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Research article

On the SEL Egyptian fraction expansion for real numbers

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Abstract: In the authors' earlier work, the SEL Egyptian fraction expansion for any real number was constructed and characterizations of rational numbers by using such expansion were established. These results yield a generalized version of the results for the Fibonacci-Sylvester and the Engel series expansions. Under a certain condition, one of such characterizations also states that the SEL Egyptian fraction expansion is finite if and only if it represents a rational number. In this paper, we obtain an upper bound for the length of the SEL Egyptian fraction expansion for rational numbers, and the exact length of this expansion for a certain class of rational numbers is verified. Using such expansion, not only is a large class of transcendental numbers constructed, but also an explicit bijection between the set of positive real numbers and the set of sequences of nonnegative integers is established.

Keywords: SEL Egyptian fraction expansion; upper bound; transcendental number; bijection
Mathematics Subject Classification: 11A67

1. Introduction

It is well known that an Egyptian fraction is a finite sum of distinct unit fractions. The first algorithm for constructing Egyptian fraction expansion, due to Fibonacci [1] and also Sylvester [2], will be referred to as the *Fibonacci-Sylvester algorithm*. Fibonacci expressed any rational number between zero and one in an Egyptian fraction, and then Sylvester among others rediscovered this algorithm and extended the work towards the representations of irrational numbers [3–6]. The expansion produced by this algorithm for any real number $A \in (0,1)$ is called the *Fibonacci-Sylvester expansion* (or *Sylvester expansion*) [2–4, 7, 8], which is of the form

$$A = \sum_{n=1}^{\infty} \frac{1}{a_n},$$

where $a_n \in \mathbb{N}, a_1 \geq 2$, and $a_{n+1} \geq a_n(a_n - 1) + 1$ for all $n \geq 1$. Moreover, a real number $A \in (0,1)$ is rational if and only if the Fibonacci-Sylvester expansion of A is finite.

We have seen in [8] that each real number can be uniquely written as an Engel series expansion, and such expansion is finite if and only if it represents a rational number. In 1973, Cohen [9] rediscovered this expansion by proving that any real number A can be uniquely represented as an Egyptian fraction expansion called *Engel series expansion*, which is of the form

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n},$$

where $a_0 \in \mathbb{Z}, a_n \in \mathbb{N}, a_1 \geq 2, a_{n+1} \geq a_n$ for all $n \geq 1$, and the infinite sequence $\{a_n\}$ does not satisfy $a_{n+1} = a_n$ for all sufficiently large n (or no term of the sequence appears infinitely often). Moreover, a real number A is rational if and only if the Engel series expansion of A is finite. Using such expansion, Cohen [9] obtained a large class of transcendental numbers and established an explicit bijection between the set of positive real numbers and the set of sequences of nonnegative integers. For more information on the Engel series expansion (or the Cohen-Egyptian fraction expansion), see [7, 8, 10, 11].

Recently, the authors [10] have introduced an algorithm for constructing an Egyptian fraction expansion for any real number, called the *SEL Egyptian fraction expansion*, and then established characterizations of rational numbers by using such expansion. These results yield a generalized version of the results for the Fibonacci-Sylvester expansion and the Engel series expansion. One result implies that the Fibonacci-Sylvester expansion for any real number A is unique provided that the infinite sequence $\{a_n\}$ does not satisfy $a_{n+1} = a_n(a_n - 1) + 1$ for all sufficiently large n. The algorithm for constructing the SEL Egyptian fraction expansion is as follows. Given any real number A, by letting $a_0 = \lfloor A \rfloor$ and $A_1 = A - a_0$, we have $0 \leq A_1 < 1$. For all $n \geq 1$ with $A_n \neq 0$, define

$$a_n = \left\lceil \frac{1}{A_n} \right\rceil \text{ and } A_{n+1} = (a_n A_n - 1)\alpha_n,$$

where $\alpha_n = \alpha_n(a_n)$ is a positive rational number, which may depend on a_n. Note that $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are the floor and the ceiling functions, respectively. The following theorem yields the SEL Egyptian fraction expansion for any real number [10].

Theorem A. If $(a_n - 1)/\alpha_n \in \mathbb{N}$ for all $n \geq 1$, then a real number A can be uniquely represented as an expansion called the *SEL Egyptian fraction expansion*, which is of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 \alpha_1 \cdots a_n \alpha_n a_{n+1}},$$

where $a_0 \in \mathbb{Z}, a_n \in \mathbb{N}, a_1 \geq 2, a_{n+1} \geq (a_n - 1)/\alpha_n + 1 \geq 2$ for all $n \geq 1$, and the infinite sequence $\{a_n\}$ does not satisfy $a_{n+1} = (a_n - 1)/\alpha_n + 1$ for all sufficiently large n.

Moreover, the following theorems provide characterizations of rational numbers by the SEL Egyptian fraction expansion [10].

Theorem B. If $1/\alpha_n \in \mathbb{N}$ for all $n \geq 1$, then the corresponding SEL Egyptian fraction expansion of a real number A is finite if and only if $A \in \mathbb{Q}$.

Theorem C. If $\alpha_n \in \mathbb{N}$ for all $n \geq 1$, then the SEL Egyptian fraction expansion of a real number A is finite or periodic if and only if $A \in \mathbb{Q}$.

Note that the results for the Fibonacci-Sylvester and Engel series expansions mentioned earlier follow immediately from Theorem A and Theorem B by setting $\alpha_n = 1/a_n$ and $\alpha_n = 1$ for all $n \geq 1$, respectively. Moreover, a new expansion called the *Lüroth Egyptian fraction expansion*, together with its characterization of rational numbers, is obtained by taking $\alpha_n = a_n - 1$ in Theorem A and Theorem C, respectively.

