



Research article

Analysis of travelling wave solutions for Eyring-Powell fluid formulated with a degenerate diffusivity and a Darcy-Forchheimer law

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Abstract: The goal of this paper is to provide analytical assessments to a fluid flowing in a porous medium with a non-linear diffusion linked to a degenerate diffusivity. The viscosity term is formulated with an Eyring-Powell law, together with a non-homogeneous diffusion typical of porous medium equations (as known in the theory of partial differential equations). Further, the equation is supplemented with an absorptive reaction term of Darcy-Forchheimer, commonly used to model flows in porous medium. The work starts by analyzing regularity, existence and uniqueness of solutions. Afterwards, the problem is transformed to study travelling wave kind of solutions. An asymptotic expansion is considered with a convergence criteria based on the geometric perturbation theory. Supported by this theory, there exists an exponential decaying rate in the travelling wave profile. Such exponential behaviour is validated with a numerical assessment. This is not a trivial result given the degenerate diffusivity induced by the non-linear diffusion of porous medium type and suggests the existence of regularity that can serve as a baseline to construct numerical or energetic approaches.

Keywords: Eyring-Powell fluid; Darcy-Forchheimer; porous medium equation; travelling waves; geometric perturbation theory

Mathematics Subject Classification: 35B65, 35Q35

Nomenclature

K_1 : Permeability of the medium.

K_2 : Inertial permeability.

L: Fluid characteristic length.
M: Hartmann number.
MHD: Magnetohydrodynamics.
PME: Porous Medium Equation.
Re: Reynolds.
 U_0 : Fluid characteristic velocity.
 ρ : Fluid density.
 μ : Fluid viscosity.
 ν : Fluid kinematic viscosity.
 B_0 : Applied magnetic field.
 σ : Electrical conductivity in the medium.

1. Introduction

Any kind of material exhibiting pores filled by a flowing fluid leads to a physical interpretation beyond the Newtonian description related with viscosity and diffusion. Currently, flow in porous medium is a source of research, not only because of purely mathematical interests, but also because of their applications in electrochemical, mechanical, geophysical, metallurgical and chemical processes. In 1856, the observations made by Henry Darcy [1] associated to fluid flow in porous medium led to the formulation of a physical law to describe the basic dynamic experienced. Such law was intended to describe the fluid principles for low Reynolds numbers, typical of flows in this kind of porous medium. Afterwards, Forchheimer in 1901 [2] and Jaeger in 1956 [3] provided an extension of Darcy's law to account for turbulent flows at higher Reynolds numbers. Based on these works, the Darcy's law was renamed as Darcy-Forchheimer's law by Mustak in [4] and [5].

It shall be noted that the above mentioned researchers considered the viscosity and diffusive terms as provided by the classical Newtonian description. Nonetheless and since then, several researchers have tried to extend the viscosity formulation to increase the modelling accuracy beyond the Newtonian scope. As a set of representative examples of the described non-Newtonian fluid models, the reader is referred to the works [6–9] together with references therein. The non-Newtonian fluids are classified into several classes. It is relevant to highlight that efforts have been made to describe diffusion phenomena based on the kinetic theory of liquids rather than on experimental or intuitive principles related with viscosity. Such efforts were conducted with success leading to a kind of non-Newtonian fluid known as Eyring-Powell. The use of the kinetic theory of liquids to describe viscosity and diffusive mechanisms makes the Eyring-Powell fluid particularly attractive to model flows given by electrically conducting particles arising in Magnetohydrodynamics (MHD). In a set of interesting analysis, Akbar et al. [10] obtained the numerical results for two dimensional MHD Eyring-Powell fluid over stretching surface. Hina et al. [11] analyzed the heat transfer for MHD Eyring-Powell fluid under the slip effect at the boundary. Bhatti et al. [12] considered two dimensional MHD Eyring-Powell fluid under permeable surface and analyzed the heat transfer by converting the equations into nonlinear ordinary equations and that are solved based on numerical principles. Other interesting results in non-Newtonian fluids are given in the references [13–18], but particularly there are some recent achievements in this regards to highlight: In [19] a modified Eyring-Powell model is introduced to improve predictions related with the Eyring-Powell parameter on flow velocity. In [20], the authors

discuss the adequacy of Eyring-Powell models for viscous dissipation in a flow past a convectively heated stretching sheet. Furthermore, in [21] and [22], an Eyring-Powell fluid is considered to model Coriolis effects over a non-uniform surfaces together with a discussion on the adequacy of the formulation. Finally, the reader is referred to [23] and [24] for discussions on other types of fluids and other types of analysis based on purely numerical schemes respectively.

Other techniques have been of help to describe flows under non-Newtonian approaches. For instance, some kinds of solutions known as travelling and solitary waves have provided interesting results from analytical and numerical perspective. In this regard, the reader is referred to the set of studies [25–27].

As a main research question driving the present study, it shall be mentioned the idea of searching for appropriate regularity conditions even when the Eyring-Powell viscosity term is formulated with a degenerate diffusion. Specially but not limited, the question of finding an exponential profile of solution.

1.1. Model formulation

Let us start by considering a one dimensional Darcy-Forchheimer flow with a diffusion term given by an Eyring-Powell principle. The velocity vector is then $V = (v_1, 0, 0)$ where $v_1(y, t)$ is the velocity profile in the x -direction that is dependant on the transversal direction y . The geometry under study is given by an open porous domain where the flow is developed through the x -direction and is initially distributed by an initial condition in the y -direction. In addition, the diffusion is further modified to account for a nonlinear term of the form $\frac{\partial^2 v_1^m}{\partial y^2}$ (refer to [28] and [29] for a complete discussion about this kind of non-linearity and to [14–16] for different applications of particular diffusion terms). Note that the basic homogeneous problem

$$\frac{\partial v_1}{\partial t} = \frac{\partial^2 v_1^m}{\partial y^2}, \quad m > 1, \quad (1.1)$$

is well known as porous medium equation. The PME is a nonlinear parabolic equation with degenerate diffusivity. Indeed, the diffusivity is given by $D(v_1) = m v_1^{m-1}$ that is null whenever $v_1 = 0$. This degeneracy leads to the loss of a positive principle for null (or slightly null) solutions, and hence to the loss of regularity as understood for gaussian diffusion problems.

Based on the exposed principles, the equation under study emerges from the MHD theory and reads

$$\frac{\partial v_1}{\partial t} = -\frac{1}{\rho} \frac{dP}{dx} + \left(\nu + \frac{1}{\beta d_1 \rho} \right) \frac{\partial^2 v_1^m}{\partial y^2} - \frac{1}{2\beta d_1^3 \rho} \left(\frac{\partial v_1}{\partial y} \right)^2 \frac{\partial^2 v_1}{\partial y^2} - \left(\frac{\sigma B_0^2}{\rho} + \frac{\nu}{K_1} \right) v_1 - \frac{v_1^2}{K_2}, \quad (1.2)$$

where $m > 1$, P is the pressure field acting along the x -direction and β and d_1 are the characteristic constants of the Eyring-Powell fluid model.

