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Research article

Common fixed point and coincidence point results for generalized α-φ_E-Geraghty contraction mappings in b-metric spaces

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Abstract: In this paper, we study the existence and uniqueness of common fixed point of $\alpha_{i,j}$-$\varphi_{E_{M,N}}$-Geraghty contraction mappings and the existence of coincidence point of $\alpha_{i,j}$-φ_{E_N}-Geraghty contraction mapping in the framework of b-metric spaces. We also give two examples to support our results.

Keywords: common fixed point; coincidence point; b-metric; $\alpha_{i,j}$-$\varphi_{E_{M,N}}$-Geraghty contraction mappings; $\alpha_{i,j}$-φ_{E_N}-Geraghty contraction mapping
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1. Introduction

In 1973, Geraghty [1] generalized the Banach contraction principle in the setting of a complete metric space by considering an auxiliary function. This interesting result has attracted the attention of a great number of researchers. In 2012, Samet et al. [2] introduced the concept of α-admissible and α-ψ-contraction mappings and presented fixed point theorems for them. In 2013, Cho et al. [3] introduced the concept of α-Geraghty contraction mappings in metric spaces and proved some fixed point results of such mappings. Karapinar et al. [4] gave the notion of an α-ζ-E-Pata contraction and proved the existence and uniqueness of a fixed point of such mappings in the setting of a complete metric space. In [5], authors established that the main result via ψ-Geraghty type contraction is equivalent to an existing related result in the literature. In the framework of a complete b-metric space, Karapinar et al. [6] investigated the existence of fixed points for α-almost Istratesc contraction of type E and of type E^*. Alghamdi et al. [7] considered a common fixed point theorem via extended Z-contraction with respect to y-simulation function over an auxiliary function x. In [8], Afshari et al. obtained a fixed point result of generalized α-ψ-Geraghty contractive type mappings. Then, the notion of Geraghty contraction of type E was introduced by Fulga and Proca [9]. Recently, some new set-valued Meir-Keeler, Geraghty and Edelstein type fixed point theorems were presented in [10]. In 2019, Aydi [11] introduced the notion of α-β_E-Geraghty contraction mappings on b-metric spaces and proved the existence and uniqueness of fixed point for such mappings. In the following year, Alqahtani [12] proved the existence and uniqueness of a common fixed point for the Geraghty contraction of type $E_{S,T}$ on complete metric spaces. In [13], Debnath et al. extended the notions of orbitally continuous and asymptotically regular mappings in the set-valued context. They introduced two new contractive inequalities one of which is of Geraghty-type and the other is of Boyd and Wong-type and proved two new existence of fixed point results corresponding to those inequalities. Okeke et al. [14] proved some theorems on the existence and uniqueness of fixed point for Reich-type contraction mappings and Geraghty-type mappings satisfying rational inequalities in modular metric spaces. For recent development on fixed point theory of Geraghty type contraction, we refer to [15–17] and the related references therein.

Czerwik [18] introduced the concept of b-metric space in 1993, which is a generalization of metric space and proved some fixed point theorems of contractive mappings in this space. Afterwards, some authors had studied the fixed point theorems of a various new type of contractive conditions in b-metric spaces (see [19–26]). Recently, Abbas et al. [27] studied the existence of fixed points of T-Ciric type mappings in the setup of partially ordered spaces. Roshan et al. [28] proved a common fixed point theorem for three mappings in G_b-metric space which is not continuous. Mustafa et al. [29] introduced the class of extended rectangular b-metric spaces as a generalization of both rectangular metric and rectangular b-metric spaces. In addition, some fixed point results connected with certain contractions are obtained. In [30], Mustafa et al. presented some coincidence point results for six mappings satisfying the generalized (ψ, φ)-weakly contractive condition in the framework of partially ordered G-metric spaces.

Throughout this paper, we aim to obtain common fixed point of $\alpha_{i,j}$-$\varphi_{E_{M,N}}$-Geraghty contraction mappings and coincidence point of $\alpha_{i,j}$-φ_{E_N}-Geraghty contraction mapping in the framework of b-metric space. Furthermore, we provide two examples that elaborate on the usability of our results.

There are some basic definitions and theorems that need to be used later. We state them as follows.

Definition 1.1. [18] Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \to [0, +\infty)$ is said to be a b-metric if and only if, for all $a, b, c \in X$, the following conditions are satisfied:

(i) $d(a, b) = 0$ if and only if $a = b$;

(ii) $d(a, b) = d(b, a)$;

(iii) $d(a, c) \leq s[d(a, b) + d(b, c)]$.

In general, (X, d, s) is called a b-metric space with parameter $s \geq 1$.

It is obvious that the class of b-metric spaces is effectively larger than that of metric spaces since any metric space is a b-metric space with $s = 1$. The following examples show that, in general, a b-metric space need not necessarily be a metric space.

Example 1.2. [31] Let $X = \mathbb{R}$ and let the mapping $d : X \times X \to \mathbb{R}^+$ be defined by

$$d(a, b) = |a - b|^2$$

for all $a, b \in X$. Then $(X, d, 2)$ is a b-metric space with parameter $s = 2$.

Example 1.3. [32] Let $X = \{0, 1, 2\}$ and $d(2, 0) = d(0, 2) = m \geq 2$, $d(0, 1) = d(1, 2) = d(1, 0) = d(2, 1) = 1$, $d(0, 0) = d(1, 1) = d(2, 2) = 0$. Then

$$d(a, b) \leq \frac{m}{2}[d(a, c) + d(c, b)],$$

for all $a, b, c \in X$. If $m > 2$, the ordinary triangle inequality does not hold.

In [33], the authors showed the generality of Example 1.2.

Example 1.4. [33] Let (X,ρ) be a metric space, and $d(a,b) = (\rho(a,b))^p$, where $p > 1$ is a real number. Then (X, d, s) is a b-metric space with $s = 2^{p-1}$.

In general, a b-metric function d for $s > 1$ is not jointly continuous in all of its two variables. Following is an example of a b-metric which is not continuous.

Example 1.5. [34] Let $X = \mathbb{N} \cup \{\infty\}$ and $d : X \times X \to \mathbb{R}^+$ be defined by

$$d(m,n) = \begin{cases} 0, & \text{if } m = n, \\ |\dfrac{1}{m} - \dfrac{1}{n}|, & \text{if } m, n \text{ are even or } mn = \infty, \\ 5, & \text{if } m \text{ and } n \text{ are odd and } m \neq n, \\ 2, & \text{otherwise.} \end{cases}$$

Then it is easy to see that for all $m, n, p \in X$, we have

$$d(m, p) \leq 3(d(m, n) + d(n, p)).$$

Thus, $(X, d, 3)$ is a b-metric space. If $a_n = 2n$, for each $n \in \mathbb{N}$, then

$$d(2n, +\infty) = \frac{1}{2n} \to 0, \text{as } n \to \infty,$$

that is, $a_n \to \infty$, but $d(x_n, 1) = 2 \nrightarrow d(\infty, 1)$, as $n \to \infty$.

Definition 1.6. [35] Let (X, d, s) be a b-metric space with parameter $s \geq 1$. Then a sequence $\{a_n\}$ in X is said to be:

(i) b-convergent if and only if there exists $a \in X$ such that $d(a_n, a) \to 0$ as $n \to \infty$;

(ii) a Cauchy sequence if and only if $d(a_n, a_m) \to 0$ when $n, m \to \infty$.

