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Research article

Linear Diophantine fuzzy graphs with new decision-making approach

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Abstract: The concept of linear Diophantine fuzzy set (LDFS) is a new mathematical tool for optimization, soft computing, and decision analysis. The aim of this article is to extend the notion of graph theory towards LDFSs. We initiate the idea of linear Diophantine fuzzy graph (LDF-graph) as a generalization of certain theoretical concepts including, q-rung orthopair fuzzy graph, Pythagorean fuzzy graph, and intuitionistic fuzzy graph. We extend certain properties of crisp graph theory towards LDF-graph including, composition, join, and union of LDF-graphs. We elucidate these operations with various illustrations. We analyze some interesting results that the composition of two LDF-graphs is a LDF-graph, cartesian product of two LDF-graphs is a LDF-graph, and the join of two LDF-graphs is a LDF-graph. We describe the idea of homomorphisms for LDF-graphs. We observe the equivalence relation via an isomorphism between LDF-graphs. Some significant results related to complement of LDF-graph are also investigated. Lastly, an algorithm based on LDFSs and LDF-relations is proposed for decision-making problems. A numerical example of medical diagnosis application is presented based on proposed approach.

Keywords: LDFSs; LDF-graphs; properties of LDF-graph; LDF-relations; decision-making
Mathematics Subject Classification: 03E72, 05C72

1. Introduction

Modeling uncertainties in real-life have a become a key factor in various life problem problems including, medical diagnosis, data analysis, computational intelligence, sustainability, etc. [1–4]. An initiative is pioneered by Zadeh [5] in terms of fuzzy set (FS) and fuzzy logic. Since then, numerous researchers have investigated into the idea of fuzzy sets theory in order to overcome a wide range of real-life problems involving uncertain circumstances. Chang [6] introduced fuzzy topology and fuzzy topological spaces. Smithson [7] studied topologies generated by relations. Feng and Qiu [8] fuzzy orders and fuzzifying topologies.

Atanassov [9, 10] initiated intuitionistic fuzzy sets (IFSs) as direct extension of FS. Palaniappan and Srinivasan [11] studied IFSs of root type with application to image processing. Szmidt and Kacprzyk [12] introduced IFS similarity measures with MCDM. Vlachos and Sergiadis [13] established pattern recognition application using IFSs. The idea of a fuzzy graph initiated by many researchers; Kaufmann [14], Rosenfeld Mordeson [15], and Bhattacharya [16]. The idea of complex fuzzy graphs suggested by Thirunavukarasu et al. [17]. Intuitionistic fuzzy graphs studied by many scholars (see [18–25]).

Burillo and Bustince [26] developed the notions of IFS relations, IFS t-norm and t-conorm, Atanassov intuitionistic operators, and composition of t-norm and t-conorm. Bustince and Burillo [27] established the structures on intuitionistic fuzzy relations and the structures of its complementary intuitionistic fuzzy relations. Further properties of intuitionistic fuzzy relations studied by numerous researchers (see [28, 29]. Deschrijver and Kerre [28] presented the notion of composition of IF relations. Hur et al. [29] developed IF equivalence relations and their properties. They defined the notion of level sets of an IF relation and IF transitive closures. See also [30–34].

The idea of Pythagorean fuzzy sets (PFSs) suggested by Yager [35], and Yager and Abbasov [36]. Later Yager [37] presented generalized orthopair fuzzy sets which is well-known as q-rung orthopair fuzzy set (q-ROFS). Naeem et al. [38] introduced novel ideas of Pythagorean m-polar fuzzy sets (P-m-PFSs) and Pythagorean fuzzy relations (PF-relations). They proposed the concept of score and accuracy functions of a Pythagorean m-polar fuzzy numbers. They investigated and proposed images and inverse images of Pythagorean m-polar fuzzy sets. They developed an application of PF-relations indecision-making and choosing the life partner. Akram et al. [39, 40] studied certain PFS-graphs and q-ROF graphs under Hamacher operators. Yin et al. [41] proposed product operations on q-ROF graphs. Sitara et al. [42] presented q-rung picture fuzzy graph structures and their properties with applications. They adopted q-rung image fuzzy graph structures to investigate relationships among developed and developing countries. Riaz and Hashmi [43] proposed linear Diophantine fuzzy set (LDFS) and their application towards multi-attribute decision-making (MADM). They proposed LDF aggregation operators for information fusion of LDFNs. Recently, LDFSs have been extended to linear Diophantine fuzzy soft rough set [44], algebraic structures of LDFS [45], LDF-relations with decision making [46], and q-LDFS [47].

Klement et al. [48] presented generalizing expected values to the case of L^*-fuzzy events. They investigated and introduced the idea of expected values of fuzzy events in the general sense. Klement and R. Mesiar [49] investigated L-fuzzy sets and isomorphic lattices. They analyzed several mathematical concepts about fuzzy set, interval-valued fuzzy set, intuitionistic fuzzy set, Pythagorean fuzzy set, isomorphic lattices, and truth values. Liu et al. [50] developed generalized Einstein averaging aggregation operators for complex q-rung orthopair fuzzy information aggregation and their application in MADM. Yaqoob et al. [51] defined complex intuitionistic fuzzy graphs and their homomorphisms with application network provider agencies.

Some objectives of this manuscript as the following.

1) Fuzzy graphs are conceptual frameworks to analyze the features that are frequently connected to a network. We proposed a novel extension of fuzzy graphs named as linear Diophantine fuzzy graph (LDF graph) which remove various strict limitations of the existing graphs.

2) The reference parameters (RPs) corresponding to membership grades are helping to analyze the best or worst grading by the decision makers. The RPs will attain a high value for best grading and a low value for worst grading by the decision makers, respectively. In fact, the RPs control the best worst situation in the decision analysis.
3) Linear Diophantine fuzzy graph provides a robust approach for fuzzy modeling in the best worst situation. Consequently, the decision-making approach becomes robust with linear Diophantine fuzzy information.
4) Linear Diophantine Fuzzy graph (LDF-graph) theory becomes superior to IF-graph, PyF-graph, and q-ROF-graph theories, due to broader space for membership and non-membership values.
5) Novel concepts of LDF-graph and certain operations on LDF-graph are introduced.
6) Certain properties of LDF-graphs are investigated including, order of a LDF-graph, degree of a vertex, cartesian product of LDF-graphs, composition of LDF-graphs, union of two LDF-graphs, and join of two LDF-graphs. Various illustrations are given to explain these concepts.
7) The idea of homomorphism, isomorphism, and weak isomorphism (co-isomorphism) between two LDF-graphs is introduced.
8) The concept of complement of LDF-graphs in proposed and related results are established.
9) A medical diagnosis application is established based on proposed decision-making technique. For this objective, we construct LDF graphs and construct corresponding LDF-relations. An algorithm is developed for decision-making based on LDFSs and LDF-relations.

The arrangement of this paper is arranged as follows. The idea of LDFSs and fundamental operation on LDFSs are reviewed in Section 2. Novel concepts of LDF-graph and certain operations on LDf-graph are introduced in Section 3. Moreover, we study various properties of LDF-graphs and their related illustrations. In Section 4, we define the idea of homomorphism and isomorphism between two LDF-graphs. In Section 5, we define the idea of complement of LDF-graphs and related results. In Section 6, we construct an algorithm based on LDFSs and LDF-relations for decision-making problems. Based on proposed algorithm an application of medical diagnosis is presented. Lastly, the specific summary of manuscript is given in Section 7.

2. Preliminaries and basic definitions

In this fragment, we study the idea of LDFSs and their fundamental operation that are essential for the study of LDF-graph theory.

Definition 2.1. *Let $\mathbb{Y}$ be the universe. A LDFS $£_{\mathfrak{R}}$ on $\mathbb{Y}$ is defined by*

$$\begin{aligned} £_{\mathfrak{R}} &= \left\{\left(\vartheta, \langle \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta)\rangle, \langle \alpha, \beta\rangle\right) : \vartheta \in \mathbb{Y}\right\} \\ &= \left\{\left\langle \vartheta, (\mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta)), (\alpha, \beta)\right\rangle : \vartheta \in \mathbb{Y}\right\} \end{aligned}$$

where, $\mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta), \alpha, \beta \in [0, 1]$ such that

$$0 \leq \alpha\mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) + \beta\mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta) \leq 1 \ \forall \vartheta \in \mathbb{Y} \tag{2.1}$$

$$0 \leq \alpha + \beta \leq 1 \tag{2.2}$$

The hesitation part can be written as

$$\xi\pi_{\mathfrak{R}} = 1 - (\alpha\mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) + \beta\mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta)) \tag{2.3}$$

where ξ is the RP.
The value of $\pounds_{\mathfrak{R}} = \left(\langle\mathfrak{M}^{\tau}_{\mathfrak{R}}, \mathfrak{N}^{\nu}_{\mathfrak{R}}\rangle, \langle\alpha,\beta\rangle\right)$ or $\pounds_{\mathfrak{R}} = \left\langle(\mathfrak{M}^{\tau}_{\mathfrak{R}}, \mathfrak{N}^{\nu}_{\mathfrak{R}}), (\alpha,\beta)\right\rangle$ is known as the linear Diophantine fuzzy number (LDFN).

Definition 2.2. *An absolute LDFS on $\mathbb{Y}$ is of the form*

$$^{1}\pounds_{\mathfrak{R}} = \left\{\left(\vartheta, \langle 1,0\rangle, \langle 1,0\rangle\right) : \vartheta \in \mathbb{Y}\right\}$$

and empty or null LDFS is of the form

$$^{0}\pounds_{\mathfrak{R}} = \left\{\left(\vartheta, \langle 0,1\rangle, \langle 0,1\rangle\right) : \vartheta \in \mathbb{Y}\right\}$$

Definition 2.3. *Let $\pounds_{\mathfrak{R}} = \left(\langle\mathfrak{M}^{\tau}_{\mathfrak{R}}, \mathfrak{N}^{\nu}_{\mathfrak{R}}\rangle, \langle\alpha,\beta\rangle\right)$ and $\pounds_{\mathfrak{P}} = \left(\langle\mathfrak{M}^{\tau}_{\mathfrak{P}}, \mathfrak{N}^{\nu}_{\mathfrak{P}}\rangle, \langle\gamma,\delta\rangle\right)$ be two LDFSs on the reference set $\mathbb{Y}$ and $\vartheta \in \mathbb{Y}$. Then*
- $\pounds^{c}_{\mathfrak{R}} = \left(\langle\mathfrak{N}^{\nu}_{\mathfrak{R}}, \mathfrak{M}^{\tau}_{\mathfrak{R}}\rangle, \langle\beta,\alpha\rangle\right)$.
- $\pounds_{\mathfrak{R}} = \pounds_{\mathfrak{P}} \Leftrightarrow \mathfrak{M}^{\tau}_{\mathfrak{R}} = \mathfrak{M}^{\tau}_{\mathfrak{P}}, \mathfrak{N}^{\nu}_{\mathfrak{R}} = \mathfrak{N}^{\nu}_{\mathfrak{P}}, \alpha = \gamma, \beta = \delta$.
- $\pounds_{\mathfrak{R}} \subseteq \pounds_{\mathfrak{P}} \Leftrightarrow \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) \leq \mathfrak{M}^{\tau}_{\mathfrak{P}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta) \geq \mathfrak{N}^{\nu}_{\mathfrak{P}}(\vartheta), \alpha \leq \gamma, \beta \geq \delta$.
- $\pounds_{\mathfrak{R}} \cup \pounds_{\mathfrak{P}} = (\langle\mathfrak{M}^{\tau}_{\mathfrak{R}\cup\mathfrak{P}}, \mathfrak{N}^{\nu}_{\mathfrak{R}\cap\mathfrak{P}}\rangle, \langle\alpha \vee \gamma, \beta \wedge \delta\rangle)$
- $\pounds_{\mathfrak{R}} \cap \pounds_{\mathfrak{P}} = (\langle\mathfrak{M}^{\tau}_{\mathfrak{R}\cap\mathfrak{P}}, \mathfrak{N}^{\nu}_{\mathfrak{R}\cup\mathfrak{P}}\rangle, \langle\alpha \wedge \gamma, \beta \vee \delta\rangle)$

where

$$\mathfrak{M}^{\tau}_{\mathfrak{R}\cup\mathfrak{P}}(\vartheta) = \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) \vee \mathfrak{M}^{\tau}_{\mathfrak{P}}(\vartheta), \quad \mathfrak{M}^{\tau}_{\mathfrak{R}\cap\mathfrak{P}}(\vartheta) = \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) \wedge \mathfrak{M}^{\tau}_{\mathfrak{P}}(\vartheta),$$
$$\mathfrak{N}^{\nu}_{\mathfrak{R}\cap\mathfrak{P}}(\vartheta) = \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta) \wedge \mathfrak{N}^{\nu}_{\mathfrak{P}}(\vartheta), \quad \mathfrak{N}^{\nu}_{\mathfrak{R}\cup\mathfrak{P}}(\vartheta) = \mathfrak{N}_{\mathfrak{R}}(\vartheta) \vee \mathfrak{N}^{\nu}_{\mathfrak{P}}(\vartheta).$$

3. LDF-graphs

In this part, we review certain concepts of LDF-graph including new operations on LDF-graphs.