To account for a single general equation, introduce the following non-dimensional quantities (see [30]) as follows

$$\begin{aligned} v_1^* &= \frac{v_1}{U_0}, \quad x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad t^* = \frac{U_0 t}{L}, \quad P^* = \frac{P}{\rho U_0^2}, \quad A_1 = \frac{1}{\nu \beta \rho d_1} \\ \epsilon &= \frac{1}{2\beta \rho d_1^3 L^3}, \quad A_2 = \frac{L\nu}{U_0 K_1}, \quad A_3 = \frac{L}{K_2}, \quad A_4 = \frac{U_0^{m-1}}{R_e}, \end{aligned} \quad (1.3)$$

where L is a characteristic length that can be given by the pores dimension. After replacement of the set of expressions in (1.3) into the Eq (1.2) (ignoring * for simplicity)

$$\frac{\partial v_1}{\partial t} = -\frac{1}{\rho} \frac{dP}{dx} + A_4(1 + A_1) \frac{\partial^2 v_1^m}{\partial y^2} - \epsilon \left(\frac{\partial v_1}{\partial y} \right)^2 \frac{\partial^2 v_1}{\partial y^2} - \left(\frac{M^2}{R_e} + A_2 \right) v_1 - A_3 v_1^2, \quad (1.4)$$

where $R_e = \frac{U_0 L}{\nu}$ is the Reynolds number and $M = B_0 \sqrt{\frac{L\sigma}{\rho}}$ refers to the Hartmann number. Making the differentiation of Eq (1.4) with respect to x

$$-\frac{1}{\rho} \frac{dP}{dx} = A_5.$$

After using the value of $\frac{dP}{dx}$ in Eq (1.4)

$$\frac{\partial v_1}{\partial t} = A_5 + A_4(1 + A_1) \frac{\partial^2 v_1^m}{\partial y^2} - \epsilon \left(\frac{\partial v_1}{\partial y} \right)^2 \frac{\partial^2 v_1}{\partial y^2} - \left(\frac{M^2}{R_e} + A_2 \right) v_1 - A_3 v_1^2. \quad (1.5)$$

In addition, consider the following initial generalized condition

$$v_1(y, 0) = v_0(y) \in L^1(R) \cap L^\infty(R), \quad (1.6)$$

where $v_0(y)$ is the initial velocity that represents any kind of non-homogeneous distribution with the only requirement to be bounded in L^∞ -norm and under a finite flowing mass quantity, i.e., a bound in L^1 -norm. In addition, note that this initial condition allows modelling any kind of irregularities or discontinuities that may exist in the porous medium involved.

2. Regularity, existence and uniqueness of solutions

Given the degeneracy in the diffusivity introduced by the non-linear term $\frac{\partial^2 v_1^m}{\partial y^2}$ and given the generalized initial velocity distribution, solutions are analyzed based on a weak formulation. To this end, assume the existence of a test function $\Omega \in C^\infty(R)$ with compact support (or at least with a decreasing behaviour over R) such that for $0 < \tau < t < T$

$$\begin{aligned} \int_R v_1(t) \Omega(t) dy &= \int_R v_1(\tau) \Omega(\tau) dy + \int_\tau^t \int_R v_1(s) \frac{\partial \Omega}{\partial s} dy ds + A_5 \int_\tau^t \int_R \Omega dy ds \\ &+ A_4(1 + A_1) \int_\tau^t \int_R v_1^m \frac{\partial^2 \Omega}{\partial y^2} dy ds + \frac{\epsilon}{3} \int_\tau^t \int_R \left(\frac{\partial v_1}{\partial y} \right)^3 \frac{\partial \Omega}{\partial y} \\ &- \left(\frac{M^2}{R_e} + A_2 \right) \int_\tau^t \int_R v_1 \Omega dy ds - A_3 \int_\tau^t \int_R \Omega v_1^2 dy ds. \end{aligned} \quad (2.1)$$

Given a finite r_0 , admit the following ball B_r centered in r_0 and with radius $r \gg r_0$, such that the following equation is defined

$$v_1(t) \frac{\partial \Omega}{\partial s} + A_5 \Omega + A_4(1 + A_1) v_1^m \frac{\partial^2 \Omega}{\partial y^2} + \frac{\epsilon}{3} \left(\frac{\partial v_1}{\partial y} \right)^3 \frac{\partial \Omega}{\partial y}$$

$$- \left(\frac{M^2}{R_e} + A_2 \right) v_1 \Omega - A_3 \Omega v_1^2 = 0, \quad (2.2)$$

in $B_r \times [0, T]$, with the following boundary like and initial conditions

$$0 < \frac{\partial \Omega}{\partial y} = \Omega \ll 1, v_1(y, 0) = v_0(y) \in L^1(R) \cap L^\infty(R). \quad (2.3)$$

Firstly, the following theorem expresses the regularity properties of any solution to (1.5), i.e., solutions exist globally.

Theorem 2.1. *Given $v_0(y) \in L^1(R) \cap L^\infty(R)$, then any non-negative solution ($v_1 \geq 0$) to (1.2) is bounded for all $(y, t) \in B_r \times [0, T]$ with $r \gg 1$.*

Proof. Consider $\lambda \in R^+$, the following supporting cut-off function is defined as

$$\begin{aligned} \psi_\lambda &\in C_0^\infty, & 0 \leq \psi_\lambda \leq 1, \\ \psi_\lambda &= 1 \text{ in } B_{r-\lambda}, \psi_\lambda = 0 \text{ in } R - B_{r-\lambda}, \end{aligned}$$

such that

$$\left| \frac{\partial \psi_\lambda}{\partial \lambda} \right| = \frac{C_c}{\lambda}.$$

Multiplying Eq (2.2) by ψ_λ and integrating over $B_r \times [\tau, T]$

$$\begin{aligned} \int_\tau^t \int_{B_r} v_1 \frac{\partial \Omega}{\partial s} \psi_\lambda dy ds + A_5 \int_\tau^t \int_R \Omega \psi_\lambda dy ds &= -A_4 (1 + A_1) \int_\tau^t \int_R v_1^m \frac{\partial^2 \Omega}{\partial y^2} \psi_\lambda dy ds \\ -\frac{\epsilon}{3} \int_\tau^t \int_R \left(\frac{\partial v_1}{\partial y} \right)^3 \frac{\partial \Omega}{\partial y} \psi_\lambda dy ds + \left(\frac{M_2}{R_e} + A_2 \right) \int_\tau^t \int_R v_1 \Omega \psi_\lambda dy ds &+ A_3 \int_\tau^t \int_R \Omega v_1^2 \psi_\lambda dy ds. \end{aligned} \quad (2.4)$$

Based on some results in [28] and [29], for some large $r \gg r_0 > 1$ the following holds

$$\int_\tau^t v_1 ds \leq \int_\tau^t v_1^m ds \leq C_1(\tau) r^{\frac{2m}{m-1}}, \quad (2.5)$$

and

$$\int_\tau^t v_1^2 ds \leq C_1(\tau) r^4. \quad (2.6)$$

Considering the spatial variable y close to ∂B_r , it can be assumed $y \sim r$. Then

$$\int_\tau^t \left(\frac{\partial v_1}{\partial y} \right)^3 ds \leq \frac{8m^3}{m-1} C_1^3(\tau) r^{\frac{3m+3}{m-1}}. \quad (2.7)$$

After using the expression (2.4) and after integration in the nonlinear diffusion term, the following holds

$$\begin{aligned} -A_4 (1 + A_1) \int_\tau^t \int_{B_r} v_1^m \frac{\partial^2 \Omega}{\partial y^2} \psi_\lambda dy ds \\ \leq A_4 (1 + A_1) C_1(\tau) r^{\frac{2m}{m-1}} \left(\left(\frac{\partial \Omega}{\partial y} \psi_\lambda \right)_{\partial B_r} - \int_{B_r} \frac{\partial \Omega}{\partial y} \frac{\partial \psi_\lambda}{\partial y} dy ds \right). \end{aligned}$$

Since $r \gg 1$ and since Ω is sufficiently small and regular in the asymptotic approach, it is possible to choose Ω in such a way $\frac{\partial\Omega}{\partial y}\psi_\lambda \ll 1$ in the proximity of the boundary ∂B_r . Then, the above expression becomes

$$\begin{aligned} -A_4(1+A_1) \int_\tau^t \int_{B_r} v_1^m \frac{\partial^2 \Omega}{\partial y^2} \psi_\lambda dy ds &\leq -A_4(1+A_1) C_1(\tau) \int_{B_r} r^{\frac{2m}{m-1}} \frac{\partial \Omega}{\partial y} \frac{\partial \psi_\lambda}{\partial y} dy ds \\ &\leq A_4(1+A_1) C_1(\tau) \int_{B_r} r^{\frac{2m}{m-1}} \frac{\partial \Omega}{\partial y} \frac{\partial \psi_\lambda}{\partial y} dy ds. \end{aligned} \quad (2.8)$$

Based on (2.7), the following integral holds

$$-\frac{\epsilon}{3} \int_{B_r} \int_\tau^t \left(\frac{\partial v_1}{\partial y} \right)^3 \frac{\partial \Omega}{\partial y} \psi_\lambda dy ds \leq \frac{8\epsilon m^3}{3(m-1)^3} C_1^3(\tau) \int_{B_r} r^{\frac{3m+3}{m-1}} \Omega \frac{\partial \psi_\lambda}{\partial y} dy. \quad (2.9)$$

