AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(8): 15120–15137.
DOI: 10.3934/math.2022829
Received: 10 March 2022
Revised: 13 May 2022
Accepted: 07 June 2022
Published: 15 June 2022

Research article

Existence results of sequential fractional Caputo sum-difference boundary value problem

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Abstract: In this article, we study the existence and uniqueness results for a sequential nonlinear Caputo fractional sum-difference equation with fractional difference boundary conditions by using the Banach contraction principle and Schaefer's fixed point theorem. Furthermore, we also show the existence of a positive solution. Our problem contains different orders and four fractional difference operators. Finally, we present an example to display the importance of these results.

Keywords: fractional sum-difference equations; boundary value problem; existence; uniqueness
Mathematics Subject Classification: 39A10, 39A13, 39A70

1. Introduction

Fractional calculus is an emerging field drawing attention from both theoretical and applied disciplines. In particular, fractional calculus is a powerful tool for explaining problems in ecology, biology, chemistry, physics, mechanics, networks, flow in porous media, electricity, control systems, viscoelasticity, mathematical biology, fitting of experimental data, and so forth. One may see the papers [1–5] and the references therein.

Fractional difference calculus or discrete fractional calculus is a very new field for mathematicians. Some real-world phenomena are being studied with the assistance of fractional difference operators. Basic definitions and properties of fractional difference calculus can be found in the book [6]. Fractional boundary value problems can be found in the books [7, 8]. Now, the studies of boundary value problems for fractional difference equations are extended to be more complex. Excellent papers related to discrete fractional boundary value problems can be found in [9–35] and references cited therein. In particular, there are some recent papers that present the Caputo fractional difference calculus [36–41]. In the literature, there are apparently few research works studying boundary value problems for Caputo fractional difference-sum equations. For example, [42] studied a boundary value problem for $p-$Laplacian Caputo fractional difference equations with fractional sum boundary conditions of the forms

$$\begin{cases} \Delta_C^{\alpha}[\phi_p(\Delta_C^{\beta}x)](t) = f(t+\alpha+\beta-1, x(t+\alpha+\beta-1)), \qquad t \in \mathbb{N}_{0,T} := \{0, 1, \ldots, T\}, \\ \Delta_C^{\beta}x(\alpha-1) = 0, \\ x(\alpha+\beta+T) = \rho\Delta^{-\gamma}x(\eta+\gamma). \end{cases} \tag{1.1}$$

In [43], investigated a nonlocal fractional sum boundary value problem for a Caputo fractional difference-sum equation of the form

$$\begin{aligned} &\Delta_C^{\alpha}u(t) = \mathcal{F}\Big[t+\alpha-1, u_{t+\alpha-1}, \Delta_C^{\beta}u(t+\alpha-\beta)\Big], \qquad t \in \mathbb{N}_{0,T}, \\ &\Delta_C^{\gamma}u(\alpha-\gamma-1) = 0, \quad u(T+\alpha) = \rho\Delta^{-\omega}u(\eta+\omega). \end{aligned} \tag{1.2}$$

In addition, [44] considered a periodic boundary value problem for Caputo fractional difference-sum equations of the form

$$\begin{aligned} &\Delta_C^{\alpha}u(t) = F\Big[t+\alpha-1, u(t+\alpha-1), \Psi^{\gamma}u(t+\alpha-1)\Big], \ \ t \in \mathbb{N}_{0,T}, \ t+\alpha-1 \neq t_k, \\ &\Delta u(t_k) = I_k\left(u(t_k-1)\right), \ \ k = 1, 2, ..., p, \\ &\Delta\left(\Delta^{-\beta}u(t_k+\beta)\right) = J_k\left(\Delta^{-\beta}u(t_k+\beta-1)\right), \ \ k = 1, 2, ..., p, \\ &Au(\alpha-1) + B\Delta^{-\beta}u(\alpha+\beta-1) = Cu(T+\alpha) + D\Delta^{-\beta}u(T+\alpha+\beta). \end{aligned} \tag{1.3}$$

We aim to fill the gaps related to the boundary value problem of Caputo fractional difference-sum equations. The goal of this paper is to enrich this new research area by using the unknown function of Caputo fractional difference and fractional sum in the problem. So, in this paper, we consider a sequential nonlinear Caputo fractional sum-difference equation with fractional difference boundary value conditions of the form

$$\begin{aligned} &{}^{C}\Delta_{\alpha}^{\alpha}\, {}^{C}\Delta_{\alpha+\beta-1}^{\beta}x(t) = \mathcal{H}\Big[t+\alpha+\beta-1, x(t+\alpha+\beta-1), {}^{C}\Delta_{\alpha+\beta-1}^{\nu}x(t+\alpha+\beta-\nu), \\ &\qquad\qquad\qquad\qquad \Psi^{\mu}(t+\alpha+\beta-1, x(t+\alpha+\beta-1))\Big], \qquad t \in \mathbb{N}_{0,T}, \\ &\rho_1 x(\alpha+\beta-2) = x(T+\alpha+\beta), \\ &\rho_2\, {}^{C}\Delta_{\alpha+\beta-2}^{\gamma}x(\alpha+\beta-\gamma-1) = {}^{C}\Delta_{\alpha+\beta-2}^{\gamma}x(T+\alpha+\beta-\gamma+1), \end{aligned} \tag{1.4}$$

where $\rho_1, \rho_2 \in \mathbb{R}, 0 < \alpha, \beta, \gamma, \nu, \mu \leq 1, 1 < \alpha+\beta \leq 2$ are given constants, $\mathcal{H} \in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \times \mathbb{R}^3, \mathbb{R})$, and for $\varphi \in C(\odot \times \mathbb{R}^2, [0, \infty)), \odot := \Big\{(t, r) \ : \ t, r \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \text{ and } r \leq t\Big\}$. The operator Ψ^{μ} is defined by

$$\Psi^{\mu}(t, x(t)) := \sum_{s=\alpha+\beta-\mu-2}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} \varphi\Big(t, s+\mu, x(s+\mu), {}^{C}\Delta_{\alpha+\beta-2}^{\nu}x(s+\mu-\nu+1)\Big).$$

The plan of this paper is as follows. In Section 2, we recall some definitions and basic lemmas. Also, we derive the solution of (1.4) by converting the problem to an equivalent equation. In Section 3, we prove existence and uniqueness results of the problem (1.4) using the Banach contraction principle and Schaefer's theorem. Furthermore, we also show the existence of a positive solution to (1.4). An illustrative example is presented in Section 4.