Definition 3.1. *A LDF-graph is described by the pair $\mathbb{G} = (\pounds_{\mathfrak{R}}, \pounds_{\mathfrak{P}})$ where, $\pounds_{\mathfrak{R}}$ is a LDFS on W and $\pounds_{\mathfrak{P}}$ is a LDFS on $E \subseteq W \times W$ as follows*

$$\begin{aligned}
\mathfrak{M}^{\tau}_{\mathfrak{P}}(wt) &\leq \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}}(w), \mathfrak{M}^{\tau}_{\mathfrak{R}}(t)\},\\
\mathfrak{N}^{\nu}_{\mathfrak{P}}(wt) &\leq \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}}(w), \mathfrak{N}^{\nu}_{\mathfrak{D}}(t)\}\\
\gamma^{wt} &\leq \min\{\alpha^{w}, \alpha^{t}\},\\
\delta^{wt} &\leq \max\{\beta^{w}, \beta^{t}\}
\end{aligned}$$

for all $w, t \in W$. Where $\alpha^{w}, \beta^{w}, \alpha^{t}, \beta^{t}$ are the reference parameters associated with the vertices w, t and γ^{wt}, δ^{wt} are the reference parameters associated with the edge wt.

Definition 3.2. *Let $\mathbb{G} = (\pounds_{\mathfrak{R}}, \pounds_{\mathfrak{P}})$ be a LDF-graph. The order of a LDF-graph is described by*

$$O(\mathbb{G}) = \left(\left\langle \sum_{w\in W}\mathfrak{M}^{\tau}_{\mathfrak{R}}(w), \sum_{w\in W}\mathfrak{N}^{\nu}_{\mathfrak{R}}(w)\right\rangle, \left\langle \sum_{w\in W}\alpha^{w}, \sum_{w\in W}\beta^{w}\right\rangle\right). \tag{3.1}$$

The degree of a vertex w in $\mathbb{G}$ is described by

$$\deg(w) = \left(\left\langle \sum_{wt\in E}\mathfrak{M}^{\tau}_{\mathfrak{P}}(wt), \sum_{wt\in E}\mathfrak{N}^{\nu}_{\mathfrak{P}}(wt)\right\rangle, \left\langle \sum_{wt\in E}\gamma^{wt}, \sum_{wt\in E}\delta^{wt}\right\rangle\right). \tag{3.2}$$

Example 3.1. *Suppose that* $E = \{ab, ac, ad, cd\}$ *and* $W = \{a, b, c, d\}$ *are associated with a graph* $\mathbb{G}^* = (W, E)$. *Let* $\pounds_{\mathfrak{R}}$ *be a LDF-subset of* W *and let* $\pounds_{\mathfrak{P}}$ *be a LDF-subset of* $E \subseteq W \times W$, *as expressed in Tables 1 and 2.*

Table 1. $(\langle \mathfrak{M}^{\tau}_{\mathfrak{R}}, \mathfrak{N}^{\nu}_{\mathfrak{R}} \rangle, \langle \alpha, \beta \rangle)$.

$\pounds_{\mathfrak{R}}$	$(\langle \mathfrak{M}^{\tau}_{\mathfrak{R}}, \mathfrak{N}^{\nu}_{\mathfrak{R}} \rangle, \langle \alpha, \beta \rangle)$
a	$(\langle 0.37, 0.61 \rangle, \langle 0.59, 0.27 \rangle)$
b	$(\langle 0.51, 0.47 \rangle, \langle 0.31, 0.33 \rangle)$
c	$(\langle 0.93, 0.52 \rangle, \langle 0.51, 0.47 \rangle)$
d	$(\langle 0.78, 0.71 \rangle, \langle 0.29, 0.21 \rangle)$

Table 2. $(\langle \mathfrak{M}^{\tau}_{\mathfrak{P}}, \mathfrak{N}^{\nu}_{\mathfrak{P}} \rangle, \langle \gamma, \delta \rangle)$.

$\pounds_{\mathfrak{P}}$	$(\langle \mathfrak{M}^{\tau}_{\mathfrak{P}}, \mathfrak{N}^{\nu}_{\mathfrak{P}} \rangle, \langle \gamma, \delta \rangle)$
ab	$(\langle 0.35, 0.59 \rangle, \langle 0.29, 0.31 \rangle)$
ac	$(\langle 0.24, 0.42 \rangle, \langle 0.49, 0.42 \rangle)$
ad	$(\langle 0.32, 0.52 \rangle, \langle 0.27, 0.24 \rangle)$
cd	$(\langle 0.75, 0.62 \rangle, \langle 0.22, 0.39 \rangle)$

(i) We see that the graph described in Figure 1 is a LDF-graph.

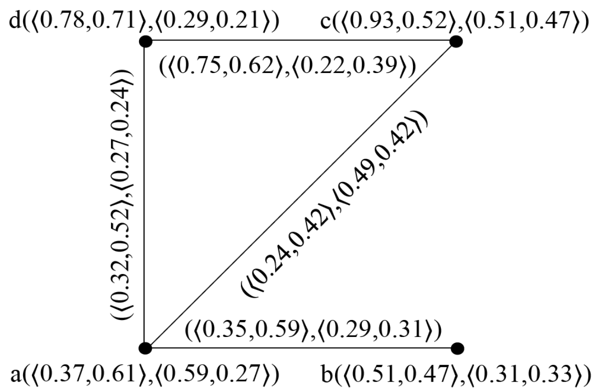

Figure 1. LDF-graph.

(ii) Using formula given by Eq 3.1 the order of LDF-graph $\mathbb{G}$ *is*

$$O(\mathbb{G}) = (\langle 2.59, 2.31 \rangle, \langle 1.70, 1.28 \rangle).$$

(iii) Using formula given by Eq 3.2 the degree of each vertex in LDF-graph $\mathbb{G}$ *is*

$$\begin{aligned} \deg(a) &= (\langle 0.91, 1.53 \rangle, \langle 1.05, 0.97 \rangle), \\ \deg(b) &= (\langle 0.35, 0.59 \rangle, \langle 0.29, 0.31 \rangle), \\ \deg(c) &= (\langle 0.99, 1.04 \rangle, \langle 0.71, 0.81 \rangle), \\ \deg(d) &= (\langle 1.07, 1.14 \rangle, \langle 0.49, 0.63 \rangle). \end{aligned}$$

Definition 3.3. *The cartesian product* $\mathbb{G}_1 \times \mathbb{G}_2$ *of two LDF-graphs is defined as a pair* $\mathbb{G}_1 \times \mathbb{G}_2 = (\pounds_{\mathfrak{R}_1} \times \pounds_{\mathfrak{R}_2}, \pounds_{\mathfrak{P}_1} \times \pounds_{\mathfrak{P}_2})$*, such that:*

1) $\mathfrak{M}^{\tau}_{\mathfrak{R}_1\times\mathfrak{R}_2}(w_1, w_2) = \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}_1}(w_1), \mathfrak{M}^{\tau}_{\mathfrak{R}_2}(w_2)\}$, $\mathfrak{N}^{\nu}_{\mathfrak{R}_1\times\mathfrak{R}_2}(w_1, w_2) = \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}_1}(w_1), \mathfrak{N}^{\nu}_{\mathfrak{R}_2}(w_2)\}$,
$(\alpha_1 \times \alpha_2)^{(w_1,w_2)} = \min\{\alpha_1^{w_1}, \alpha_2^{w_2}\}$, $(\beta_1 \times \beta_2)^{(w_1,w_2)} = \max\{\beta_1^{w_1}, \beta_2^{w_2}\}$,
for all $w_1, w_2 \in W$,

2) $\mathfrak{M}^{\tau}_{\mathfrak{P}_1\times\mathfrak{P}_2}((w, w_2)(w, t_2)) = \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}_1}(w), \mathfrak{M}^{\tau}_{\mathfrak{P}_2}(w_2t_2)\}$, $\mathfrak{N}^{\nu}_{\mathfrak{P}_1\times\mathfrak{P}_2}((w, w_2)(w, t_2)) = \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}_1}(w), \mathfrak{N}^{\nu}_{\mathfrak{P}_2}(w_2t_2)\}$,
$(\gamma_1 \times \gamma_2)^{((w,w_2)(w,t_2))} = \min\{\alpha_1^{w}, \gamma_2^{w_2t_2}\}$, $(\delta_1 \times \delta_2)^{((w,w_2)(w,t_2))} = \max\{\beta_1^{w}, \delta_2^{w_2t_2}\}$,
for all $w \in W_1$*, and* $w_2t_2 \in E_2$,

3) $\mathfrak{M}^{\tau}_{\mathfrak{P}_1\times\mathfrak{P}_2}((w_1, t)(t_1, t)) = \min\{\mathfrak{M}^{\tau}_{\mathfrak{P}_1}(w_1t_1), \mathfrak{M}^{\tau}_{\mathfrak{R}_2}(t)\}$, $\mathfrak{N}^{\nu}_{\mathfrak{P}_1\times\mathfrak{P}_2}((w_1, t)(t_1, t)) = \max\{\mathfrak{N}^{\nu}_{\mathfrak{P}_1}(w_1t_1), \mathfrak{N}^{\nu}_{\mathfrak{R}_2}(t)\}$,
$(\gamma_1 \times \gamma_2)^{((w_1,t)(t_1,t))} = \min\{\gamma_1^{w_1t_1}, \alpha_2^{t}\}$, $(\delta_1 \times \delta_2)^{((w_1,t)(t_1,t))} = \max\{\delta_1^{w_1t_1}, \beta_2^{t}\}$,
for all $t \in W_2$*, and* $w_1t_1 \in E_1$.

Definition 3.4. *Let* $\mathbb{G}_1$ *and* $\mathbb{G}_2$ *be two LDF-graphs. Then the degree of vertex in* $\mathbb{G}_1 \times \mathbb{G}_2$ *is described as follows: for any* $(w_1, w_2) \in W_1 \times W_2$,

$$d_{\mathbb{G}_1\times\mathbb{G}_2}(w_1, w_2) = \begin{pmatrix} \left\langle \sum\limits_{(w_1,w_2)(t_1,t_2)\in E} \mathfrak{M}^{\tau}_{\mathfrak{P}_1\times\mathfrak{P}_2}((w_1, w_2)(t_1, t_2)), \sum\limits_{(w_1,w_2)(t_1,t_2)\in E} \mathfrak{N}^{\nu}_{\mathfrak{P}_1\times\mathfrak{P}_2}((w_1, w_2)(t_1, t_2)) \right\rangle, \\ \left\langle \sum\limits_{(w_1,w_2)(t_1,t_2)\in E} (\gamma_1 \times \gamma_2)^{((w_1,w_2)(t_1,t_2))}, \sum\limits_{(w_1,w_2)(t_1,t_2)\in E} (\delta_1 \times \delta_2)^{((w_1,w_2)(t_1,t_2))} \right\rangle \end{pmatrix}.$$

Example 3.2. *Consider the two LDF-graphs* $\mathbb{G}_1$ *and* $\mathbb{G}_2$*, as shown in Figures 2 and 3.*

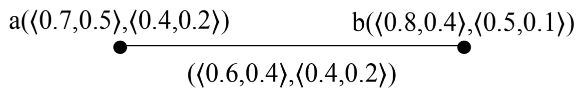

Figure 2. LDF-graph $\mathbb{G}_1$.

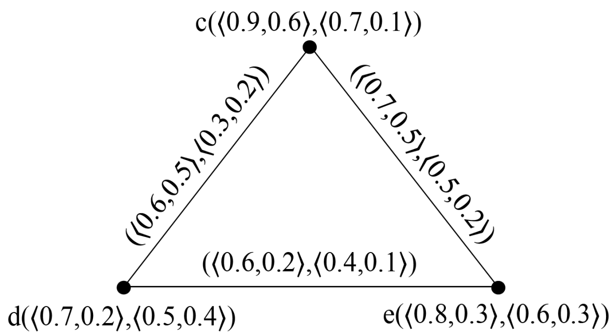

Figure 3. LDF-graph $\mathbb{G}_2$.

