frac{1}{k},$$
$$|E_{11}(u_k, v_{k-1}) + \mu_k u_k|_{H_0^1} \le \frac{1}{k}, \quad |E_{22}(u_k, v_k) + \gamma_k v_k|_{H_0^1} \le \frac{1}{k},$$

where

$$\mu_k = \begin{cases} -\frac{1}{R_1^2}(E_{11}(u_k, v_{k-1}), u_k)_{H_0^1}, & \text{if } |u_k|_{H_0^1} = R_1 \text{ and } (E_{11}(u_k, v_{k-1}), u_k)_{H_0^1} < 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\gamma_k = \begin{cases} -\frac{1}{R_2^2}(E_{22}(u_k, v_k), v_k)_{H_0^1}, & \text{if } |v_k|_{H_0^1} = R_2 \text{ and } (E_{22}(u_k, v_k), v_k)_{H_0^1} < 0 \\ 0, & \text{otherwise.} \end{cases}$$

**Step 4:** Convergence to zero of the sequences $(\mu_k)$ and $(\gamma_k)$.

Assume the contrary. Then, passing eventually to subsequences, we may assume that $\mu_k \to \mu > 0$ or $\gamma_k \to \gamma > 0$. Using the expressions of $E_{11}$ and $E_{22}$ and denoting

$$\alpha_k := E_{11}(u_k, v_{k-1}) + \mu_k u_k, \quad \beta_k := E_{22}(u_k, v_k) + \gamma_k v_k, \tag{5.13}$$

we have

$$\begin{aligned} u_k &= S\left(f_1 + g_1\left(\cdot, u_k, v_{k-1}\right)\right) + \frac{\mu_k}{a + b\,|u_k|_{H_0^1}^2}, \\ v_k &= S\left(f_2 + g_2\left(\cdot, u_k, v_k\right)\right) + \frac{\gamma_k}{a + b\,|v_k|_{H_0^1}^2}. \end{aligned} \tag{5.14}$$

The sequences $(u_k)$, $(v_k)$ being bounded and the operators $S\left(f_1 + g_1\left(\cdot, u, v\right)\right)$, $S\left(f_2 + g_2\left(\cdot, u, v\right)\right)$ being compact, we have that the two sequences from the right-hand sides in (5.14) are compact; thus $(u_k)$ and $(v_k)$ have convergent subsequences $\left(u_{k_j}\right)$, $\left(v_{k_j}\right)$. The same reasoning applied to the second formula in (5.14) with $k_j - 1$ instead of $k$ allows us, passing again to subsequence, to assume that the sequences $\left(u_{k_j}\right)$, $\left(v_{k_j}\right)$ and $\left(v_{k_j-1}\right)$ are convergent. Let $u, v, \overline{v}$ be their limits. If we take the limit in (5.13)

$$E_{11}\left(u, \overline{v}\right) + \mu u = 0, \quad E_{22}\left(u, v\right) + \gamma v = 0,$$

where $|u|_{H_0^1} = R_1$ if $\mu > 0$ and $|v|_{H_0^1} = R_2$ if $\gamma > 0$. In each case, one of the two Leray-Schauder conditions (5.12) is contradicted. Therefore $\mu_k \to 0$ and $\gamma_k \to 0$ as $k \to +\infty$.

**Step 5:** Estimations for $(u_k)$ and $(v_k)$.

We can proceed similarly to Theorem 11, Step 4, to obtain inequalities (5.6) and (5.7). Under the notations from Step 4 in the proof of the previous theorem, and the additional notations $c_{k,p} = \left|\mu_{k+p} - \mu_k\right|$, $d_{k,p} = \left|\gamma_{k+p} - \gamma_k\right|$, we arrive to the matrix inequality

$$M_k' \begin{bmatrix} x_{k,p} \\ y_{k,p} \end{bmatrix} \le \frac{2}{k} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - (M_k - M_k') \begin{bmatrix} x_{k-1,p} \\ y_{k-1,p} \end{bmatrix} + \begin{bmatrix} c_{k,p} R_1 \\ d_{k,p} R_2 \end{bmatrix},$$

where now

$$M_k = \begin{bmatrix} m_{11} + \mu_k & -m_{12} \\ -m_{21} & m_{22} + \mu_k \end{bmatrix}, \quad M'_k = \begin{bmatrix} m_{11} + \mu_k & 0 \\ -m_{21} & m_{22} + \mu_k \end{bmatrix}.$$