2. Preliminaries

In the following, there are notations, definitions and lemmas which are used in the main results.

Definition 2.1. *[10] We define the generalized falling function by* $t^{\underline{\alpha}} := \dfrac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$, *for any* t *and* α *for which the right-hand side is defined. If* $t+1-\alpha$ *is a pole of the Gamma function and* $t+1$ *is not a pole, then* $t^{\underline{\alpha}} = 0$.

Lemma 2.1. *[9] Assume the following factorial functions are well defined. If* $t \leq r$, *then* $t^{\underline{\alpha}} \leq r^{\underline{\alpha}}$ *for any* $\alpha > 0$.

Definition 2.2. *[10] For* $\alpha > 0$ *and* f *defined on* $\mathbb{N}_a := \{a, a+1, \ldots\}$, *the* α*-order fractional sum of* f *is defined by*

$$\Delta_a^{-\alpha} f(t) = \Delta^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+\alpha}$ *and* $\sigma(s) = s+1$.

Definition 2.3. *[11] For* $\alpha > 0$ *and* f *defined on* $\mathbb{N}_a$, *the* α*-order Caputo fractional difference of* f *is defined by*

$${}^{C}\Delta_a^{\alpha} f(t) = \Delta_C^{\alpha} f(t) := \Delta_a^{-(N-\alpha)} \Delta^N f(t) = \frac{1}{\Gamma(N-\alpha)} \sum_{s=a}^{t-(N-\alpha)} (t-\sigma(s))^{\underline{N-\alpha-1}} \Delta^N f(s),$$

where $t \in \mathbb{N}_{a+N-\alpha}$ *and* $N \in \mathbb{N}$ *is chosen so that* $0 \leq N-1 < \alpha < N$. *If* $\alpha = N$, *then* $\Delta_C^{\alpha} f(t) = \Delta^N f(t)$.

Lemma 2.2. *[11] Assume that* $\alpha > 0$ *and* $0 \leq N-1 < \alpha \leq N$. *Then,*

$$\Delta_{a+N-\alpha}^{-\alpha} {}^{C}\Delta_a^{\alpha} y(t) = y(t) + C_0 + C_1 t^{\underline{1}} + C_2 t^{\underline{2}} + \ldots + C_{N-1} t^{\underline{N-1}},$$

for some $C_i \in \mathbb{R}$, $0 \leq i \leq N-1$.

To study the solution of the boundary value problem (1.4), we need the following lemma that deals with a linear variant of the boundary value problem (1.4) and gives a representation of the solution.

Lemma 2.3. *Let* $\Lambda(\rho_1 - 1) \neq 0$, $0 < \alpha, \beta, \gamma, \nu, \mu \leq 1$, $1 < \alpha + \beta \leq 2$ *and* $h \in C\left(\mathbb{N}_{\alpha+\beta-1, T+\alpha+\beta-1}, \mathbb{R}\right)$ *be given. Then, the problem*

$${}^{C}\Delta_{\alpha}^{\alpha} {}^{C}\Delta_{\alpha+\beta-1}^{\beta} x(t) = h(t+\alpha+\beta-1), \quad t \in \mathbb{N}_{0,T} \tag{2.1}$$

$$\begin{cases} \rho_1 x(\alpha+\beta-2) = x(T+\alpha+\beta), \\ \rho_2 {}^{C}\Delta_{\alpha+\beta-2}^{\gamma} x(\alpha+\beta-\gamma-1) = {}^{C}\Delta_{\alpha+\beta-2}^{\gamma} x(T+\alpha+\beta-\gamma+1), \end{cases} \tag{2.2}$$

has the unique solution

$$x(t)$$

$$= \frac{T + (\rho_1 - 1)(t-\alpha-\beta) + 2\rho_1}{\Lambda(\rho_1 - 1)\Gamma(1-\gamma)\Gamma(\beta-1)\Gamma(\alpha)} \sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} (T+\alpha+\beta-\gamma+1-\sigma(s))^{\underline{-\gamma}} \times$$

$$(s-\sigma(r))^{\underline{\beta-2}}(r-\sigma(\xi))^{\underline{\alpha-1}}h(\xi+\alpha+\beta-1)$$
$$+\frac{1}{(\rho_1-1)\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{T+\alpha}\sum_{\xi=0}^{s-\alpha}(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}h(\xi+\alpha+\beta-1)$$
$$+\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{t-\beta}\sum_{\xi=0}^{s-\alpha}(t-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}h(\xi+\alpha-1), \tag{2.3}$$

where

$$\Lambda = \rho_2-\frac{\Gamma(T-\gamma+4)}{\Gamma(2-\gamma)\Gamma(T+3)}. \tag{2.4}$$

Proof. Using the fractional sum of order $\alpha\in(0,1]$ for (2.1) and from Lemma 2.2, we obtain

$$^{C}\Delta^{\beta}_{\alpha+\beta-2}x(t)=C_1+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}h(s+\alpha+\beta-1), \tag{2.5}$$

for $t\in\mathbb{N}_{\alpha-1,T+\alpha}$.
Using the fractional sum of order $0<\beta\le 1$ for (2.5), we obtain

$$x(t)=C_2+C_1t+\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{t-\beta}\sum_{\xi=0}^{s-\alpha}(t-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}h(\xi+\alpha+\beta-1), \tag{2.6}$$

for $t\in\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$.

By substituting $t=\alpha+\beta-2, T+\alpha+\beta$ into (2.6) and employing the first condition of (2.2), we obtain

$$-C_2(\rho_1-1)+C_1\left[(T-(\rho_1-1)(\alpha+\beta)+2\rho_1\right] \tag{2.7}$$
$$= -\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{T+\alpha}\sum_{\xi=0}^{s-\alpha}(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}h(\xi+\alpha+\beta-1).$$