Then, their cartesian product $\mathbb{G}_1 \times \mathbb{G}_2$ *is described in Figure 4.*

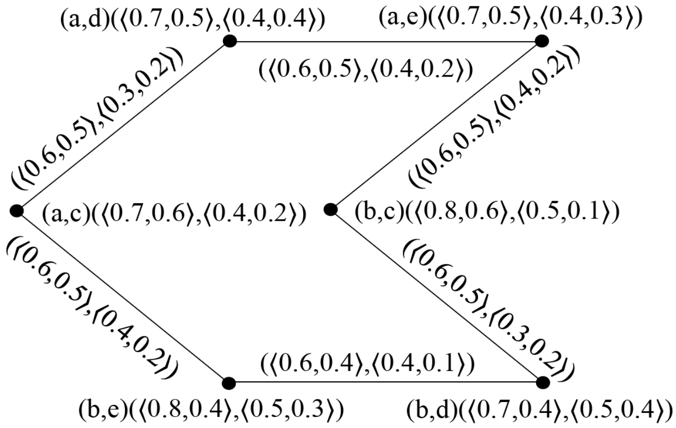

Figure 4. LDF-graph $\mathbb{G}_1 \times \mathbb{G}_2$.

Proposition 3.1. *The cartesian product of two LDF-graphs is a LDF-graph.*

Proof. The conditions for $\pounds_{\mathfrak{R}_1} \times \pounds_{\mathfrak{R}_2}$ are obvious, therefore, we investigate only conditions for $\pounds_{\mathfrak{P}_1} \times \pounds_{\mathfrak{P}_2}$. Let $w \in W_1$, and $w_2t_2 \in E_2$. Then

$$\begin{aligned}
\mathfrak{M}^{\tau}_{\mathfrak{P}_1\times\mathfrak{P}_2}((w,w_2)(w,t_2)) &= \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}_1}(w), \mathfrak{M}^{\tau}_{\mathfrak{P}_2}(w_2t_2)\} \\
&\leq \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}_1}(w), \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}_2}(w_2), \mathfrak{M}^{\tau}_{\mathfrak{R}_2}(t_2)\}\} \\
&= \min\{\min\{\mathfrak{M}^{\tau}_{\mathfrak{R}_1}(w), \mathfrak{M}^{\tau}_{\mathfrak{D}_2}(w_2)\}, \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}_1}(w), \mathfrak{M}^{\tau}_{\mathfrak{R}_2}(t_2)\}\} \\
&= \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}_1\times\mathfrak{R}_2}(w,w_2), \mathfrak{M}^{\tau}_{\mathfrak{R}_1\times\mathfrak{R}_2}(w,t_2)\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{N}^{\nu}_{\mathfrak{P}_1\times\mathfrak{P}_2}((w,w_2)(w,t_2)) &= \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}_1}(w), \mathfrak{N}^{\nu}_{\mathfrak{P}_2}(w_2t_2)\} \\
&\leq \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}_1}(w), \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}_2}(w_2), \mathfrak{N}^{\nu}_{\mathfrak{R}_2}(t_2)\}\} \\
&= \max\{\max\{\mathfrak{N}^{\nu}_{\mathfrak{R}_1}(w), \mathfrak{N}^{\nu}_{\mathfrak{D}_2}(w_2)\}, \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}_1}(w), \mathfrak{N}^{\nu}_{\mathfrak{R}_2}(t_2)\}\} \\
&= \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}_1\times\mathfrak{R}_2}(w,w_2), \mathfrak{N}^{\nu}_{\mathfrak{R}_1\times\mathfrak{R}_2}(w,t_2)\},
\end{aligned}$$

$$\begin{aligned}
(\gamma_1 \times \gamma_2)^{((w,w_2)(w,t_2))} &= \min\{\alpha_1^{w}, \gamma_2^{w_2t_2}\} \\
&\leq \min\{\alpha_1^{w}, \min\{\alpha_2^{w_2}, \alpha_2^{t_2}\}\} \\
&= \min\{\min\{\alpha_1^{w}, \alpha_2^{w_2}\}, \min\{\alpha_1^{w}, \alpha_2^{t_2}\}\} \\
&= \min\{(\alpha_1 \times \alpha_2)^{(w,w_2)}, (\alpha_1 \times \alpha_2)^{(w,t_2)}\},
\end{aligned}$$

$$\begin{aligned}
(\delta_1 \times \delta_2)^{((w,w_2)(w,t_2))} &= \max\{\beta_1^{w}, \delta_2^{w_2t_2}\} \\
&\leq \max\{\beta_1^{w}, \max\{\beta_2^{w_2}, \beta_2^{t_2}\}\} \\
&= \max\{\max\{\beta_1^{w}, \beta_2^{w_2}\}, \max\{\beta_1^{w}, \beta_2^{t_2}\}\} \\
&= \max\{(\beta_1 \times \beta_2)^{(w,w_2)}, (\beta_1 \times \beta_2)^{(w,t_2)}\}.
\end{aligned}$$

Likewise, we can verify it for $t \in W_2$, and $w_1t_1 \in E_1$. □

Definition 3.5. *The composition $\mathbb{G}_1 \circ \mathbb{G}_2$ of two LDF-graphs is described as a pair $\mathbb{G}_1 \circ \mathbb{G}_2 = (\pounds_{\mathfrak{R}_1} \circ \pounds_{\mathfrak{R}_2}, \pounds_{\mathfrak{P}_1} \circ \pounds_{\mathfrak{P}_2})$, such that:*

1) $\mathfrak{M}^{\tau}_{\mathfrak{R}_1 \circ \mathfrak{R}_2}(w_1, w_2) = \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}_1}(w_1), \mathfrak{M}^{\tau}_{\mathfrak{R}_2}(w_2)\}$, $\mathfrak{N}^{\nu}_{\mathfrak{R}_1 \circ \mathfrak{R}_2}(w_1, w_2) = \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}_1}(w_1), \mathfrak{N}^{\nu}_{\mathfrak{R}_2}(w_2)\}$,
   $(\alpha_1 \circ \alpha_2)^{(w_1,w_2)} = \min\{\alpha_1^{w_1}, \alpha_2^{w_2}\}$, $(\beta_1 \circ \beta_2)^{(w_1,w_2)} = \max\{\beta_1^{w_1}, \beta_2^{w_2}\}$, *for all* $w_1, w_2 \in W$,
2) $\mathfrak{M}^{\tau}_{\mathfrak{P}_1 \circ \mathfrak{P}_2}((w, w_2)(w, t_2)) = \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}_1}(w), \mathfrak{M}^{\tau}_{\mathfrak{P}_2}(w_2 t_2)\}$, $\mathfrak{N}^{\nu}_{\mathfrak{P}_1 \circ \mathfrak{P}_2}((w, w_2)(w, t_2)) = \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}_1}(w), \mathfrak{N}^{\nu}_{\mathfrak{P}_2}(w_2 t_2)\}$,
   $(\gamma_1 \circ \gamma_2)^{((w,w_2)(w,t_2))} = \min\{\alpha_1^{w}, \gamma_2^{w_2 t_2}\}$, $(\delta_1 \circ \delta_2)^{((w,w_2)(w,t_2))} = \max\{\beta_1^{w}, \delta_2^{w_2 t_2}\}$,
   for all $w \in W_1$, *and* $w_2 t_2 \in E_2$,
3) $\mathfrak{M}^{\tau}_{\mathfrak{P}_1 \circ \mathfrak{P}_2}((w_1, t)(t_1, t)) = \min\{\mathfrak{M}^{\tau}_{\mathfrak{P}_1}(w_1 t_1), \mathfrak{M}^{\tau}_{\mathfrak{R}_2}(t)\}$, $\mathfrak{N}^{\nu}_{\mathfrak{P}_1 \circ \mathfrak{P}_2}((w_1, t)(t_1, t)) = \max\{\mathfrak{N}^{\nu}_{\mathfrak{P}_1}(w_1 t_1), \mathfrak{N}^{\nu}_{\mathfrak{R}_2}(t)\}$,
   $(\gamma_1 \circ \gamma_2)^{((w_1,t)(t_1,t))} = \min\{\gamma_1^{w_1 t_1}, \alpha_2^{t}\}$, $(\delta_1 \circ \delta_2)^{((w_1,t)(t_1,t))} = \max\{\delta_1^{w_1 t_1}, \beta_2^{t}\}$,
   for all $t \in W_2$, *and* $w_1 t_1 \in E_1$.
4) $\mathfrak{M}^{\tau}_{\mathfrak{P}_1 \circ \mathfrak{P}_2}((w_1, w_2)(t_1, t_2)) = \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}_2}(w_2), \mathfrak{M}^{\tau}_{\mathfrak{R}_2}(t_2), \mathfrak{M}^{\tau}_{\mathfrak{P}_1}(w_1 t_1)\}$
   $\mathfrak{N}^{\nu}_{\mathfrak{P}_1 \circ \mathfrak{P}_2}((w_1, w_2)(t_1, t_2)) = \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}_2}(w_2), \mathfrak{N}^{\nu}_{\mathfrak{R}_2}(t_2), \mathfrak{N}^{\nu}_{\mathfrak{P}_1}(w_1 t_1)\}$
   $(\gamma_1 \circ \gamma_2)^{((w_1,w_2)(t_1,t_2))} = \min\{\alpha_2^{w_2}, \alpha_2^{t_2}, \gamma_1^{w_1 t_1}\}$
   $(\delta_1 \circ \delta_2)^{((w_1,w_2)(t_1,t_2))} = \max\{\beta_2^{w_2}, \beta_2^{t_2}, \delta_1^{w_1 t_1}\}$
   for all $w_2, t_2 \in W_2$, $w_2 \neq t_2$ *and* $w_1 t_1 \in E_1$.

Definition 3.6. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two LDF-graphs. The degree of a vertex in $\mathbb{G}_1 \circ \mathbb{G}_2$ can be described as follows: for any $(w_1, w_2) \in W_1 \times W_2$,*

$$d_{\mathbb{G}_1 \circ \mathbb{G}_2}(w_1, w_2) = \left(\begin{array}{c} \left\langle \sum\limits_{(w_1,w_2)(t_1,t_2) \in E} \mathfrak{M}^{\tau}_{\mathfrak{P}_1 \circ \mathfrak{P}_2}((w_1, w_2)(t_1, t_2)), \sum\limits_{(w_1,w_2)(t_1,t_2) \in E} \mathfrak{N}^{\nu}_{\mathfrak{P}_1 \circ \mathfrak{P}_2}((w_1, w_2)(t_1, t_2)) \right\rangle, \\ \left\langle \sum\limits_{(w_1,w_2)(t_1,t_2) \in E} (\gamma_1 \circ \gamma_2)^{((w_1,w_2)(t_1,t_2))}, \sum\limits_{(w_1,w_2)(t_1,t_2) \in E} (\delta_1 \circ \delta_2)^{((w_1,w_2)(t_1,t_2))} \right\rangle \end{array} \right).$$

Example 3.3. *Consider the two LDF-graphs, as shown in Figure 5.*

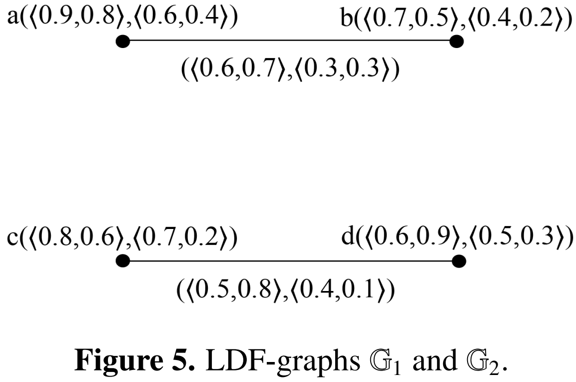

Figure 5. LDF-graphs $\mathbb{G}_1$ and $\mathbb{G}_2$.

Then, their composition $\mathbb{G}_1 \circ \mathbb{G}_2$ is shown in Figure 6.