After using the expressions (2.5), (2.6), (2.8) and (2.9) into (2.4), the following holds

$$\begin{aligned} \int_\tau^t \int_{B_r} v_1 \frac{\partial \Omega}{\partial s} \psi_\lambda dy ds + A_5 \int_\tau^t \int_R \Omega \psi_\lambda dy ds &\leq A_4(1+A_1) C_1(\tau) \int_{B_r} r^{\frac{2m}{m-1}} \frac{\partial \Omega}{\partial y} \frac{\partial \psi_\lambda}{\partial y} dy \\ &+ \frac{8\epsilon m^3}{3(m-1)^3} C_1^3(\tau) \int_{B_r} r^{\frac{3m+3}{m-1}} \Omega \frac{\partial \psi_\lambda}{\partial y} dy \\ &+ \left(\frac{M_2}{R_e} + A_2 \right) C_1(\tau) \int_{B_r} r^{\frac{2m}{m-1}} \Omega \psi_\lambda dy \\ &+ A_3 C_1(\tau) \int_{B_r} r^4 \Omega \psi_\lambda dy \\ &\leq A_4(1+A_1) C_1(\tau) \int_{B_r} r^{\frac{2m}{m-1}} \frac{\partial \Omega}{\partial y} \frac{C_c}{\lambda} dy \\ &+ \frac{8\epsilon m^3}{3(m-1)^3} C_1^3(\tau) \int_{B_r} r^{\frac{3m+3}{m-1}} \Omega \frac{C_c}{\lambda} dy \\ &+ \left(\frac{M_2}{R_e} + A_2 \right) C_1(\tau) \int_{B_r} r^{\frac{2m}{m-1}} \Omega \psi_\lambda dy \\ &+ A_3 C_1(\tau) \int_{B_r} r^4 \Omega \psi_\lambda dy, \end{aligned}$$

which implies that

$$\begin{aligned} \int_\tau^t \int_{B_r} v_1 \frac{\partial \Omega}{\partial s} dy ds + A_5 \int_\tau^t \int_R \Omega dy ds &\leq A_4(1+A_1) C_c C_1(\tau) \int_{B_r} r^{\frac{m+1}{m-1}} \frac{\partial \Omega}{\partial y} dy \\ &+ \frac{8\epsilon m^3}{3(m-1)^3} C_c C_1^3(\tau) \int_{B_r} r^{\frac{2m+4}{m-1}} \Omega dy \\ &+ \left(\frac{M_2}{R_e} + A_2 \right) C_1(\tau) \int_{B_r} r^{\frac{2m}{m-1}} \Omega dy \\ &+ A_3 C_1(\tau) \int_{B_r} r^4 \Omega dy. \end{aligned} \quad (2.10)$$

Consider a test function $\Omega \in C^\infty(R)$ of the form

$$\Omega(r, s) = e^{-ls} (1 + r^2)^{-\alpha}. \quad (2.11)$$

Choose α in a way that the expression (2.10) is convergent, therefore

$$\begin{aligned} \int_{\tau}^t \int_{B_r} v_1 \frac{\partial \Omega}{\partial s} dy ds + A_5 \int_{\tau}^t \int_R \Omega dy ds &\leq 2A_4 (1 + A_1) C_c C_1(\tau) \int_{B_r} e^{-ls} r^{\frac{2}{m-1}-2\alpha} dy \\ &+ \frac{8\epsilon m^3}{3(m-1)^3} C_c C_1^3(\tau) \int_{B_r} e^{-ls} r^{\frac{2m+4}{m-1}-2\alpha} dy \\ &+ \left(\frac{M_2}{R_e} + A_2 \right) C_1(\tau) \int_{B_r} e^{-ls} r^{\frac{2m}{m-1}-2\alpha} dy \\ &+ A_3 C_1(\tau) \int_{B_r} e^{-ls} r^{4-2\alpha} dy. \end{aligned} \quad (2.12)$$

For convergence, take $\alpha > \frac{m+2}{m-1}$ and $r \rightarrow \infty$, so that the expression (2.12) becomes

$$0 \leq \int_{\tau}^t \int_{B_r} v_1 \frac{\partial \Omega}{\partial s} dy ds + A_5 \int_{\tau}^t \int_R \Omega dy ds \leq \Upsilon, \quad (2.13)$$

where Υ is a suitable value obtained for each B_r and is sufficiently small for $r \rightarrow \infty$; hence, a finite value of Υ holds. Note that Ω and v_1 are not negative, then the integrals in the expression (2.13) are finite in $\tau < s < t \leq T$, which shows that the solution v_1 is bounded in $R \times [0, T]$. \square

Now, it is the aim to show additional regularity results, in particular that $\frac{\partial v_1}{\partial y}$ is bounded.

Theorem 2.2. Assume that $v_1(y)$ is a non-negative solution to Eq (1.5), then $\frac{\partial v_1}{\partial y}$ is bounded for all $(y, t) \in R \times [0, T]$.

Proof. Multiplying Eq (1.5) by v_1 and applying integration by parts, the following holds

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_R |v_1|^2 dy &= A_5 \int_R v_1 dy - mA_4 (1 + A_1) \int_R v_1^{m-1} \left(\frac{\partial v_1}{\partial y} \right)^2 dy \\ &+ \frac{\epsilon}{3} \int_R \left(\frac{\partial v_1}{\partial y} \right)^4 dy - \left(\frac{M^2}{R_e} + A_2 \right) \int_R v_1^2 dy - A_3 \int_R v_1^3 dy, \end{aligned}$$

which implies that

$$\begin{aligned} \int_R \left(\frac{\partial v_1}{\partial y} \right)^2 \left[\frac{\epsilon}{3} \left(\frac{\partial v_1}{\partial y} \right)^2 - mA_4 (1 + A_1) v_1^{m-1} \right] dy &= \frac{1}{2} \frac{d}{dt} \int_R |v_1|^2 dy - A_5 \int_R v_1 dy \\ &+ \left(\frac{M^2}{R_e} + A_2 \right) \int_R v_1^2 dy + A_3 \int_R v_1^3 dy. \end{aligned}$$

Now, applying the integration done

$$\int_0^t \int_R \left(\frac{\partial v_1}{\partial y} \right)^2 \left[\frac{\epsilon}{3} \left(\frac{\partial v_1}{\partial y} \right)^2 - mA_4 (1 + A_1) v_1^{m-1} \right] dy ds = \frac{1}{2} \int_R |v_1|^2 dy - \int_R |v_0(y)|^2 dy$$

$$-A_5 \int_0^t \int_R v_1 dy ds + \left(\frac{M^2}{R_e} + A_2 \right) \int_0^t \int_R v_1^2 dy ds + A_3 \int_R v_1^3 dy ds. \quad (2.14)$$

The Theorem 2.1 established the bound properties of non-negative solutions, as a consequence the right hand side of Eq (2.14) is bounded. Choose a finite bounding constant A_5 such that

$$\int_0^t \int_R \left(\frac{\partial v_1}{\partial y} \right)^2 \left[\frac{\epsilon}{3} \left(\frac{\partial v_1}{\partial y} \right)^2 - mA_4 (1 + A_1) v_1^{m-1} \right] dy ds \leq A_5,$$

leading to conclude on the bound of $\frac{\partial v_1}{\partial y}$.

□

The degeneracy introduced by the non-linear diffusion term may lead to the loss of uniqueness of non-negative solutions. Hence, a result about uniqueness of solutions is required to be shown.