Using the fractional Caputo difference of order $0<\gamma\le 1$ for (2.6), we obtain

$$^{C}\Delta^{\gamma}_{\alpha+\beta-2}x(t)$$
$$= \frac{C_1}{\Gamma(1-\gamma)}\sum_{s=\alpha+\beta-2}^{t+\gamma-1}(t-\sigma(s))^{\underline{-\gamma}}+\frac{1}{\Gamma(1-\gamma)\Gamma(\beta)\Gamma(\alpha)}\times$$
$$\sum_{s=\alpha+\beta-2}^{t+\gamma-1}(t-\sigma(s))^{\underline{-\gamma}}{}_s\Delta\left[\sum_{r=\alpha}^{s-\beta}\sum_{\xi=0}^{r-\alpha}(s-\sigma(r))^{\underline{\beta-1}}(r-\sigma(\xi))^{\underline{\alpha-1}}h(\xi+\alpha+\beta-1)\right]$$
$$= \frac{C_1}{\Gamma(1-\gamma)}\sum_{s=\alpha+\beta-2}^{t+\gamma-1}(t-\sigma(s))^{\underline{-\gamma}}+\frac{1}{\Gamma(1-\gamma)\Gamma(\beta-1)\Gamma(\alpha)}\times \tag{2.8}$$
$$\sum_{s=\alpha+\beta-2}^{t+\gamma-1}\sum_{r=\alpha}^{s-\beta+1}\sum_{\xi=0}^{r-\alpha}(t-\sigma(s))^{\underline{-\gamma}}(s-\sigma(r))^{\underline{\beta-2}}(r-\sigma(\xi))^{\underline{\alpha-1}}h(\xi+\alpha+\beta-1),$$

for $\mathbb{N}_{\alpha+\beta-\gamma-1,T+\alpha+\beta-\gamma+1}$.
By substituting $t = \alpha+\beta-\gamma-1, T+\alpha+\beta-\gamma+1$ into (2.8) and employing the second condition of (2.2), it implies

$$\begin{aligned} C_1 &= -\frac{1}{\Lambda\Gamma(1-\gamma)\Gamma(\beta-1)\Gamma(\alpha)} \sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} (T+\alpha+\beta-\gamma+1-\sigma(s))^{\underline{-\gamma}} \times \\ & (s-\sigma(r))^{\underline{\beta-2}} (r-\sigma(\xi))^{\underline{\alpha-1}} h(\xi+\alpha+\beta-1). \end{aligned}$$

The constant C_2 can be obtained by substituting C_1 into (2.7). Then, we get

$$\begin{aligned} C_2 &= \frac{T-(\rho_1-1)(\alpha+\beta)+2\rho_1}{(\rho_1-1)\Lambda\Gamma(1-\gamma)\Gamma(\beta-1)\Gamma(\alpha)} \sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} (T+\alpha+\beta-\gamma+1-\sigma(s))^{\underline{-\gamma}} \times \\ & (s-\sigma(r))^{\underline{\beta-2}} (r-\sigma(\xi))^{\underline{\alpha-1}} h(\xi+\alpha+\beta-1) \\ & + \frac{1}{(\rho_1-1)\Gamma(\beta)\Gamma(\alpha)} \sum_{s=\alpha}^{T+\alpha} \sum_{\xi=0}^{s-\alpha} (T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}} (s-\sigma(\xi))^{\underline{\alpha-1}} h(\xi+\alpha+\beta-1), \end{aligned}$$

where Λ is defined by (2.4). Substituting the constants C_1 and C_2 into (2.6), we obtain (2.3). □

3. Main results

In this section, we wish to establish the existence results for the problem (1.4). We denote $C = C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}, \mathbb{R})$ as the Banach space of all functions x with the norm defined by

$$\|x\|_C = \|x\| + \|\Delta_C^{\nu} x\|,$$

where $\|x\| = \max_{t\in\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}} |x(t)|$ and $\|\Delta_C^{\nu} x\| = \max_{t\in\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}} \left|\Delta_C^{\nu} x(t-\nu+1)\right|$.

The following assumptions are assumed:

(A1) $\mathcal{H}[t,x,y,z] : \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function.

(A2) There exist constants $K_1, K_2 > 0$ such that for each $t \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$ and all $x_i, y_i, z_i \in \mathbb{R},\ i = 1,2$, we have
$$\left|\mathcal{H}[t,x_1,y_1,z_1] - \mathcal{H}[t,x_2,y_2,z_2]\right| \le K_1 \left[|x_1-x_2| + |y_1-y_2| + |z_1-z_2|\right],$$
and
$$K_2 = \max_{t\in\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}} \left|\mathcal{H}[t,0,0,\Psi^{\mu}(t,0)]\right|,$$
where $\Psi^{\mu}(t,0) := \dfrac{1}{\Gamma(\mu)} \displaystyle\sum_{s=\alpha+\beta-\mu-2}^{t-\mu} (t-\sigma(s))^{\underline{\mu-1}} \varphi\big(t, s+\mu, 0, 0\big)$.

(A3) $\varphi : \odot \times \mathbb{R}^2 \to \mathbb{R}$ is continuious for $(t,s) \in \odot$, and there exists a constant $L > 0$, such that for each $(t,s) \in \odot$ and all $x_i, y_i \in C,\ i = 1,2$ we have
$$\left|\varphi(t,s+\mu,x_1,y_1) - \varphi(t,s+\mu,x_2,y_2)\right| \le L\left[|x_1-x_2| + |y_1-y_2|\right].$$

Let us define the operator $\widetilde{\mathcal{H}}[t,x(t)]$ by

$$\widetilde{\mathcal{H}}[t,x(t)] := \mathcal{H}\Big[t, x(t), \Delta_C^{\nu} x(t-\nu+1), \Psi^{\mu}(t,x(t))\Big], \tag{3.1}$$

for each $t \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$ and $x \in \mathcal{C}$.

Note that $\Delta^{-\beta}\Delta^{-\alpha}\widetilde{\mathcal{H}}[t,x(t)]$ and $\Delta_C^{\nu}\Delta^{-\beta}\Delta^{-\alpha}\widetilde{\mathcal{H}}[t,x(t)]$ exist when $\nu < \alpha+\beta \leq 2$.