Recall that a rational number a/b with $1 \leq a < b$ can be uniquely written as a finite Fibonacci-Sylvester expansion and a finite Engel series expansion. Let $\mathrm{FS}(a,b)$ and $\mathrm{E}(a,b)$ denote the lengths (or the number of terms) in the Fibonacci-Sylvester and Engel series expansions of a/b, respectively. It is interesting to estimate these lengths by finding bounds in terms of a and b. In 1958, Erdős, Rényi, and Szüsz [7] proved in the last section that $\mathrm{FS}(a,b) \leq a$ and $\mathrm{E}(a,b) \leq a$. In 1991, Erdős and Shallit [12] obtained an improved bound for $\mathrm{E}(a,b)$, namely $\mathrm{E}(a,b) = O(b^{1/3+\epsilon})$ for all $\epsilon > 0$, and proved that there exists a constant $c > 0$ such that $\mathrm{E}(a,b) > c\log b$ infinitely often. For the case of the Fibonacci-Sylvester expansion, Tongron, Kanasri, and Laohakosol [13] improved the upper bound for $\mathrm{FS}(a,b)$ mentioned above by showing that

$$\mathrm{FS}(a,b) \leq a\left\lceil \frac{b}{a} \right\rceil - b + 1 \tag{1.1}$$

for all positive integers a and b with $a < b$ and $\gcd(a,b) = 1$. The fact that $1 \leq a < b$ implies that $-b = aq + r$ for some integers q, r with $q < 0$ and $0 \leq r < a$. Then $b = a(-q) - r$ and $0 < -q = \lceil b/a \rceil$, and thus $a\lceil b/a \rceil - b + 1 = r + 1 \leq a$. They also proved that if $\{a_i\}$ is a sequence of positive integers defined by $a_1 = 2$ and $a_{i+1} = a_i(a_i - 1) + 1$ for $i \geq 1$, then

$$\mathrm{FS}(a_{n+1} - 2, a_{n+1} - 1) = n \quad (n \geq 1), \tag{1.2}$$

which yields the exact length of this expansion for a class of rational numbers.

In this work, we are interested in studying the length of the SEL Egyptian fraction expansion for rational numbers only in the case $1/\alpha_n \in \mathbb{N}$. We prove that the upper bound for $\mathrm{FS}(a,b)$ in (1.1) is also an upper bound for the length of the SEL Egyptian fraction expansion for rational numbers. Moreover, we obtain the exact length of such expansion for a certain class of rational numbers, which is similar to the one of $\mathrm{FS}(a,b)$ in (1.2). In a similar way to the Engel series expansion, the SEL Egyptian fraction expansion of the real numbers leads us to construct a large class of transcendental numbers and to obtain an explicit bijection between the set of positive real numbers and the set of sequences of nonnegative integers.

2. Bounds for the length of the SEL Egyptian fraction expansion

In this section, we assume that $1/\alpha_k \in \mathbb{N}$ for all $k \geq 1$. By Theorem B, the SEL Egyptian fraction expansion is finite if and only if it represents a rational number. By the algorithm for constructing the SEL Egyptian fraction expansion, it suffices to consider only the rational numbers in the interval $(0, 1)$. Let a and b be two natural numbers such that $a < b$. Let $\mathrm{SEL}(a,b)$ denote the *length* of the SEL Egyptian fraction expansion for a/b. Then $\mathrm{SEL}(a,b) = n$ if and only if

$$\frac{a}{b} = \frac{1}{a_1} + \sum_{k=1}^{n-1} \frac{1}{a_1\alpha_1 \cdots a_k\alpha_k a_{k+1}}, \tag{2.1}$$

where $a_1 \geq 2, \alpha_k = \alpha_k(a_k) \in \mathbb{Q}^+$, and $a_{k+1} \geq (a_k - 1)/\alpha_k + 1$ for all $k = 1, 2, \ldots, n-1$. Note that if $a/b = c/d$ with $1 \leq a < b, 1 \leq c < d$, and $\gcd(c, d) = 1$, then $\mathrm{SEL}(a, b) = \mathrm{SEL}(c, d)$.

The algorithm of Fibonacci and Sylvester for Egyptian fractions of rationals can be considered as the iteration of the following lemma, which is a modified version of the classical division algorithm [14].

Lemma 1. *(Modified division algorithm) For all $a, b \in \mathbb{Z}$ with $a > 0$, there exist unique $q, r \in \mathbb{Z}$ such that*

$$b = aq - r \quad \text{with} \quad 0 \leq r < a.$$

(Note that $q = \lceil b/a \rceil$.)

In the next theorem, we illustrate the use of Lemma 1 to explicitly construct the SEL Egyptian fraction expansion for any rational number $a/b \in \mathbb{Q} \cap (0, 1)$ and then determine an upper bound for $\mathrm{SEL}(a, b)$.

