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Research article

Periodic solutions and limit cycles of mixed Lienard-type differential equations

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Abstract: In the attractive research field of nonlinear differential equations, there are a few studies devoted to finding exact and explicit harmonic and isochronous periodic solutions and limit cycles. In this contribution, we present some classes of polynomial mixed Lienard-type differential equations that can generate many equations with exact solutions. These classes of equations constitute counterexamples of the classical existence theorems.

Keywords: polynomial mixed Lienard-type differential equations; exact harmonic and isochronous periodic solutions; exact algebraic limit cycles; classical existence theorems
Mathematics Subject Classification: 34A05, 34C05, 34C25, 34C15

1. Introduction

Although there is a large body of literature on limit cycles, a very limited number of studies are devoted to their exact and explicit formula. This fact is not surprising since the problem is to find solutions of nonlinear differential equations that are in general not explicitly integrable. A salient example is the Van der Pol equation [1, 2]

$$\ddot{x} + \beta(x^2 - 1)\dot{x} + x = 0, \tag{1.1}$$

where overdot means differentiation with respect to time, and β is a constant. The exact solution is currently unknown. The Van der Pol equation belongs to the class of Lienard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1.2}$$

where $f(x)$ and $g(x)$ are functions of x. The conditions for the existence of a stable and unique limit cycle for Eq (1.1) primarily [1–3] require that $g(x)$ be odd, $g(0) = 0$, $xg(x) > 0$ for $| x |> 0$ and $f(x)$ be even and $f(0) < 0$, are well known for many years. The periodic solutions of a more generalized form of Eq (1.2) in the form

$$\ddot{x} + h(x, \dot{x})\dot{x} + g(x) = 0, \tag{1.3}$$

called the mixed Lienard-type differential equation or Lienard-Levinson-Smith system [3, 4], where $h(x, \dot{x})$ is function of its arguments, are known to be difficult to exactly calculate. Accordingly, this type of equation is mainly investigated using qualitative theory of differential equations and approximate analytical techniques [1–7]. The conditions for the existence of at least one limit cycle for Eq (1.3) primarily require [1–4, 6, 7] as in the previous case, that
(i) $g(0) = 0$, $xg(x) > 0$ when $| x |> 0$,
(ii) $h(0, 0) < 0$,
(iii) there exists $x_0 > 0$ such that $h(x, \dot{x}) \geq 0$ for $| x |\geq x_0$.
The conditions $g(0) = 0$, $xg(x) > 0$ for $| x |> 0$ ensure the existence of a single equilibrium point at the origin [1]. Equation (1.3) or its equivalent planar dynamical system

$$\dot{x} = y, \quad \dot{y} = -yh(x, y) - g(x), \tag{1.4}$$

has been additionally subject to a rich literature for counting the maximum number of limit cycles when $h(x, y)$ and $g(x)$ are polynomials using averaging theory in connection with the second part of the Hilbert 16*th* problem [5]. The hybrid Rayleigh-Van der Pol oscillator

$$\ddot{x} + \beta(\dot{x}^2 + x^2 - 1)\dot{x} + x = 0, \tag{1.5}$$

is the typical example of equations of the form (1.3), where β is a constant. Equation (1.5) has the exact harmonic solution *cost* and exhibits in the $(x, y = \dot{x})$ phase plane the algebraic limit cycle of degree 2 given by

$$x^2 + y^2 - 1 = 0. \tag{1.6}$$

However, we have seen now in the literature that in many cases, qualitative and approximation theories of nonlinear differential equations are not satisfactory to predict the existence of periodic and nonperiodic solutions. Thus, the search for differential equations with exact and explicit solutions is of utmost importance. In other words, the knowledge of exact and explicit solutions is an irreplaceable necessity for practical applications. Even when it is sometimes possible to calculate an exact periodic solution, this often consists of a complicated expression in terms of special functions that are not always easy to implement in practical applications. Conversely, the mathematical properties of sinusoidal functions are well known and well mastered for their implementation in engineering and industrial practices. Therefore, one can understand the vital importance of a nonlinear differential equation with an exact sinusoidal solution. On the other hand, one can say that nonlinear equations having many terms with the exact harmonic periodic solution are not extensively investigated in the literature. Thus, it is necessary to investigate in a significant and intensive way such equations that can be exploited for numerical simulation of oscillations in nonlinear dynamic systems and test of the effectiveness and reliability of numerical methods implemented in the ODE solvers. The present study fails within this perspective. Now, note that the problem of finding exact algebraic limit cycles for equations of type (1.3) has been considered in recent papers [8–11]. The works additionally show the existence of classes of polynomial and nonpolynomial differential equations that can exhibit many equations with exact algebraic limit cycles [8–11]. In [12], the authors proved that the equation

$$\ddot{x} + x(\dot{x}^2 + x^2 - 1)\dot{x} + x = 0, \tag{1.7}$$

has the exact harmonic periodic solution. In this regard, the question is to ask if one can modify Eq (1.7) to build interesting classes of equations that can generate many equations with exact harmonic periodic solutions and limit cycles. From this perspective, the objective is to formulate some classes of polynomial differential equations of type (1.3) that can generate many conservative and nonconservative equations with exact harmonic solutions and algebraic limit cycles by conveniently modifying Eq (1.7). Therefore, we can prove the following result.