Lemma 3.1. *Assume that* ***(A1)–(A3)*** *hold. Then, the following property holds:*

(A4) *There exits a positive constant* Θ *such that*

$$\Big|\widetilde{\mathcal{H}}[t,x_1(t)] - \widetilde{\mathcal{H}}[t,x_2(t)]\Big| \leq \Theta \,\|x_1 - x_2\|_{\mathcal{C}},$$

for each $t \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$ *and* $x_1, x_2 \in \mathcal{C}$*, where*

$$\Theta := K_1\left[1 + \frac{L\Gamma(T+\mu+3)}{\Gamma(\mu+1)\Gamma(T+3)}\right]. \tag{3.2}$$

Proof. By **(A3)**, for each $t \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$ and $x_1, x_2 \in \mathcal{C}$, we obtain

$$\begin{aligned}
&\Big|(\Psi^{\mu}x_1)(t) - (\Psi^{\mu}x_2)(t)\Big| \\
&\leq \frac{1}{\Gamma(\mu)} \sum_{s=\alpha+\beta-\mu-2}^{t-\mu} (t-\sigma(s))^{\underline{\mu-1}} \Big|\varphi\Big(t, s+\mu, x_1(s+\mu), \Delta_C^{\nu}x_1(s+\mu-\nu+1)\Big) \\
&\quad - \varphi\Big(t, s+\mu, x_2(s+\mu), \Delta_C^{\nu}x_2(s+\mu-\nu+1)\Big)\Big| \\
&\leq \frac{1}{\Gamma(\mu)} \sum_{s=\alpha+\beta-\mu-2}^{T+\alpha+\beta-\mu} (T+\alpha+\beta-\sigma(s))^{\underline{\mu-1}} \times \\
&\quad L\Big[\Big|x_1(s+\mu) - x_2(s+\mu)\Big| + \Big|\Delta_C^{\nu}x_1(s+\mu-\nu+1) - \Delta_C^{\nu}x_2(s+\mu-\nu+1)\Big|\Big] \\
&\leq \frac{L\Gamma(T+\mu+3)}{\Gamma(\mu+1)\Gamma(T+3)} \Big\{\Big\|x_1 - x_2\Big\| + \Big\|\Delta_C^{\nu}x_1 - \Delta_C^{\nu}x_2\Big\|\Big\},
\end{aligned}$$

and hence

$$\begin{aligned}
&\Big|\widetilde{\mathcal{H}}[t,x_1(t)] - \widetilde{\mathcal{H}}[t,x_2(t)]\Big| \\
&\leq K_1\Big[\Big|x_1(t) - x_2(t)\Big| + \Big|\Delta_C^{\nu}x_1(t-\nu+1) - \Delta_C^{\nu}x_2(t-\nu+1)\Big|\Big] \\
&\quad + \frac{K_1 L\Gamma(T+\mu+3)}{\Gamma(\mu+1)\Gamma(T+3)} \Big\{\Big\|x_1 - x_2\Big\| + \Big\|\Delta_C^{\nu}x_1 - \Delta_C^{\nu}x_2\Big\|\Big\} \\
&= \Theta \,\|x_1 - x_2\|_{\mathcal{C}}.
\end{aligned}$$

□

Next, we define the operator $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{C}$ by

$$(\mathcal{F}x)(t)$$

$$
\begin{aligned}
&= \sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} \mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi)\, \widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)] \\
&\quad + \sum_{s=\alpha}^{T+\alpha} \sum_{\xi=0}^{s-\alpha} \frac{(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}}{(\rho_1-1)\,\Gamma(\beta)\Gamma(\alpha)}\, \widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)] \\
&\quad + \sum_{s=\alpha}^{t-\beta} \sum_{\xi=0}^{s-\alpha} \frac{(t-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}}{\Gamma(\beta)\Gamma(\alpha)}\, \widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)],
\end{aligned}
\tag{3.3}
$$

where

$$
\begin{aligned}
\mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi) := {}& \frac{T+(\rho_1-1)(t-\alpha-\beta)+2\rho_1}{\Lambda\,(\rho_1-1)\,\Gamma(1-\gamma)\Gamma(\beta-1)\Gamma(\alpha)} \times \\
& (T+\alpha+\beta-\gamma+1-\sigma(s))^{\underline{-\gamma}}(s-\sigma(r))^{\underline{\beta-2}}(r-\sigma(\xi))^{\underline{\alpha-1}}.
\end{aligned}
\tag{3.4}
$$

By Lemma 2.3, we find that any solution of the problem (1.4) is the fixed point of the operator $\mathcal{F}$.

Lemma 3.2. *Assume that the function $\mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi)$ satisfies the following properties:*

(A5) *$\mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi)$ is a continuous function for all $(t,s,r,\xi) \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \times \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \times \mathbb{N}_{\alpha-1,T+\alpha+1} \times \mathbb{N}_{0,T+2} =: \mathcal{D}$, and there exist two constants $\Omega_1, \Omega_2 > 0$, such that*

$$
\sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} \left| \mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi) \right| \le \Omega_1,
$$

$$
\sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} \left| {}_t\Delta_C^{\nu}\, \mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi) \right| \le \Omega_2,
$$

where

$$
\Omega_1 := \left[\frac{(1+|\rho_1-1|)\,T+2p_1}{|\rho_1-1||\Lambda|} \right] \frac{\Gamma(T+\gamma+3)\Gamma(T+\alpha+\beta+1)}{\Gamma(2-\gamma)\Gamma(\alpha+\beta)[\Gamma(T+2)]^2}, \tag{3.5}
$$

$$
\Omega_2 := \left| \frac{1-\alpha-\beta}{\Lambda} \right| \frac{\Gamma(T-\gamma+3)[\Gamma(T+\alpha+\beta+1)]^2}{\Gamma(2-\nu)\Gamma(2-\gamma)\Gamma(\alpha+\beta)[\Gamma(T+2)]^2\Gamma(T+\alpha+\beta-\nu)}, \tag{3.6}
$$

and ${}_t\Delta_C^{\nu}$ is the Caputo fractional difference with respect to t.

Proof. It is obvious that $\mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi)$ is a continuous function for all $(t,s,r,\xi) \in \mathcal{D}$. Next, we consider

$$
\begin{aligned}
&\sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} \left| \mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi) \right| \\
&\le \max_{t\in\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}} \sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} \left| \mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi) \right|
\end{aligned}
$$

$$= \max_{t\in\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}} \left| \frac{T+(\rho_1-1)(t-\alpha-\beta)+2\rho_1}{\Lambda(\rho_1-1)\Gamma(1-\gamma)\Gamma(\beta-1)\Gamma(\alpha)} \right| \times$$
$$\sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} (T+\alpha+\beta-\gamma+1-\sigma(s))^{\underline{-\gamma}} (s-\sigma(r))^{\underline{\beta-2}} (r-\sigma(\xi))^{\underline{\alpha-1}}$$
$$\leq \left[\frac{(1+|\rho_1-1|)\,T+2p_1}{|\rho_1-1||\Lambda|} \right] \sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} (T+\alpha+\beta-\gamma+1-\sigma(s))^{\underline{-\gamma}} (s-\sigma(r))^{\underline{\beta-2}} (r-\sigma(\xi))^{\underline{\alpha-1}}$$
$$\leq \left[\frac{(1+|\rho_1-1|)\,T+2p_1}{|\rho_1-1||\Lambda|} \right] \frac{\Gamma(T+\gamma+3)\Gamma(T+\alpha+\beta+1)}{\Gamma(T+2)\Gamma(T+2)\Gamma(2-\gamma)\Gamma(\alpha+\beta)} = \Omega_1,$$