Theorem 1. *Let $a/b \in \mathbb{Q} \cap (0, 1)$ with* $\gcd(a, b) = 1$. *If $1/\alpha_n \in \mathbb{N}$ for all $n \geq 1$, then*

$$\mathrm{SEL}(a, b) \leq a\left\lceil \frac{b}{a} \right\rceil - b + 1.$$

Proof. Let $a/b \in \mathbb{Q} \cap (0, 1)$ with $\gcd(a, b) = 1$ and assume that $1/\alpha_n \in \mathbb{N}$ for all $n \geq 1$. By successively applying Lemma 1, we find that

$$\begin{aligned}
b &= aq_1 - s_1, & 0 &< s_1 < a, \\
\frac{b}{\alpha_1} &= s_1 q_2 - s_2, & 0 &< s_2 < s_1, \\
\frac{b}{\alpha_1 \alpha_2} &= s_2 q_3 - s_3, & 0 &< s_3 < s_2, \\
&\ \vdots \\
\frac{b}{\alpha_1 \cdots \alpha_{N-1}} &= s_{N-1} q_N - s_N, & 0 &< s_N < s_{N-1}, \\
\frac{b}{\alpha_1 \cdots \alpha_N} &= s_N q_{N+1}, & s_{N+1} &= 0.
\end{aligned}$$

The last step occurs since $\{s_i\}$ is a sequence of nonnegative integers such that $0 \leq \cdots < s_2 < s_1 < a$. Writing these equations in the fractional form, we have

$$\begin{aligned}
\frac{a}{b} &= \frac{1}{q_1} + \frac{s_1}{bq_1}, \\
\frac{s_1}{bq_1} &= \frac{1}{q_1 \alpha_1 q_2} + \frac{s_2}{bq_1 q_2}, \\
\frac{s_2}{bq_1 q_2} &= \frac{1}{q_1 \alpha_1 q_2 \alpha_2 q_3} + \frac{s_3}{bq_1 q_2 q_3}, \\
&\ \vdots \\
\frac{s_{N-1}}{bq_1 \cdots q_{N-1}} &= \frac{1}{q_1 \alpha_1 \cdots q_{N-1} \alpha_{N-1} q_N} + \frac{s_N}{bq_1 \cdots q_N},
\end{aligned}$$

$$\frac{s_N}{bq_1\cdots q_N} = \frac{1}{q_1\alpha_1\cdots q_N\alpha_N q_{N+1}}.$$

Combining the first two equations, we obtain

$$\frac{a}{b} = \frac{1}{q_1} + \frac{1}{q_1\alpha_1 q_2} + \frac{s_2}{bq_1q_2}. \tag{2.2}$$

Similarly, combining the third equation with (2.2), we obtain

$$\frac{a}{b} = \frac{1}{q_1} + \frac{1}{q_1\alpha_1 q_2} + \frac{1}{q_1\alpha_1 q_2\alpha_2 q_3} + \frac{s_3}{bq_1q_2q_3}.$$

Continuing in this manner, we find that

$$\frac{a}{b} = \frac{1}{q_1} + \frac{1}{q_1\alpha_1 q_2} + \frac{1}{q_1\alpha_1 q_2\alpha_2 q_3} + \cdots + \frac{1}{q_1\alpha_1\cdots q_N\alpha_N q_{N+1}}.$$

We now prove that $\mathrm{SEL}(a,b) = N+1$ by showing that $q_1 \geq 2$ and $q_{k+1} \geq (q_k - 1)/\alpha_k + 1$ for all $n = 1, 2, \ldots, N$. By Lemma 1 and the fact that $1 \leq a < b$, we have $q_1 = \lceil b/a \rceil \geq 2$. Moreover, for all $i = 1, 2, \ldots, N$, it follows from Lemma 1 that

$$q_n = \left\lceil \frac{b}{\alpha_1\cdots\alpha_{n-1}s_{n-1}} \right\rceil, \text{ and thus } \frac{1}{q_n} \leq \frac{\alpha_1\cdots\alpha_{n-1}s_{n-1}}{b} < \frac{1}{q_n - 1}.$$

Then for all $i = 1, 2, \ldots, N$, we have

$$\begin{aligned}
\frac{1}{q_{n+1}} &\leq \frac{\alpha_1\cdots\alpha_n s_n}{b} \\
&= \left(\frac{s_n}{bq_1\cdots q_n}\right) q_1\alpha_1\cdots q_n\alpha_n \\
&= \left(\frac{s_{n-1}}{bq_1\cdots q_{n-1}} - \frac{1}{q_1\alpha_1\cdots q_{n-1}\alpha_{n-1}q_n}\right) q_1\alpha_1\cdots q_{n-1}\alpha_{n-1}q_n\alpha_n \\
&= \left(\frac{\alpha_1\cdots\alpha_{n-1}s_{n-1}}{b} - \frac{1}{q_n}\right)\alpha_n q_n \\
&< \left(\frac{1}{q_n - 1} - \frac{1}{q_n}\right) q_n\alpha_n \\
&= \frac{\alpha_n}{q_n - 1},
\end{aligned}$$

yielding $q_{n+1} > (q_n - 1)/\alpha_n$. Since $(q_n - 1)/\alpha_n \in \mathbb{N}$, we have $q_{n+1} \geq (q_n - 1)/\alpha_n + 1$ for all $n = 1, 2, \ldots, N$. This shows that $\mathrm{SEL}(a,b) = N+1$.

Finally, we note that

$$\begin{aligned}
s_1 &= aq_1 - b = a\lceil b/a \rceil - b, \\
s_2 &\leq s_1 - 1 = a\lceil b/a \rceil - b - 1, \\
s_3 &\leq s_2 - 1 \leq a\lceil b/a \rceil - b - 2, \\
&\vdots \\
0 = s_{N+1} &\leq s_N - 1 \leq a\lceil b/a \rceil - b - N.
\end{aligned}$$

The last inequality implies that $N \leq a\lceil b/a \rceil - b$, and hence $\mathrm{SEL}(a,b) = N + 1 \leq a\lceil b/a \rceil - b + 1$. □

We conclude this section with the exact length of the SEL Egyptian fraction expansions for a certain class of rational numbers.