As usual, a b-metric space is called complete if and only if each Cauchy sequence in this space is b-convergent.

As b-metric is not continuous in general, so we need the following simple lemma about the b-convergent sequences.

Lemma 1.7. [33] Let (X, d, s) be a b-metric space with parameter $s \geq 1$. Assume that $\{a_n\}$ and $\{b_n\}$ are b-convergent to a and b, respectively. Then we have

$$\frac{1}{s^2} d(a, b) \leq \liminf_{n\to\infty} d(a_n, b_n) \leq \limsup_{n\to\infty} d(a_n, b_n) \leq s^2 d(a, b).$$

In particular, if $a = b$, then we have $\lim_{n\to\infty} d(a_n, b_n) = 0$. Moreover, for each $c \in X$, we have

$$\frac{1}{s} d(a, c) \leq \liminf_{n\to\infty} d(a_n, c) \leq \limsup_{n\to\infty} d(a_n, c) \leq s d(a, c).$$

Definition 1.8. [36] Let M and N be two self-mappings on a nonempty set X. If $a^* = Ma = Na$, for some $a \in X$, then a is said to be the coincidence point of M and N, where a^* is called the point of coincidence of M and N. Let $C(M, N)$ denote the set of all coincidence points of M and N.

Definition 1.9. [36] Let M and N be two self-mappings defined on a nonempty set X. Then M and N are said to be weakly compatible if they commute at every coincidence point, that is, $Ma = Na \Rightarrow MNa = NMa$ for every $a \in C(M, N)$.

Inspired the concept α admissible mapping introduced by [2, 32], Popescu [37] gave the following definition:

Definition 1.10. [37] Let $M : X \to X$ and $\alpha : X \times X \to \mathbb{R}$ be given mappings. We say that M is α-orbital admissible if for all $a \in X$, we have

$$\alpha(a, Ma) \geq 1 \Rightarrow \alpha(Ma, M^2a) \geq 1.$$

Definition 1.11. [37] Let $M : X \to X$ and $\alpha : X \times X \to \mathbb{R}$ be given mappings. A mapping M is called a triangular α-orbital admissible mapping if

(i) M is α-orbital admissible;

(ii) $\alpha(a, b) \geq 1$ and $\alpha(b, Mb) \geq 1 \Rightarrow \alpha(a, Mb) \geq 1, a, b \in X$.

Definition 1.12. [1] Let (X, d) be a complete metric space. An operator $M : X \to X$ is called a Geraghty contraction if there exists a function $\varphi : [0, \infty) \to [0, 1)$ which satisfies the following condition

$$\lim_{n\to\infty} \varphi(\nu_n) = 1 \text{ implies that } \lim_{n\to\infty} \nu_n = 0 \tag{1.1}$$

such that

$$d(Ma, Nb) \leq \varphi(d(a, b))d(a, b), \text{ for all } a, b \in X.$$

In the following, we denote the class of functions $\varphi : [0, \infty) \to [0, 1)$ which satisfies the condition (1.1) by Φ.

Definition 1.13. [12] Suppose that M and N are two self-mappings on a metric space (X, d). Suppose that there is a $\varphi \in \Phi$ such that the inequality

$$d(Ma, Nb) \leq \varphi(E_{M,N}(a, b))E_{M,N}(a, b), \text{ for all } a, b \in X,$$

is satisfied, where

$$E_{M,N}(a, b) = d(a, b) + |d(a, Ma) - d(b, Nb)|.$$

Then, we say that the mappings M and N satisfy the Geraghty contraction of type $E_{M,N}$.

Now, for $s \geq 1$, let Φ_s denotes the family of functions $\varphi : [0, \infty) \to [0, \frac{1}{s})$ satisfying the condition:

$$\lim_{n\to\infty} \varphi(\nu_n) = \frac{1}{s} \text{ implies that } \lim_{n\to\infty} \nu_n = 0.$$

Definition 1.14. [11] Let (X, d) be a metric space and $\alpha : X \times X \to R$ be a function. A mapping $M : X \to X$ is said to be an α-φ_E-Geraghty contraction mapping if there exists $\varphi \in \Phi_s$ such that

$$\alpha(a, b) \geq 1 \Rightarrow d(Ma, Mb) \leq \varphi(E(a, b))E(a, b)$$

for all $a, b \in X$, where

$$E(a, b) = d(a, b) + |d(a, Ma) - d(b, Mb)|.$$

2. Main results

In this section, we introduce some new definitions and concepts and prove some new common fixed point theorems and coincidence point theorems in a b-metric space which is not required the continuity of b-metric. Meanwhile, we provide two examples to support our results.

In the following, we assume that i, j are two arbitrary positive integers unless otherwise state.

Definition 2.1. Let (X, d, s) be a b-metric space with parameter $s \geq 1$ and $\alpha_{i,j} : X \times X \to [0, \infty)$ be a function. Two mappings $M, N : X \to X$ are called $\alpha_{i,j}$-$\varphi_{E_{M,N}}$-Geraghty contraction mappings if there exists a function $\varphi \in \Phi_s$ such that

$$\alpha_{i,j}(a, b) \geq s^p \Rightarrow \alpha_{i,j}(a, b)d(M^i a, N^j b) \leq \varphi(E(a, b))E(a, b), \text{for all } a, b \in X, \tag{2.1}$$

where

$$E(a, b) = d(a, b) + |d(a, M^i a) - d(b, N^j b)|$$

and $p \geq 2$ is a constant.

Remark 2.2.

(i) If $s = 1, \alpha_{i,j}(a, b) = 1$ and $i = j = 1$, we can get the Geraghty contraction of type $E_{M,N}$.

(ii) If $M = N$ and $i = j = 1$, we can get an α-φ_E-Geraghty contraction.

Definition 2.3. Let (X, d, s) be a b-metric space with parameter $s \geq 1$ and $\alpha_{i,j} : X \times X \to [0, \infty)$ be a function. The self-mappings $M, N : X \to X$ are said to be $\alpha_{i,j}$-orbital admissible, if the following conditions hold:

$$\alpha_{i,j}(a, M^i a) \geq s^p \Rightarrow \alpha_{i,j}(M^i a, N^j M^i a) \geq s^p,$$
$$\alpha_{i,j}(a, N^j a) \geq s^p \Rightarrow \alpha_{i,j}(N^j a, M^i N^j a) \geq s^p,$$

for $a \in X$, where $p \geq 2$ is a constant.

Definition 2.4. Let (X, d, s) be a b-metric space with parameter $s \geq 1$ and $\alpha_{i,j} : X \times X \to [0, \infty)$ be a function. Let $M, N : X \to X$ be two given self-mappings. The pair (M, N) is said to be triangular $\alpha_{i,j}$-orbital admissible, if

(i) M, N are $\alpha_{i,j}$-orbital admissible;

(ii) $\alpha_{i,j}(a, b) \geq s^p, \alpha_{i,j}(b, M^i b) \geq s^p$ and $\alpha_{i,j}(b, N^j b) \geq s^p$ implies $\alpha_{i,j}(a, M^i b) \geq s^p$ and $\alpha_{i,j}(a, N^j b) \geq s^p$, where $p \geq 2$ is a constant.