and

$$\sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} \left| {}_t\Delta_C^{\nu}\, \mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi) \right|$$
$$\leq \max_{t\in\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}} \sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} \left| {}_t\Delta_C^{\nu}\, \mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi) \right|$$
$$= \max_{t\in\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}} \left| \frac{\sum_{s=\alpha+\beta-2}^{t+\nu-1} (t-\sigma(s))^{\underline{-\nu}} {}_s\Delta\left[T+(\rho_1-1)(s-\alpha-\beta)+2\rho_1\right]}{\Lambda\Gamma(1-\nu)\Gamma(1-\gamma)\Gamma(\beta-1)\Gamma(\alpha)} \right| \times$$
$$\sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} (T+\alpha+\beta-\gamma+1-\sigma(s))^{\underline{-\gamma}} (s-\sigma(r))^{\underline{\beta-2}} (r-\sigma(\xi))^{\underline{\alpha-1}}$$
$$\leq \left| \frac{\sum_{s=\alpha+\beta-2}^{T+\alpha+\beta+\nu-1} (T+\alpha+\beta-\sigma(s))^{\underline{-\nu}} (1-\alpha-\beta)}{\Lambda\Gamma(1-\nu)} \right| \frac{\Gamma(T-\gamma+3)\Gamma(T+\alpha+\beta+1)}{[\Gamma(T+2)]^2\Gamma(2-\gamma)\Gamma(\alpha+\beta)}$$
$$\leq \left| \frac{1-\alpha-\beta}{\Lambda} \right| \frac{\Gamma(T-\gamma+3)[\Gamma(T+\alpha+\beta+1)]^2}{\Gamma(2-\nu)\Gamma(2-\gamma)\Gamma(\alpha+\beta)[\Gamma(T+2)]^2\Gamma(T+\alpha+\beta-\nu)} = \Omega_2.$$

Thus, the condition **(A5)** holds. □

In what follows, we consider the existence and uniqueness of a solution to the problem (1.4) using the Banach contraction principle.

Theorem 3.1. *Assume that **(A1)–(A5)** hold. If*

$$\Theta[\,\Omega_1+\Omega_2+\phi_1+\phi_2\,] < 1, \tag{3.7}$$

where Ω_1, Ω_2 *are defined as (3.5)–(3.6), and*

$$\phi_1 = \left[\frac{1+|\rho_1-1|}{|\rho_1-1|} \right] \frac{\Gamma(T+\alpha+\beta+1)}{\Gamma(T+1)\Gamma(\alpha+\beta+1)}, \tag{3.8}$$
$$\phi_2 = \frac{\Gamma(T+2)\Gamma(T+\alpha+\beta+\nu)}{\Gamma(2-\nu)\Gamma(\alpha+\beta+\nu+1)\left[\Gamma(T+\nu+1)\right]^2}, \tag{3.9}$$

then, the problem (1.4) *has a unique solution in* $\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$.

Proof. Choose a constant R satisfying

$$R \geq \frac{K_2(\,\Omega_1 + \Omega_2 + \phi_1 + \phi_2\,)}{1 - \Theta(\,\Omega_1 + \Omega_2 + \phi_1 + \phi_2\,)}.$$

We will show that $\mathcal{F}(B_R) \subset B_R$, where $B_R = \{x \in C : \|x\|_C \leq R\}$. For all $x \in B_R$, we have

$$\begin{aligned}
&|(\mathcal{F}x)(t)| \\
&\leq \sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} \left|\mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi)\right| \Big(\Big|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)] \\
&\quad - \widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, 0]\Big| + \Big|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, 0]\Big|\Big) \\
&\quad + \Bigg| \frac{1}{|\rho_1 - 1|\,\Gamma(\beta)\Gamma(\alpha)} \sum_{s=\alpha}^{T+\alpha} \sum_{\xi=0}^{s-\alpha} (T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}} (s-\sigma(\xi))^{\underline{\alpha-1}} \times \\
&\quad \Big(\Big|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)] - \widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, 0]\Big| + \Big|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, 0]\Big|\Big) \\
&\quad + \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \sum_{s=\alpha}^{t-\beta} \sum_{\xi=0}^{s-\alpha} (t-\sigma(s))^{\underline{\beta-1}} (s-\sigma(\xi))^{\underline{\alpha-1}} \Big(\Big|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)] \\
&\quad - \widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, 0]\Big| + \Big|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, 0]\Big|\Big) \Bigg| \\
&\leq (\Theta\|x\|_C + K_2)\ \Omega_1 + (\Theta\|x\|_C + K_2) \left(\frac{1 + |\rho_1 - 1|}{\Gamma(\beta)\Gamma(\alpha)\,|\rho_1 - 1|} \right) \times \\
&\quad \sum_{s=\alpha}^{T+\alpha} \sum_{\xi=0}^{s-\alpha} (T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}} (s-\sigma(\xi))^{\underline{\alpha-1}} \\
&\leq (\Theta\,\|x\|_C + K_2) \left\{ \Omega_1 + \left(\frac{1 + |\rho_1 - 1|}{|\rho_1 - 1|} \right) \frac{\Gamma(T+\alpha+\beta+1)}{\Gamma(T+1)\Gamma(\alpha+\beta+1)} \right\} \\
&\leq (\Theta\, R + K_2)\,[\,\Omega_1 + \phi_1\,],
\end{aligned}$$