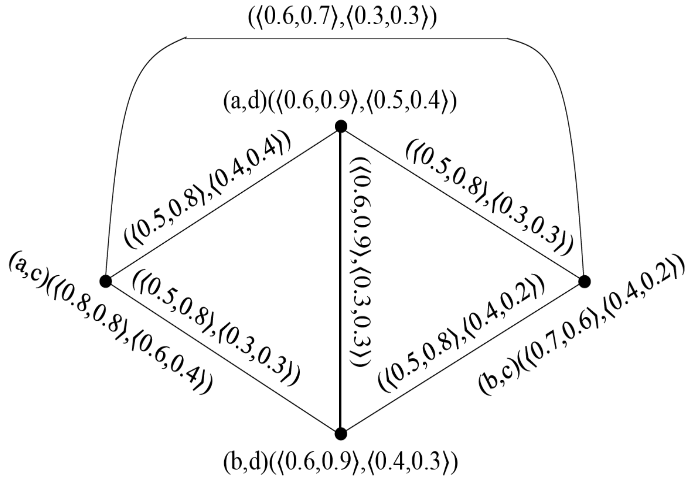

Figure 6. LDF-graph $\mathbb{G}_1 \circ \mathbb{G}_2$.

Proposition 3.2. *The composition of two LDF-graphs is a LDF-graph.*

Definition 3.7. *The union $\mathbb{G}_1 \cup \mathbb{G}_2 = (\pounds_{\Re_1} \cup \pounds_{\Re_2}, \pounds_{\mathfrak{P}_1} \cup \pounds_{\mathfrak{P}_2})$ of two LDF-graphs is defined as follows:*

1) $\mathfrak{M}^{\tau}_{\Re_1\cup\Re_2}(w) = \mathfrak{M}^{\tau}_{\Re_1}(w)$, $\mathfrak{N}^{\nu}_{\Re_1\cup\Re_2}(w) = \mathfrak{N}^{\nu}_{\Re_1}(w)$,
   $(\alpha_1 \cup \alpha_2)^w = \alpha_1^w$, $(\beta_1 \cup \beta_2)^w = \beta_1^w$,
   for $w \in W_1$ and $w \notin W_2$.
2) $\mathfrak{M}^{\tau}_{\Re_1\cup\Re_2}(w) = \mathfrak{M}^{\tau}_{\Re_2}(w)$, $\mathfrak{N}^{\nu}_{\Re_1\cup\Re_2}(w) = \mathfrak{N}^{\nu}_{\Re_2}(w)$,
   $(\alpha_1 \cup \alpha_2)^w = \alpha_2^w$, $(\beta_1 \cup \beta_2)^w = \beta_2^w$,
   for $w \in W_2$ and $w \notin W_1$.
3) $\mathfrak{M}^{\tau}_{\Re_1\cup\Re_2}(w) = \max\{\mathfrak{M}^{\tau}_{\Re_1}(w), \mathfrak{M}^{\tau}_{\Re_2}(w)\}$, $\mathfrak{N}^{\nu}_{\Re_1\cup\Re_2}(w) = \min\{\mathfrak{N}^{\nu}_{\Re_1}(w), \mathfrak{N}^{\nu}_{\Re_2}(w)\}$,
   $(\alpha_1 \cup \alpha_2)^w = \max\{\alpha_1^w, \alpha_2^w\}$, $(\beta_1 \cup \beta_2)^w = \min\{\beta_1^w, \beta_2^w\}$,
   for $w \in W_1 \cap W_2$.
4) $\mathfrak{M}^{\tau}_{\mathfrak{P}_1\cup\mathfrak{P}_2}(wt) = \mathfrak{M}^{\tau}_{\mathfrak{P}_1}(wt)$, $\mathfrak{N}^{\nu}_{\mathfrak{P}_1\cup\mathfrak{P}_2}(wt) = \mathfrak{N}^{\nu}_{\mathfrak{P}_1}(wt)$,
   $(\gamma_1 \cup \gamma_2)^{wt} = \gamma_1^{wt}$, $(\beta_1 \cup \beta_2)^{wt} = \beta_1^{wt}$, *for $wt \in E_1$ and $wt \notin E_2$.*
5) $\mathfrak{M}^{\tau}_{\mathfrak{P}_1\cup\mathfrak{P}_2}(wt) = \mathfrak{M}^{\tau}_{\mathfrak{P}_2}(wt)$, $\mathfrak{N}^{\nu}_{\mathfrak{P}_1\cup\mathfrak{P}_2}(wt) = \mathfrak{N}^{\nu}_{\mathfrak{P}_2}(wt)$,
   $(\gamma_1 \cup \gamma_2)^{wt} = \gamma_2^{wt}$, $(\beta_1 \cup \beta_2)^{wt} = \beta_2^{wt}$,
   for $wt \in E_2$ and $wt \notin E_1$.
6) $\mathfrak{M}^{\tau}_{\mathfrak{P}_1\cup\mathfrak{P}_2}(wt) = \max\{\mathfrak{M}^{\tau}_{\mathfrak{P}_1}(wt), \mathfrak{M}^{\tau}_{\mathfrak{P}_2}(wt)\}$, $\mathfrak{N}^{\nu}_{\mathfrak{P}_1\cup\mathfrak{P}_2}(wt) = \min\{\mathfrak{N}^{\nu}_{\mathfrak{P}_1}(wt), \mathfrak{N}^{\nu}_{\mathfrak{P}_2}(wt)\}$,
   $(\gamma_1 \cup \gamma_2)^{wt} = \max\{\gamma_1^{wt}, \gamma_2^{wt}\}$, $(\beta_1 \cup \beta_2)^{wt} = \min\{\beta_1^{wt}, \beta_2^{wt}\}$,
   for $wt \in E_1 \cap E_2$.

Example 3.4. *Consider the two LDF-graphs, as shown in Figures 7 and 8.*

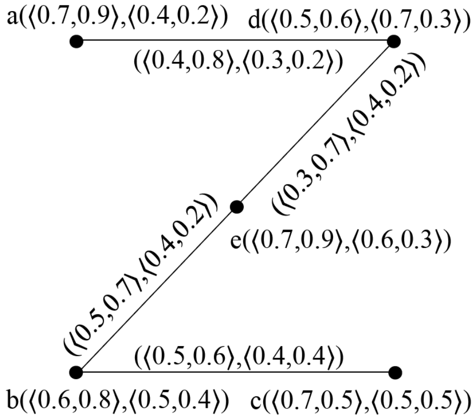

Figure 7. LDF-graph $\mathbb{G}_1$.

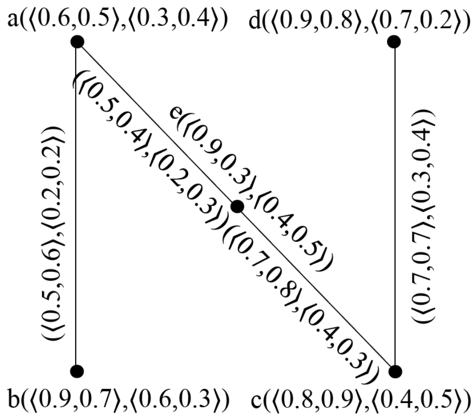

Figure 8. LDF-graph $\mathbb{G}_2$.

Then, their corresponding union $\mathbb{G}_1 \cup \mathbb{G}_2$ *is shown in Figure 9.*

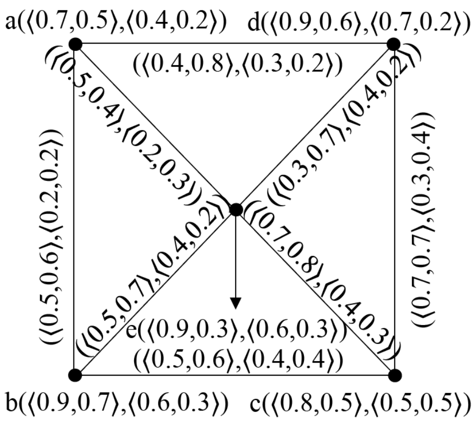

Figure 9. LDF-graph $\mathbb{G}_1 \cup \mathbb{G}_2$.

Proposition 3.3. *The union of two LDF-graphs is a LDF-graph.*

Definition 3.8. *The join* $\mathbb{G}_1 + \mathbb{G}_2 = (\pounds_{\mathfrak{R}_1} + \pounds_{\mathfrak{R}_2}, \pounds_{\mathfrak{P}_1} + \pounds_{\mathfrak{P}_2})$ *of two LDF-graphs, where,* $W_1 \cap W_2 = \emptyset$*, is defined as follows:*

$$\textit{1) } \begin{cases} \mathfrak{M}^{\tau}_{\mathfrak{R}_1+\mathfrak{R}_2}(w) = \mathfrak{M}^{\tau}_{\mathfrak{D}_1\cup\mathfrak{R}_2}(w),\ \mathfrak{N}^{\nu}_{\mathfrak{R}_1+\mathfrak{R}_2}(w) = \mathfrak{N}^{\nu}_{\mathfrak{R}_1\cup\mathfrak{R}_2}(w) \\ (\alpha_1 + \alpha_2)^w = (\alpha_1 \cup \alpha_2)^w,\ (\beta_1 + \beta_2)^w = (\beta_1 \cup \beta_2)^w \end{cases} \quad \textit{if } w \in W_1 \cup W_2,$$

$$\textit{2) } \begin{cases} \mathfrak{M}^{\tau}_{\mathfrak{P}_1+\mathfrak{P}_2}(wt) = \mathfrak{M}^{\tau}_{\mathfrak{P}_1\cup\mathfrak{P}_2}(wt),\ \mathfrak{N}^{\nu}_{\mathfrak{P}_1+\mathfrak{P}_2}(wt) = \mathfrak{N}^{\nu}_{\mathfrak{P}_1\cup\mathfrak{P}_2}(wt) \\ (\gamma_1 + \gamma_2)^{wt} = (\gamma_1 \cup \gamma_2)^{wt},\ (\delta_1 + \delta_2)^{wt} = (\delta_1 \cup \delta_2)^{wt} \end{cases} \quad \textit{if } wt \in E_1 \cup E_2,$$

$$\textit{3) } \begin{cases} \mathfrak{M}^{\tau}_{\mathfrak{P}_1+\mathfrak{P}_2}(wt) = \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}_1}(w), \mathfrak{M}^{\tau}_{\mathfrak{R}_2}(t)\}, \\ \mathfrak{N}^{\nu}_{\mathfrak{P}_1+\mathfrak{P}_2}(wt) = \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}_1}(w), \mathfrak{N}^{\nu}_{\mathfrak{R}_2}(t)\} \\ (\gamma_1 + \gamma_2)^{wt} = \min\{\alpha_1^w, \alpha_2^t\},\ (\delta_1 + \delta_2)^{wt} = \max\{\beta_1^w, \beta_2^t\} \end{cases} \quad \textit{if } wt \in \acute{E},$$

where $\acute{E}$ *is the set of all edges joining the vertices of* W_1 *and* W_2*.*

Example 3.5. *Consider the two LDF-graphs, as shown in Figure 10.*

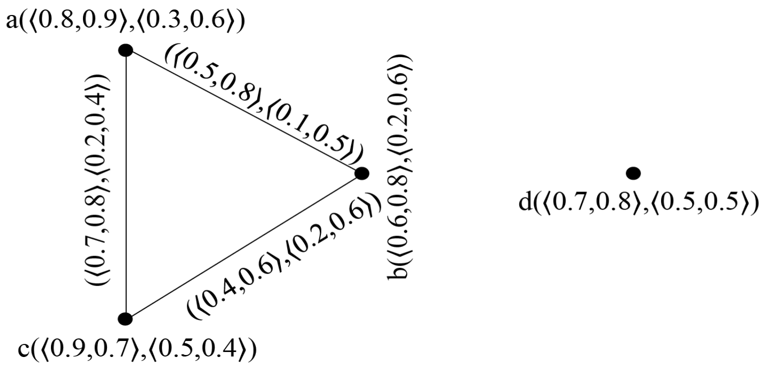

Figure 10. LDF-graphs $\mathbb{G}_1$ and $\mathbb{G}_2$.

Then, their corresponding join $\mathbb{G}_1 + \mathbb{G}_2$ *is shown in Figure 11.*

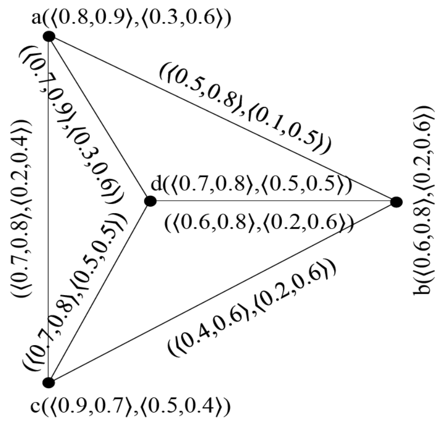

Figure 11. LDF-graph $\mathbb{G}_1 + \mathbb{G}_2$.