Lemma 2.5. Let (X, d, s) be a complete b-metric space with parameter $s \geq 1$. Let $M, N : X \to X$ be two self-mappings such that the pair (M, N) is triangular $\alpha_{i,j}$-orbital admissible. Assume that there exists $a_0 \in X$ such that $\alpha_{i,j}(a_0, M^i a_0) \geq s^p$. Define a sequence $\{a_n\}$ in X by $a_{2n} = N^j a_{2n-1}, a_{2n+1} = M^i a_{2n}$ where $n = 0, 1, 2, \cdots$. Then for $n, m \in \mathbb{N} \cup \{0\}$ with $m > n$, we have $\alpha_{i,j}(a_n, a_m) \geq s^p$.

Proof. Since $\alpha_{i,j}(a_0, M^i a_0) = \alpha_{i,j}(a_0, a_1) \geq s^p$ and (M, N) is triangular $\alpha_{i,j}$-orbital admissible, we obtain

$$\alpha_{i,j}(a_0, M^i a_0) \geq s^p \text{ implies } \alpha_{i,j}(M^i a_0, N^j M^i a_0) = \alpha_{i,j}(a_1, N^j a_1) = \alpha_{i,j}(a_1, a_2) \geq s^p,$$
$$\alpha_{i,j}(a_1, N^j a_1) \geq s^p \text{ implies } \alpha_{i,j}(N^j a_1, M^i N^j a_1) = \alpha_{i,j}(a_2, M^i a_2) = \alpha_{i,j}(a_2, a_3) \geq s^p,$$
$$\alpha_{i,j}(a_2, M^i a_2) \geq s^p \text{ implies } \alpha_{i,j}(M^i a_2, N^j M^i a_2) = \alpha_{i,j}(a_3, N^j a_3) = \alpha_{i,j}(a_3, a_4) \geq s^p.$$

Applying the above argument repeatedly, it follows that $\alpha_{i,j}(a_n, a_{n+1}) \geq s^p$ for all $n \in \mathbb{N} \cup \{0\}$. Since (M, N) is triangular $\alpha_{i,j}$-orbital admissible, $\alpha_{i,j}(a_n, a_m) \geq s^p$ for all $n, m \in \mathbb{N} \cup \{0\}$ with $m > n$. □

Theorem 2.6. Let (X, d, s) be a complete b-metric space with parameter $s \geq 1$, $\alpha_{i,j} : X \times X \to [0, \infty)$ be a symmetrical function and $M, N : X \to X$ be two given mappings. Suppose that the following conditions are satisfied:

(i) The pair (M, N) is triangular $\alpha_{i,j}$-orbital admissible;

(ii) M, N are $\alpha_{i,j}$-$\varphi_{E_{M,N}}$-Geraghty contraction mappings;

(iii) there is $a_0 \in X$ satisfying $\alpha_{i,j}(a_0, M^i a_0) \geq s^p$;

(iv) if $\{a_n\}$ is a sequence in X such that $\alpha_{i,j}(a_n, a_{n+1}) \geq s^p$ for all $n \in \mathbb{N}$ and $a_n \to a$ as $n \to \infty$, then there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\alpha_{i,j}(a_{n_k}, a) \geq s^p$ for all $k \in \mathbb{N}$;

(v) for all $a, b \in Fix(M^i)$ or $Fix(N^j)$, we have $\alpha_{i,j}(a, b) \geq s^p$, where $Fix(M^i)$ denotes the set of fixed points of M^i.

Then M and N have a unique common fixed point.

Proof. Let $a_0 \in X$ satisfy $\alpha_{i,j}(a_0, M^i a_0) \geq s^p$. Define a sequence $\{a_n\}$ in X by $a_{2n} = N^j a_{2n-1}$, $a_{2n+1} = M^i a_{2n}$ for $n = 0, 1, 2, \cdots$. Suppose there exists a $n_0 \in \mathbb{N}$ such that $a_{n_0} = a_{n_0+1}$. We consider two cases:

(i) n_0 is odd. We have $a_{n_0+1} = N^j a_{n_0} = a_{n_0}$, that is, a_{n_0} is a fixed point of N^j. Next, we will prove that $a_{n_0} = a_{n_0+1} = N^j a_{n_0} = M^i a_{n_0+1}$. Considering $N^j a_{n_0} \neq M^i a_{n_0+1}$ and according to Lemma 2.5, we get $\alpha_{i,j}(a_{n_0}, a_{n_0+1}) = \alpha_{i,j}(a_{n_0+1}, a_{n_0}) \geq s^p$ and

$$d(M^i a_{n_0+1}, N^j a_{n_0}) \leq \alpha_{i,j}(a_{n_0+1}, a_{n_0}) d(M^i a_{n_0+1}, N^j a_{n_0}) \leq \varphi(E(a_{n_0+1}, a_{n_0})) E(a_{n_0+1}, a_{n_0}),$$

where

$$E(a_{n_0+1}, a_{n_0}) = d(a_{n_0+1}, a_{n_0}) + |d(a_{n_0+1}), M^i a_{n_0+1}) - d(a_{n_0}, N^j a_{n_0}) = d(a_{n_0+1}, M^i a_{n_0+1}).$$

It follows that

$$d(M^i a_{n_0+1}, N^j a_{n_0}) \leq \varphi(d(a_{n_0+1}, M^i a_{n_0+1})) d(a_{n_0+1}, M^i a_{n_0+1}) < \frac{1}{s} d(N^j a_{n_0}, M^i a_{n_0+1}),$$

which is a contradiction. Hence, $d(N^j a_{n_0}, M^i a_{n_0+1}) = 0$ and $a_{n_0} = a_{n_0+1} = N^j a_{n_0} = M^i a_{n_0+1}$. Then a_{n_0} is a fixed point of M^i, that is, a_{n_0} is a common fixed point of M^i and N^j.

(ii) n_0 is even. We have $a_{n_0+1} = M^i a_{n_0} = a_{n_0}$, that is, a_{n_0} is a fixed point of M^i. By the same way, we obtain that a_{n_0} is a common fixed point of M^i and N^j.

Consequently, throughout the proof, we assume $a_n \neq a_{n+1}$ for all $n \geq 0$. We consider the following cases:

Case(1)**.** In (2.1), let $a = a_{2n}$ and $b = a_{2n-1}$. Then we have $\alpha_{i,j}(a_{2n}, a_{2n-1}) \geq s^p$. Hence, we get

$$\begin{aligned} d(a_{2n}, a_{2n+1}) &= d(N^j a_{2n-1}, M^i a_{2n}) \\ &\leq \alpha_{i,j}(a_{2n}, a_{2n-1}) d(M^i a_{2n}, N^j a_{2n-1}) \\ &\leq \varphi(E(a_{2n}, a_{2n-1})) E(a_{2n}, a_{2n-1}), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} E(a_{2n}, a_{2n-1}) &= d(a_{2n}, a_{2n-1}) + |d(a_{2n}, M^i a_{2n}) - d(a_{2n-1}, N^j a_{2n-1})| \\ &= d(a_{2n}, a_{2n-1}) + |d(a_{2n}, a_{2n+1}) - d(a_{2n-1}, a_{2n})|. \end{aligned} \tag{2.3}$$

If $d(a_{2n}, a_{2n+1}) \geq d(a_{2n-1}, a_{2n})$, we have $E(a_{2n}, a_{2n-1}) = d(a_{2n}, a_{2n+1})$ and

$$d(a_{2n}, a_{2n+1}) \leq \varphi(d(a_{2n}, a_{2n+1}))d(a_{2n}, a_{2n+1}) < \frac{1}{s}d(a_{2n}, a_{2n+1}),$$

which is a contradiction. Therefore, $d(a_{2n}, a_{2n+1}) < d(a_{2n-1}, a_{2n})$.