and

$$\begin{aligned}
&|(\Delta_C^{\nu} \mathcal{F}x)(t-\nu+1)| \\
&\leq \sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} \left|{}_t\Delta_C^{\nu}\, \mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi)\right| \Big(\Big|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)] \\
&\quad - \widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, 0]\Big| + \Big|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, 0]\Big|\Big) \\
&\quad + \frac{1}{\Gamma(1-\nu)\Gamma(\beta)\Gamma(\alpha)} \sum_{s=\alpha+\beta-2}^{t+\nu-1} (t-\sigma(s))^{\underline{-\nu}} \Delta_s \Bigg[\sum_{r=\alpha}^{s-\beta} \sum_{\xi=0}^{r-\alpha} (s-\sigma(r))^{\underline{\beta-1}} \times \\
&\quad (r-\sigma(\xi))^{\underline{\alpha-1}} \Big(\Big|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)] - \widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, 0]\Big| \\
&\quad + \Big|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, 0]\Big|\Big) \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&\le\ (\Theta\|x\|_C + K_2)\ \Omega_2 + \frac{(\Theta\|x\|_C + K_2)}{\Gamma(1-\nu)\Gamma(\beta)\Gamma(\alpha)}\times \\
&\quad \sum_{s=\alpha+\beta-1}^{T+\alpha+\beta+\nu-1}\ \sum_{r=\alpha}^{s-\beta+1}\ \sum_{\xi=0}^{r-\alpha}(T+\alpha+\beta-\sigma(s))^{\underline{-\nu}}(s-\sigma(r))^{\underline{\beta-2}}(r-\sigma(\xi))^{\underline{\alpha-1}} \\
&\le\ (\Theta\|x\|_C + K_2)\ \left\{\Omega_2 + \frac{\Gamma(T+2)\Gamma(T+\alpha+\beta+\nu)}{\Gamma(2-\nu)\Gamma(\alpha+\beta+\nu+1)\,[\Gamma(T+\nu+1)]^2}\right\} \\
&\le\ (\Theta R + K_2)\ [\,\Omega_2 + \phi_2\,].
\end{aligned}
$$

Thus,

$$
\|\mathcal{F}x\|_C\ \le\ (\Theta R + K_2)\ [\,\Omega_1 + \Omega_2 + \phi_1 + \phi_2\,]\ \le\ R,
$$

and hence, $\mathcal{F}(B_R) \subset B_R$.

We next show that $\mathcal{F}$ is a contraction. For all $x, y \in C$ and for each $t \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$, we have

$$
\begin{aligned}
&\left|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\right| \\
&\le \sum_{s=\alpha+\beta-1}^{T+\alpha+\beta}\ \sum_{r=\alpha}^{s-\beta+1}\ \sum_{\xi=0}^{r-\alpha}\left|\mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi)\right|\left|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)]\right. \\
&\quad \left.-\,\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, y(\xi+\alpha+\beta-1)]\right| \\
&\quad + \frac{1}{|\rho_1-1|\,\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{T+\alpha}\sum_{\xi=0}^{s-\alpha}(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}\times \\
&\qquad \left|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)] - \widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, y(\xi+\alpha+\beta-1)]\right| \\
&\quad + \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{t-\beta}\sum_{\xi=0}^{s-\alpha}(t-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}\left|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)]\right. \\
&\quad \left.-\,\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, y(\xi+\alpha+\beta-1)]\right| \\
&\le \sum_{s=\alpha+\beta-1}^{T+\alpha+\beta}\ \sum_{r=\alpha}^{s-\beta+1}\ \sum_{\xi=0}^{r-\alpha}\left|\mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi)\right|\left|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)]\right. \\
&\quad \left.-\,\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, y(\xi+\alpha+\beta-1)]\right| \\
&\quad + \left(\frac{1+|\rho_1-1|}{|\rho_1-1|\,\Gamma(\beta)\Gamma(\alpha)}\right)\sum_{s=\alpha}^{T+\alpha}\sum_{\xi=0}^{s-\alpha}(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}\times \\
&\qquad \left|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)] - \widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, y(\xi+\alpha+\beta-1)]\right| \\
&\le \Theta\,\|x-y\|_C\Omega_1 + \Theta\,\|x-y\|_C\left(\frac{1+|\rho_1-1|}{|\rho_1-1|\,\Gamma(\beta)\Gamma(\alpha)}\right)\times \\
&\quad \sum_{s=\alpha}^{T+\alpha}\sum_{\xi=0}^{s-\alpha}(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}} \\
&\le \Theta\,[\,\Omega_1 + \phi_1\,]\,\|x-y\|_C,
\end{aligned}
$$

and

$$
\begin{aligned}
&|(\Delta_C^{\nu}\mathcal{F}x)(t-\nu+1)| \\
\leq &\sum_{s=\alpha+\beta-1}^{T+\alpha+\beta}\sum_{r=\alpha}^{s-\beta+1}\sum_{\xi=0}^{r-\alpha}\left|{}_t\Delta_C^{\nu}\mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi)\right|\Big|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1,x(\xi+\alpha+\beta-1)] \\
&-\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1,y(\xi+\alpha+\beta-1)]\Big| \\
&+\frac{1}{\Gamma(1-\nu)\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha+\beta-1}^{t+\nu-1}\sum_{r=\alpha}^{s-\beta+1}\sum_{\xi=0}^{r-\alpha}(t-\sigma(s))^{\underline{-\nu}}(s-\sigma(r))^{\underline{\beta-2}}(r-\sigma(\xi))^{\underline{\alpha-1}}\times \\
&\Big|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1,x(\xi+\alpha+\beta-1)]-\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1,y(\xi+\alpha+\beta-1)]\Big| \\
\leq &\,\Theta\,\|x-y\|_C\,\Omega_2+\Theta\,\|x-y\|_C\,\frac{1}{\Gamma(1-\nu)\Gamma(\beta)\Gamma(\alpha)}\times \\
&\sum_{s=\alpha+\beta-1}^{T+\alpha+\beta+\nu-1}\sum_{r=\alpha}^{s-\beta+1}\sum_{\xi=0}^{r-\alpha}(T+\alpha+\beta-\sigma(s))^{\underline{-\nu}}(s-\sigma(r))^{\underline{\beta-2}}(r-\sigma(\xi))^{\underline{\alpha-1}} \\
\leq &\,\Theta\,[\,\Omega_2+\phi_2\,]\,\|x-y\|_C.
\end{aligned}
$$

Thus,

$$\|\mathcal{F}x-\mathcal{F}y\|_C \leq \Theta\,[\,\Omega_1+\Omega_2+\phi_1+\phi_2\,]\,\|x-y\|_C \leq \|x-y\|_C.$$

Therefore, $\mathcal{F}$ is a contraction. Hence, by using Banach fixed point theorem, we get that $\mathcal{F}$ has a fixed point which is a unique solution of the problem (1.4). □

We next deduce the existence of a solution to (1.4) by using the following Schaefer's fixed point theorem.

Theorem 3.2. *[45] (Arzelá-Ascoli Theorem) A set of functions in* $C[a,b]$ *with the sup norm is relatively compact if and only if it is uniformly bounded and equicontinuous on* $[a,b]$.