Theorem 2.3. *Let us consider two non-negative solutions to the Eq (1.5), $v_{11}(y, t)$ and $v_{12}(y, t)$. Then, if both solutions have the same initial distribution, uniqueness holds, i.e., $v_{11}(y, t) = v_{12}(y, t)$.*

Proof. First of all, define a test function $\Psi(y, t) \in C^\infty(Q_T)$, where $Q_T = R \times [0, T]$. The weak formulations for both solutions v_{11} and v_{22} are given as

$$\begin{aligned} \int_R v_{11}(y, t) \Psi(y, t) dy &= \int_R v_{11}(y, 0) \Psi(y, 0) dy + \int_0^t \int_R v_{11} \frac{\partial \Psi}{\partial s} dy ds \\ &+ A_4 (1 + A_1) \int_0^t \int_R v_{11}^m \frac{\partial^2 \Psi}{\partial y^2} dy ds + A_5 \int_0^t \int_R \Psi dy dt \\ &+ \frac{\epsilon}{3} \int_0^t \int_R \left(\frac{\partial v_{11}}{\partial y} \right)^4 \frac{\partial \Psi}{\partial y} dy ds - \left(\frac{M^2}{R_e} + A_2 \right) \int_0^t \int_R v_{11} \Psi dy dt \\ &- A_3 \int_0^t \int_R v_{11}^2 \Psi dy dt, \end{aligned} \quad (2.15)$$

and

$$\begin{aligned} \int_R v_{12}(y, t) \Psi(y, t) dy &= \int_R v_{12}(y, 0) \Psi(y, 0) dy + \int_0^t \int_R v_{12} \frac{\partial \Psi}{\partial s} dy ds \\ &+ A_4 (1 + A_1) \int_0^t \int_R v_{12}^m \frac{\partial^2 \Psi}{\partial y^2} dy ds + A_5 \int_0^t \int_R \Psi dy dt \\ &+ \frac{\epsilon}{3} \int_0^t \int_R \left(\frac{\partial v_{12}}{\partial y} \right)^4 \frac{\partial \Psi}{\partial y} dy ds - \left(\frac{M^2}{R_e} + A_2 \right) \int_0^t \int_R v_{12} \Psi dy dt \\ &- A_3 \int_0^t \int_R v_{12}^2 \Psi dy dt. \end{aligned} \quad (2.16)$$

Assume $v_{11} \geq v_{12}$. After subtraction, the following holds

$$\int_R (v_{11} - v_{12})(y, t) \Psi(y, t) dy = \int_0^t \int_R (v_{11} - v_{12})(y, t) \frac{\partial \Psi}{\partial s} dy ds$$

$$\begin{aligned}
& + A_4 (1 + A_1) \int_0^t \int_R (v_{11}^m - v_{12}^m)(y, t) \frac{\partial^2 \Psi}{\partial y^2} dy ds \\
& + \frac{\epsilon}{3} \int_0^t \int_R \left(\left(\frac{\partial v_{11}}{\partial y} \right)^4 - \left(\frac{\partial v_{12}}{\partial y} \right)^4 \right) \frac{\partial \Psi}{\partial y} dy ds \\
& - \left(\frac{M^2}{R_e} + A_2 \right) \int_0^t \int_R (v_{11} - v_{12})(y, t) \Psi(y, t) dy dt \\
& - A_3 \int_0^t \int_R (v_{11}^2 - v_{12}^2)(y, t) \Psi(y, t) dy dt.
\end{aligned} \tag{2.17}$$

For $v_{11}(y, t) > v_{12}(y, t)$, the following holds

$$v_{11}^m - v_{12}^m \leq m v_{11}^{m-1} (v_{11} - v_{12}) \leq m C_2^{m-1} (v_{11} - v_{12}), \tag{2.18}$$

where $C_2 = \max_{(y,t) \in Q_T} (v_{11})$. Now, consider

$$\begin{aligned}
\left(\left(\frac{\partial v_{11}}{\partial y} \right)^4 - \left(\frac{\partial v_{12}}{\partial y} \right)^4 \right) &= \left(\frac{\partial v_{11}}{\partial y} - \frac{\partial v_{12}}{\partial y} \right) \left(\frac{\partial v_{11}}{\partial y} + \frac{\partial v_{12}}{\partial y} \right) \left(\left(\frac{\partial v_{11}}{\partial y} \right)^2 + \left(\frac{\partial v_{12}}{\partial y} \right)^2 \right) \\
&\leq C_3 \left(\frac{\partial v_{11}}{\partial y} - \frac{\partial v_{12}}{\partial y} \right),
\end{aligned} \tag{2.19}$$

where C_3 is a suitable bounding constant that can be defined supported by the regularity results shown in Theorem 2.2.

Based on the expressions (2.18) and (2.19), the Eq (2.17) becomes

$$\begin{aligned}
\int_R (v_{11} - v_{12})(y, t) \Psi(y, t) dy &= \int_0^t \int_R (v_{11} - v_{12})(y, t) \frac{\partial \Psi}{\partial s} dy ds \\
&+ \left(m C_2^{m-1} A_4 (1 + A_1) - \frac{\epsilon C_3}{3} \right) \int_0^t \int_R (v_{11} - v_{12})(y, t) \frac{\partial^2 \Psi}{\partial y^2} dy ds \\
&- \left(\frac{M^2}{R_e} + A_2 - 2 A_3 C_2 \right) \int_0^t \int_R (v_{11} - v_{12})(y, t) \Psi dy ds.
\end{aligned} \tag{2.20}$$

Assume the following smooth shape test function defined as

$$\Psi(y, s) = e^{-as} (1 + y^2)^{-\gamma_1}, \tag{2.21}$$

where γ_1 can be chosen such that

$$\int_R \Psi(y, t) dy < \infty. \tag{2.22}$$

For simplicity

$$\int_R \Psi(y, t) dy = 1. \tag{2.23}$$

For $|y| \rightarrow \infty$, the integral mass shall be null. Then, for $R \gg 1$

$$\int_{|y|=R \rightarrow \infty} \Psi(y, t) dy = 0. \tag{2.24}$$

In the asymptotic approximation $|y| \rightarrow \infty$

$$|y|^{2\gamma_1}|y| \rightarrow 0, \quad (2.25)$$

then

$$\gamma_1 > \frac{1}{2}. \quad (2.26)$$

Now, define

$$\int_R \psi(y, s) dy = e^{-as} \int_R \frac{1}{(1+y^2)^{\gamma_1}} dy = e^{-as} \chi(y), \quad (2.27)$$

where

$$\chi(y) = \int_{y \rightarrow \infty} \frac{1}{(1+y^2)^{\gamma_1}} dy. \quad (2.28)$$

The integral (2.28) is finite under the condition for γ_1 in (2.26).

Now, differentiate (2.21) with respect to y , so that

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial y^2} &= \frac{4\gamma_1(\gamma_1 + 1)y^2 e^{-as}}{(1+y^2)^{\gamma_1+2}} - \frac{2\gamma_1}{(1+y^2)^{\gamma_1+1}} \\ &\leq \frac{4\gamma_1(\gamma_1 + 1)e^{-as}}{(1+y^2)^{\gamma_1}} \\ &= 4\gamma_1(\gamma_1 + 1)\Psi. \end{aligned}$$

After integration

$$\int_R \frac{\partial^2 \Psi}{\partial y^2} \leq C_4(\gamma_1)\Psi = C_4(\gamma_1)e^{-as}\chi(y),$$

where $C_4(\gamma_1) = 4\gamma_1(\gamma_1 + 1)$.