Proposition 3.4. *The join of two LDF-graphs is a LDF-graph.*

Proposition 3.5. *Let $\mathbb{G}_1 = (\pounds_{\mathfrak{R}_1}, \pounds_{\mathfrak{P}_1})$ and $\mathbb{G}_2 = (\pounds_{\mathfrak{R}_2}, \pounds_{\mathfrak{P}_2})$ be LDF-graphs and let $W_1 \cap W_2 = \emptyset$. Then union $\mathbb{G}_1 \cup \mathbb{G}_2 = (\pounds_{\mathfrak{R}_1} \cup \pounds_{\mathfrak{D}_2}, \pounds_{\mathfrak{P}_1} \cup \pounds_{\mathfrak{P}_2})$ is a LDF-graph if and only if $\mathbb{G}_1$ and $\mathbb{G}_2$ are LDF-graphs, respectively.*

Proof. Suppose that $\mathbb{G}_1 \cup \mathbb{G}_2$ is a LDF-graph. Let $wt \in E_1$. Then $wt \notin E_2$ and $w, t \in W_1 - W_2$. Thus

$$\begin{aligned}
\mathfrak{M}^{\tau}_{\mathfrak{P}_1}(wt) &= \mathfrak{M}^{\tau}_{\mathfrak{P}_1 \cap \mathfrak{P}_2}(wt) \\
&\leq \min(\mathfrak{M}^{\tau}_{\mathfrak{R}_1 \cap \mathfrak{R}_2}(w), \mathfrak{M}^{\tau}_{\mathfrak{R}_1 \cap \mathfrak{R}_2}(t)) \\
&= \min(\mathfrak{M}^{\tau}_{\mathfrak{R}_1}(w), \mathfrak{M}^{\tau}_{\mathfrak{R}_1}(t)).
\end{aligned}$$

$$\begin{aligned}
\mathfrak{N}^{\nu}_{\mathfrak{P}_1}(wt) &= \mathfrak{N}^{\nu}_{\mathfrak{P}_1 \cap \mathfrak{P}_2}(wt) \\
&\leq \max(\mathfrak{N}^{\nu}_{\mathfrak{R}_1 \cap \mathfrak{R}_2}(w), \mathfrak{N}^{\nu}_{\mathfrak{R}_1 \cap \mathfrak{R}_2}(t)) \\
&= \max(\mathfrak{N}^{\nu}_{\mathfrak{R}_1}(w), \mathfrak{N}^{\nu}_{\mathfrak{R}_1}(t)).
\end{aligned}$$

$$\begin{aligned}
\gamma_1^{wt} &= (\gamma_1 \cap \gamma_2)^{wt} \\
&\leq \min((\alpha_1 \cap \alpha_2)^{w}, (\alpha_1 \cap \alpha_2)^{t}) \\
&= \min(\alpha_1^{w}, \alpha_2^{t}).
\end{aligned}$$

$$\begin{aligned}
\delta_1^{wt} &= (\delta_1 \cap \delta_2)^{wt} \\
&\leq \max((\beta_1 \cap \beta_2)^{w}, (\beta_1 \cap \beta_2)^{t}) \\
&= \max(\beta_1^{w}, \beta_2^{t}).
\end{aligned}$$

This shows that $\mathbb{G}_1 = (\pounds_{\mathfrak{R}_1}, \pounds_{\mathfrak{P}_1})$ is a LDF-graph. Similarly, we can show that $\mathbb{G}_2 = (\pounds_{\mathfrak{R}_2}, \pounds_{\mathfrak{P}_2})$ is a LDF-graph. The converse part is obvious. □

Proposition 3.6. *Let $\mathbb{G}_1 = (\pounds_{\mathfrak{R}_1}, \pounds_{\mathfrak{P}_1})$ and $\mathbb{G}_2 = (\pounds_{\mathfrak{R}_2}, \pounds_{\mathfrak{P}_2})$ be LDF-graphs and let $W_1 \cap W_2 = \emptyset$. Then join $\mathbb{G}_1 + \mathbb{G}_2 = (\pounds_{\mathfrak{R}_1} + \pounds_{\mathfrak{R}_2}, \pounds_{\mathfrak{P}_1} + \pounds_{\mathfrak{P}_2})$ is a LDF-graph if and only if $\mathbb{G}_1$ and $\mathbb{G}_2$ are LDF-graphs, respectively.*

Proof. The proof is obvious and identical with a proof of Proposition 3.5. □

4. Isomorphisms of LDF-graphs

Now we introduce the idea of isomorphisms of LDF-graphs.

Definition 4.1. *Let $\mathbb{G}_1 = (\pounds_{\mathfrak{R}_1}, \pounds_{\mathfrak{P}_1})$ and $\mathbb{G}_2 = (\pounds_{\mathfrak{R}_2}, \pounds_{\mathfrak{P}_2})$ be two LDF-graphs. A homomorphism $g : \mathbb{G}_1 \to \mathbb{G}_2$ is a mapping $g : W_1 \to W_2$ such that:*

1) $\begin{cases} \mathfrak{M}^{\tau}_{\mathfrak{R}_1}(w_1) \leq \mathfrak{M}^{\tau}_{\mathfrak{R}_2}(g(w_1)),\ \mathfrak{N}^{\nu}_{\mathfrak{R}_1}(w_1) \leq \mathfrak{N}^{\nu}_{\mathfrak{R}_2}(g(w_1)) \\ \alpha_1^{w_1} \leq \alpha_2^{g(w_1)},\ \beta_1^{w_1} \leq \beta_2^{g(w_1)} \end{cases}$ *for all* $w_1 \in W_1$,

2) $\begin{cases} \mathfrak{M}^{\tau}_{\mathfrak{P}_1}(w_1t_1) \leq \mathfrak{M}^{\tau}_{\mathfrak{P}_2}(g(w_1)g(t_1)),\ \mathfrak{N}^{\nu}_{\mathfrak{P}_1}(w_1t_1) \leq \mathfrak{N}^{\nu}_{\mathfrak{P}_2}(g(w_1)g(t_1)) \\ \gamma_1^{w_1t_1} \leq \gamma_2^{g(w_1)g(t_1)},\ \delta_1^{w_1t_1} \leq \delta_2^{g(w_1)g(t_1)} \end{cases}$ *for all* $w_1t_1 \in E_1$.

A bijective homomorphism with the property.

3) $\begin{cases} \mathfrak{M}^{\tau}_{\mathfrak{R}_1}(w_1) = \mathfrak{M}^{\tau}_{\mathfrak{R}_2}(g(w_1)),\ \mathfrak{N}^{\nu}_{\mathfrak{R}_1}(w_1) = \mathfrak{N}^{\nu}_{\mathfrak{R}_2}(g(w_1)) \\ \alpha_1^{w_1} = \alpha_2^{g(w_1)},\ \beta_1^{w_1} = \beta_2^{g(w_1)} \end{cases}$ *for all* $w_1 \in W_1$,

is called a weak isomorphism. A bijective homomorphism $g : \mathbb{G}_1 \to \mathbb{G}_2$ *such that:*

4) $\begin{cases} \mathfrak{M}^{\tau}_{\mathfrak{P}_1}(w_1t_1) = \mathfrak{M}^{\tau}_{\mathfrak{P}_2}(g(w_1)g(t_1)),\ \mathfrak{N}^{\nu}_{\mathfrak{P}_1}(w_1t_1) = \mathfrak{N}^{\nu}_{\mathfrak{P}_2}(g(w_1)g(t_1)) \\ \gamma_1^{w_1t_1} = \gamma_2^{g(w_1)g(t_1)},\ \delta_1^{w_1t_1} = \delta_2^{g(w_1)g(t_1)} \end{cases}$ *for all* $w_1t_1 \in E_1$,

is called a strong co-isomorphism. A bijective mapping $g : \mathbb{G}_1 \to \mathbb{G}_2$ *satisfying* 3) *and* 4) *is called an isomorphism.*

Example 4.1. *Consider two LDF-graphs, as shown in Figure 12.*

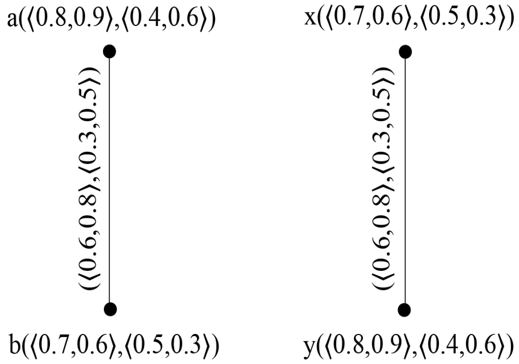

Figure 12. LDF-graphs $\mathbb{G}_1$ and $\mathbb{G}_2$.

Then, it is easy to see that the mapping $g : W_1 \to W_2$ *defined by* $g(a) = y$ *and* $g(b) = x$ *is a strong co-isomorphism.*

Proposition 4.1. *An isomorphism between LDF-graphs is an equivalence relation.*

Proof. The reflexivity and symmetry are obvious. We know that the composition mapping $g_\lambda \circ h_\lambda : W_1 \to W_3$ is a bijective from W_1 to W_3, where $h_\lambda : W_1 \to W_2$ is the isomorphisms of $\mathbb{G}_1$ onto $\mathbb{G}_2$, and $g_\lambda : W_2 \to W_3$ be the isomorphisms $\mathbb{G}_2$ onto $\mathbb{G}_3$. For transitivity, we let $h_\lambda : W_1 \to W_2$ and $g_\lambda : W_2 \to W_3$ be the isomorphisms of $\mathbb{G}_1$ onto $\mathbb{G}_2$ and $\mathbb{G}_2$ onto $\mathbb{G}_3$, respectively. Since a map $h_\lambda : W_1 \to W_2$ defined by $h_\lambda\,(w_1) = w_2$ for $w_1 \in W_1$ is an isomorphism, so we have

$$\begin{aligned} \mathfrak{M}^{\tau}_{\mathfrak{R}_1}(w_1) &= \mathfrak{M}^{\tau}_{\mathfrak{R}_2}(h_\lambda(w_1)) \\ &= \mathfrak{M}^{\tau}_{\mathfrak{R}_2}(w_2) \text{ for all } w_1 \in W_1 \cdots (A_1), \end{aligned}$$

$$\begin{aligned} \mathfrak{N}^{\nu}_{\mathfrak{R}_1}(w_1) &= \mathfrak{N}^{\nu}_{\mathfrak{R}_2}(h_\lambda(w_1)) \\ &= \mathfrak{N}^{\nu}_{\mathfrak{R}_2}(w_2) \text{ for all } w_1 \in W_1 \cdots (A_2), \end{aligned}$$

$$\begin{aligned} \mathfrak{M}^{\tau}_{\mathfrak{P}_1}(w_1t_1) &= \mathfrak{M}^{\tau}_{\mathfrak{P}_2}(h_\lambda(w_1)h_\lambda(t_1)) \\ &= \mathfrak{M}^{\tau}_{\mathfrak{P}_2}(w_2t_2) \text{ for all } w_1t_1 \in E_1 \cdots (B_1). \end{aligned}$$

$$\mathfrak{N}^{\nu}_{\mathfrak{P}_1}(w_1t_1) = \mathfrak{N}^{\nu}_{\mathfrak{P}_2}(h_\lambda(w_1)h_\lambda(t_1))$$

$$= \mathfrak{N}^{\nu}_{\mathfrak{P}_2}(w_2t_2) \text{ for all } w_1t_1 \in E_1 \cdots (B_2).$$

Since a map $g_\lambda : W_2 \to W_3$ defined by $g_\lambda(w_2) = w_3$ for $w_2 \in W_2$ is an isomorphism, so

$$\begin{aligned} \mathfrak{M}^{\tau}_{\mathfrak{R}_2}(w_2) &= \mathfrak{M}^{\tau}_{\mathfrak{R}_3}(g_\lambda(w_2)) \\ &= \mathfrak{M}^{\tau}_{\mathfrak{R}_3}(w_3) \text{ for all } w_2 \in W_2 \cdots (C_1), \end{aligned}$$

$$\begin{aligned} \mathfrak{N}^{\nu}_{\mathfrak{R}_2}(w_2) &= \mathfrak{N}^{\nu}_{\mathfrak{R}_3}(g_\lambda(w_2)) \\ &= \mathfrak{N}^{\nu}_{\mathfrak{R}_3}(w_3) \text{ for all } w_2 \in W_2 \cdots (C_2), \end{aligned}$$

$$\begin{aligned} \mathfrak{M}^{\tau}_{\mathfrak{P}_2}(w_2t_2) &= \mathfrak{M}^{\tau}_{\mathfrak{P}_3}(g_\lambda(w_2)g_\lambda(t_2)) \\ &= \mathfrak{M}^{\tau}_{\mathfrak{P}_3}(w_3t_3) \text{ for all } w_2t_2 \in E_2 \cdots (D_1). \end{aligned}$$

$$\begin{aligned} \mathfrak{N}^{\nu}_{\mathfrak{P}_2}(w_2t_2) &= \mathfrak{N}^{\nu}_{\mathfrak{P}_3}(g_\lambda(w_2)g_\lambda(t_2)) \\ &= \mathfrak{N}^{\nu}_{\mathfrak{P}_3}(w_3t_3) \text{ for all } w_2t_2 \in E_2 \cdots (D_2). \end{aligned}$$