Theorem 1.1. *Consider the equation*

$$\ddot{x} + x\left(\dot{x}^2 + \dot{x}\sum_{\ell=0}^{n} x^{2\ell} + x^2 - 1\right)\dot{x} + x^{2n+3} = 0, \tag{1.8}$$

where $n \geq 0$ is an integer. Then, Eq (1.8) has the exact harmonic solution

$$x(t) = cost, \tag{1.9}$$

for $n = 0, 1, 2, \ldots$.

Remark 1. Let

$$h(x, \dot{x}) = x\left(\dot{x}^2 + \dot{x}\sum_{\ell=0}^{n} x^{2\ell} + x^2 - 1\right). \tag{1.10}$$

Then, $h(0, 0) = 0$, and Eq (1.8) does not satisfy the classical theorem for the existence of at least one periodic solution for $n = 0, 1, 2, \ldots$. Figures 1–3 show the phase portraits and vector field of Eq (1.8) exhibiting closed trajectories corresponding to periodic solutions when $n = 0, 1$ and 2, respectively. We can also prove the following theorem.

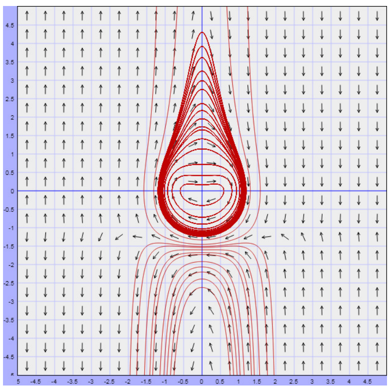

Figure 1. Phase portrait and vector field of Eq (1.8) for $n = 0$.

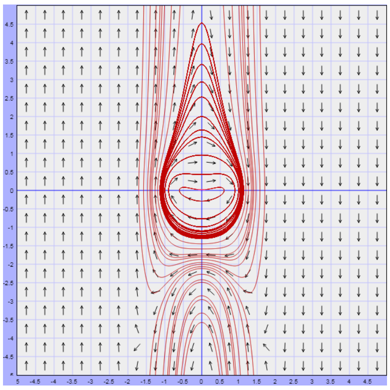

Figure 2. Phase portrait and vector field of Eq (1.8) for $n = 1$.

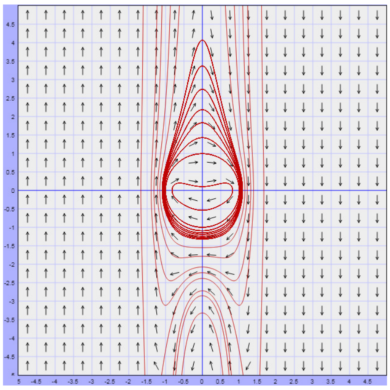

Figure 3. Phase portrait and vector field of Eq (1.8) for $n = 2$.

Theorem 1.2. *Consider the equation*

$$\ddot{x} + (x\dot{x}^2 + \dot{x} + x^3 - x)\dot{x} + x + x^2 - 1 = 0. \tag{1.11}$$

Then, Eq (1.11) has the exact harmonic solution

$$x(t) = cost. \tag{1.12}$$

Remark 2. Let

$$h(x, \dot{x}) = x\dot{x}^2 + \dot{x} + x^3 - x, \tag{1.13}$$

and

$$g(x) = x + x^2 - 1. \tag{1.14}$$

Then, $h(0, 0) = 0$, and condition (ii) is not satisfied. $g(0) = -1 \neq 0$ and $g(x)$ is not odd and condition (i) is not satisfied. Thus Eq (1.11) does not satisfy the Lienard-Levinson-Smith theorem [1–4, 6, 7] for the existence of at least one limit cycle. But, Figure 4 exhibits the phase portrait and vector field of Eq (1.11) showing the existence of an algebraic limit cycle of degree 2 given by

$$x^2 + y^2 - 1 = 0. \tag{1.15}$$

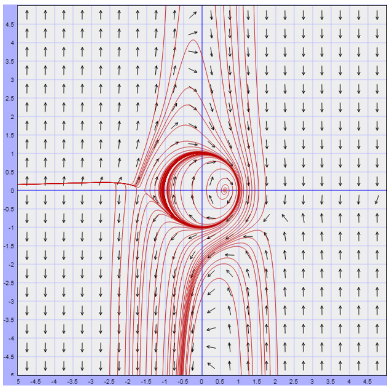

Figure 4. Phase portrait and vector field of Eq (1.11).