Now, the intention is to assess each of the integrals involved in the right hand side of Eq (2.20)

$$\begin{aligned} \int_0^t \int_R (v_{11} - v_{12})(y, t) \frac{\partial \Psi}{\partial s} dy ds &= -a \int_0^t \int_R (v_{11} - v_{12})(y, t) \Psi dy ds \\ &\leq a \int_0^t \int_R (v_{11} - v_{12})(y, t) \Psi dy ds \\ &\leq a \sup |v_{11} - v_{12}| \chi(y) \int_0^t e^{-as} ds \\ &= a \sup |v_{11} - v_{12}| \chi(y) (1 - e^{-at}). \end{aligned} \quad (2.29)$$

$$\begin{aligned} &\left(mC_2^{m-1} A_4 (1 + A_1) - \frac{\epsilon C_3}{3} \right) \int_0^t \int_R (v_{11} - v_{12})(y, t) \frac{\partial^2 \Psi}{\partial y^2} dy ds \\ &\leq \left(mC_2^{m-1} A_4 (1 + A_1) - \frac{\epsilon C_3}{3} \right) \sup |v_{11} - v_{12}| C_4(\gamma_1) \chi(y) \int_0^t \int_R \Psi(y, t) dy ds \end{aligned}$$

$$\begin{aligned}
&= \left(mC_2^{m-1}A_4(1+A_1) - \frac{\epsilon C_3}{3} \right) \sup |v_{11} - v_{12}| C_4(\gamma_1)\chi(y) \int_0^t e^{-as} ds \\
&= \left(mC_2^{m-1}A_4(1+A_1) - \frac{\epsilon C_3}{3} \right) \sup |v_{11} - v_{12}| C_4(\gamma_1)\chi(y) (1 - e^{-at}), \tag{2.30}
\end{aligned}$$

and similarly

$$\begin{aligned}
&-\left(\frac{M^2}{R_e} + A_2 - 2A_3C_2 \right) \int_0^t \int_R (v_{11} - v_{12})(y, t)\Psi dy ds \\
&\leq \left(\frac{M^2}{R_e} + A_2 - 2A_3C_2 \right) \sup |v_{11} - v_{12}| \chi(y) (1 - e^{-at}). \tag{2.31}
\end{aligned}$$

Combining the assessments in (2.29), (2.30) and (2.31) with the Eq (2.20), the following holds

$$\begin{aligned}
\int_R (v_{11} - v_{12})(y, t)\Psi(y, t) dy &\leq a \sup |v_{11} - v_{12}| \chi(y) (1 - e^{-as}) \\
&+ \left(mC_2^{m-1}A_4(1+A_1) - \frac{\epsilon C_3}{3} \right) \sup |v_{11} - v_{12}| C_4(\gamma_1)\chi(y) (1 - e^{-at}) \\
&+ \left(\frac{M^2}{R_e} + A_2 - 2A_3C_2 \right) \sup |v_{11} - v_{12}| \chi(y) (1 - e^{-at}). \tag{2.32}
\end{aligned}$$

Since $|\chi(y)| < \infty$ and for $t > 0$, with $\sup |v_{11} - v_{12}| \rightarrow 0$

$$\int_R (v_{11} - v_{12})(y, t)\Psi(y, t) dy \leq 0,$$

which implies that $v_{11} \leq v_{12}$. This result contradicts the initial assumption $v_{11} \geq v_{12}$. As a consequence, the only compatible result is to consider $v_{11} = v_{12}$, leading to conclude on solutions uniqueness. \square

3. Travelling waves analysis

The travelling wave profiles of solutions are given as per the following definitions: $v_1(y, t) = k(\zeta)$, where $\zeta = y - a_1t$, a_1 denotes travelling wave speed and the profile $k : R \rightarrow (0, \infty)$ belongs to $L^\infty(R)$. The Eq (1.5) is then transformed in terms of the travelling wave profile as

$$-a_1k' = A_5 + A_4(1+A_1)(k^m)'' - \epsilon k'k'' - \left(\frac{M^2}{R_e} + A_2 \right) k - A_3k^2. \tag{3.1}$$

Let us introduce the following new variables

$$Y = k(\zeta), \quad Z = (k^m)', \tag{3.2}$$

such that the Eq (3.1) can be analyzed in a phase space driven by the following set of equations

$$Y' = \frac{1}{m} Y^{1-m} Z$$

$$Z' = \frac{m^3 \left[-\frac{a_1}{m} Y^{1-m} Z + \frac{\epsilon(1-m)}{m^3} Y^{2-3m} Z^3 - A_5 + \left(\frac{M^2}{R_e} + A_2 \right) Y + A_3 Y^2 \right]}{m^3 + A_4 (1 + A_1) - \epsilon Y^{3-3m} Z^2}. \quad (3.3)$$

Making $Y' = 0$ and $Z' = 0$ to determine the critical points, the following is obtained

$$Y^2 + \left(\frac{M^2}{A_3 R_e} + \frac{A_2}{A_3} \right) Y - \frac{A_5}{A_3} = 0. \quad (3.4)$$

with solutions

$$Y_1 = -\frac{1}{2} \left(\frac{M^2}{R_e} + \frac{A_2}{A_3} \right) + \frac{1}{2} \sqrt{\left(\frac{M^2}{A_3 R_e} + \frac{A_2}{A_3} \right)^2 + \frac{4A_5}{A_3}} \quad (3.5)$$

and

$$Y_2 = -\frac{1}{2} \left(\frac{M^2}{R_e} + \frac{A_2}{A_3} \right) - \frac{1}{2} \sqrt{\left(\frac{M^2}{A_3 R_e} + \frac{A_2}{A_3} \right)^2 + \frac{4A_5}{A_3}}. \quad (3.6)$$

Consequently, the system critical points are given by $(Y_1, 0)$ and $(Y_2, 0)$. The next intention is to characterize such critical points together with the bundles of orbits in their proximity.

3.1. Geometric perturbation theory

The Geometric perturbation theory is used to show the asymptotic behavior of specifically defined manifolds in the proximity of the system critical points. Firstly, assume the following manifold I_0 defined as

$$I_0 = \left[(Y, Z) / \begin{aligned} Y' &= \frac{1}{m} Y^{1-m} Z; \\ Z' &= \frac{m^3 \left[-\frac{a_1}{m} Y^{1-m} Z + \frac{\epsilon(1-m)}{m^3} Y^{2-3m} Z^3 - A_5 + \left(\frac{M^2}{R_e} + A_2 \right) Y + A_3 Y^2 \right]}{m^3 + A_4 (1 + A_1) - \epsilon Y^{3-3m} Z^2} \end{aligned} \right],$$

with the system critical points $(Y_1, 0)$ and $(Y_2, 0)$.

A perturbed manifold I_β close to I_0 in the critical point $(Y_1, 0)$ is defined as

$$I_\beta = \left[(Y, Z) / \begin{aligned} Y' &= \beta Z; \\ Z' &= C_5 \beta (Y - Y_2), \end{aligned} \right], \quad (3.7)$$

where β denotes a perturbation parameter close to the equilibrium $(Y_1, 0)$ and C_5 is a suitable constant obtained after root factorization. For the coming assessments, assume $Y_3 = Y - Y_2$.

The intention is to apply the Fenichel invariant manifold theorem [31] as formulated in [32] and [33]. To this end, it shall be shown that I_0 is a normally hyperbolic manifold, i.e., the eigenvalues associated to I_0 in the linear frame proximal to the critical points, and transversal to the tangent space, have non-zero real part. This is shown based on the following equivalent linear flow associated to I_β .

$$\begin{bmatrix} Y_3' \\ Z' \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ C_5 \beta & 0 \end{bmatrix} \begin{bmatrix} Y_3 \\ Z \end{bmatrix}.$$

The related eigenvalues are both real. Consequently, I_0 is a hyperbolic manifold. Now, it is shown that the manifold I_β is locally invariant under the flow (3.3), so that the manifold I_0 can be shown as an asymptotic approach to I_β . For this basis, consider the functions

$$\begin{aligned}\psi^{I_0} &= \frac{m^3 \left[-\frac{a_1}{m} Y^{1-m} Z + \frac{\epsilon(1-m)}{m^3} Y^{2-3m} Z^3 - A_5 + \left(\frac{M^2}{R_e} + A_2 \right) Y + A_3 Y^2 \right]}{m^3 + A_4 (1 + A_1) - \epsilon Y^{3-3m} Z^2}, \\ \psi^{I_\beta} &= C_5 \beta Y_3,\end{aligned}$$

which are $C^i(R \times [0, \delta_1])$, $i \geq 1$, in the proximity of the critical point $(Y_1, 0)$. In this case, δ_1 is determined based on the following flows that are considered to be measurable a.e. in R

$$\|\psi^{I_\beta} - \psi^{I_0}\| \leq \|C_5 \beta Y_3 - \epsilon_0\| \leq C_5 \beta \|Y_3\| \leq \delta_1 \beta,$$

where ϵ_0 is the infinitesimal distance between the critical point $(Y_1, 0)$ and the flow ψ^{I_0} , which exists given the normal hyperbolic condition of I_0 . As the solutions have been shown to be bounded, it is possible to conclude on a finite δ_1 . So the distance between the manifolds holds the normal hyperbolic condition for $\delta_1 \in (0, \infty)$ and β sufficiently small in the proximity of the critical point $(Y_1, 0)$.