From (A_1), (C_1) and $h_\lambda\,(w_1) = w_2, w_1 \in W_1$, we have

$$\begin{aligned} \mathfrak{M}^{\tau}_{\mathfrak{R}_1}(w_1) &= \mathfrak{M}^{\tau}_{\mathfrak{R}_2}(h_\lambda(w_1)) = \mathfrak{M}^{\tau}_{\mathfrak{R}_2}(w_2) \\ &= \mathfrak{M}^{\tau}_{\mathfrak{R}_3}(g_\lambda(w_2)) \\ &= \mathfrak{M}^{\tau}_{\mathfrak{R}_3}(g_\lambda(h_\lambda(w_1))). \end{aligned}$$

From (A_2), (C_2) and $h_\lambda\,(w_1) = w_2$, $w_1 \in W_1$, we have

$$\begin{aligned} \mathfrak{N}^{\nu}_{\mathfrak{R}_1}(w_1) &= \mathfrak{N}^{\nu}_{\mathfrak{R}_2}(h_\lambda(w_1)) = \mathfrak{N}^{\nu}_{\mathfrak{R}_2}(w_2) \\ &= \mathfrak{N}^{\nu}_{\mathfrak{R}_3}(g_\lambda(w_2)) \\ &= \mathfrak{N}^{\nu}_{\mathfrak{R}_3}(g_\lambda(h_\lambda(w_1))). \end{aligned}$$

From (B_1) and (D_1) we have

$$\begin{aligned} \mathfrak{M}^{\tau}_{\mathfrak{P}_1}(w_1t_1) &= \mathfrak{M}^{\tau}_{\mathfrak{P}_2}(h_\lambda(w_1)h_\lambda(t_1)) = \mathfrak{M}^{\tau}_{\mathfrak{P}_2}(w_2t_2) \\ &= \mathfrak{M}^{\tau}_{\mathfrak{P}_3}(g_\lambda(w_2)g_\lambda(t_2)) \\ &= \mathfrak{M}^{\tau}_{\mathfrak{P}_3}(g_\lambda(h_\lambda(w_1))g_\lambda(h_\lambda(t_1))). \end{aligned}$$

From (B_2) and (D_2), we have

$$\begin{aligned} \mathfrak{N}^{\nu}_{\mathfrak{P}_1}(w_1t_1) &= \mathfrak{N}^{\nu}_{\mathfrak{P}_1}(h_\lambda(w_1)h_\lambda(t_1)) = \mathfrak{N}^{\nu}_{\mathfrak{P}_2}(w_2t_2) \\ &= \mathfrak{N}^{\nu}_{\mathfrak{P}_3}(g_\lambda(w_2)g_\lambda(t_2)) \\ &= \mathfrak{N}^{\nu}_{\mathfrak{P}_3}(g_\lambda(h_\lambda(w_1))g_\lambda(h_\lambda(t_1))), \end{aligned}$$

for all $w_1t_1 \in E_1$. Also,

$$\alpha_1^{w_1} = \alpha_2^{h_\lambda(w_1)} = \alpha_2^{w_2} \text{ for all } w_1 \in W_1 \cdots (AA_1),$$

$$\beta_1^{w_1} = \beta_2^{h_\lambda(w_1)} = \beta_2^{w_2} \text{ for all } w_1 \in W_1 \cdots (AA_2),$$
$$\gamma_1^{w_1t_1} = \gamma_2^{h_\lambda(w_1)h_\lambda(t_1)} = \gamma_2^{w_2t_2} \text{ for all } w_1t_1 \in E_1 \cdots (BB_1).$$
$$\delta_1^{w_1t_1} = \delta_2^{h_\lambda(w_1)h_\lambda(t_1)} = \delta_2^{w_2t_2} \text{ for all } w_1t_1 \in E_1 \cdots (BB_2).$$

Since mapping $g_\lambda : W_2 \to W_3$ is described by $g_\lambda(w_2) = w_3$ for $w_2 \in W_2$ is an isomorphism, so

$$\alpha_2^{w_2} = \alpha_3^{g_\lambda(w_2)} = \alpha_3^{w_3} \text{ for all } w_2 \in W_2 \cdots (CC_1),$$

$$\beta_2^{w_2} = \beta_3^{g_\lambda(w_2)} = \beta_3^{w_3} \text{ for all } w_2 \in W_2 \cdots (CC_2),$$
$$\gamma_2^{w_2t_2} = \gamma_3^{g_\lambda(w_2)g_\lambda(t_2)} = \gamma_3^{w_3t_3} \text{ for all } w_2t_2 \in E_2 \cdots (DD_1).$$
$$\delta_2^{w_2t_2} = \delta_3^{g_\lambda(w_2)g_\lambda(t_2)} = \delta_3^{w_3t_3} \text{ for all } w_2t_2 \in E_2 \cdots (DD_2).$$

From (AA_1), (CC_1) and $h_\lambda(w_1) = w_2$, $w_1 \in W_1$, we have

$$\begin{aligned} \alpha_1^{w_1} &= \alpha_2^{h_\lambda(w_1)} = \alpha_2^{w_2} = \alpha_3^{g_\lambda(w_2)} \\ &= \alpha_3^{g_\lambda(h_\lambda(w_1))}. \end{aligned}$$

From (AA_2), (CC_2) and $h_\lambda(w_1) = w_2$, $w_1 \in W_1$, we have

$$\begin{aligned} \beta_1^{w_1} &= \beta_2^{h_\lambda(w_1)} = \beta_2^{w_2} = \beta_3^{g_\lambda(w_2)} \\ &= \beta_3^{g_\lambda(h_\lambda(w_1))}. \end{aligned}$$

From (BB_1) and (DD_1) we have

$$\begin{aligned} \gamma_1^{w_1t_1} &= \gamma_2^{h_\lambda(w_1)h_\lambda(t_1)} = \gamma_2^{w_2t_2} = \gamma_3^{g_\lambda(w_2)g_\lambda(t_2)} \\ &= \gamma_3^{g_\lambda(h_\lambda(w_1))g_\lambda(h_\lambda(t_1))}. \end{aligned}$$

From (BB_2) and (DD_2), we have

$$\begin{aligned} \delta_1^{w_1t_1} &= \delta_1^{h_\lambda(w_1)h_\lambda(t_1)} = \delta_2^{w_2t_2} = \delta_3^{g_\lambda(w_2)g_\lambda(t_2)} \\ &= \delta_3^{g_\lambda(h_\lambda(w_1))g_\lambda(h_\lambda(t_1))}, \end{aligned}$$

for all $w_1t_1 \in E_1$. Therefore, $g_\lambda \circ h_\lambda$ is an isomorphism between $\mathbb{G}_1$ and $\mathbb{G}_3$. This completes the proof. □

5. Complement of LDF-graphs

Now we introduce the concept of complements of LDF-graphs.

Definition 5.1. *The complement of a LDF-graph $\mathbb{G} = (£_{\mathfrak{R}}, £_{\mathfrak{P}})$ is a LDF-graph $\overline{\mathbb{G}} = (\overline{£_{\mathfrak{R}}}, \overline{£_{\mathfrak{P}}})$, is defined by*

(i) $\overline{W} = W$,

(ii) $\begin{cases} \overline{\mathfrak{M}}^{\tau}_{\mathfrak{R}}(w) = \mathfrak{M}^{\tau}_{\mathfrak{R}}(w), \ \overline{\mathfrak{N}}^{\nu}_{\mathfrak{R}}(w) = \mathfrak{N}^{\nu}_{\mathfrak{D}}(w) \\ \overline{\alpha}^{w} = \alpha^{w}, \ \overline{\beta}^{w} = \beta^{w} \end{cases}$ *for all $w \in W$,*

$$(iii)\begin{cases} \overline{\mathfrak{M}}^{\tau}_{\mathfrak{P}}(wt) = \begin{cases} 0 & if\, \mathfrak{M}^{\tau}_{\mathfrak{P}}(wt) \neq 0 \\ \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}}(w), \mathfrak{M}^{\tau}_{\mathfrak{R}}(t)\} & if\, \mathfrak{M}^{\tau}_{\mathfrak{P}}(wt) = 0. \end{cases} \\ \overline{\mathfrak{N}}^{\nu}_{\mathfrak{P}}(wt) = \begin{cases} 0 & if\, \mathfrak{N}^{\nu}_{\mathfrak{P}}(wt) \neq 0 \\ \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}}(w), \mathfrak{N}^{\nu}_{\mathfrak{R}}(t)\} & if\, \mathfrak{N}^{\nu}_{\mathfrak{P}}(wt) = 0. \end{cases} \\ \overline{\gamma}^{wt} = \begin{cases} 0 & if\, \gamma^{wt} \neq 0 \\ \min\{\alpha^{w}, \alpha^{t}\} & if\, \gamma^{wt} = 0. \end{cases} \\ \overline{\delta}^{wt} = \begin{cases} 0 & if\, \delta^{wt} \neq 0 \\ \max\{\beta^{w}, \beta^{t}\} & if\, \delta^{wt} = 0. \end{cases} \end{cases}$$

Example 5.1. *Consider a LDF-graph $\mathbb{G}$, as shown in Figure 13.*

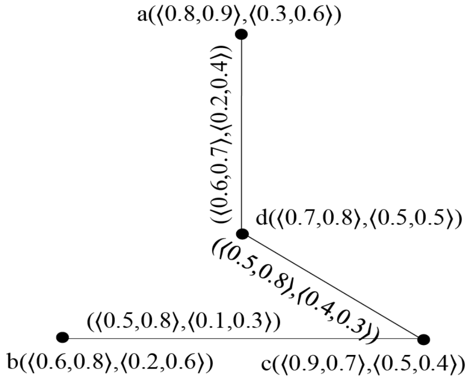

Figure 13. LDF-graph $\mathbb{G}$.

Then, the complement $\overline{\mathbb{G}}$ of $\mathbb{G}$ is shown in Figure 14.

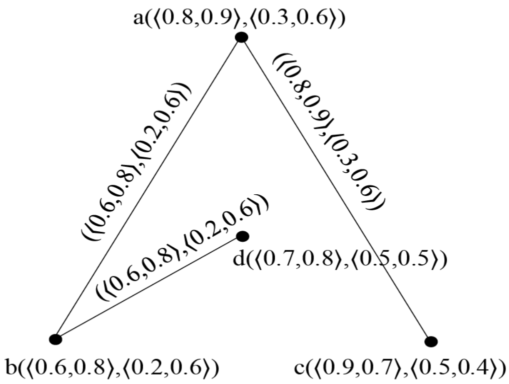

Figure 14. LDF-graph $\overline{\mathbb{G}}$.

Definition 5.2. *A LDF-graph G is called self complementary if $\overline{\mathbb{G}} \approx \mathbb{G}$.*

Proposition 5.1. *Let $\mathbb{G} = (\pounds_{\mathfrak{R}}, \pounds_{\mathfrak{P}})$ be a self complementary LDF-graph. Then*

$$\begin{aligned} \sum_{x\neq y} \mathfrak{M}^{\tau}_{\mathfrak{P}}(wt) &= \frac{1}{2}\sum_{x\neq y} \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}}(w), \mathfrak{M}_{\mathfrak{R}}(t)\},\ \sum_{x\neq y} \mathfrak{N}^{\nu}_{\mathfrak{P}}(wt) = \frac{1}{2}\sum_{x\neq y} \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}}(w), \mathfrak{N}^{\nu}_{\mathfrak{R}}(t)\}, \\ \sum_{x\neq y} \gamma^{wt} &= \frac{1}{2}\sum_{x\neq y} \min\{\alpha^{w}, \alpha^{t}\},\ \sum_{x\neq y} \delta^{wt} = \frac{1}{2}\sum_{x\neq y} \max\{\beta^{w}, \beta^{t}\}. \end{aligned}$$

Proposition 5.2. *Let $\mathbb{G} = (\pounds_{\mathfrak{R}}, \pounds_{\mathfrak{P}})$ be a LDF-graph. If*

$$\begin{aligned} \mathfrak{M}^{\tau}_{\mathfrak{P}}(wt) &= \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}}(w), \mathfrak{M}^{\tau}_{\mathfrak{R}}(t)\},\ \mathfrak{N}^{\nu}_{\mathfrak{P}}(wt) = \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}}(w), \mathfrak{N}^{\nu}_{\mathfrak{R}}(t)\}, \\ \gamma^{wt} &= \min\{\alpha^{w}, \alpha^{t}\},\ \delta^{wt} = \max\{\beta^{w}, \beta^{t}\}. \end{aligned}$$

$w, t \in W$, then G is self complementary.