Consider now the following obtained result.

Theorem 1.3. *Let*

$$\ddot{x} + \left[x\dot{x}^2 + \dot{x}\left(1 + \sum_{\ell=0}^{n} x^{2\ell+1}\right) + x^3 - x\right]\dot{x} + x^{2n+3} + x^2 - 1 = 0, \tag{1.16}$$

where $n \geq 0$ is an integer. Then, Eq (1.16) has the exact harmonic solution

$$x(t) = cost, \tag{1.17}$$

for $n = 0, 1, 2, \ldots$.

Remark 3. Let

$$h(x, \dot{x}) = \left[x\dot{x}^2 + \dot{x}\left(1 + \sum_{\ell=0}^{n} x^{2\ell+1}\right) + x^3 - x\right], \tag{1.18}$$

and

$$g(x) = x^{2n+3} + x^2 - 1. \tag{1.19}$$

Then, $h(0, 0) = 0$, is not negative, and condition (ii) is not satisfied. $g(x)$ is not odd, and $g(0) = -1 \neq 0$, and condition (i) is not satisfied. Consequently, Eq (1.16) does not satisfy the Lienard-Levinson-Smith theorem for the existence of at least one limit cycle. However, Figures 5–7 show the phase portraits and vector field of Eq (1.16) exhibiting algebraic limit cycles of degree 2 given by Eq (1.15) for $n = 0, 1$ and 2.

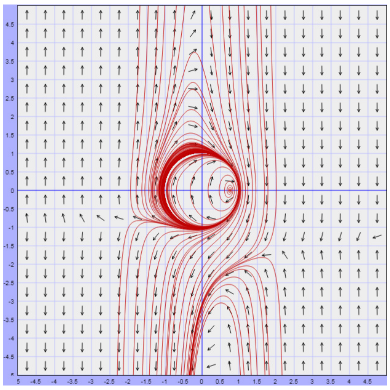

Figure 5. Phase portrait and vector field of Eq (1.16) for $n = 0$.

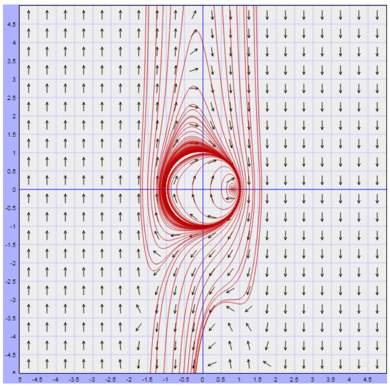

Figure 6. Phase portrait and vector field of Eq (1.16) for $n = 1$.

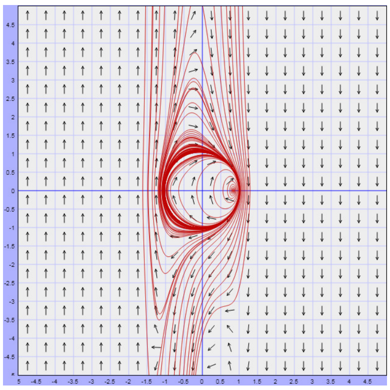

Figure 7. Phase portrait and vector field of Eq (1.16) for $n = 2$.

Theorem 1.4. *Consider the equation*

$$\ddot{x} + \left[(x+1)\dot{x}^2 + \dot{x} \sum_{\ell=0}^{n} x^{2\ell+1} + x^3 + x^2 - x - 1 \right] \dot{x} + x^{2n+3} = 0, \tag{1.20}$$

where $n \geq 0$ is an integer. Then, Eq (1.20) has the exact harmonic solution

$$x(t) = cost, \tag{1.21}$$

for $n = 0, 1, 2, \ldots$.

Remark 4. It is easy to note that Eq (1.20) does not satisfy the Lienard-Levinson-Smith theorem [1–4, 6, 7]. In this regard let

$$h(x, \dot{x}) = \left[(x+1)\dot{x}^2 + \dot{x}\sum_{\ell=0}^{n} x^{2\ell+1} + x^3 + x^2 - x - 1\right].$$

Then $h(x, \dot{x}) \geq 0$ for $\dot{x}x \sum_{\ell=0}^{n} x^{2\ell} \geq 0$ or $\dot{x}x \geq 0$ that is for $\pm x\sqrt{1-x^2} \geq 0$ under Theorem 1.4 that is under $x^2 + y^2 - 1 = 0$, with $y = \dot{x}$, such that $x_0 = 0$, and condition (iii) is not satisfied. Thus Eq (1.20) does not satisfy the Lienard-Levinson-Smith theorem. Figures 8–10 exhibit the phase portraits and vector field of Eq (1.20) showing the existence of algebraic limit cycles of degree 2 given by Eq (1.15) for $n = 0, 1$ and 2, respectively.