Case(2)**.** In (2.1), let $a = a_{2n}$ and $b = a_{2n+1}$. Then we get $\alpha_{i,j}(a_{2n}, a_{2n+1}) \geq s^p$. It follows that

$$\begin{aligned}
d(a_{2n+1}, a_{2n+2}) =& d(M^i a_{2n}, N^j a_{2n+1}) \\
\leq& \alpha_{i,j}(a_{2n}, a_{2n+1}) d(M^i a_{2n}, N^j a_{2n+1}) \\
\leq& \varphi(E(a_{2n}, a_{2n+1}))E(a_{2n}, a_{2n+1}),
\end{aligned}$$

where

$$\begin{aligned}
E(a_{2n}, a_{2n+1}) =& d(a_{2n}, a_{2n+1}) + |d(a_{2n}, M^i a_{2n}) - d(a_{2n+1}, N^j a_{2n+1})| \\
=& d(a_{2n}, a_{2n+1}) + |d(a_{2n}, a_{2n+1}) - d(a_{2n+1}, a_{2n+2})|.
\end{aligned}$$

If $d(a_{2n+1}, a_{2n+2}) \geq d(a_{2n}, a_{2n+1})$, we have $E(a_{2n}, a_{2n+1}) = d(a_{2n+1}, a_{2n+2})$ and

$$d(a_{2n+1}, a_{2n+2}) \leq \varphi(d(a_{2n+1}, a_{2n+2}))d(a_{2n+1}, a_{2n+2}) < \frac{1}{s}d(a_{2n+1}, a_{2n+2}),$$

which is a contradiction. Therefore, $d(a_{2n+1}, a_{2n+2}) < d(a_{2n}, a_{2n+1})$.

To sum up, $\{d(a_n, a_{n+1})\}$ is non-increasing. Thus, there exists a γ such that

$$\lim_{n\to\infty} d(a_n, a_{n+1}) = \gamma.$$

We claim that $\gamma = 0$. In fact, by taking limits as $n \to \infty$ in (2.3), we get

$$\lim_{n\to\infty} E(a_{2n}, a_{2n-1}) = \lim_{n\to\infty}(2d(a_{2n}, a_{2n-1}) - d(a_{2n}, a_{2n+1})) = \gamma.$$

Letting $n \to \infty$ in (2.2) and combining the above equality, we obtain

$$\begin{aligned}
\frac{\gamma}{s} &= \frac{1}{s} \lim_{n\to\infty} d(a_{2n}, a_{2n+1}) \\
&\leq \lim_{n\to\infty} d(a_{2n}, a_{2n+1}) \\
&\leq \lim_{n\to\infty} \varphi(E(a_{2n}, a_{2n-1}))E(a_{2n}, a_{2n-1}) \\
&\leq \lim_{n\to\infty} \frac{1}{s}E(a_{2n}, a_{2n-1}) \\
&= \frac{\gamma}{s}.
\end{aligned}$$

As a result, we get

$$\lim_{n\to\infty} \varphi(E(a_{2n}, a_{2n-1}))E(a_{2n}, a_{2n-1}) = \frac{\gamma}{s}.$$

It follows that

$$\lim_{n\to\infty} \varphi(E(a_{2n}, a_{2n-1})) = \frac{1}{s}$$

which implies $\lim_{n\to\infty} E(a_{2n}, a_{2n-1}) = \gamma = 0$, that is,

$$\lim_{n\to\infty} d(a_n, a_{n+1}) = 0.$$

Next, we shall prove that $\{a_n\}$ is a Cauchy sequence in X. To verify this, it is sufficient to prove that $\{a_{2n}\}$ is Cauchy. Instead, we assume that there exists $\ell > 0$ for which one can find subsequences $\{a_{2m_k}\}$ and $\{a_{2n_k}\}$ of $\{a_{2n}\}$ satisfying m_k is the smallest index for which $m_k > n_k > k$, and

$$d(a_{2m_k}, a_{2n_k}) \geq \ell \text{ and } d(a_{2m_k-2}, a_{2n_k}) < \ell.$$

In view of the triangle inequality, one can deduce that

$$\ell \leq d(a_{2m_k}, a_{2n_k}) \leq sd(a_{2m_k}, a_{2n_k+1}) + sd(a_{2n_k+1}, a_{2n_k}).$$

So, we have

$$\frac{\ell}{s} \leq \liminf_{k\to\infty} d(a_{2m_k}, a_{2n_k+1}) \leq \limsup_{k\to\infty} d(a_{2m_k}, a_{2n_k+1}). \tag{2.4}$$

From Lemma 2.5, we have $\alpha_{i,j}(a_{2m_k-1}, a_{2n_k}) \geq s^p$ and

$$\begin{aligned} d(a_{2m_k}, a_{2n_k+1}) &\leq \alpha_{i,j}(a_{2m_k-1}, a_{2n_k}) d(M^i a_{2n_k}, N^j a_{2m_k-1}) \\ &\leq \varphi(E(a_{2n_k}, a_{2m_k-1}))E(a_{2n_k}, a_{2m_k-1}), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} E(a_{2n_k}, a_{2m_k-1}) =& d(a_{2n_k}, a_{2m_k-1}) + |d(a_{2n_k}, M^i a_{2n_k}) - d(a_{2m_k-1}, N^j a_{2m_k-1})| \\ =& d(a_{2n_k}, a_{2m_k-1}) + |d(a_{2n_k}, a_{2n_k+1}) - d(a_{2m_k-1}, a_{2m_k})|. \end{aligned}$$

So there is

$$\begin{aligned} \liminf_{k\to\infty} E(a_{2n_k}, a_{2m_k-1}) &= \liminf_{k\to\infty} d(a_{2m_k-1}, a_{2n_k}) \\ &\leq \liminf_{k\to\infty} [sd(a_{2m_k-1}, a_{2m_k-2}) + sd(a_{2m_k-2}, a_{2n_k})] \\ &\leq s\ell. \end{aligned}$$

Taking $k \to \infty$ in inequality (2.5) and combining (2.4), we obtain

$$\begin{aligned} \ell &= s \cdot \frac{\ell}{s} \\ &\leq s \liminf_{k\to\infty} d(a_{2m_k}, a_{2n_k+1}) \\ &\leq s^p \liminf_{k\to\infty} d(a_{2m_k}, a_{2n_k+1}) \\ &\leq \alpha_{i,j}(a_{2m_k-1}, a_{2n_k}) \liminf_{k\to\infty} d(M^i a_{2n_k}, N^j a_{2m_k-1}) \\ &\leq \liminf_{k\to\infty} \varphi(E(a_{2n_k}, a_{2m_k-1}))E(a_{2n_k}, a_{2m_k-1}) \\ &\leq \liminf_{k\to\infty} \frac{1}{s} \cdot s\ell \\ &= \ell. \end{aligned}$$

At the same time, we get

$$\begin{aligned}\ell &\leq \alpha_{i,j}(a_{2m_k-1}, a_{2n_k}) \limsup_{k\to\infty} d(M^i a_{2n_k}, N^j a_{2m_k-1})\\ &\leq \limsup_{k\to\infty} \varphi(E(a_{2n_k}, a_{2m_k-1}))E(a_{2n_k}, a_{2m_k-1})\\ &= \ell.\end{aligned}$$