6. Decision making application with LDF-relations

A medical diagnosis application is established based on proposed decision-making technique. For this objective, we construct complete bipartite graphs and construct corresponding LDF-relations. An algorithm is developed for decision-making based on LDFSs and LDF-relations.

Definition 6.1. *[46] A LDF-relation $\mathcal{R}_{\eth}$ from $\mathbb{Y}_1$ to $\mathbb{Y}_2$ is an expression of the following form:*

$$\mathcal{R}_{\eth} = \{((\hbar_1, \hbar_2), < \delta^{\tau}_{\mathcal{R}_{\eth}}(\hbar_1, \hbar_2), \delta^{\nu}_{\mathcal{R}_{\eth}}(\hbar_1, \hbar_2) >, < \alpha(\hbar_1, \hbar_2), \beta(\hbar_1, \hbar_2) >) : \hbar_1 \in \mathbb{Y}_1, \hbar_2 \in \mathbb{Y}_2\}$$

where the mappings

$$\delta^{\tau}_{\mathcal{R}_{\eth}}, \delta^{\nu}_{\mathcal{R}_{\eth}} : \mathbb{Y}_1 \times \mathbb{Y}_2 \to [0, 1]$$

and $\alpha(\hbar_1, \hbar_2), \beta(\hbar_1, \hbar_2) \in [0, 1]$ such that

$$0 \leq \alpha(\hbar_1, \hbar_2)\delta^{\tau}_{\mathcal{R}_{\eth}}(\hbar_1, \hbar_2) + \beta(\hbar_1, \hbar_2)\delta^{\nu}_{\mathcal{R}_{\eth}}(\hbar_1, \hbar_2) \leq 1$$

for all $(\hbar_1, \hbar_2) \in \mathbb{Y}_1 \times \mathbb{Y}_2$ with $0 \leq \alpha(\hbar_1, \hbar_2) + \beta(\hbar_1, \hbar_2) \leq 1$. For a LDF-relation from $\mathbb{Y}_1$ to $\mathbb{Y}_2$, we shall use

$$\mathcal{R}_{\eth} = (< \delta^{\tau}_{\mathcal{R}_{\eth}}(\hbar_1, \hbar_2), \delta^{\nu}_{\mathcal{R}_{\eth}}(\hbar_1, \hbar_2) >, < \alpha(\hbar_1, \hbar_2), \beta(\hbar_1, \hbar_2) >) \tag{6.1}$$

For F-relation $\pi_{\mathfrak{R}} : \mathbb{Y}_1 \times \mathbb{Y}_2 \to [0, 1]$ associated with each LDF-relation 6.1, where

$$\gamma_{\mathfrak{R}}(\hbar_1, \hbar_2)\pi_{\mathfrak{R}}(\hbar_1, \hbar_2) = 1 - (\alpha(\hbar_1, \hbar_2)\delta^{\tau}_{\mathcal{R}_{\eth}}(\hbar_1, \hbar_2) + \beta(\hbar_1, \hbar_2)\delta^{\nu}_{\mathcal{R}_{\eth}}(\hbar_1, \hbar_2))$$

The number $\pi_{\mathfrak{R}}(\hbar_1, \hbar_2)$ is hesitation degree of $(\hbar_1, \hbar_2)$ wether $\hbar_1$ and $\hbar_2$ are the relation $\mathcal{R}_{\eth}$ or not, and $\gamma_{\mathfrak{R}}(\hbar_1, \hbar_2)$ is the degree of hesitation of RPs.

Definition 6.2. *[46] Let $\mathbb{Y}_1 = \{\mathfrak{w}_1, \mathfrak{w}_2, ..., \mathfrak{w}_m\}$, and $\mathbb{Y}_2 = \{\mathfrak{t}_1, \mathfrak{t}_2, ..., \mathfrak{t}_n\}$ be two universes. Let $\mathcal{R}_{\eth} = (< \delta^{\tau}_{\mathcal{R}_{\eth}}(\mathfrak{w}_i, \mathfrak{w}_j), \delta^{\nu}_{\mathcal{R}_{\eth}}(\mathfrak{w}_i, \mathfrak{w}_j) >, < \alpha(\mathfrak{w}_i, \mathfrak{w}_j), (\mathfrak{w}_i, \mathfrak{w}_j) >)$ be an LDF-relation from $\mathbb{Y}_1$ to $\mathbb{Y}_2$.*
Then, an LDF-relation $\mathcal{R}_{\eth}$ can be expressed in terms of matrices of MG, NMGs, and RPs as follows:

$$\delta^{\tau}_{\mathcal{R}_{\eth}} = (\mathfrak{a}_{ij})_{m\times n} = \begin{pmatrix} \mathfrak{k}_{11} & \mathfrak{k}_{12} & ... & \mathfrak{k}_{1n} \\ \mathfrak{k}_{21} & \mathfrak{k}_{22} & ... & \mathfrak{k}_{2n} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ \mathfrak{k}_{m1} & \mathfrak{k}_{m2} & ... & \mathfrak{k}_{mn} \end{pmatrix}, \delta^{\nu}_{\mathcal{R}_{\eth}} = (\mathfrak{l}_{ij})_{m\times n} = \begin{pmatrix} \mathfrak{l}_{11} & \mathfrak{l}_{12} & ... & \mathfrak{l}_{1n} \\ \mathfrak{l}_{21} & \mathfrak{l}_{22} & ... & \mathfrak{l}_{2n} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ \mathfrak{l}_{m1} & \mathfrak{l}_{m2} & ... & \mathfrak{l}_{mn} \end{pmatrix}.$$

$$\alpha = (\alpha_{ij})_{m\times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & ... & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & ... & \alpha_{2n} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ \alpha_{m1} & \alpha_{m2} & ... & \alpha_{mn} \end{pmatrix}, \beta = (\beta_{ij})_{m\times n} = \begin{pmatrix} \beta_{11} & \beta_{12} & ... & \beta_{1n} \\ \beta_{21} & \beta_{22} & ... & \beta_{2n} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ \beta_{m1} & \beta_{m2} & ... & \beta_{mn} \end{pmatrix}.$$

Or in form of one matrix as follows:

$$\mathcal{R}_D = \begin{pmatrix} ((\mathfrak{k}_{11}, \mathfrak{l}_{11}), (\alpha_{11}, \beta_{11})) & ((\mathfrak{k}_{12}, \mathfrak{l}_{12}), (\alpha_{12}, \beta_{12})) & ... & ((\mathfrak{k}_{1n}, \mathfrak{l}_{1n}), (\alpha_{1n}, \beta_{1n})) \\ ((\mathfrak{k}_{21}, \mathfrak{l}_{21}), (\alpha_{21}, \beta_{21})) & ((\mathfrak{k}_{22}, \mathfrak{l}_{22}), (\alpha_{22}, \beta_{22})) & ... & ((\mathfrak{k}_{2n}, \mathfrak{l}_{2n}), (\alpha_{2n}, \beta_{2n})) \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ ((\mathfrak{k}_{m1}, \mathfrak{l}_{m1}), (\alpha_{m1}, \beta_{m1})) & ((\mathfrak{k}_{m2}, \mathfrak{l}_{m2}), (\alpha_{m2}, \beta_{m2})) & ... & ((a_{mn}, \mathfrak{l}_{mn}), (\alpha_{mn}, \beta_{mn})) \end{pmatrix},$$

where $\mathcal{R}_D = (<\delta^{\tau}_{\mathcal{R}_{\eqcirc}}(\mathfrak{w}_i, \mathfrak{w}_j), \delta^{\nu}_{\mathcal{R}_{\eqcirc}}(\mathfrak{w}_i, \mathfrak{w}_j)>, <\alpha(\mathfrak{w}_i, \mathfrak{w}_j), \beta(\mathfrak{w}_i, \mathfrak{w}_j)>) = (<\mathfrak{k}_{ij}, \mathfrak{l}_{ij}>, <\alpha_{ij}, \beta_{ij}>)_{m\times n}$.

Definition 6.3. *[46] Let* $\mathcal{R}_{\eqcirc} = (<\delta^{\tau}_{\mathcal{R}_{\eqcirc}}(\hbar_1, \hbar_2), \delta^{\nu}_{\mathcal{R}_{\eqcirc}}(\hbar_1, \hbar_2)>, <\alpha(\hbar_1, \hbar_2), \beta(\hbar_1, \hbar_2)>)$ *be a* $\mathcal{LFR}$*−relation from* $\mathbb{Y}_1$ *to* $\mathbb{Y}_2$*. Define the score function on* $\mathcal{R}_{\eqcirc}$ *by a map*

$$\mathfrak{R} : \mathcal{LDFR}(\mathbb{Y}_1 \times \mathbb{Y}_2) \to [-1, 1]$$

given as follows:

$$\mathfrak{R}(\mathcal{R}_{\eqcirc}) = \frac{1}{2}[(\delta^{\tau}_{\mathcal{R}_{\eqcirc}}(\hbar_1, \hbar_2) - \delta^{\nu}_{\mathcal{R}_{\eqcirc}}(\hbar_1, \hbar_2)) + (\alpha(\hbar_1, \hbar_2) - \beta(\hbar_1, \hbar_2))]$$

We extend the algorithm developed by Ayub et al. [46] for MCDM approach using LDF-relations.

Algorithm:

(1) Consider multi-criterion for the objects in the universes $\mathbb{Y}_1$, $\mathbb{Y}_2$ and $\mathbb{Y}_3$. Construct LDF bipartite graph from $\mathbb{Y}_1$ and $\mathbb{Y}_2$ and from $\mathbb{Y}_2$ to $\mathbb{Y}_3$.
(2) Construct two LDF-relations $\mathcal{R}_{\eqcirc}$ from $\mathbb{Y}_1$ to $\mathbb{Y}_2$, and $\mathcal{P}_{\mathfrak{R}}$ from $\mathbb{Y}_2$ to $\mathbb{Y}_3$.
(3) Compute composition $\mathcal{R}_{\eqcirc}\hat{\circ}\mathcal{P}_{\mathfrak{R}}$.
(4) Calculate the hesitation values by using

$$\eta_{ik} = 1 - (\delta^{\tau}_{\mathcal{R}_{\eqcirc}}(\hbar_1, \hbar_2)\alpha(\hbar_1, \hbar_2) + \delta^{\nu}_{\mathcal{R}_{\eqcirc}}(\hbar_1, \hbar_2)\beta(\hbar_1, \hbar_2))$$

(5) Estimate the association grades for the elements of $\mathbb{Y}_1$ and $\mathbb{Y}_3$ by using

$$\ddot{\mathcal{A}} = \delta^{\tau}_{\mathcal{R}_{\eqcirc}}(\hbar_1, \hbar_2) - \delta^{\nu}_{\mathcal{R}_{\eqcirc}}(\hbar_1, \hbar_2)\eta_{ik}$$

(6) determine the pair $(\mathfrak{q}_i, \mathfrak{q}_k)$, where $\mathfrak{q}_i \in \mathbb{Y}_1$, $\mathfrak{q}_k \in \mathbb{Y}_3$ having the highest association grade value $\ddot{\mathcal{A}}_{ik}$.
(7) The pair $(\mathfrak{q}_i, \mathfrak{q}_k)$ is the selected object.

For elaboration, we employ the extended Algorithm in the following illustration.

6.1. Numerical example

Now we discuss a decision making application of LDFSs, LDF graph and LDF relations to medical diagnosis. In order to diagnose a patient having multiple symptoms, we utilize the above proposed algorithm.

Step 1. Let $\mathbb{Y}_1 = \{p_1, p_2, p_3, p_4\}$ be a set of patients, $\mathbb{Y}_2 = \{s_1, s_2, s_3, s_4, s_5\}$ be the symptoms of the diagnosis, where
s_1 = Muscle pain,
s_2 = Fever,
s_3 = Weakness,
s_4 = Shortness of breath,
s_5 = Chest pain, and
$\mathbb{Y}_3 = \{D_1 = \text{Pneumonia}, D_2 = \text{Influenza}, D_3 = \text{Corona virus}\}$ be the diagnosis set (set of diseases).
Construct LDF graphs from $\mathbb{Y}_1$ and $\mathbb{Y}_2$ and from $\mathbb{Y}_2$ to $\mathbb{Y}_3$. See the graph in Figure 15.