Since for any $k \in \mathbb{N}$, $M'_k$ is invertible and

$$M'^{-1}_k = \begin{bmatrix} \frac{1}{m_{11}+\mu_k} & 0 \\ \frac{m_{21}}{(m_{11}+\mu_k)(m_{22}+\gamma_k)} & \frac{1}{m_{22}+\gamma_k} \end{bmatrix}$$

we obtain

$$\begin{bmatrix} x_{k,p} \\ y_{k,p} \end{bmatrix} \leq \frac{2}{k} M'^{-1}_k \begin{bmatrix} 1 \\ 1 \end{bmatrix} - M'^{-1}_k \begin{bmatrix} 0 & -m_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{k-1,p} \\ y_{k-1,p} \end{bmatrix} + M'^{-1}_k \begin{bmatrix} c_{k,p} R_1 \\ d_{k,p} R_2 \end{bmatrix}.$$

Thus

$$\begin{aligned} x_{k,p} &\leq \frac{2}{k(m_{11} + \mu_k)} + \frac{m_{12}}{m_{11} + \mu_k} y_{k-1,p} + c_{k,p} R_1 \frac{1}{m_{11} + \mu_k} \\ &\leq \frac{2}{k m_{11}} + \frac{m_{12}}{m_{11}} y_{k-1,p} + c_{k,p} R_1 \frac{1}{m_{11}}, \\ y_{k,p} &\leq \frac{2}{k} \left( \frac{1}{m_{22} + \gamma_k} \right) \\ &\quad + \frac{1}{(m_{11} + \mu_k)(m_{22} + \gamma_k)} \left[ \frac{2}{k} m_{12} + m_{12} m_{21} y_{k-1,p} + m_{21} c_{k,p} R_1 \right] + \frac{1}{m_{22} + \gamma_k} d_{k,p} R_2 \\ &\leq \frac{2}{k} \left( \frac{m_{12}}{m_{11} m_{22}} + \frac{1}{m_{22}} \right) + \frac{m_{12} m_{21}}{m_{11} m_{12}} y_{k-1,p} + \frac{m_{21}}{m_{11} m_{22}} c_{k,p} R_1 + \frac{1}{m_{22}} d_{k,p} R_2. \end{aligned}$$

Hence

$$y_{k,p} \leq \alpha y_{k-1,p} + \frac{2}{k} \left( \frac{m_{12}}{m_{11} m_{22}} + \frac{1}{m_{22}} \right) + \frac{m_{21}}{m_{11} m_{22}} c_{k,p} R_1 + \frac{1}{m_{22}} d_{k,p} R_2 \ ,$$

where $\alpha := \frac{m_{12} m_{21}}{m_{11}} < 1$ and $c_{k,p},\ d_{k,p}$ converge to zero uniformly with respect to $p$. Now the conclusion follows as in the proof of Theorem 11 with the limits $u^*$ and $v^*$ of the sequences $(u_k)$ and $(v_k)$ satisfying

$$E_{11}(u^*, v^*) = 0, \quad E_{22}(u^*, v^*) = 0$$

and

$$E_1(u^*, v^*) = \inf_{B_{R_1}} E_1(\cdot, v^*), \quad E_{22}(u^*, v^*) = \inf_{B_{R_2}} E_2(u^*, \cdot).$$