Therefore

$$\lim_{k\to\infty} \varphi(E(a_{2n_k}, a_{2m_k-1}))E(a_{2n_k}, a_{2m_k-1}) = \ell. \tag{2.6}$$

Similarly, we have

$$\begin{aligned}s\ell &= s^2 \cdot \frac{\ell}{s}\\ &\leq s^2 \liminf_{k\to\infty} d(a_{2m_k}, a_{2n_k+1})\\ &\leq s^p \liminf_{k\to\infty} d(a_{2m_k}, a_{2n_k+1})\\ &\leq \alpha_{i,j}(a_{2m_k-1}, a_{2n_k}) \liminf_{k\to\infty} d(M^i a_{2n_k}, N^j a_{2m_k-1})\\ &\leq \liminf_{k\to\infty} \varphi(E(a_{2n_k}, a_{2m_k-1}))E(a_{2n_k}, a_{2m_k-1})\\ &\leq \liminf_{k\to\infty} E(a_{2n_k}, a_{2m_k-1})\\ &\leq s\ell,\end{aligned}$$

and

$$\begin{aligned}s\ell &\leq \alpha_{i,j}(a_{2m_k-1}, a_{2n_k}) \limsup_{k\to\infty} d(M^i a_{2n_k}, N^j a_{2m_k-1})\\ &\leq \limsup_{k\to\infty} E(a_{2n_k}, a_{2m_k-1})\\ &\leq s\ell.\end{aligned}$$

It follows that

$$\lim_{k\to\infty} E(a_{2n_k}, a_{2m_k-1}) = s\ell. \tag{2.7}$$

Combining (2.6) and (2.7), we get

$$\lim_{k\to\infty} \varphi(E(a_{2n_k}, a_{2m_k-1})) = \frac{1}{s} \text{ and therefore } \lim_{k\to\infty} E(a_{2n_k}, a_{2m_k-1}) = 0,$$

which is a contradiction. Hence, $\{a_n\}$ is a Cauchy sequence. The completeness of (X, d, s) ensures that there exists an a^* in X such that

$$\lim_{n\to\infty} a_n = a^*.$$

Then from condition (iv), we can deduce that there exists a subsequence $\{a_{2n_k}\}$ of $\{a_n\}$ such that $\alpha_{i,j}(a_{2n_k}, a^*) \geq s^p$ and

$$d(a_{2n_k+1}, N^j a^*) \leq \alpha_{i,j}(a_{2n_k}, a^*)d(M^i a_{2n_k}, N^j a^*) \leq \varphi(E(a_{2n_k}, a^*))E(a_{2n_k}, a^*), \tag{2.8}$$

where

$$E(a_{2n_k}, a^*) = d(a_{2n_k}, a^*) + |d(a_{2n_k}, M^i a_{2n_k}) - d(a^*, N^j a^*)|$$
$$= d(a_{2n_k}, a^*) + |d(a_{2n_k}, a_{2n_k+1}) - d(a^*, N^j a^*)|$$

and

$$\lim_{k\to\infty} E(a_{2n_k}, a^*) = d(a^*, N^j a^*).$$

In view of the triangle inequality, we have

$$\frac{1}{s} d(a^*, N^j a^*) - d(a^*, a_{2n_k+1}) \leq d(a_{2n_k+1}, N^j a^*). \tag{2.9}$$

Combining (2.8) and (2.9) and letting $k \to \infty$, we conclude

$$\frac{1}{s} d(a^*, N^j a^*) \leq \lim_{k\to\infty} d(a_{2n_k+1}, N^j a^*) \leq \lim_{k\to\infty} \varphi(E(a_{2n_k}, a^*))E(a_{2n_k}, a^*) \leq \frac{1}{s} d(a^*, N^j a^*).$$

Thus, we obtain

$$\lim_{k\to\infty} \varphi(E(a_{2n_k}, a^*)) = \frac{1}{s} \text{ and therefore } \lim_{k\to\infty} E(a_{2n_k}, a^*) = 0,$$

that is,

$$d(a^*, N^j a^*) = 0.$$

In the same method, we get $d(a^*, M^i a^*) = 0$. Therefore, a^* is a common fixed point of M^i and N^j.

Next, we prove the uniqueness of the common fixed point. Suppose there exists another $b^* \in X$ such that $b^* = M^i b^*$. We have $\alpha_{i,j}(b^*, a^*) \geq s^p$ by condition (v). Consequently, we get

$$d(b^*, a^*) \leq \alpha_{i,j}(b^*, a^*) d(M^i b^*, N^j a^*) \leq \varphi(E(b^*, a^*))E(b^*, a^*),$$

where

$$E(b^*, a^*) = d(b^*, a^*) + |d(b^*, M^i b^*) - d(a^*, N^j a^*)| = d(b^*, a^*).$$

Hence, we have $d(b^*, a^*) < \frac{1}{s} d(b^*, a^*)$ which is contradictory. Then, we can obtain a^* which is a unique fixed point of M^i. By the similar method, one can obtain a^* is a unique fixed point of N^j. Since

$$Ma^* = MM^i a^* = M^i M a^* \text{ and } Na^* = NN^j a^* = N^j N a^*,$$

we find a^* is a common fixed point of M and N because of the uniqueness. It's easy to prove that a^* is a unique common fixed point of M and N. □

Example 2.7. Let $X = [0, 1]$ and $d : X \times X \to \mathbb{R}^+$ be defined as $d(a, b) = |a - b|^2$. It is clear that (X, d, s) forms a b-metric space with $s = 2$. Put $p = 2, i = 4, j = 2$ and

$$\alpha_{i,j}(a, b) = \begin{cases} s^p, & a, b \in [0, 1], \\ 0, & \text{others}. \end{cases}$$

Let $M, N : X \to X$ be defined by $M(a) = \frac{a}{2}, N(a) = \frac{a}{4}$ and take $\varphi(t) = \frac{1}{64}$ for all $t > 0$. According to the definition of M and N, it is easy to get that the pair (M, N) is triangular $\alpha_{i,j}$-orbital admissible. Next, we prove that M, N are $\alpha_{i,j}$-$\varphi_{E_{M,N}}$-Geraghty contraction mappings. Indeed, we have

$$\alpha_{i,j}(a,b) = 4, d(M^i(a), N^j(b)) = d(\frac{a}{16}, \frac{b}{16}) = \frac{1}{16^2}|a-b|^2,$$

$$E(a,b) = d(a,b) + |d(a, M^i a) - d(b, N^j b)| = |a-b|^2 + ||a - \frac{a}{16}|^2 - |b - \frac{b}{16}|^2|$$
$$= |a-b|^2 + |(\frac{15}{16}a)^2 - (\frac{15}{16}b)^2|.$$

Hence, we obtain

$$4 \cdot \frac{1}{16^2}|a-b|^2 \leq \frac{1}{64}[|a-b|^2 + |(\frac{15}{16}a)^2 - (\frac{15}{16}b)^2|],$$

which satisfies (2.1). In conclusion, for any $a, b \in X$, all the presumptions of Theorem 2.6 are satisfied. It follows that M and N have exactly one common fixed point in X. It is obvious that 0 is the unique common fixed point of M and N.