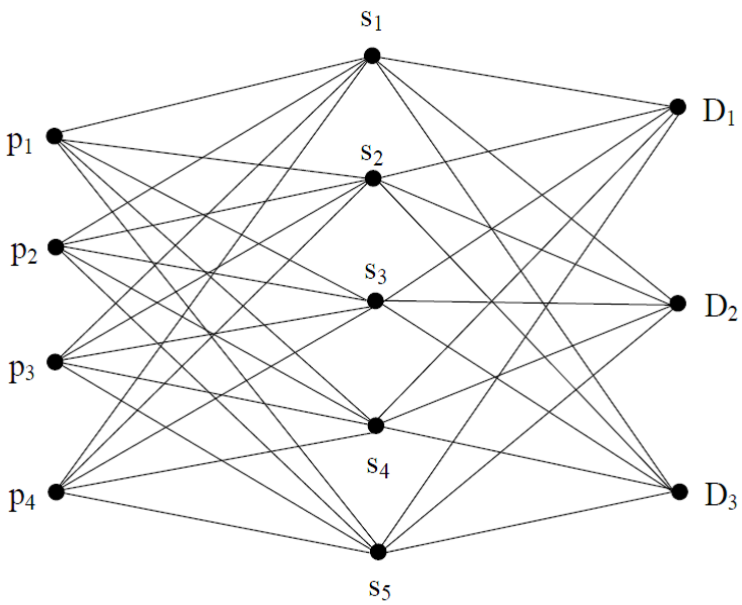

Figure 15. A bipartite graph.

Step 2. Construct the LDF-relation $\mathcal{R}_{\eqsim}$ from $\mathbb{Y}_1$ to $\mathbb{Y}_2$ from LDF graphs in Step 1, which describe the presence and non-presence of symptoms in patient to certain membership and non-membership degrees together with the parametric values α = Strong symptom and β = not strong symptom, respectively. The membership and non-membership fuzzy relations $\delta^{\tau}_{\mathcal{R}_{\eqsim}}$ and $\delta^{\nu}_{\mathcal{R}_{\eqsim}}$, together with their parametric values α and β are presented in the following matrix forms:

$$\delta^{\tau}_{\mathcal{R}_{\eqsim}} = \begin{pmatrix} 0.86 & 0.56 & 0.78 & 0.25 & 0.12 \\ 0.75 & 0.46 & 0.45 & 0.67 & 0.58 \\ 0.56 & 0.34 & 0.78 & 0.89 & 0.76 \\ 0.95 & 0.99 & 0.86 & 0.89 & 0.75 \end{pmatrix}, \delta^{\nu}_{\mathcal{R}_{\eqsim}} = \begin{pmatrix} 0.34 & 0.49 & 0.35 & 0.76 & 0.89 \\ 0.34 & 0.74 & 0.41 & 0.32 & 0.28 \\ 0.44 & 0.66 & 0.59 & 0.31 & 0.45 \\ 0.11 & 0.21 & 0.35 & 0.21 & 0.41 \end{pmatrix},$$

$$\alpha = \begin{pmatrix} 0.75 & 0.50 & 0.65 & 0.20 & 0.10 \\ 0.60 & 0.28 & 0.32 & 0.56 & 0.51 \\ 0.48 & 0.25 & 0.61 & 0.72 & 0.60 \\ 0.80 & 0.88 & 0.75 & 0.72 & 0.60 \end{pmatrix}, \beta = \begin{pmatrix} 0.24 & 0.37 & 0.25 & 0.60 & 0.72 \\ 0.24 & 0.60 & 0.27 & 0.24 & 0.21 \\ 0.26 & 0.53 & 0.49 & 0.22 & 0.28 \\ 0.10 & 0.08 & 0.24 & 0.08 & 0.27 \end{pmatrix}.$$

Now, construct the LDF-relation $\mathcal{P}_{\mathfrak{R}}$ from $\mathbb{Y}_2$ to $\mathbb{Y}_3$ which describes the relationship among the symptoms and diagnosis by the membership and non-membership fuzzy relations $\delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{R}}}, \delta^{\mathcal{N}}_{\mathcal{P}_{\mathfrak{R}}}$ together with parametric values α' = Serious symptoms, β' = Not serious symptoms in the following matrix notations:

$$\delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{R}}} = \begin{pmatrix} 0.86 & 0.86 & 0.75 \\ 0.65 & 0.78 & 0.75 \\ 0.70 & 0.86 & 0.89 \\ 0.95 & 0.66 & 0.98 \\ 0.78 & 0.42 & 0.89 \end{pmatrix}, \delta^{\mathcal{N}}_{\mathcal{P}_{\mathfrak{R}}} = \begin{pmatrix} 0.50 & 0.42 & 0.31 \\ 0.42 & 0.32 & 0.27 \\ 0.40 & 0.21 & 0.10 \\ 0.31 & 0.20 & 0.001 \\ 0.31 & 0.45 & 0.2 \end{pmatrix},$$

$$\alpha' = \begin{pmatrix} 0.70 & 0.75 & 0.65 \\ 0.60 & 0.62 & 0.65 \\ 0.41 & 0.48 & 0.56 \\ 0.75 & 0.51 & 0.95 \\ 0.70 & 0.61 & 0.85 \end{pmatrix}, \beta' = \begin{pmatrix} 0.25 & 0.20 & 0.20 \\ 0.18 & 0.17 & 0.15 \\ 0.28 & 0.21 & 0.10 \\ 0.25 & 0.42 & 0.00001 \\ 0.27 & 0.21 & 0.01 \end{pmatrix}.$$

Step 3. By simple calculations of the composition LDF-relation, we get the following:

$$\delta^{\tau}_{\mathcal{R}_{\eqsim}} \hat{\circ} \delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{R}}} = \begin{pmatrix} 0.86 & 0.86 & 0.78 \\ 0.75 & 0.75 & 0.75 \\ 0.89 & 0.78 & 0.89 \\ 0.89 & 0.86 & 0.89 \end{pmatrix}, \delta^{\nu}_{\mathcal{R}_{\eqsim}} \hat{\circ} \delta^{\mathcal{N}}_{\mathcal{P}_{\mathfrak{R}}} = \begin{pmatrix} 0.40 & 0.35 & 0.34 \\ 0.31 & 0.32 & 0.28 \\ 0.31 & 0.31 & 0.31 \\ 0.31 & 0.21 & 0.21 \end{pmatrix},$$

$$\alpha \hat{\circ} \alpha' = \begin{pmatrix} 0.70 & 0.75 & 0.65 \\ 0.60 & 0.60 & 0.60 \\ 0.72 & 0.60 & 0.72 \\ 0.72 & 0.75 & 0.72 \end{pmatrix}, \beta \hat{\circ} \beta' = \begin{pmatrix} 0.25 & 0.24 & 0.24 \\ 0.25 & 0.21 & 0.21 \\ 0.25 & 0.26 & 0.22 \\ 0.18 & 0.17 & 0.08 \end{pmatrix}.$$

The resulting LDF-relation $\mathcal{R}_{\eqsim} \hat{\circ} \mathcal{P}_{\mathfrak{R}}$ from $\mathbb{Y}_1$ to $\mathbb{Y}_3$ given in Table 3.

Table 3. LDF-relation $\mathcal{R}_{\eqsim} \hat{\circ} \mathcal{P}_{\mathfrak{R}}$.

$\mathcal{R}_{\eqsim} \hat{\circ} \mathcal{P}_{\mathfrak{R}}$	D_1	D_2	D_3
p_1	$((0.86, 0.40), (0.70, 0.25))$	$((0.86, 0.35), (0.75, 0.24))$	$((0.78, 0.34), (0.65, 0.24))$
p_2	$((0.75, 0.31), (0.60, 0.25))$	$((0.75, 0.32), (0.60, 0.21))$	$((0.75, 0.28), (0.60, 0.21))$
p_3	$((0.89, 0.31), (0.72, 0.25))$	$((0.78, 0.31), (0.60, 0.26))$	$((0.89, 0.31), (0.72, 0.22))$
p_4	$((0.89, 0.31), (0.72, 0.18))$	$((0.86, 0.21), (0.75, 0.17))$	$((0.89, 0.21), (0.72, 0.08))$

This composition basically describes the diagnosis of the diseases in the given patients.

Step 4. By using the Definition 6.1, hesitation degrees can be evaluated by the formulae $\eta_{ik} = 1 - (\delta^{\tau}_{\mathcal{R}_{\Xi}}(\hbar_1, \hbar_2)\alpha(\hbar_1, \hbar_2) + \delta^{\nu}_{\mathcal{R}_{\Xi}}(\hbar_1, \hbar_2)\beta(\hbar_1, \hbar_2))$ of the LDF-relation $\mathcal{R}_{\Xi}\hat{\circ}\mathcal{P}_{\mathfrak{R}}$ from $\mathbb{Y}_1$ to $\mathbb{Y}_3$, are given in Table 4.

Table 4. LDF-relation $\mathcal{R}_{\Xi}\hat{\circ}\mathcal{P}_{\mathfrak{R}}$.

η_{ik}	D_1	D_2	D_3
p_1	0.298	0.271	0.4114
p_2	0.4725	0.4828	0.4912
p_3	0.2817	0.4514	0.291
p_4	0.3034	0.3193	0.3424

Step 5. The association grades among objects of $\mathcal{P}$ and $\mathcal{D}$ can be evaluated by using the formulae $\ddot{\mathscr{A}} = \delta^{\tau}_{\mathcal{R}_{\Xi}}(\hbar_1, \hbar_2) - \delta^{\nu}_{\mathcal{R}_{\Xi}}(\hbar_1, \hbar_2)\eta_{ik}$ are given in Table 5.

Table 5. The association grades among objects of $\mathcal{P}$ and $\mathcal{D}$.

$\ddot{\mathscr{A}}_{ik}$	D_1	D_2	D_3
p_1	0.7408	0.76515	0.640124
p_2	0.603525	0.595504	0.612464
p_3	0.802673	0.640066	0.79979
p_4	0.795946	0.792947	0.818096

Step 6. Clearly, in first row the pair (p_1, D_2), in second row the pair (p_2, D_3), in third row the pair (p_3, D_1), and in the last row the pair (p_4, D_3) having the highest association grades.

Step 7. Decision is, p_1 has the disease D_2, p_2 has D_3, p_3 is suffering from D_1, and p_4 is suffering from D_3. For confirmation, the score values among $\mathbb{Y}_1$ and $\mathbb{Y}_3$ by using the Definition 6.3, are calculated in Table 6.

Table 6. Score values among $\mathbb{Y}_1$ and $\mathbb{Y}_3$.

$\mathfrak{N}^{\nu}_{ik}$	D_1	D_2	D_3
p_1	0.455	0.51	0.425
p_2	0.395	0.41	0.43
p_3	0.56	0.405	0.54
p_4	0.56	0.615	0.66

It can be easily seen that, in first row the pair (p_1, D_2), in second row pair (p_2, D_3), in third row the pair (p_3, D_1), and in the last row the pair (p_4, D_3) having the highest score values. Thus, our above decisions are true. Hence, our proposed algorithm is reliable and employs valid results.

7. Conclusions

Uncertain optimization, modeling uncertainty, and optimization problems have been studied by researchers. A Linear Diophantine fuzzy graph provides a robust approach for modeling uncertainty in the best worst situation. Consequently, the decision-making approach becomes robust with linear

Diophantine fuzzy information. We introduced the idea of LDF-graph as a generalization of certain existing concepts including, q-ROF graph, PF-graph, and IF-graph. We introduced certain properties of LDF-graph including, join, union, and composition of LDF-graphs. We elucidate these operations with various illustrations. We analyzed some interesting results that the composition of two LDF-graphs is a LDF-graph, cartesian product of two LDF-graphs is a LDF-graph, and the join of two LDF-graphs is a LDF-graph. We described the idea of homomorphisms and isomorphism for LDF-graphs. We proved the result that an isomorphism between LDF-graphs is an equivalence relation. many other significant results related to complement of LDF-graph are also established. Lastly, an algorithm based on LDFSs and LDF-relations is proposed for decision-making problems. Based on proposed algorithm an application of medical diagnosis is presented.

In future we may work on the following topics:

1) Linear Diophantine fuzzy soft graphs.

2) Linear Diophantine fuzzy plannar graphs.

3) Linear Diophantine fuzzy hypergraphs.

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Conflict of interest

The authors of this paper declare that they have no conflict of interest.

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