Theorem 2. *Let* $\{a_n\}$ *be a sequence of positive integers defined by*

$$a_1 = 2 \text{ and } a_{n+1} = (a_n - 1)/\alpha_n + 1 \quad (n \geq 1), \tag{2.3}$$

where $\alpha_n = \alpha_n(a_n) \in \mathbb{Q}^+$ *for all* $n \geq 1$*. Then*

$$\mathrm{SEL}(a_1 \cdots a_n - 1, a_1 \cdots a_n) = n \quad (n \geq 1).$$

Proof. We first show by induction on n that

$$\frac{1}{a_1} + \frac{1}{a_1\alpha_1 a_2} + \cdots + \frac{1}{a_1\alpha_1 \cdots a_{n-1}\alpha_{n-1}a_n} = \frac{a_1 \cdots a_n - 1}{a_1 \cdots a_n} \quad (n \geq 1). \tag{2.4}$$

For $n = 1$, we have $1/a_1 = 1/2 = (a_1 - 1)/a_1$. By (2.3), we obtain

$$\frac{1}{\alpha_n} = \frac{a_{n+1} - 1}{a_n - 1} \quad (n \geq 1). \tag{2.5}$$

Assume that (2.4) holds for some $n \geq 1$. It follows from (2.5) that

$$\begin{aligned}
&\frac{1}{a_1} + \frac{1}{a_1\alpha_1 a_2} + \cdots + \frac{1}{a_1\alpha_1 \cdots a_{n-1}\alpha_{n-1}a_n} + \frac{1}{a_1\alpha_1 \cdots a_n\alpha_n a_{n+1}} \\
&\quad = \frac{a_1 \cdots a_n - 1}{a_1 \cdots a_n} + \frac{1}{a_1\alpha_1 \cdots a_n\alpha_n a_{n+1}} \\
&\quad = \frac{a_1 \cdots a_n - 1}{a_1 \cdots a_n} + \frac{1}{a_1 \cdots a_{n+1}} \cdot \frac{a_2 - 1}{a_1 - 1} \cdot \frac{a_3 - 1}{a_2 - 1} \cdots \frac{a_n - 1}{a_{n-1} - 1} \cdot \frac{a_{n+1} - 1}{a_n - 1} \\
&\quad = \frac{a_1 \cdots a_n - 1}{a_1 \cdots a_n} + \frac{a_{n+1} - 1}{a_1 \cdots a_{n+1}} \\
&\quad = \frac{a_1 \cdots a_{n+1} - a_{n+1} + a_{n+1} - 1}{a_1 \cdots a_{n+1}} \\
&\quad = \frac{a_1 \cdots a_{n+1} - 1}{a_1 \cdots a_{n+1}}.
\end{aligned}$$

Using (2.1) and (2.4), we conclude that $\mathrm{SEL}(a_1 \cdots a_n - 1, a_1 \cdots a_n) = n$ for all $n \geq 1$, as desired. □

From Theorem 2, by letting $\alpha_n = 1/a_n$ for all $n \geq 1$, we obtain

$$a_{n+1} - 1 = (a_n - 1)a_n \quad (n \geq 1). \tag{2.6}$$

Since $a_1 = 2$, it follows from (2.6) that

$$\begin{aligned}
\frac{a_1 \cdots a_n - 1}{a_1 \cdots a_n} &= \frac{(a_1 - 1)a_1 a_2 \cdots a_n - 1}{(a_1 - 1)a_1 a_2 \cdots a_n} \\
&= \frac{(a_2 - 1)a_2 \cdots a_n - 1}{(a_2 - 1)a_2 \cdots a_n} \\
&\ \ \vdots
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(a_{n-1}-1)a_{n-1}a_n - 1}{(a_{n-1}-1)a_{n-1}a_n} \\
&= \frac{(a_n-1)a_n - 1}{(a_n-1)a_n} \\
&= \frac{a_{n+1}-2}{a_{n+1}-1}.
\end{aligned}
$$

This shows that $\mathrm{FS}(a_{n+1}-2, a_{n+1}-1) = n$ for all $n \geq 1$, by Theorem 2.

3. Transcendental numbers arising from SEL Egyptian fraction expansions

A complex number α is called an *algebraic number* if it is a root of some nonzero polynomial $f(x) \in \mathbb{Q}[x]$. Any complex number that is not algebraic is said to be *transcendental*. Transcendental numbers were first explicitly constructed by Liouville via the following theorem on rational approximation to algebraic numbers.

Theorem D. *(Liouville's theorem) [15] Let α be an irrational algebraic number of degree d. Then there exists a positive constant c depending only on α such that*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^d}$$

for all rational numbers p/q.

The first number shown to be transcendental by using Liouville's theorem [15] is

$$\sum_{n=1}^{\infty} 10^{-n!} = 0.110001000000000000000001000 0\ldots.$$

Liouville's result can be restated as the following theorem.

Theorem E. *(Liouville's theorem restated) [15] Let α be a real number. Suppose that there exists an infinite sequence of rational numbers p_n/q_n satisfying the inequality*

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^n}.$$

Then α is transcendental.