If (X, d) is a metric space and let $p = 1, i = j = 1$, and $\alpha_{i,j} = s^p = 1$ in Theorem 2.6, we obtain Theorem 5 in [12] immediately:

Corollary 2.8. Let (X, d) be a complete metric space, and $M, N : X \to X$ be two given mappings. If the pair (M, N) forms a Geraghty contraction of type $E_{M,N}$, then the pair of mappings M, N has a unique common fixed point.

If $p = 1, M = N$ and $i = j = 1$ in Theorem 2.6, we obtain Theorem 2.2 in [11] immediately :

Corollary 2.9. Let (X, d, s) be a complete b-metric space with parameter $s \geq 1$, $\alpha : X \times X \to [0, \infty)$ be a function and $M : X \to X$ be a given mapping. Suppose that the following conditions are satisfied:

(i) M is triangular $\alpha-$orbital admissible;

(ii) M is an $\alpha - \varphi_E-$Geraghty contraction mapping;

(iii) there is $a_0 \in X$ satisfying $\alpha(a_0, Ma_0) \geq 1$;

(iv) if $\{a_n\}$ is a sequence in X such that $\alpha(a_n, a_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $a_n \to a$ as $n \to \infty$, then there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\alpha(a_{n_k}, a) \geq 1$ for all $k \in \mathbb{N}$.

Then M has a fixed point $a^* \in X$.

Definition 2.10. Let M and N be two self-mappings defined on a nonempty set. Then, M and N are said to be (i, j)-weakly compatible if $M^i a = N^j a \Rightarrow MN^j a = NM^i a$ for every $a \in C(M, N)$.

Remark 2.11. For $i = j = 1$, the definition reduces to the definition of weakly compatible.

Definition 2.12. Let (X, d, s) be a $b-$metric space with parameter $s \geq 1$, and let $M, N : X \to X$ and $\alpha_{i,j} : X \times X \to [0, \infty)$ be given mappings satisfying

$$\alpha_{i,j}(a,b) \geq s^p \text{ and } \alpha_{i,j}(b,c) \geq s^p \Rightarrow \alpha_{i,j}(a,c) \geq s^p$$

for all $a, b, c \in X$, where $p \geq 1$ is an arbitrary constant. The mapping M is said to be N-$\alpha_{i,j}$-admissible if, for all $a, b \in X$,

$$\alpha_{i,j}(N^j a, N^j b) \geq s^p \text{ implies } \alpha_{i,j}(M^i a, M^i b) \geq s^p.$$

Remark 2.13.
(i) For $i = j = 1$, the definition reduces to an N-α_{s^p}-admissible mapping in a b-metric space.
(ii) For $i = j = 1$ and $N = I$, the definition reduces to an α_{s^p}-admissible mapping in a b-metric space.
(iii) For $s = p = i = j = 1$ and $N = I$, the definition reduces to the definition of an α-admissible mapping in a metric space.

Lemma 2.14. Let (X, d, s) be a b-metric space with parameter $s \geq 1$. Let $M : X \to X$ be a self-mapping such that M is N-$\alpha_{i.j}$-admissible. Assume that there exists $a_0 \in X$ such that $\alpha_{i,j}(N^j a_0, M^i a_0) \geq s^p$. Define sequences $\{a_n\}$ and $\{b_n\}$ in X by $b_n = M^i a_n = N^j a_{n+1}$ where $n = 0, 1, 2, \cdots$. Then for $n, m \in \mathbb{N} \cup \{0\}$ with $m > n$, we have $\alpha_{i,j}(b_n, b_m) \geq s^p$.

Proof. Since M is N-$\alpha_{i,j}$-admissible, then we have

$$\begin{aligned}
&\alpha_{i,j}(N^j a_0, N^j a_1) = \alpha_{i,j}(N^j a_0, M^i a_0) \geq s^p \Rightarrow \alpha_{i,j}(N^j a_1, N^j a_2) = \alpha_{i,j}(M^i a_0, M^i a_1) \geq s^p,\\
&\alpha_{i,j}(N^j a_1, N^j a_2) = \alpha_{i,j}(M^i a_0, M^i a_1) \geq s^p \Rightarrow \alpha_{i,j}(N^j a_2, N^j a_3) = \alpha_{i,j}(M^i a_1, M^i a_2) \geq s^p,\\
&\cdots.
\end{aligned}$$

Eventually, we get

$$\alpha_{i,j}(N^j a_n, N^j a_{n+1}) = \alpha_{i,j}(M^i a_{n-1}, M^i a_n) \geq s^p.$$

Hence, we get $\alpha_{i,j}(b_n, b_{n+1}) \geq s^p$ for all $n \in \mathbb{N}$. Since $\alpha_{i,j}(b_n, b_{n+1}) \geq s^p$ and $\alpha_{i,j}(b_{n+1}, b_{n+2}) \geq s^p$, we deduce $\alpha_{i,j}(b_n, b_{n+2}) \geq s^p$. It follows that one can get $\alpha_{i,j}(b_n, b_m) \geq s^p$ for all $n, m \in \mathbb{N} \cup \{0\}$ with $m > n$. □

Definition 2.15. Let (X, d, s) be a b-metric space with parameter $s \geq 1$, and let $M, N : X \to X$ be two self-mappings and $\alpha_{i,j} : X \times X \to [0, \infty)$ be a function. A mapping M is called a $\alpha_{i,j}$-φ_{E_N}-Geraghty contraction mapping, if there exists a function $\varphi \in \Phi_s$ satisfying the following:

$$\alpha_{i,j}(N^j a, N^j b) \geq s^p \Rightarrow \alpha_{i,j}(N^j a, N^j b) d(M^i a, M^i b) \leq \varphi(E(a, b))E(a, b), \text{for all } a, b \in X, \tag{2.10}$$

where

$$E(a, b) = d(N^j a, N^j b) + |d(M^i a, N^j a) - d(M^i b, N^j b)| \text{ and } p \geq 1 \text{ is a constant.}$$

Theorem 2.16. Let (X, d, s) be a complete b-metric space with parameter $s \geq 1$ and $\alpha_{i,j} : X \times X \to [0, \infty)$ be a function. Let $M, N : X \to X$ be two given self-mappings satisfying $M^i(X) \subset N^j(X)$ and $N^j(X)$ is closed. Suppose that the following conditions are satisfied:
(i) M is N-$\alpha_{i,j}$-admissible;
(ii) M is a $\alpha_{i,j}$-φ_{E_N}-Geraghty contraction mapping;
(iii) there is $a_0 \in X$ satisfying $\alpha_{i,j}(N^j a_0, M^i a_0) \geq s^p$;
(iv) if $\{a_n\}$ is a sequence in X such that $N^j a_n \to N^j a$ as $n \to \infty$, then there exists a subsequence $\{N^j a_{n_k}\}$ of $\{N^j a_n\}$ such that $\alpha_{i,j}(N^j a_{n_k}, N^j a) \geq s^p$ for all $k \in \mathbb{N}$;
(v) M and N are (i, j)-weakly compatible.
Then M and N have a coincidence point in X.