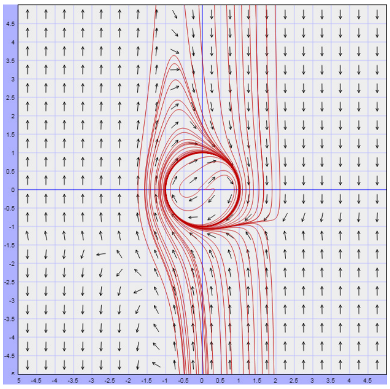

Figure 8. Phase portrait and vector field of Eq (1.20) for $n = 0$.

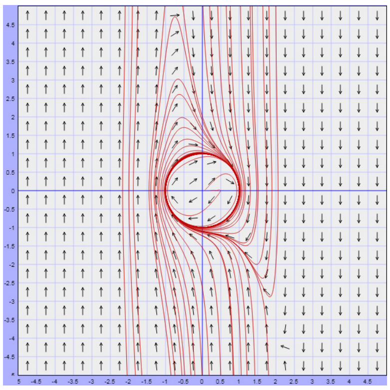

Figure 9. Phase portrait and vector field of Eq (1.20) for $n = 1$.

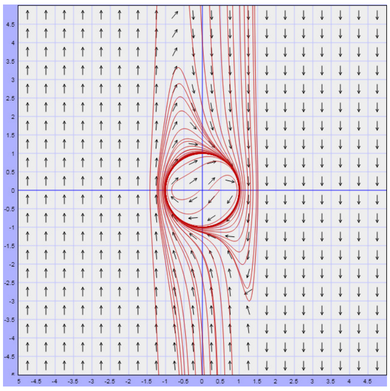

Figure 10. Phase portrait and vector field of Eq (1.20) for $n = 2$.

Let us consider the following theorems.

Theorem 1.5. *Let*

$$\ddot{x} + \left[x\dot{x}^2 + x^3 - x + \dot{x}\left(1 + \sum_{l=0}^{n} x^{2l+2}\right)\right]\dot{x} + x^{2n+4} + x - 1 = 0, \tag{1.22}$$

where $n \geq 0$ is an integer. Then, Eq (1.22) posseses the exact harmonic solution

$$x(t) = \cos t. \tag{1.23}$$

Remark 5. Equation (1.22) is of type (1.3) such that

$$h(x, \dot{x}) = x\dot{x}^2 + x^3 - x + \dot{x}\left(1 + \sum_{l=0}^{n} x^{2l+2}\right), \tag{1.24}$$

and

$$g(x) = x^{2n+4} + x - 1. \tag{1.25}$$

Since $h(0, 0) = 0$ is not negative and $g(x)$ is not odd, with $g(0) = -1 \neq 0$, therefore conditions (ii) and (i) are not respectively satisfied by $h(x, \dot{x})$ and $g(x)$. In this way, Eq (1.22) does not satisfy the classical theorems for the existence of at least one periodic solution. As an example of illustration, Eq (1.22) can be reduced to

$$\ddot{x} + \left[x\dot{x}^2 + x^3 - x + \dot{x}\left(1 + x^2\right)\right]\dot{x} + x^4 + x - 1 = 0, \tag{1.26}$$

when $n = 0$. Figure 11 shows the phase paths and vector field of Eq (1.26).

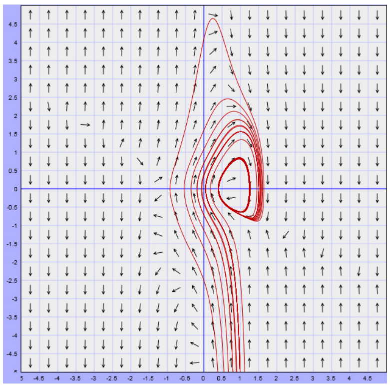

Figure 11. Phase portrait and vector field of Eq (1.22) for $n = 0$.

Theorem 1.6. *Let us consider*

$$\ddot{x} + \left[x\dot{x}^2 + x^3 - x + \dot{x}\left(1 - \dot{x}^{2n+2} - x^2 \sum_{l=0}^{n} \dot{x}^{2l}\right)\right]\dot{x} + x = 0, \tag{1.27}$$

where $n \geq 0$ is an integer. Then, Eq (1.27) admits the exact and explicit harmonic solution

$$x(t) = \cos t. \tag{1.28}$$

Remark 6. Equation (1.27) has the form of the mixed Lienard-type Eq (1.3) where

$$h(x, \dot{x}) = x\dot{x}^2 + x^3 - x + \dot{x}\left(1 - \dot{x}^{2n+2} - x^2 \sum_{l=0}^{n} \dot{x}^{2l}\right), \tag{1.29}$$

such that $h(0, 0) = 0$ is not negative. Thus condition (ii) is not satisfied and Eq (1.27) does not satisfy the classical theorems for the existence of at least one periodic solution. For example, when $n = 0$, Eq (1.27) leads to

$$\ddot{x} + \left[x\dot{x}^2 + x^3 - x + \dot{x}\left(1 - \dot{x}^2 - x^2\right)\right]\dot{x} + x = 0. \tag{1.30}$$

Figure 12 exhibits the phase portrait and vector field of Eq (1.30).