Now, consider a perturbed manifold I_γ associated to I_0 , but for the critical point $(Y_2, 0)$ and defined as

$$I_\gamma = [Y, Z/Y' = \gamma Z; Z' = C_6 \gamma (Y - Y_1)], \quad (3.8)$$

where γ denotes a perturbation parameter in the proximity of the equilibrium $(Y_2, 0)$ and C_6 is a suitable constant obtained after root factorization. Suppose $Y_4 = Y - Y_1$, then the following flow associated to I_γ is given by

$$\begin{bmatrix} Y_4' \\ Z' \end{bmatrix} = \begin{bmatrix} 0 & \gamma \\ C_6 \gamma & 0 \end{bmatrix} \begin{bmatrix} Y_4 \\ Z \end{bmatrix}.$$

The associated eigenvalues are both real, leading to state that I_0 is, indeed, a hyperbolic manifold. Now, it is shown that the manifold I_γ is locally invariant under the flow (3.3), so that the manifold I_0 can be shown as an asymptotic approach to I_γ . For this basis consider the functions

$$\begin{aligned}\psi^{I_0} &= \frac{m^3 \left[-\frac{a_1}{m} Y^{1-m} Z + \frac{\epsilon(1-m)}{m^3} Y^{2-3m} Z^3 - A_5 + \left(\frac{M^2}{R_e} + A_2 \right) Y + A_3 Y^2 \right]}{m^3 + A_4 (1 + A_1) - \epsilon Y^{3-3m} Z^2}, \\ \psi^{I_\gamma} &= C_6 \gamma Y_4,\end{aligned}$$

which are $C^i(R \times [0, \delta_1])$, $i \geq 1$, in the proximity of the critical point $(Y_2, 0)$. In this case, δ_1 is determined based on the following flows that are considered to be measurable a.e. in R :

$$\|\psi^{I_\gamma} - \psi^{I_0}\| \leq \|C_6 \gamma Y_4 - \epsilon_1\| \leq C_6 \gamma \|Y_4\| \leq \delta_1 \beta,$$

where ϵ_1 is the infinitesimal distance between the critical point $(Y_2, 0)$ and the flow ψ^{I_0} , which exists given the normal hyperbolic condition of I_0 . As the solutions are bounded, δ_1 is finite. Hence, the distance between the manifolds holds the normal hyperbolic condition for $\delta_1 \in (0, \infty)$ and γ sufficiently small, in the proximity of the critical point $(Y_2, 0)$.

3.2. Construction of travelling wave profiles

Since the manifold I_0 has been shown to satisfy the normal hyperbolic condition under the flow (3.3), it is possible to obtain asymptotic profiles operating in the linearized flows I_β and I_γ .

For this purpose, consider the Eq (3.3) such that a family of trajectories in the phase plane (Y, Z) can be obtained by

$$\frac{dZ}{dY} = \frac{m^4 \left[-\frac{a_1}{m} Y^{1-m} Z + \frac{\epsilon(1-m)}{m^3} Y^{2-3m} Z^3 - A_5 + \left(\frac{M^2}{R_e} + A_2 \right) Y + A_3 Y^2 \right]}{Y^{1-m} Z [m^3 + A_4 (1 + A_1) - \epsilon Y^{3-3m} Z^2]} = H(Y, Z). \quad (3.9)$$

For $Y \gg 1$, the function $H > 0$ while for $0 < Y \ll 1$, the function $H < 0$. In addition, as the function H is singular for $Z = 0$, a trajectory in the proximity of the critical condition $(Y_1, 0)$ shall require

$$-\frac{a_1}{m} Y_1^{1-m} Z_1 + \frac{\epsilon(1-m)}{m^3} Y_1^{2-3m} Z_1^3 - A_5 + \left(\frac{M^2}{R_e} + A_2 \right) Y_1 + A_3 Y_1^2 = 0. \quad (3.10)$$

After using the Eq (3.3) into Eq (3.10), the following holds

$$\epsilon(1-m)(Y_1')^3 - a_1 Y_1 Y_1' - A_5 Y_1 + \left(\frac{M^2}{R_e} + A_2 \right) Y_1^2 + A_3 Y_1^3 = 0. \quad (3.11)$$

Typically, the flow velocity in a porous medium permits the hypothesis $U_0 \ll 1$. In addition, in the assumption of a sufficiently large characteristic length, $L \gg 1$, the parameter $\epsilon \ll 1$ (see (1.3)) for typical fluid densities. As a consequence, such ϵ is considered to construct a solution as an asymptotic expansion in terms of the mentioned parameter ϵ , understood as a perturbation and as given in (1.3). The selection of this parameter permits to account for a physical nuisance as ϵ is related with some of the constants involved in the fluid problem. Then, the following expansion holds for suitable regular functions Y_{1j} , $j = 0, 1, 2, \dots$

$$Y_1 = Y_{10} + \epsilon Y_{11} + \epsilon^2 Y_{12} + \dots \quad (3.12)$$

After using (3.12) into (3.11), the following holds

$$\begin{aligned} \epsilon(1-m)(Y_{10}' + \epsilon Y_{11}' + \epsilon^2 Y_{12}' + \dots)^3 &= a_1 (Y_{10} + \epsilon Y_{11} + \epsilon^2 Y_{12} + \dots)(Y_{10}' + \epsilon Y_{11}' + \epsilon^2 Y_{12}' + \dots) \\ &+ A_5 (Y_{10} + \epsilon Y_{11} + \epsilon^2 Y_{12} + \dots) \\ &- \left(\frac{M^2}{R_e} + A_2 \right) (Y_{10} + \epsilon Y_{11} + \epsilon^2 Y_{12} + \dots)^2 \\ &- A_3 (Y_{10} + \epsilon Y_{11} + \epsilon^2 Y_{12} + \dots)^3. \end{aligned} \quad (3.13)$$

Comparing the terms of orders ϵ^0 and ϵ^1

$$Y_{10}' = -\frac{A_5}{a_1} + \left(\frac{M^2}{R_e a_1} + \frac{A_2}{a_1} \right) Y_{10} + \frac{A_3}{a_1} (Y_{10})^2, \quad (3.14)$$

and

$$Y'_{11} + \left(\frac{Y'_{10}}{Y_{10}} + \frac{A_5}{a_1 Y_{10}} - \frac{2M^2}{R_e a_1} + \frac{2A_2}{a_1} - \frac{3A_3 Y_{10}}{a_1} \right) Y_{11} = \frac{(1-m)(Y'_{10})^3}{a_1 Y_{10}}. \quad (3.15)$$

The standard resolution of (3.14) leads to

$$Y_{10} = \frac{1}{1 - e^{-\frac{2A_3 b}{a_1} \zeta}} \left[\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right) + \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}} \right. \\ \left. + \left(\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right) - \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}} \right) e^{-\frac{2A_3 \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}}}{a_1} \zeta} \right]. \quad (3.16)$$

Differentiate (3.16) to get

$$Y'_{10} = \frac{\left(\frac{M^2}{A_3 R_e} + \frac{A_2}{A_3} \right) \frac{2A_3 \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}}}{a_1} e^{-\frac{2A_3 \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}}}{a_1} \zeta}}{\left(1 - e^{-\frac{2A_3 \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}}}{a_1} \zeta} \right)^2}. \quad (3.17)$$

Upon substitution of (3.16) and (3.17) in (3.15), it holds

$$Y'_{11} + \left[-\frac{2abe^{-b\zeta}}{(1 - e^{-b\zeta})(c + de^{-b\zeta})} + \frac{A_5(1 - e^{-b\zeta})}{a_1(c + de^{-b\zeta})} + f - \frac{3A_3(c + de^{-b\zeta})}{a_1(1 - e^{-b\zeta})} \right] Y_{11} \\ = -\frac{(1-m)8a^3 b^3 e^{-3b\zeta}}{a_1(c + de^{-b\zeta})(1 - e^{-b\zeta})^5}, \quad (3.18)$$

where $a = \left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)$, $b = \frac{2A_3 \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}}}{a_1}$, $c = \left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right) + \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}}$,
 $d = \left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right) - \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}}$ and $f = -\frac{2M^2}{R_e a_1} + \frac{2A_2}{a_1}$.