Proof. According to condition (iii), there exists an $a_0 \in X$ such that $\alpha_{i,j}(N^j a_0, M^i a_0) \geq s^p$. Now, we define sequences $\{a_n\}$ and $\{b_n\}$ in X by $b_n = M^i a_n = N^j a_{n+1}$ for all $n \in \mathbb{N}$. If $b_n = b_{n+1}$ for some $n \in \mathbb{N}$, then we deduce $N^j a_{n+1} = b_n = b_{n+1} = M^i a_{n+1}$ and M^i and N^j have a coincidence point a_{n+1}. Since M and N are (i, j)-weakly compatible mappings, then we get

$$M^i a_{n+1} = N^j a_{n+1} \Rightarrow MN^j a_{n+1} = NM^i a_{n+1}.$$

Therefore,

$$Mb_n = MM^i a_{n+1} = MN^j a_{n+1} = NM^i a_{n+1} = Nb_n,$$

that is, b_n is a coincidence point of M and N.

Without loss of generality, we assume that $b_n \neq b_{n+1}$ for all $n \in \mathbb{N}$. It follows from Lemma 2.14 that $\alpha_{i,j}(b_{n-1}, b_n) = \alpha_{i,j}(N^j a_n, N^j a_{n+1}) \geq s^p$. In light of condition (ii), we obtain

$$\begin{aligned} d(b_n, b_{n+1}) &= d(M^i a_n, M^i a_{n+1}) \\ &\leq s^p d(M^i a_n, M^i a_{n+1}) \\ &\leq \alpha_{i,j}(N^j a_n, N^j a_{n+1}) d(M^i a_n, M^i a_{n+1}) \\ &\leq \varphi(E(a_n, a_{n+1}))E(a_n, a_{n+1}), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} E(a_n, a_{n+1}) =& d(N^j a_n, N^j a_{n+1}) + |d(M^i a_n, N^j a_n) - d(M^i a_{n+1}, N^j a_{n+1})| \\ =& d(b_{n-1}, b_n) + |d(b_n, b_{n-1}) - d(b_{n+1}, b_n)|. \end{aligned} \tag{2.12}$$

If $d(b_{n+1}, b_n) \geq d(b_n, b_{n-1})$, we have $E(a_n, a_{n+1}) = d(b_n, b_{n+1}) > 0$, which means

$$d(b_n, b_{n+1}) \leq \varphi(d(b_n, b_{n+1}))d(b_n, b_{n+1}) < \frac{1}{s} d(b_n, b_{n+1}),$$

a contradiction. Hence, $d(b_{n+1}, b_n) < d(b_n, b_{n-1})$, that is, $\{d(b_n, b_{n+1})\}$ is a non-increasing sequence and so there exists $\gamma \geq 0$ such that

$$\lim_{n\to\infty} d(b_n, b_{n+1}) = \gamma.$$

In view of (2.12), we obtain

$$E(a_n, a_{n+1}) = 2d(b_n, b_{n-1}) - d(b_n, b_{n+1}) \text{ and } \lim_{n\to\infty} E(a_n, a_{n+1}) = \gamma. \tag{2.13}$$

Substituting (2.13) into (2.11) and taking the limit as $n \to \infty$, we get

$$\begin{aligned} \frac{\gamma}{s} &= \frac{1}{s} \lim_{n\to\infty} d(b_n, b_{n+1}) \\ &\leq \lim_{n\to\infty} d(b_n, b_{n+1}) \\ &\leq \lim_{n\to\infty} \varphi(E(a_n, a_{n+1}))E(a_n, a_{n+1}) \\ &\leq \lim_{n\to\infty} \frac{1}{s} E(a_n, a_{n+1}) \\ &= \frac{\gamma}{s}. \end{aligned}$$

The above formula means

$$\lim_{n\to\infty}\varphi(E(a_n,a_{n+1}))E(a_n,a_{n+1})=\frac{\gamma}{s} \text{ and } \lim_{n\to\infty}\varphi(E(a_n,a_{n+1}))=\frac{1}{s}.$$

So there is

$$\lim_{n\to\infty}E(a_n,a_{n+1})=\gamma=\lim_{n\to\infty}d(b_n,b_{n+1})=0.$$

Next, we aim to prove that $\{b_n\}$ is a Cauchy sequence. Suppose on the contrary, there exists $\ell>0$ for which one can find sequences $\{b_{m_k}\}$ and $\{b_{n_k}\}$ of $\{b_n\}$ satisfying n_k is the smallest index for which $n_k>m_k>k$, and

$$d(b_{n_k},b_{m_k})\geq\ell \text{ and } d(b_{n_k-1},b_{m_k})<\ell.$$

Since $\alpha_{i,j}(b_{n_k-1},b_{m_k-1})=\alpha_{i,j}(N^ja_{n_k},N^ja_{m_k})\geq s^p$, we have

$$\begin{aligned}\ell&\leq d(b_{n_k},b_{m_k})\\&=d(M^ia_{n_k},M^ia_{m_k})\\&\leq s^pd(M^ia_{n_k},M^ia_{m_k})\\&\leq\alpha_{i,j}(N^ja_{n_k},N^ja_{m_k})d(M^ia_{n_k},M^ia_{m_k})\\&\leq\varphi(E(a_{n_k},a_{m_k}))E(a_{n_k},a_{m_k}),\end{aligned}\tag{2.14}$$

where

$$\begin{aligned}E(a_{n_k},a_{m_k})=&d(N^ja_{n_k},N^ja_{m_k})+|d(M^ia_{n_k},N^ja_{n_k})-d(M^ia_{m_k},N^ja_{m_k})|\\=&d(b_{n_k-1},b_{m_k-1})+|d(b_{n_k},b_{n_k-1})-d(b_{m_k},b_{m_k-1})|\\\leq&sd(b_{n_k-1},b_{m_k})+sd(b_{m_k},b_{m_k-1})+|d(b_{n_k},b_{n_k-1})-d(b_{m_k},b_{m_k-1})|\\<&s\ell+sd(b_{m_k},b_{m_k-1})+|d(b_{n_k},b_{n_k-1})-d(b_{m_k},b_{m_k-1})|.\end{aligned}$$

Passing to the limit as $k\to\infty$ in the above inequality and (2.14), we obtain

$$\lim_{k\to\infty}E(a_{n_k},a_{m_k})=s\ell$$

and

$$\ell\leq\lim_{k\to\infty}\varphi(E(a_{n_k},a_{m_k}))E(a_{n_k},a_{m_k})\leq\frac{1}{s}E(a_{n_k},a_{m_k})\leq\frac{1}{s}s\ell=\ell.$$

We deduce

$$\lim_{k\to\infty}\varphi(E(a_{n_k},a_{m_k}))E(a_{n_k},a_{m_k})=\ell.$$

Therefore,

$$\lim_{k\to\infty}\varphi(E(a_{n_k},a_{m_k}))=\frac{1}{s} \text{ and } \lim_{k\to\infty}E(a_{n_k},a_{m_k})=0,$$

which is a contradiction. Hence, $\{b_n\}$ is a Cauchy sequence. The completeness of (X, d, s) ensures that there exists a $b^* \in X$ such that $b_n \to b^*$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} d(b_n, b^*) = \lim_{n\to\infty} d(M^i a_n, b^*) = \lim_{n\to\infty} d(N^j a_{n+1}, b^*) = 0.$$

Since $N^j(A)$ is closed, we have $b^* \in N^j(A)$. It follows that one can choose an $a^* \in X$ such that $b^* = N^j a^*$, and we can write

$$\lim_{n\to\infty} d(b_n, N^j a^*) = \lim_{n\to\infty} d(M^i a_n, N^j a^*) = \lim_{n\to\infty} d(N^j a_{n+1}, N^j a^*) = 0.$$