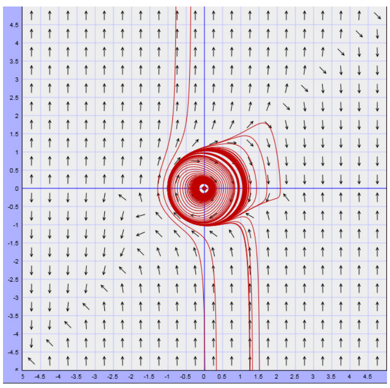

Figure 12. Phase portrait and vector field of Eq (1.27) for $n = 0$.

Theorem 1.7. *Consider the equation*

$$\ddot{x} + \left[x \left(\dot{x}^2 + x^2 - 1 + x \sum_{l=0}^{n} \dot{x}^{2l+1} \right) + \dot{x}^{2n+3} \right] \dot{x} + x^2 + x - 1 = 0, \tag{1.31}$$

where $n \geq 0$ is an integer. Then, Eq (1.31) has the exact solution

$$x(t) = \cos t. \tag{1.32}$$

Remark 7. From Eq (1.31), according to Eq (1.3), we can have

$$h(x, \dot{x}) = x \left(\dot{x}^2 + x^2 - 1 + x \sum_{l=0}^{n} \dot{x}^{2l+1} \right) + \dot{x}^{2n+3}, \tag{1.33}$$

and

$$g(x) = x^2 + x - 1. \tag{1.34}$$

Thus $h(0, 0) = 0$ is not negative and $g(0) = -1$. Additionally $g(x)$ is not odd. In this way, Eq (1.31) does not satisfy the classical theorems for the existence of at least one periodic solution. We can reduce Eq (1.31) to

$$\ddot{x} + [x(\dot{x}^2 + x^2 - 1 + x\dot{x}) + \dot{x}^3]\dot{x} + x^2 + x - 1 = 0, \tag{1.35}$$

when $n = 0$ as an example. The phase portrait and vector field of Eq (1.35) is represented in Figure 13.

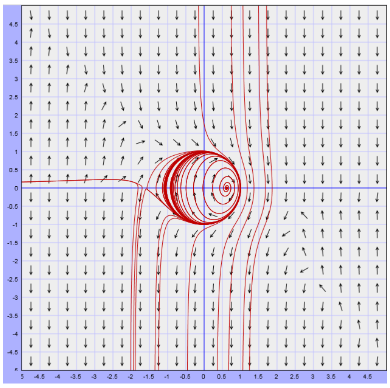

Figure 13. Phase portrait and vector field of Eq (1.31) for $n = 0$.

Theorem 1.8. *Let us consider*

$$\ddot{x} + \left(\dot{x}^2 + x^2 - 1 + \dot{x}^{2n+3} + x^2 \sum_{l=0}^{n} \dot{x}^{2l+1} \right) \dot{x} + x^2 + x - 1 = 0, \tag{1.36}$$

where $n \geq 0$ is an integer. Then, Eq (1.36) possesses the exact sinusoidal solution

$$x(t) = \cos t. \tag{1.37}$$

Remark 8. From Eq (1.36), we can write

$$h(x,\dot{x}) = \dot{x}^2 + x^2 - 1 + \dot{x}^{2n+3} + x^2 \sum_{l=0}^{n} \dot{x}^{2l+1}, \tag{1.38}$$

and

$$g(x) = x^2 + x - 1. \tag{1.39}$$

It is obvious that Eq (1.36) does not satisfy the classical theorems for the existence of at least one periodic solution since $g(0) = -1 \neq 0$ and $g(x)$ is not odd. Figure 14 exhibits the phase portrait and vector field of Eq (1.36) for $n = 0$.

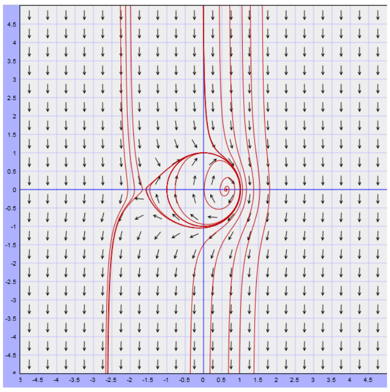

Figure 14. Phase portrait and vector field of Eq (1.36) for $n = 0$.