**Step 6:** Uniqueness.
Similar to the proof in Theorem 11. □

**Example 16.** Consider the Dirichlet problem for the system of Kirchhoff type

$$\begin{cases} -\left(2 + \int_0^1 |u'|^2\right) u'' = -u^3 + u - \sin v + \pi^2 \sin(\pi x) \\ -\left(2 + \int_0^1 |v'|^2\right) v'' = -v^3 + v + \sin u & \text{on } (0,1) \\ u(0) = v(0) = u(1) = v(1) = 0. \end{cases} \tag{5.15}$$

For $R_1 = R_2 = 1$, we apply Theorem 15, where

$$\Omega = (0,1),\ \ a = 2,\ \ b = 1,\ \ f_1(x) = \pi^2 \sin(\pi x),\ \ f_2 \equiv 0,$$

$$g_1(x,u,v) = -u^3 + u - \sin v,\ \ g_2(x,u,v) = -v^3 + v + \sin u.$$

Since

$$\left(-u^3 + \overline{u}^3\right)(u - \overline{u}) = -(u-\overline{u})^2\left(u^2 + u\overline{u} + \overline{u}^2\right) \leq 0,$$

one has

$$\begin{aligned}(g_1(x,u,v) - g_1(x,\overline{u},\overline{v}))(u-\overline{u}) &\leq (u - \sin v - \overline{u} + \sin\overline{v})(u-\overline{u}) \\ &\leq |u-\overline{u}|^2 + |u-\overline{u}|\,|v-\overline{v}|.\end{aligned}$$

Similarly

$$(g_2(x,u,v) - g_2(x,\overline{u},\overline{v}))(v-\overline{v}) \leq |u-\overline{u}|\,|v-\overline{v}| + |v-\overline{v}|^2.$$

Hence, condition (5.3) holds with $a_{ij} = 1$ for $i, j = 1, 2$. In addition, since $\lambda_1 = \pi^2$, condition (5.2) also holds. Thus assumption (h1) is satisfied. Next we check condition (h2'). We have $|f_2|_{H^{-1}} = 0$ and that the function $u_0(x) = \sin(\pi x)$ is the solution of the Dirichlet problem $-u'' = f_1$ in $(0,1)$, $u(0) = u(1) = 0$. Then

$$|f_1|_{H^{-1}} = |u_0|_{H_0^1} = \left|u_0'\right|_{L^2} = \left(\int_0^1 \pi^2\cos^2(\pi x)\right)^{\frac{1}{2}} = \frac{\pi}{\sqrt{2}}.$$

Now, condition (h2') is verified since both $2\,/\,\pi^2 + \pi\,/\,\sqrt{2}$ and $2\,/\,\pi^2$ are less than 3.

Therefore, the Dirichlet problem (5.15) has a unique solution

$$(u^*, v^*) \in \left\{u \in H_0^1(0,1)\ :\ |u|_{H_0^1} \leq 1\right\} \times \left\{v \in H_0^1(0,1)\ :\ |v|_{H_0^1} \leq 1\right\},$$

which is a Nash equilibrium for the corresponding energy functionals.

## 6. Conclusions

In this paper, we have studied the existence, uniqueness, localization and variational properties of solutions for some equations and systems of Kirchhoff type. First we have defined the solution operator associated to nonhomogeneous equations subjected to the Dirichlet boundary condition and we have made the connexion with the corresponding energy functional. Next, we have considered equations with a reaction term and using Banach contraction principle and Schaefer's fixed point theorem we have established sufficient conditions so that a solution exist and be localized in some bounded sets. For a system of two Kirchhoff equations, under appropriate conditions, we have proved the existence of a unique solution which appears as a Nash equilibrium for the associated energy functionals. Both global Nash equilibrium, in the whole space, and local Nash equilibrium, in balls, have been obtained by using an iterative procedure simulating a noncooperative game and based on Ekeland's variational principle.

## Conflict of interest

The authors declare no conflict of interest.

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