By using the condition (iv), we get that there exists a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ so that $\alpha_{i,j}(b_{n_k-1}, N^j a^*) = \alpha_{i,j}(N^j a_{n_k}, N^j a^*) \geq s^p$. Applying contractive condition (2.10), we have

$$d(M^i a_{n_k}, M^i a^*) \leq \alpha_{i,j}(N^j a_{n_k}, N^j a^*) d(M^i a_{n_k}, M^i a^*) \leq \varphi(E(a_{n_k}, a^*))E(a_{n_k}, a^*)$$

where

$$\begin{aligned} E(a_{n_k}, a^*) =& d(N^j a_{n_k}, N^j a^*) + |d(M^i a_{n_k}, N^j a_{n_k}) - d(M^i a^*, N^j a^*)| \\ =& d(b_{n_k-1}, N^j a^*) + |d(b_{n_k}, b_{n_k-1}) - d(M^i a^*, N^j a^*)| \end{aligned}$$

and

$$\lim_{k\to\infty} E(a_{n_k}, a^*) = d(M^i a^*, N^j a^*).$$

Since $d(M^i a^*, N^j a^*) \leq sd(M^i a^*, M^i a_{n_k}) + sd(M^i a_{n_k}, N^j a^*)$, then

$$\frac{1}{s} d(M^i a^*, N^j a^*) - d(M^i a_{n_k}, N^j a^*) \leq d(M^i a_{n_k}, M^i a^*) \leq \varphi(E(a_{n_k}, a^*))E(a_{n_k}, a^*).$$

Taking the limit as $k \to \infty$ in the above inequality, we obtain

$$\frac{1}{s} d(M^i a^*, N^j a^*) \leq \lim_{k\to\infty} \varphi(E(a_{n_k}, a^*))E(a_{n_k}, a^*) \leq \lim_{k\to\infty} \frac{1}{s} E(a_{n_k}, a^*) = \frac{1}{s} d(M^i a^*, N^j a^*).$$

As a result, we get

$$\lim_{k\to\infty} \varphi(E(a_{n_k}, a^*)) = \frac{1}{s} \text{ and } \lim_{k\to\infty} E(a_{n_k}, a^*) = d(M^i a^*, N^j a^*) = 0.$$

Thus, $b^* = M^i a^* = N^j a^*$ is a point of coincidence for M^i and N^j. Since M and N are (i, j)-weakly compatible mappings, then we get

$$M^i a^* = N^j a^* \Rightarrow MN^j a^* = NM^i a^*.$$

Therefore

$$Mb^* = MM^i a^* = MN^j a^* = NM^i a^* = Nb^*,$$

that is, b^* is a coincidence point of M and N. □

Example 2.17. Let $X = \{0, 1, 2\}$ and defined $d : X \times X \to \mathbb{R}$ by $d(a, b) = |a - b|^2$ for all $a, b \in X$. Therefore, (X, d, s) is a b-metric space with $s = 2$. Put $p = 2, i = 2, j = 3$ and

$$\alpha_{i,j}(a, b) = \begin{cases} s^p, & a, b \in X, \\ 0, & otherwise. \end{cases}$$

Let $M, N : X \to X$ be two mappings defined by $M(0) = 1, M(1) = 1, M(2) = 0$, $N(0) = 2, N(1) = 1$, and $N(2) = 0$. Take $\varphi(t) = \frac{1}{2s}$ for all $t \geq 0$.

It is obvious that M is N-α_{s^p}-admissible. Meanwhile, M is a $\alpha_{i,j}$-φ_{E_N}-Geraghty contraction mapping. Indeed, for $a = b$, we get $\alpha_{i,j}(N^j a, N^j b)d(M^i a, M^i b) \leq \varphi(E(a, b))E(a, b)$. On the other hand, we have

$$E(0, 1) = d(2, 1) + |d(1, 0) - d(1, 1)| = 2,$$

$$E(0, 2) = d(2, 0) + |d(1, 2) - d(1, 0)| = 4,$$

$$E(1, 2) = d(1, 0) + |d(1, 1) - d(1, 0)| = 2.$$

So, we obtain the following cases:

(a) $a = 0$ and $b = 1$. Then,

$$\alpha_{i,j}(N^j a, N^j b)d(M^i a, M^i b) = 4d(1, 1) = 0 \leq \frac{1}{2s}E(0, 1) = \frac{1}{2}.$$

(b) $a = 0$ and $b = 2$. Then,

$$\alpha_{i,j}(N^j a, N^j b)d(M^i a, M^i b) = 4d(1, 1) = 0 \leq \frac{1}{2s}E(0, 2) = 1.$$

(c) $a = 1$ and $b = 2$. Then,

$$\alpha_{i,j}(N^j a, N^j b)d(M^i a, M^i b) = 4d(1, 1) = 0 \leq \frac{1}{2s}E(1, 2) = \frac{1}{2}.$$

Considering the symmetry of d, we have (3.10) is true for all $a, b \in X$ and

$$M^i(1) = N^j(1) \Rightarrow NM^i(1) = MN^j(1),$$

which means M and N are (i, j)-weakly compatible. All hypotheses of Theorem 2.16 are satisfied. So, M and N have a coincidence point in X. Obviously, 1 is the coincidence point of M and N.

If (X, d) is a metric space and let $i = j = 1, p = 1, N = I$ in Theorem 2.16, we obtain the following result immediately:

Corollary 2.18. Let (X, d) be a complete metric space and $\alpha : X \times X \to [0, \infty)$ be a function. Let $M : X \to X$ be a given self-mapping. Suppose that the following conditions are satisfied:

(i) M is α-admissible;

(ii) M is a α-φ_{E_N}-Geraghty contraction mapping;

(iii) there is $a_0 \in X$ with satisfying $\alpha(Na_0, Ma_0) \geq 1$;

(iv) if $\{a_n\}$ is a sequence in X such that $a_n \to a$ as $n \to \infty$, then there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\alpha(a_{n_k}, a) \geq 1$ for all $k \in \mathbb{N}$;

Then M has a fixed point in X.

If $i = j = 1$ in Theorem 2.16, we have the following result:
Corollary 2.19. Let (X, d, s) be a complete b-metric space with parameter $s \geq 1$ and $\alpha : X\times X \to [0, \infty)$ be a function. Let $M, N : X \to X$ be two given self-mappings satisfying $M(X) \subset N(X)$ and $N(X)$ is closed. Suppose that the following conditions are satisfied:

(i) M is N-α_{s^p}-admissible;

(ii) M is a α-φ_{E_N}-Geraghty contraction mapping;

(iii) there is $a_0 \in X$ satisfying $\alpha(Na_0, Ma_0) \geq s^p$;

(iv) if $\{a_n\}$ is a sequence in X such that $Na_n \to Na$ as $n \to \infty$, then there exists a subsequence $\{Na_{n_k}\}$ of $\{Na_n\}$ such that $\alpha(Na_{n_k}, Na) \geq s^p$ for all $k \in \mathbb{N}$;

(v) M and N are weakly compatible.

Then M and N have a coincidence point in X.

3. Conclusions

In this manuscript, we introduced two new classes of Geraghty contraction mappings and established common fixed point and coincidence point results involving these new classes of mappings in the framework of b-metric spaces. Further, we provided two examples that elaborated on the usability of our results.

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Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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