Theorem 3.3. *[45] If a set is closed and relatively compact, then it is compact.*

Theorem 3.4. *[46] Let* X *be a Banach space and* $T: X \to X$ *be a continuous and compact mapping. If the set*

$$\{x \in X \;:\; x = \lambda T(x), \textit{ for some } \lambda \in (0,1)\}$$

is bounded, then T *has a fixed point.*

Theorem 3.5. *Suppose that* **(A1)–(A5)** *hold. Then, the problem (1.4) has at least one solution on* $\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$.

Proof. We shall use Schaefer's fixed point theorem to prove that the operator F defined by (3.3) has a fixed point. It is clear that $\mathcal{F}: C \longrightarrow C$ is completely continuous. So, it remains to show that the set

$$E = \left\{u \in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}) : u = \lambda\mathcal{F}u \text{ for some } 0<\lambda<1\right\} \text{ is bounded.}$$

Let $u \in E$. Then,

$$u(t) = \lambda(Fu)(t) \;\text{ for some }\; 0<\lambda<1.$$

Thus, for each $t \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$, we have

$$
\begin{aligned}
&|\lambda(\mathcal{F}x)(t)| \\
&\leq \lambda \sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} \left|\mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi)\right| \left|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)]\right| \\
&\quad + \frac{\lambda}{|\rho_1-1|\,\Gamma(\beta)\Gamma(\alpha)} \sum_{s=\alpha}^{T+\alpha} \sum_{\xi=0}^{s-\alpha} (T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}} (s-\sigma(\xi))^{\underline{\alpha-1}} \times \\
&\quad \left|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)]\right| \\
&\quad + \frac{\lambda}{\Gamma(\beta)\Gamma(\alpha)} \sum_{s=\alpha}^{t-\beta} \sum_{\xi=0}^{s-\alpha} (t-\sigma(s))^{\underline{\beta-1}} (s-\sigma(\xi))^{\underline{\alpha-1}} \left|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)]\right| \\
&< (\Theta R + K_2)\,[\,\Omega_1 + \phi_1\,],
\end{aligned}
$$

and

$$
\begin{aligned}
&|\lambda(\Delta_C^{\nu}\mathcal{F}x)(t-\nu+1)| \\
&\leq \lambda \sum_{s=\alpha+\beta-1}^{T+\alpha+\beta} \sum_{r=\alpha}^{s-\beta+1} \sum_{\xi=0}^{r-\alpha} \left|{}_t\Delta_C^{\nu}\, \mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi)\right| \left|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)]\right| \\
&\quad + \frac{\lambda}{\Gamma(1-\nu)\Gamma(\beta)\Gamma(\alpha)} \sum_{s=\alpha+\beta-2}^{t+\nu-1} (t-\sigma(s))^{\underline{-\nu}} \Delta_s \Bigg[\sum_{r=\alpha}^{s-\beta} \sum_{\xi=0}^{r-\alpha} (s-\sigma(r))^{\underline{\beta-1}} \times \\
&\quad (r-\sigma(\xi))^{\underline{\alpha-1}} \left|\widetilde{\mathcal{H}}[\xi+\alpha+\beta-1, x(\xi+\alpha+\beta-1)]\right| \Bigg] \\
&< (\Theta R + K_2)\ [\,\Omega_2 + \phi_2\,].
\end{aligned}
$$

Hence,

$$
\left\|\lambda(\mathcal{F}x)(t)\right\| < \widetilde{\Theta}_R\,[\,\Omega_1+\Omega_2+\phi_1+\phi_2\,] < R.
$$

This shows that E is bounded. By Schaefer's fixed point theorem, we conclude that the problem (1.4) has at least one solution. □

In the sequel, we discuss the positivity of the obtained solution $x \in C$. To this end, we add adequate assumptions and provide the following theorem.

We note that a positive solution of (1.4) in C is a function $x(t) > 0$ which has $\Delta_C^{\nu} x(t-\nu+1) > 0$ for all $t \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$.

Theorem 3.6. *Suppose that **(A1)–(A5)** are fulfilled in* $\mathbb{R}^+$, *where* $\mathcal{H} \in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ *and* $\varphi \in C(\odot \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$. *If condition* (3.7) *is satisfied, for* $\alpha,\beta,\gamma,\nu,\mu \in (0,1)$, *and in addition*

$$
\rho_1 > 1 \ \ \text{and} \ \ \rho_2 > \frac{\Gamma(T+\gamma+4)}{\Gamma(2-\gamma)\Gamma(T+3)},
$$

then a solution in C *of the problem* (1.4) *is positive.*

Proof. By Theorem 3.1 and the fact that, for $\rho_1 > 1$ and $\rho_2 > \dfrac{\Gamma(T+\gamma+4)}{\Gamma(2-\gamma)\Gamma(T+3)}$, the condition (3.7) is a particular case, the problem (1.4) admits a unique solution in $\mathcal{C}$.
Moreover, since $\alpha, \beta, \gamma, \nu, \mu \in (0,1)$, we obtain for each $(t, s, r, \xi) \in \mathcal{D}$,

$$\mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi) = \left[\frac{T + (\rho_1 - 1)(t-\alpha-\beta) + 2\rho_1}{(\rho_1 - 1)\left(\rho_2 - \frac{\Gamma(T+\gamma+4)}{\Gamma(2-\gamma)\Gamma(T+3)}\right)\Gamma(1-\gamma)\Gamma(\beta)\Gamma(\alpha)}\right] \times$$
$$(T+\alpha+\beta-\gamma+1-\sigma(s))^{\underline{-\gamma}}(s-\sigma(r))^{\underline{\beta-2}}(r-\sigma(\xi))^{\underline{\alpha-1}} > 0,$$

and

$$\begin{aligned} &{}_t\Delta_C^{\nu}\, \mathcal{A}_{\alpha,\beta,\gamma,\rho_1,\rho_2}(t,s,r,\xi) \\ &= \sum_{s=\alpha+\beta-2}^{t+\nu-1}\left[\frac{(t-\sigma(s))^{\underline{-\nu}}(s-\alpha-\beta)}{\left(\rho_2 - \frac{\Gamma(T+\gamma+4)}{\Gamma(2-\gamma)\Gamma(T+3)}\right)\Gamma(1-\nu)\Gamma(1-\gamma)\Gamma(\beta)\Gamma(\alpha)}\right]\times \\ &\quad (T+\alpha+\beta-\gamma+1-\sigma(s))^{\underline{-\gamma}}(s-\sigma(r))^{\underline{\beta-2}}(r-\sigma(\xi))^{\underline{\alpha-1}} > 0. \end{aligned}$$