Cohen [9] constructed a large class of transcendental numbers by imposing the following restriction on the sequence $\{n_i\}$: Let $n_1 \geq 2$ and let n_{i+1} satisfy the inequality

$$n_{i+1} \geq (n_1 \cdots n_i)^i n_i + 1 \quad (i \geq 1). \tag{3.1}$$

Applying Liouville's theorem, he found that the resulting real number α with the Engel series expansion

$$\alpha = \frac{1}{n_1} + \frac{1}{n_1 n_2} + \frac{1}{n_1 n_2 n_3} + \cdots$$

is transcendental.

In this section, we construct a large class of transcendental numbers by using the SEL Egyptian fraction expansion. The ingredient of the proof consists of the following lemmas used in the proof of Theorem A.

Lemma 2. *[10] Any infinite series*

$$\frac{1}{b_1} + \sum_{n=1}^{\infty} \frac{1}{b_1\beta_1 \cdots b_n\beta_n b_{n+1}},$$

where $b_n \in \mathbb{N}, b_1 \geq 2, b_{n+1} \geq (b_n - 1)/\beta_n + 1 \geq 2$, *and* $\beta_n = \beta_n(b_n) \in \mathbb{Q}^+$ *for all* $n \geq 1$, *converges to a real number* B_1 *such that* $b_1 = 1 + \lfloor 1/B_1 \rfloor$.

Lemma 3. *[10] For all* $n \geq 1$, *if* $b_1 \geq 2, b_{i+1} \geq (b_i - 1)/\beta_i + 1 \geq 2$, *and* $\beta_i = \beta_i(b_i) \in \mathbb{Q}^+$ *for all* $i = 1, 2, \ldots, n$, *then*

$$\frac{1}{b_i} \leq \frac{1}{b_i} + \frac{1}{b_i\beta_i b_{i+1}} + \cdots + \frac{1}{b_i\beta_i \cdots b_{n-1}\beta_{n-1}b_n} < \frac{1}{b_i - 1} \quad (1 \leq i \leq n).$$

The following two theorems are our second main results.

Theorem 3. *Let* $a_1 \geq 2, \alpha_i = \alpha_i(a_i) \in \mathbb{N}$ *with* $(a_i - 1)/\alpha_i \in \mathbb{N}$, *and let* a_{i+1} *satisfy the inequality*

$$a_{i+1} \geq \frac{(a_1\alpha_1 \cdots a_{i-1}\alpha_{i-1}a_i)^i}{\alpha_i} + 1 \qquad (i \geq 1). \tag{3.2}$$

Then the real number $\dfrac{1}{a_1} + \sum_{i=1}^{\infty} \dfrac{1}{a_1\alpha_1 \cdots a_i\alpha_i a_{i+1}}$ *is transcendental.*

Proof. For all $i \geq 1$, we have

$$a_{i+1} \geq \frac{(a_1\alpha_1 \cdots a_{i-1}\alpha_{i-1}a_i)^i}{\alpha_i} + 1 \geq \frac{a_i}{\alpha_i} + 1 > \frac{a_i - 1}{\alpha_i} + 1 \geq 2.$$

By Lemma 2, the series $1/a_1 + \sum_{i=1}^{\infty} 1/(a_1\alpha_1 \cdots a_i\alpha_i a_{i+1})$ converges to a real number x such that $a_1 = 1 + \lfloor 1/x \rfloor$. It follows that $0 < 1/a_1 < x \leq 1/(a_1 - 1) \leq 1$. Let n be an arbitrary positive integer and consider the rational number

$$\frac{p_n}{q_n} = \frac{1}{a_1} + \sum_{i=1}^{n-1} \frac{1}{a_1\alpha_1 \cdots a_i\alpha_i a_{i+1}} = \frac{\alpha_1 a_2 \cdots \alpha_{n-1}a_n + \cdots + \alpha_{n-1}a_n + 1}{a_1\alpha_1 \cdots a_{n-1}\alpha_{n-1}a_n}$$

with $p_n, q_n \in \mathbb{N}$ and $\gcd(p_n, q_n) = 1$. By Lemma 3, we have

$$0 < \frac{1}{a_1} \leq \frac{p_n}{q_n} < \frac{1}{a_1 - 1} \leq 1,$$

so $q_n > 1$. Note that q_n must divide $a_1\alpha_1 \cdots a_{n-1}\alpha_{n-1}a_n$, implying that $q_n \leq a_1\alpha_1 \cdots a_{n-1}\alpha_{n-1}a_n$. It follows from (3.2) that

$$a_{n+1} - 1 \geq \frac{(a_1\alpha_1 \cdots a_{n-1}\alpha_{n-1}a_n)^n}{\alpha_n} \geq \frac{q_n^n}{\alpha_n}. \tag{3.3}$$

Again, Lemma 2 implies that

$$\frac{1}{a_{n+1}} < \frac{1}{a_{n+1}} + \frac{1}{a_{n+1}\alpha_{n+1}a_{n+2}} + \cdots \leq \frac{1}{a_{n+1} - 1}.$$