After solving the Eq (3.18)

$$Y_{11} = \frac{(1 - e^{-b\zeta})^{C_3}}{(c + de^{-b\zeta})^{C_1} e^{C_2 \zeta}} \left[\frac{c^{C_1-1} e^{(C_2-1)\zeta}}{C_2 - 3} + \frac{c^{C_1} [cC_3 + C_1 d - d + 5] e^{(C_2-b-3)\zeta}}{C_2 - b - 3} + \dots \right], \quad (3.19)$$

where $C_1 = \frac{2a}{d-c} - \frac{2A_3 A_5 (c+d)}{a_1 b c d}$, $C_2 = \frac{A_5}{c} + \frac{3(2A_3 a + a_1 b)}{2a_1} + f$ and $C_3 = \frac{2a}{d-c} - \frac{12aA_3}{a_1^2 b}$.

Introducing the expressions (3.16) and (3.19) into (3.12), the following holds

$$Y_1 = \frac{1}{1 - e^{-\frac{2A_3 b}{a_1} \zeta}} \left[\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right) + \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}} \right]$$

$$\begin{aligned}
& + \left[\left(\frac{M^2}{2A_3R_e} + \frac{A_2}{2A_3} \right) - \sqrt{\left(\frac{M^2}{2A_3R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}} \right] e^{-\frac{2A_3 \sqrt{\left(\frac{M^2}{2A_3R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}}}{a_1} \zeta}. \\
& + \frac{\epsilon (1 - e^{-b\zeta})^{C_3}}{(c + de^{-b\zeta})^{C_1} e^{C_2\zeta}} \left[\frac{c^{C_1-1} e^{(C_2-1)\zeta}}{C_2 - 3} + \frac{c^{C_1} [cC_3 + C_1d - d + 5] e^{(C_2-b-3)\zeta}}{C_2 - b - 3} + \dots \right] + O(\epsilon^2),
\end{aligned} \tag{3.20}$$

which implies that

$$\begin{aligned}
v_1(y, t) &= \frac{1}{1 - e^{-\frac{2A_3b}{a_1}(y-a_1t)}} \left[\left(\frac{M^2}{2A_3R_e} + \frac{A_2}{2A_3} \right) + \sqrt{\left(\frac{M^2}{2A_3R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}} \right. \\
& + \left. \left(\frac{M^2}{2A_3R_e} + \frac{A_2}{2A_3} \right) - \sqrt{\left(\frac{M^2}{2A_3R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}} \right] e^{-\frac{2A_3 \sqrt{\left(\frac{M^2}{2A_3R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}}}{a_1} (y-a_1t)} \\
& + \frac{\epsilon (1 - e^{-b(y-a_1t)})^{C_3}}{(c + de^{-b(y-a_1t)})^{C_1} e^{C_2(y-a_1t)}} \left[\frac{c^{C_1-1} e^{(C_2-1)(y-a_1t)}}{C_2 - 3} + \frac{c^{C_1} [cC_3 + C_1d - d + 5] e^{(C_2-b-3)(y-a_1t)}}{C_2 - b - 3} \right. \\
& \left. + \dots \right] + O(\epsilon^2).
\end{aligned} \tag{3.21}$$

The Eq (3.21) shows the existence of an exponential decaying under the travelling wave profile asymptotic expansion. This results is not trivial in this case given the degeneracy in the diffusivity under the non-linear diffusion of porous medium kind.

Now, the intention is to show that the defined supporting manifold I_β preserves the exponential behavior shown in the proximity of the associated critical point. For this purpose, consider the expression (3.7), so that

$$\frac{dZ}{dY} = \frac{C_5(Y - Y_2)}{Z}. \tag{3.22}$$

After solving (3.22)

$$Z = -\sqrt{C_5}(Y - Y_2). \tag{3.23}$$

Introducing the expression (3.7) into (3.23), the following holds

$$Y' = -\beta \sqrt{C_5}(Y - Y_2), \tag{3.24}$$

such that a solution is

$$Y = Y_2 + e^{-\beta \sqrt{C_5} \zeta}, \tag{3.25}$$

which implies that

$$v_1(y, t) = \frac{1}{2} \left(\frac{M^2}{R_e} + \frac{A_2}{A_3} \right) + \frac{1}{2} \sqrt{\left(\frac{M^2}{A_3R_e} + \frac{A_2}{A_3} \right)^2 + \frac{4A_5}{A_3}} + e^{-\beta \sqrt{C_5}(y-a_1t)} \tag{3.26}$$

This last expression shows the conservation of the exponential profile in the proximity of the critical point kept by the asymptotic manifold I_β .

The existence and exact assessments of an orbit in the proximity of the critical point $(Y_2, 0)$ follows the same procedure as discussed above for the point $(Y_1, 0)$. Indeed, a trajectory in the phase plane is given as

$$\epsilon(1-m)(Y_2')^3 - a_1 Y_2 Y_2' - A_5 Y_2 + \left(\frac{M^2}{R_e} + A_2\right) Y_2^2 + A_3 Y_2^3 = 0. \quad (3.27)$$

Solving the Eq (3.27) leads to obtain

$$\begin{aligned} Y_2 = & \frac{1}{1 - e^{-\frac{2A_3 b}{a_1} \zeta}} \left[\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right) + \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}} \right. \\ & + \left. \left(\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right) - \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}} \right) e^{-\frac{2A_3 \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}}}{a_1} \zeta} \right] \\ & + \frac{\epsilon(1 - e^{-b\zeta})^{C_3}}{(c + de^{-b\zeta})^{C_1} e^{C_2 \zeta}} \left[\frac{c^{C_1-1} e^{(C_2-1)\zeta}}{C_2 - 3} + \frac{c^{C_1} [cC_3 + C_1 d - d + 5] e^{(C_2-b-3)\zeta}}{C_2 - b - 3} + \dots \right] + O(\epsilon^2), \end{aligned} \quad (3.28)$$

where $a = \left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)$, $b = \frac{2A_3 \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}}}{a_1}$,

$$c = \left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right) + \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}},$$

$$d = \left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right) - \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}},$$

$$f = -\frac{2M^2}{R_e a_1} + \frac{2A_2}{a_1}, C_1 = \frac{2a}{d-c} - \frac{2A_3 A_5 (c+d)}{a_1 b c d}, C_2 = \frac{A_5}{c} + \frac{3(2A_3 a + a_1 b)}{2a_1} + f \text{ and } C_3 = \frac{2a}{d-c} - \frac{12aA_3}{a_1^2 b}.$$

As a consequence, a solution is given by the following expression

$$\begin{aligned} v_1(y, t) = & \frac{1}{1 - e^{-\frac{2A_3 b}{a_1} (y-a_1 t)}} \left[\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right) + \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}} \right. \\ & + \left. \left(\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right) - \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}} \right) e^{-\frac{2A_3 \sqrt{\left(\frac{M^2}{2A_3 R_e} + \frac{A_2}{2A_3} \right)^2 + \frac{A_5}{A_3}}}{a_1} (y-a_1 t)} \right] \\ & + \frac{\epsilon(1 - e^{-b(y-a_1 t)})^{C_3}}{(c + de^{-b(y-a_1 t)})^{C_1} e^{C_2 (y-a_1 t)}} \left[\frac{c^{C_1-1} e^{(C_2-1)(y-a_1 t)}}{C_2 - 3} + \frac{c^{C_1} [cC_3 + C_1 d - d + 5] e^{(C_2-b-3)(y-a_1 t)}}{C_2 - b - 3} \right. \\ & \left. + \dots \right] + O(\epsilon^2). \end{aligned} \quad (3.29)$$

Again, the intention now is to show that the defined supporting manifold I_γ preserves the exponential behavior in the proximity of the critical point $(Y_2, 0)$. For this purpose, consider the expression (3.8) so that

$$\frac{dZ}{dY} = \frac{C_5(Y - Y_1)}{Z}. \quad (3.30)$$

After solving (3.30), the following holds

$$Z = -\sqrt{C_5}(Y - Y_1). \quad (3.31)$$

Introducing the expression (3.8) into (3.31)

$$Y_1' = -\gamma \sqrt{C_6}(Y - Y_1), \quad (3.32)$$

such that a solution is given by

$$Y = Y_1 + e^{-\gamma \sqrt{C_6} \zeta}, \quad (3.33)$$

which implies that

$$v_1(y, t) = -\frac{1}{2} \left(\frac{M^2}{R_e} + \frac{A_2}{A_3} \right) + \frac{1}{2} \sqrt{\left(\frac{M^2}{A_3 R_e} + \frac{A_2}{A_3} \right)^2 + \frac{4A_5}{A_3}} + e^{-\gamma \sqrt{C_6}(y-a_1 t)}. \quad (3.34)$$

This last expression permits to show the conservation of the exponential profile in the proximity of the critical point under study by the asymptotic manifold I_γ .