In the sequel of this work, we prove the above theorems. Therefore, we prove **Theorem** 1.1 (section 2), **Theorem** 1.2 (section 3), **Theorem** 1.3 (section 4) and **Theorem** 1.4 (section 5). Finally, we prove **Theorem** 1.5 (section 6), **Theorem** 1.6 (section 7), **Theorem** 1.7 (section 8) and **Theorem** 1.8 (section 9) and give a conclusion for the work.

2. Proof of Theorem 1.1

From Eq (1.9)

$$\dot{x}(t) = -\sin t, \tag{2.1}$$

and

$$\ddot{x}(t) = -cost. \tag{2.2}$$

Substituting Eqs (1.9), (2.1) and (2.2) into Eq (1.8) and taking into account

$$cos^2t + sin^2t = 1, \tag{2.3}$$

leads to

$$
\begin{aligned}
&\ddot{x} + x(\dot{x}^2 + \dot{x}\sum_{l=0}^{n} x^{2l} + x^2 - 1)\dot{x} + x^{2n+3} \\
=& - cost - costsint(sin^2t - sint\sum_{l=0}^{n} cos^{2l}t + cos^2t - 1) + cos^{2n+3}t \\
=& - cost + costsin^2t\sum_{l=0}^{n} cos^{2l}t + cos^{2n+3}t \\
=& - cost + cost(1 - cos^2t)\sum_{l=0}^{n} cos^{2l}t + cos^{2n+3}t \\
=& - cost + cost\sum_{l=0}^{n} cos^{2l}t - cos^3t\sum_{l=0}^{n} cos^{2l}t + cos^{2n+3}t \\
=& - cost + cost(1 + \sum_{l=1}^{n} cos^{2l}t) - cos^3t\sum_{l=0}^{n} cos^{2l}t + cos^{2n+3}t \\
=&cost\sum_{l=1}^{n} cos^{2l}t - \sum_{l=1}^{n} cos^{2l+1}t - cos^{2n+3}t + cos^{2n+3}t \\
=&0,
\end{aligned} \tag{2.4}
$$

proving Theorem 1.1. In the following, we prove Theorem 1.2.

3. Proof of Theorem 1.2

Consider Eqs (1.9), (2.1)–(2.3). Then, Eq (1.11) involves

$$
\begin{aligned}
& - cost + [(cost)sin^2t - sint + cos^3t - cost](-sint) + cost + cos^2t - 1 \\
=& - cost + [(cost)(1 - cos^2t) - sint + cos^3t - cost](-sint) + cost + cos^2t - 1 \\
=& - cost + [cost - cos^3t - sint + cos^3t - cost](-sint) + cost + cos^2t - 1 \\
=& - cost + sin^2t + cost + cos^2t - 1 \\
=& - cost + 1 - cos^2t + cost + cos^2t - 1 \\
=&0.
\end{aligned} \tag{3.1}
$$

Therefore, the proof of Theorem 1.2 is performed.

4. Proof of Theorem 1.3

Applying Eqs (1.9), (2.1)–(2.3) we can obtain

$$
\begin{aligned}
&\Big[x\dot{x}^2 + \dot{x}(1 + \sum_{l=0}^{n} x^{2l+1}) + x^3 - x\Big]\dot{x} \\
=& - costsin^3t + sin^2t(1 + \sum_{l=0}^{n} cos^{2l+1}t) - sintcos^3t + sintcost \\
=& - cost(1 - cos^2t)sint + (1 - cos^2t)(1 + \sum_{l=0}^{n} cos^{2l+1}t) - sintcos^3t + sintcost \\
=&(1 - cos^2t) + (1 - cos^2t)\sum_{l=0}^{n} cos^{2l+1}t \\
=&1 - cos^2t + cost + \sum_{l=1}^{n} cos^{2l+1}t - \sum_{l=1}^{n} cos^{2l+1}t - cos^{2n+3}t \\
=&1 - cos^2t + cost - cos^{2n+3}t.
\end{aligned}
\tag{4.1}
$$

Therefore, we immediately obtain

$$
\ddot{x} + \Big[x\dot{x}^2 + \dot{x}(1 + \sum_{l=0}^{n} x^{2l+1}) + x^3 - x\Big]\dot{x} + x^{2n+3} + x^2 - 1 = 0, \tag{4.2}
$$

and Theorem 1.3 is verified.