Using Lemma 2, Lemma 3, and (3.3), we finally have

$$\begin{aligned}
0 < \left|x - \frac{p_n}{q_n}\right| &= \left|\frac{1}{a_1\alpha_1\cdots a_n\alpha_n a_{n+1}} + \frac{1}{a_1\alpha_1\cdots a_{n+1}\alpha_{n+1}a_{n+2}} + \cdots\right| \\
&= \frac{1}{a_1\alpha_1\cdots a_n\alpha_n}\left(\frac{1}{a_{n+1}} + \frac{1}{a_{n+1}\alpha_{n+1}a_{n+2}} + \cdots\right) \\
&\le \frac{1}{a_1\alpha_1\cdots a_n\alpha_n}\left(\frac{1}{a_{n+1}-1}\right) \\
&\le \frac{1}{a_1\alpha_1\cdots a_n\alpha_n}\cdot\frac{\alpha_n}{q_n^n} \\
&= \frac{1}{a_1\alpha_1\cdots a_{n-1}\alpha_{n-1}a_n}\cdot\frac{1}{q_n^n} \\
&< \left(\frac{1}{a_1} + \frac{1}{a_1\alpha_1 a_2} + \cdots + \frac{1}{a_1\alpha_1\cdots a_{n-1}\alpha_{n-1}a_n}\right)\cdot\frac{1}{q_n^n} \\
&< \frac{1}{a_1-1}\cdot\frac{1}{q_n^n} \\
&\le \frac{1}{q_n^n}.
\end{aligned}$$

By Theorem E, we conclude that x is transcendental. □

Applying Theorem 3 with $\alpha_i = 1$ for all $i \ge 1$, we obtain a class of transcendental numbers, which also contains the class derived by Cohen [9].

Theorem 4. *Let $a_1 \ge 2$, $1/\alpha_i \in \mathbb{N}$ with $(a_i-1)/\alpha_i \in \mathbb{N}$, and let a_{i+1} satisfy the inequality*

$$a_{i+1} \ge \frac{(a_1\cdots a_i)^i}{\alpha_i} + 1 \qquad (i \ge 1).$$

Then the real number $\frac{1}{a_1} + \sum_{i=1}^{\infty}\frac{1}{a_1\alpha_1\cdots a_i\alpha_i a_{i+1}}$ is transcendental.

Proof. For all $i \ge 1$, we have

$$a_{i+1} \ge \frac{(a_1\cdots a_i)^i}{\alpha_i} + 1 > \frac{a_i - 1}{\alpha_i} + 1 \ge 2. \tag{3.4}$$

By Lemma 2, the series $1/a_1 + \sum_{i=1}^{\infty} 1/(a_1\alpha_1\cdots a_i\alpha_i a_{i+1})$ converges to a real number x such that $a_1 = 1 + \lfloor 1/x \rfloor$. It follows that $0 < 1/a_1 < x \le 1/(a_1-1) \le 1$. Let n be an arbitrary positive integer and consider the rational number

$$\frac{p_n}{q_n} = \frac{1}{a_1} + \sum_{i=1}^{n-1}\frac{1}{a_1\alpha_1\cdots a_i\alpha_i a_{i+1}}$$

with $p_n, q_n \in \mathbb{N}$ and $\gcd(p_n, q_n) = 1$. By Lemma 3, we have

$$0 < \frac{1}{a_1} \le \frac{p_n}{q_n} < \frac{1}{a_1 - 1} \le 1,$$

so $q_n > 1$. Set $1/\alpha_i = b_i \in \mathbb{N}$ $(i \geq 1)$. Then

$$\frac{p_n}{q_n} = \frac{1}{a_1} + \frac{b_1}{a_1a_2} + \frac{b_1b_2}{a_1a_2a_3} + \cdots + \frac{b_1b_2\cdots b_{n-1}}{a_1\cdots a_n} = \frac{a_2\cdots a_n + b_1a_3\cdots a_n + \cdots + b_1b_2\cdots b_{n-1}}{a_1a_2\cdots a_n}.$$

Since $\gcd(p_n, q_n) = 1$, we have $q_n \leq a_1a_2\cdots a_n$. It follows from (3.4) that

$$a_{n+1} - 1 \geq \frac{(a_1a_2\cdots a_n)^n}{\alpha_n} \geq \frac{q_n^n}{\alpha_n} = b_nq_n^n. \tag{3.5}$$

Using Lemma 2, Lemma 3, and (3.5), we finally have

$$\begin{aligned}
0 < \left|x - \frac{p_n}{q_n}\right| &= \left|\frac{1}{a_1\alpha_1\cdots a_n\alpha_n a_{n+1}} + \frac{1}{a_1\alpha_1\cdots a_{n+1}\alpha_{n+1}a_{n+2}} + \cdots\right| \\
&= \frac{1}{a_1\alpha_1\cdots a_n\alpha_n}\left(\frac{1}{a_{n+1}} + \frac{1}{a_{n+1}\alpha_{n+1}a_{n+2}} + \cdots\right) \\
&\leq \frac{b_n}{a_1\alpha_1\cdots a_n}\left(\frac{1}{a_{n+1}-1}\right) \\
&= \frac{1}{a_1\alpha_1\cdots a_{n-1}\alpha_{n-1}a_n}\cdot\frac{1}{q_n^n} \\
&< \left(\frac{1}{a_1} + \frac{1}{a_1\alpha_1a_2} + \cdots + \frac{1}{a_1\alpha_1\cdots a_{n-1}\alpha_{n-1}a_n}\right)\cdot\frac{1}{q_n^n} \\
&< \frac{1}{a_1-1}\cdot\frac{1}{q_n^n} \\
&\leq \frac{1}{q_n^n}
\end{aligned}$$

for all $n \geq 1$. It follows from Theorem E that x is transcendental, which completes the proof. □

4. A bijection arising from the SEL Egyptian fraction expansion

We now proceed to the last main result, where we use the SEL Egyptian fraction expansion, we construct a bijection between the set of positive real numbers and the set of sequences of nonnegative integers. Let S be the set of sequences of nonnegative integers and x be any positive real number. Define a function $\Phi : \mathbb{R}^+ \to S$ depending on the following cases.