4. Numerical validation

According to expressions (3.25) and (3.33), the analytical assessments have provided an exponential tail in the proximity of the system critical points. To validate the assessments, it is the intention to compare such analytical exponential tail with a numerical simulation of the Eq (2.1). Note that the numerical exploration is not intended to introduce a wide parametrical analysis with each of the involved fluid constants, on the contrary the objective is to provide evidences on the accuracy of the analytical process followed. In addition, for the validation exercise the expression (3.33) has been considered, under the form $Y - Y_1 = e^{-\gamma \sqrt{C_6} \zeta}$. The numerical assessment is executed with the `bvp4c` in Matlab (refer to [34] for further details on this software tool).

The R_e number has been considered as a free parameter to establish the accuracy of solutions. In addition, the following particular values in the other fluid constant have been considered for the sake of simplicity, but with no impact in the conclusions

$$0 < A_5 \ll 1, U_0 = 1, A_1 = 1, 0 < \epsilon \ll 1, M = 1, \nu = 1, K_1 = 1, K_2 = 1. \quad (4.1)$$

Note that $L = 1000$ as the domain of integration has been considered as $(-500, 500)$. This domain has been selected after some numerical trials and provides an efficient compromise between low impact of the collocation methods at the borders and a moderate computational time. In addition, the global absolute error allowed is of 10^{-4} .

Note that the R_e in porous media is typically low, hence the considered values are $R_e = 1; 100$.

As it can be noticed from Figures 1 and 2, the analytical asymptotic solution (3.33) based on the geometric perturbation theory and the numerical solution to (2.1) fit an exponential behaviour with an order deficiency in the decreasing rate. Nonetheless, solutions are very close for increasing values of ζ , this is in the asymptotic approach where the analytical solution has been obtained. This is not a trivial conclusion for the degenerate diffusion given in (2.1) and represents a finding to remark.

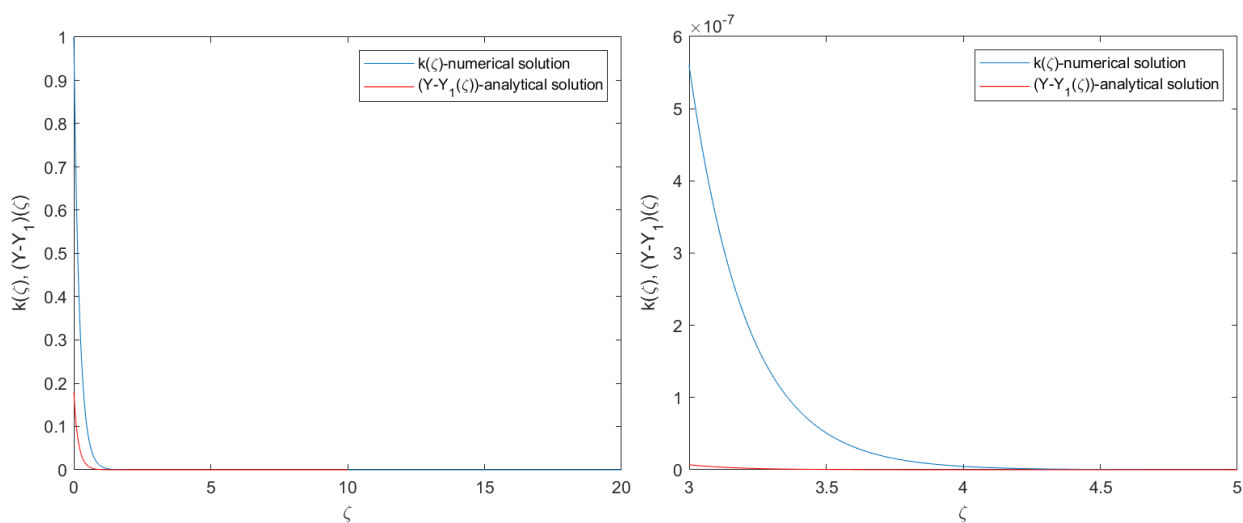


Figure 1. Solutions for $Re = 1$. The left hand figure is a global representation. Both solutions preserves the exponential behaviour. The picture on the right represents a zoom in the interval $[3, 5]$. The reader can observe a certain order deficiency between the analytical and numerical assessments. Nonetheless, both solutions are close, the distance is of the order 10^{-7} .

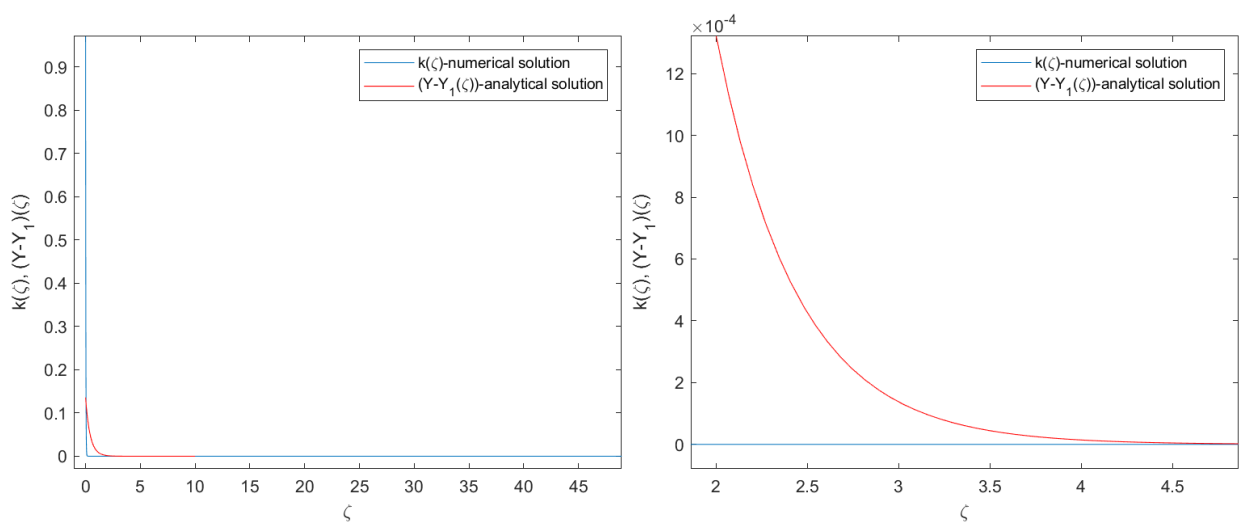


Figure 2. Idem for $Re = 100$. For $\zeta \geq 2$, the distance between the analytical and numerical solution is of order 10^{-4} .

5. Conclusions

The set of assessments and discussions presented in this paper have provided evidences on the regularity of solutions to a kind of fluid flow with degenerate diffusivity and formulated with Darcy-Forchheimer porosity principles and Eyring-Powell viscosity terms. Afterwards, asymptotic travelling waves solutions have been explored based on a perturbation technique. Particularly, the Geometric Perturbation Theory has shown that an exponential decreasing rate holds even when the equation discussed presents a degenerate diffusivity. Furthermore, two linearized manifolds in the proximity of the critical points have been shown to hold and an exponential behaviour has been obtained accordingly. Eventually, a numerical exercise has been introduced to account for the validation of the analytical assessments done.

Conflict of interest

The authors declare that there are not any conflict of interests.

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