5. Proof of Theorem 1.4

Applying Eqs (1.9), (2.1)–(2.3), Eq (1.20) becomes

$$
\begin{aligned}
&\ddot{x} + [(1 + x)\dot{x}^2 + \dot{x}\sum_{l=0}^{n} x^{2l+1} + x^3 + x^2 - x - 1]\dot{x} + x^{2n+3} \\
=& - \cos t - \sin^3 t\,(1 + \cos t) + \left(1 - \cos^2 t\right)\sum_{l=0}^{n} \cos^{2l+1} t - \sin t\cos^3 t - \sin t\cos^2 t + \sin t\cos t + \sin t \\
&+ \cos^{2n+3} t \\
=& - \cos t - \sin t\cos t\left(\cos^2 t + \sin^2 t - 1\right) - \sin t\left(\sin^2 t + \cos^2 t - 1\right) + \sum_{l=0}^{n} \cos^{2l+1} t - \cos^2 t\sum_{l=0}^{n} \cos^{2l+1} t \\
&+ \cos^{2n+3} t \\
=& - \cos t + \sum_{l=0}^{n} \cos^{2l+1} t - \cos^2 t\sum_{l=0}^{n} \cos^{2l+1} t + \cos^{2n+3} t \\
=& - \cos t + \cos t + \sum_{l=1}^{n} \cos^{2l+1} t - \cos^2 t\left(\cos t + \cos^3 t + \cos^5 t + ... + \cos^{2(n-1)+1} t + \cos^{2n+1} t\right) + \cos^{2n+3} t \\
=& \sum_{l=1}^{n} \cos^{2l+1} t - \left(\cos^3 t + \cos^5 t + \cos^7 t + ... + \cos^{2n+1} t + \cos^{2n+3} t\right) + \cos^{2n+3} t
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{l=1}^{n}\cos^{2l+1}t-\left(\cos^{3}t+\cos^{5}t+\cos^{7}t+...+\cos^{2n+1}t\right)\\
&=\sum_{l=1}^{n}\cos^{2l+1}t-\sum_{l=1}^{n}\cos^{2l+1}t\\
&=0. \qquad (5.1)
\end{aligned}
$$

Theorem 1.4 is proved.

6. Proof of Theorem 1.5

Using Eqs (1.9), (2.1)–(2.3), Eq (1.22) becomes

$$
\begin{aligned}
&\ddot{x}+\left[x\dot{x}^{2}+x^{3}-x+\dot{x}\left(1+\sum_{l=0}^{n}x^{2l+2}\right)\right]\dot{x}+x^{2n+4}+x-1\\
&=-\cos t+\left[\cos t\sin^{2}t+\cos^{3}t-\cos t-\sin t\left(1+\sum_{l=0}^{n}\cos^{2l+2}t\right)\right](-\sin t)+\cos^{2n+4}t+\cos t-1\\
&=-\cos t\sin t\left(1-\cos^{2}t\right)-\sin t\cos^{3}t+\sin t\cos t+\left(1-\cos^{2}t\right)+\left(1-\cos^{2}t\right)\sum_{l=0}^{n}\cos^{2l+2}t\\
&\quad+\cos^{2n+4}t-1\\
&=-\sin t\cos^{3}t+\sum_{l=0}^{n}\cos^{2l+2}t-\cos^{2}t\sum_{l=0}^{n}\cos^{2l+2}t+\cos^{2n+4}t-1-\sin t\cos t+\sin t\cos^{3}t+\sin t\cos t\\
&\quad+1-\cos^{2}t\\
&=\cos^{2}t+\sum_{l=1}^{n}\cos^{2l+2}t-\cos^{2}t\left(\cos^{2}t+\cos^{4}t+...+\cos^{2(n-1)+2}t+\cos^{2n+2}t\right)+\cos^{2n+4}t-\cos^{2}t\\
&=\sum_{l=1}^{n}\cos^{2l+2}t-\left(\cos^{4}t+\cos^{6}t+...+\cos^{2n+2}t\right)-\cos^{2n+4}t+\cos^{2n+4}t\\
&=\sum_{l=1}^{n}\cos^{2l+2}t-\sum_{l=1}^{n}\cos^{2l+2}t\\
&=0, \qquad (6.1)
\end{aligned}
$$

proving Theorem 1.5.

7. Proof of Theorem 1.6

Using Eqs (1.9), (2.1)–(2.3), Eq (1.27) leads to

$$
\ddot{x}+\left[x\dot{x}^{2}+x^{3}-x+\dot{x}\left(1-\dot{x}^{2n+2}-x^{2}\sum_{l=0}^{n}\dot{x}^{2l}\right)\right]\dot{x}+x
$$