Case I: $x \in \mathbb{Q}$. Then $x/(x+1)$ can be uniquely represented as a finite SEL Egyptian fraction expansion of the form

$$\frac{x}{x+1} = \frac{1}{a_1} + \frac{1}{a_1\alpha_1a_2} + \cdots + \frac{1}{a_1\alpha_1\cdots a_{m-1}\alpha_{m-1}a_m},$$

where $m \in \mathbb{N}, a_i \in \mathbb{N}, a_1 \geq 2$, and $a_{i+1} \geq (a_i - 1)/\alpha_i + 1 \geq 2$ for all $i = 1, 2, \ldots, m-1$.

If $m = 1$, then $x/(x+1) = 1/a_1$, and we define

$$\Phi(x) = \{0, a_1 - 2, 0, 0, \ldots\}.$$

If $m > 1$ and $a_m > (a_{m-1} - 1)/\alpha_{m-1} + 1$, then we define

$$\Phi(x) = \{0, a_1 - 2, a_2 - (a_1 - 1)/\alpha_1 - 1, \ldots, a_m - (a_{m-1} - 1)/\alpha_{m-1} - 1, 0, 0, \ldots\}.$$

If $m > 1$ and there exist $k, a_0 \in \mathbb{N}$ such that $k + a_0 = m$, $a_k > (a_{k-1} - 1)/\alpha_{k-1} + 1$ (if $k \geq 2$), and $a_{k+i} = (a_{k+i-1} - 1)/\alpha_{k+i-1} + 1$ for all $i = 1, 2, \ldots, a_0$, then we define

$$\Phi(x) = \{a_0, a_1 - 2, a_2 - (a_1 - 1)/\alpha_1 - 1, \ldots, a_k - (a_{k-1} - 1)/\alpha_{k-1} - 1, 0, 0, \ldots\}.$$

Case II: $x \in \mathbb{Q}^c$. Then x has the infinite SEL Egyptian fraction expansion of the form

$$x = b_0 + \frac{1}{b_1} + \sum_{i=1}^{\infty} \frac{1}{b_1\alpha_1 \cdots b_i\alpha_i b_{i+1}}.$$

Define

$$\Phi(x) = \{b_0, b_1 - 2, b_2 - (b_1 - 1)/\alpha_1 - 1, b_3 - (b_2 - 1)/\alpha_2 - 1, \ldots\}.$$

Then the authors' earlier work [10, Proposition 2.7] implies that the above sequence has infinitely many positive terms.

We now show that Φ is a bijection. Since the SEL Egyptian fraction expansion of any real number x is unique, the function Φ is well defined. To show that Φ is surjective, let $\{a_0, a_1, a_2, \ldots\} \in S$ and consider the following two possible cases:

Case 1: $\{a_0, a_1, a_2, \ldots\}$ has infinitely many positive terms. Set $b_0 = a_0$, $b_1 = 2 + a_1$, and $b_{n+1} = (b_n - 1)/\beta_n + 1 + a_{n+1}$, where $1/\beta_n \in \mathbb{N}$ $(n \geq 1)$. By Lemma 2, Theorem A, and Theorem B, there exists $x \in \mathbb{Q}^c$ such that $x = b_0 + 1/b_1 + \sum_{i=1}^{\infty} 1/(b_1\beta_1 \cdots b_i\beta_i b_{i+1})$ is its SEL Egyptian fraction expansion. Hence, we have

$$\Phi(x) = \{b_0, b_1 - 2, b_2 - (b_1 - 1)/\beta_1 - 1, b_3 - (b_2 - 1)/\beta_2 - 1, \ldots\} = \{a_0, a_1, a_2, a_3, \ldots\}.$$

Case 2: $\{a_0, a_1, a_2, \ldots\}$ has finitely many positive terms with the last positive term a_k. We consider the following four possible subcases.

Subcase 2.1: $k = 1$ and $a_0 = 0$. Set $y = 1/(2 + a_1)$ and $x = y/(1 - y)$. Then $y = x/(x + 1)$, so

$$\Phi(x) = \{0, a_1, 0, 0, \ldots\}.$$

Subcase 2.2: $k = 1$ and $a_0 \geq 1$. Set $b_1 = 2 + a_1$ and $b_{n+1} = (b_n - 1)/\beta_n + 1$, where $\beta_n = \beta_n(b_n) \in \mathbb{Q}^+$ and $(b_n - 1)/\beta_n \in \mathbb{N}$ $(1 \leq n \leq a_0)$. Let $y = 1/b_1 + \sum_{i=1}^{a_0-1} 1/(b_1\beta_1 \cdots b_i\beta_i b_{i+1})$ and $x = y/(1 - y)$. Then $y = x/(x + 1)$, and thus

$$\Phi(x) = \{a_0, b_1 - 2, 0, 0, \ldots\} = \{a_0, a_1, 0, 0, \ldots\}.$$

Subcase 2.3: $k \geq 2$ and $a_0 = 0$. Set $b_1 = 2 + a_1$ and $b_{n+1} = (b_n - 1)/\beta_n + 1 + a_{n+1}$, where $\beta_n = \beta_n(b_n) \in \mathbb{Q}^+$ and $(b_n - 1)/\beta_n \in \mathbb{N}$ $(1 \leq n \leq k - 1)$. Let $y = 1/b_1 + \sum_{i=1}^{k-1} 1/(b_1\beta_1 \cdots b_i\beta_i b_{i+1})$ and $x = y/(1 - y)$. Then $y = x/(x + 1)$, so

$$\begin{aligned}\Phi(x) &= \{0, b_1 - 2, b_2 - (b_1 - 1)/\beta_1 - 1, \ldots, b_k - (b_{k-1} - 1)/\beta_{k-1} - 1, 0, 0, \ldots\} \\ &= \{0, a_1, a_2, \ldots, a_k, 0, 0, \ldots\}.\end{aligned}$$