It results that the unique solution $x(t)$ of problem (1.4) which satisfies with (3.3) is positive for each $t \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$. □

4. An example

In this section, we present an example to illustrate our results.
Example. Consider the following fractional difference boundary value problem:

$$\begin{aligned} {}^C\Delta^{\frac{2}{3}}_{\frac{2}{3}}\, {}^C\Delta^{\frac{5}{6}}_{\frac{1}{2}} x(t) &= \frac{x\left(t+\frac{1}{2}\right)}{\left((t+\frac{1}{2})+5\right)^5\left[1+|x\left(t+\frac{1}{2}\right)|\right]} + \frac{{}^C\Delta^{\frac{1}{2}}_{\frac{1}{2}} x(t-1)}{\left((t+\frac{1}{2})+5\right)^5\left[1+|{}^C\Delta^{\frac{1}{2}}_{\frac{1}{2}} x(t-1)|\right]} \\ &\quad + \Psi^{\frac{1}{4}}\left(t+\frac{1}{2}, x\left(t+\frac{1}{2}\right)\right), \quad t \in N_{0,4}, \\ 2x\left(-\frac{1}{2}\right) &= x\left(\frac{11}{2}\right), \quad 20\Delta^{\frac{1}{3}} x\left(\frac{1}{6}\right) = \Delta^{\frac{1}{3}} x\left(\frac{37}{6}\right), \end{aligned} \tag{4.1}$$

where
$$\Psi^{\frac{1}{4}}\left(t+\frac{1}{2}, x\left(t+\frac{1}{2}\right)\right) = \sum_{s=-\frac{3}{4}}^{t-\frac{1}{4}} \frac{(t-\sigma(s))^{-\frac{3}{4}}}{\Gamma\left(\frac{1}{4}\right)}\left[\frac{e^{-(s+\frac{1}{4})}[x\left(s+\frac{1}{4}\right)+1]}{\left((t+\frac{1}{2})+5\right)^2[3+|x(s+\frac{1}{4})|]} + \frac{e^{-(s+\frac{1}{4})}[{}^C\Delta^{\frac{1}{2}}_{-\frac{1}{2}} x\left(s+\frac{3}{4}\right)+1]}{\left((t+\frac{1}{2})+5\right)^2[3+|{}^C\Delta^{\frac{1}{2}}_{-\frac{1}{2}} x(s+\frac{3}{4})|]}\right].$$

By letting $\alpha = \frac{2}{3}$, $\beta = \frac{5}{6}$, $\gamma = \frac{1}{3}$, $\nu = \frac{1}{2}$, $\mu = \frac{1}{4}$, $T = 4$, $\rho_1 = 2$, $\rho_2 = 20$, $\mathcal{H}[t,x,y,z] = \frac{1}{(t+5)^5}\left[\frac{x}{1+|x|} + \frac{y}{1+|y|} + z\right]$ and $\varphi[t+\frac{1}{2}, s+\frac{1}{4}, x, y] = \frac{e^{-s}}{(t+5)^2}\left[\frac{x+1}{3+|x|} + \frac{y+1}{3+|y|}\right]$, we can show that

$$\Lambda \approx 16.0098, \quad \Theta \approx 0.000199, \quad \Omega_1 \approx 35.0489, \quad \Omega_2 \approx 19.7664,$$
$$\Phi_1 \approx 18.0469 \quad \text{and} \quad \Phi_2 \approx 2.9653.$$

Observe that **(A1)–(A5)** hold for all $x_i, y_i, z_i \in \mathbb{R},\ i = 1, 2$, and for each $t \in \mathbb{N}_{\frac{-1}{2}, \frac{11}{2}}$, we obtain

$$\left|\mathcal{H}[t, x_1, y_1, z_1] - \mathcal{H}[t, x_2, y_2, z_2]\right| \leq \frac{1}{(t+5)^5}\Big[|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\Big].$$

So, $K_1 = \left(\frac{2}{11}\right)^5 \approx 0.000199$, and $K_2 = \max\limits_{t \in \mathbb{N}_{\frac{-1}{2}, \frac{11}{2}}} \widetilde{\mathcal{H}}[t, 0] \approx 0.0000394$.
Next, for all $x_i, y_i \in \mathbb{R},\ i = 1, 2$, and each $(t, s) \in \mathbb{N}_{\frac{-1}{2}, \frac{11}{2}} \times \mathbb{N}_{\frac{-1}{2}, \frac{11}{2}}$, we obtain

$$\left|\varphi[t, s + \frac{3}{4}, x_1, y_1] - \mathcal{H}[t, s + \frac{3}{4}, x_2, y_2]\right| \leq \frac{e^{-s}}{(t+5)^5}\Big[|x_1 - x_2| + |y_1 - y_2|\Big].$$

So, $K_2 = e^{-\frac{1}{2}} \left(\frac{2}{11}\right)^5 \approx 0.000121$.
Finally, we can show that

$$\Theta\left[\Omega_1 + \Omega_2 + \Phi_1 + \Phi_2\right] \approx 0.0151 < 1.$$

Hence, by Theorem 3.1, the problem (4.1) has a unique solution. □
Moreover, by Theorem 3.5, the problem (4.1) has at least one solution on $\mathbb{N}_{\frac{-1}{2}, \frac{11}{2}}$. □
Furthermore, $\mathcal{H}, \varphi \in \mathbb{R}^+$, and

$$\rho_1 = 2 > 1, \quad \rho_2 = 20 > \frac{\Gamma\left(\frac{25}{3}\right)}{\Gamma\left(\frac{5}{3}\right)\Gamma(7)} \approx 15.293.$$

Therefore, the solution of the problem (4.1) is positive on $\mathbb{N}_{\frac{-1}{2}, \frac{11}{2}}$ by Theorem 3.6. □

5. Conclusions

In the present research, we considered a sequential nonlinear Caputo fractional sum-difference equation with fractional difference boundary conditions. Notice that the unknown function of this problem is in the form of Caputo fractional difference and fractional sum with different orders, which expands the research scope of the problems in [42–44]. Existence results are established by a Banach contraction principle and Schaefer's fixed point theorem. The results of the paper are new and enrich the subject of boundary value problems for Caputo fractional difference-sum equations. In future work, we may extend this work by considering new boundary value problems.

Acknowledgments

This research was funded by the National Science, Research and Innovation Fund (NSRF) and Suan Dusit University with Contract no. 64-FF-06.

Conflict of interest

The authors declare no conflict of interest.

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