$$
\begin{aligned}
&= -\cos t - \cos t \sin^3 t - \sin t \cos^3 t + \sin t \cos t + \sin^2 t \left[1 - \sin^{2n+2} t - \left(1 - \sin^2 t\right) \sum_{l=0}^{n} \sin^{2l} t\right] + \cos t \\
&= -\sin t \cos t \left(\sin^2 t + \cos^2 t - 1\right) + \sin^2 t - \sin^{2n+4} t - \sum_{l=0}^{n} \sin^{2l+2} t + \sin^4 t \sum_{l=0}^{n} \sin^{2l} t \\
&= \sin^2 t - \sin^{2n+4} t - \sin^2 t - \sum_{l=1}^{n} \sin^{2l+2} t + \sin^4 t \left(1 + \sin^2 t + \sin^4 t + \sin^6 t + ... + \sin^{2(n-1)} t + \sin^{2n} t\right) \\
&= -\sin^{2n+4} t - \sin^2 t \sum_{l=1}^{n} \sin^{2l} t + \left(\sin^4 t + \sin^6 t + \sin^8 t + ... + \sin^{2n+2} t + \sin^{2n+4} t\right) \\
&= -\sin^2 t \sum_{l=1}^{n} \sin^{2l} t + \left(\sin^4 t + \sin^6 t + \sin^8 t + ... + \sin^{2n+2} t\right) \\
&= -\sin^2 t \sum_{l=1}^{n} \sin^{2l} t + \sum_{l=1}^{n} \sin^{2l+2} t \\
&= 0.
\end{aligned} \tag{7.1}
$$

Theorem 1.6 is proved.

8. Proof of Theorem 1.7

Taking into account Eqs (1.9),(2.1)–(2.3), Eq (1.31) yields

$$
\begin{aligned}
&\ddot{x} + \left[x\left(\dot{x}^2 + x^2 - 1 + x \sum_{l=0}^{n} \dot{x}^{2l+1}\right) + \dot{x}^{2n+3}\right] \dot{x} + x^2 + x - 1 \\
&= -\cos t + \left[\cos t \left(\sin^2 t + \cos^2 t - 1 - \cos t \sum_{l=0}^{n} \sin^{2l+1} t\right) - \sin^{2n+3} t\right] (-\sin t) + \cos^2 t + \cos t - 1 \\
&= \sin t \left(1 - \sin^2 t\right) \sum_{l=0}^{n} \sin^{2l+1} t + \sin^{2n+4} t - \sin^2 t \\
&= \sin t \left(\sin t + \sum_{l=1}^{n} \sin^{2l+1} t\right) - \sin^3 t \left(\sin t + \sin^3 t + \sin^5 t + ... + \sin^{2(n-1)+1} t + \sin^{2n+1} t\right) + \\
&\quad \sin^{2n+4} t - \sin^2 t \\
&= \sin t \sum_{l=1}^{n} \sin^{2l+1} t - \left(\sin^4 t + \sin^6 t + \sin^8 t + ... + \sin^{2n+2} t\right) - \sin^{2n+4} t + \sin^{2n+4} t \\
&= \sin t \sum_{l=1}^{n} \sin^{2l+1} t - \sum_{l=1}^{n} \sin^{2l+2} t \\
&= 0.
\end{aligned} \tag{8.1}
$$

Theorem 1.7 is proved.

9. Proof of Theorem 1.8

Substituting Eqs (1.9),(2.1)–(2.3), into Eq (1.36) yields

$$
\begin{aligned}
&\ddot{x} + \left(\dot{x}^2 + x^2 - 1 + \dot{x}^{2n+3} + x^2 \sum_{l=0}^{n} \dot{x}^{2l+1}\right)\dot{x} + x^2 + x - 1 \\
&= -\cos t + \left[\sin^2 t + \cos^2 t - 1 - \sin^{2n+3} t - \cos^2 t \sum_{l=0}^{n} \sin^{2l+1} t\right](-\sin t) + \cos^2 t + \cos t - 1 \\
&= -\cos t + \sin^{2n+4} t + \left(1 - \sin^2 t\right)\sum_{l=0}^{n} \sin^{2l+2} t + \cos^2 t + \cos t - 1 \\
&= \sin^{2n+4} t + \sum_{l=0}^{n} \sin^{2l+2} t - \sin^2 t \sum_{l=0}^{n} \sin^{2l+2} t - \sin^2 t \\
&= \sin^{2n+4} t + \sin^2 t + \sum_{l=1}^{n} \sin^{2l+2} t - \sin^2 t\left(\sin^2 t + \sin^4 t + \sin^6 t + ... + \sin^{2(n-1)+2} t + \sin^{2n+2} t\right) - \sin^2 t \\
&= \sin^{2n+4} t + \sum_{l=1}^{n} \sin^{2l+2} t - \left(\sin^4 t + \sin^6 t + \sin^8 t + ... + \sin^{2n+2} t\right) - \sin^{2n+4} t \\
&= \sum_{l=1}^{n} \sin^{2l+2} t - \sum_{l=1}^{n} \sin^{2l+2} t \\
&= 0.
\end{aligned}
\tag{9.1}
$$

In this context, Theorem 1.8 is proved.

Therefore, a conclusion can be carried out for the work.

10. Conclusions

In this contribution, we have succeeded in highlighting the existence of classes of polynomial mixed Lienard-type differential equations that can generate many, that is $(n+1)$ equations with exact harmonic and isochronous periodic solutions and limit cycles in contrast to the predictions of classical existence theorems.

Conflict of interest

The authors declare that they have no conflicts of interest.

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