Subcase 2.4: $k \geq 2$ and $a_0 \geq 1$. Set $b_1 = 2 + a_1$, $b_{n+1} = (b_n - 1)/\beta_n + 1 + a_{n+1}$ $(1 \leq n \leq k - 1)$, and $b_{n+1} = (b_n - 1)/\beta_n + 1$ $(k \leq n \leq k + a_0 - 1)$, where $\beta_n = \beta_n(b_n) \in \mathbb{Q}^+$ and $(b_n - 1)/\beta_n \in \mathbb{N}$ $(1 \leq n \leq k + a_0 - 1)$. Let $y = 1/b_1 + \sum_{i=1}^{k+a_0-1} 1/(b_1\beta_1 \cdots b_i\beta_i b_{i+1})$ and $x = y/(1 - y)$. Then $y = x/(x + 1)$, so

$$\Phi(x) = \{a_0, b_1 - 2, b_2 - (b_1 - 1)/\beta_1 - 1, \ldots, b_k - (b_{k-1} - 1)/\beta_{k-1} - 1, 0, 0, \ldots\}$$
$$= \{a_0, a_1, a_2, \ldots, a_k, 0, 0, \ldots\}.$$

This shows that Φ is surjective.

Finally, we show that Φ is injective. Let $x, y \in \mathbb{R}^+$ be such that $\Phi(x) = \Phi(y)$. It is clear that both x and y are either rational or irrational. We consider the following two possible cases:

Case 1: $x \in \mathbb{Q}^+$ and $y \in \mathbb{Q}^+$. Let

$$\frac{x}{x+1} = \frac{1}{a_1} + \sum_{i=1}^{k+a_0-1} \frac{1}{a_1\alpha_1 \cdots a_i\alpha_i a_{i+1}} \quad \text{and} \quad \frac{y}{y+1} = \frac{1}{b_1} + \sum_{j=1}^{l+b_0-1} \frac{1}{b_1\beta_1 \cdots b_j\beta_j b_{j+1}}$$

be SEL Egyptian fraction expansions such that $\alpha_i = \alpha_i(a_i) \in \mathbb{Q}^+$ $(1 \le i \le k + a_0 - 1), \beta_j = \alpha_j(b_j) \in \mathbb{Q}^+$ $(1 \le j \le l + b_0 - 1), a_0, b_0 \ge 0, a_{i+1} = (a_i - 1)/\alpha_i + 1$ $(k \le i \le k + a_0 - 1), b_{j+1} = (b_j - 1)/\beta_j + 1$ $(l \le j \le l + b_0 - 1), a_k > (a_{k-1} - 1)/\alpha_{k-1} + 1$ (if $k \ge 2$), and $b_l > (b_{l-1} - 1)/\beta_{l-1} + 1$ (if $l \ge 2$). Since $\Phi(x) = \Phi(y)$, we have

$$\{a_0, a_1 - 2, a_2 - (a_1 - 1)/\alpha_1 - 1, \ldots, a_k - (a_{k-1} - 1)/\alpha_{k-1} - 1, 0, 0, \ldots\}$$
$$= \{b_0, b_1 - 2, b_2 - (b_1 - 1)/\beta_1 - 1, \ldots, b_l - (b_{l-1} - 1)/\beta_{l-1} - 1, 0, 0, \ldots\}.$$

It is clear that $k = l$ and $a_0 = b_0, a_1 = b_1, \ldots, a_k = b_k$. It follows that $a_i = b_i$ and $\alpha_i = \alpha_i(a_i) = \alpha_i(b_i) = \beta_i$ $(k + 1 \le i \le k + a_0)$. This implies that $x/(x + 1) = y/(y + 1)$, and thus $x = y$.

Case 2: $x \in \mathbb{Q}^c$ and $y \in \mathbb{Q}^c$. Let

$$x = a_0 + \frac{1}{a_1} + \sum_{i=1}^{\infty} \frac{1}{a_1\alpha_1 \cdots a_i\alpha_i a_{i+1}} \quad \text{and} \quad y = b_0 + \frac{1}{b_1} + \sum_{i=1}^{\infty} \frac{1}{b_1\beta_1 \cdots b_i\beta_i b_{i+1}}$$

be SEL Egyptian fraction expansions such that $\alpha_i = \alpha_i(a_i) \in \mathbb{Q}^+$ and $\beta_i = \alpha_i(b_i) \in \mathbb{Q}^+$ for all $i \ge 1$. Since $\Phi(x) = \Phi(y)$, we have

$$\{a_0, a_1 - 2, a_2 - (a_1 - 1)/\alpha_1 - 1, \ldots\} = \{b_0, b_1 - 2, b_2 - (b_1 - 1)/\beta_1 - 1, \ldots\},$$

implying that $a_i = b_i$ $(i \ge 0)$. Hence $x = y$, which completes the proof.

Note that the bijection Φ is a generalization of the bijection constructed by Cohen [9].

5. Conclusions

In this paper, we obtain an upper bound for the length of the SEL Egyptian fraction expansion for rational numbers. In addition, the exact length of this expansion for a certain class of rational numbers is verified. Using such expansion, not only do we obtain a large class of transcendental numbers, but also an explicit bijection between the set of positive real numbers and the set of sequences of nonnegative integers is established.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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