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Research article

On some vector variational inequalities and optimization problems

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Abstract: This paper establishes connections between the solutions of some new vector controlled variational inequalities and (proper) efficient solutions of the corresponding multiobjective controlled variational problem. More precisely, under the assumptions of invexity and Fréchet differentiability of the involved curvilinear integral functionals, and by using the notion of invex set with respect to some given functions, we derive the characterization results.

Keywords: (proper) efficient solution; invex set; Fréchet differentiability; curvilinear integral
Mathematics Subject Classification: 49K20, 49J21

1. Introduction

Over time, it was necessary to introduce several concepts of *efficient solutions* for multiobjective optimization problems. Geoffrion [5] defined a rather narrow definition of efficiency, named *proper efficiency*. Klinger [13] introduced *improper solutions* for a given vector maximization problem. Kazmi [11] used vector variational-like inequalities to prove the existence of a *weak minimum* for some constrained vector optimization problems. Ghaznavi-ghosoni and Khorram [6] considered *approximate solutions* of the corresponding scalarized problems to establish efficiency conditions for approximating (weakly, properly) efficient points associated with general multi-objective optimization problems.

The concept of *convexity* is almost inevitable in optimization theory. However, since convexity is no longer sufficient in certain real-life problems, its generalization was a necessity. Thus, Hanson [8] defined *invex functions*. Over time, many other various extensions have been considered: preinvexity, univexity, pseudoinvexity, approximate convexity, quasiinvexity and so on (see, for instance, Antczak [2, 3], Arana-Jiménez et al. [4], Mishra et al. [14], Ahmad et al. [1]). In addition, these concepts have been converted for the multidimensional case defined by multiple/curvilinear integrals (see, for instance, Mititelu and Treanţă [15], Treanţă [18, 20]).

The crucial role of variational inequalities in engineering or traffic analysis is well known. Remarkable results for the vector case were developed by Giannessi [7]. Under suitable hypotheses, vector variational inequalities give the existence of solutions for multiobjective/vector optimization problems. Many papers centered on the links between the solutions of these types of inequalities and efficient solutions of mutiobjective optimization problems (see, for instance, Ruiz-Garzón et al. [16, 17], Jayswal et al. [9]). Recently, Treanţă [19] defined and studied a class of controlled variational inequalities defined by functionals of the curvilinear integral type.

Kim [12] formulated some relations between vector continuous-time programs and vector variational inequalities. As is well known, optimal control problems, regarded as continuous-time variational problems, represent a powerful ingredient for investigating many engineering problems and processes coming from game theory, economics and operations research. For this, Treanţă [21, 22] and Jha et al. [10] have contributed and proved necessary and sufficient optimality (efficiency) conditions, a saddle-point criterion, well-posedness and a modified objective function method for various multidimensional control problems determined by functionals of the multiple or curvilinear integral type.

As a natural continuation of the above-mentioned advances, in the current paper we introduce vector controlled variational inequalities and the corresponding multiobjective controlled variational problem, determined by functionals of the curvilinear integral type, which are independent of the path. By considering a new form of the notion of an invex set with respect to some given functions, we establish relations between the solutions of the considered multidimensional variational problems.

In the following, the paper is continued with the problem formulation and preliminaries. In Section 3, we establish the characterization results for the solutions associated with the considered variational problems. Section 4 contains the conclusions of this paper.

2. Problem description

In this paper, we begin with $\mathcal{B}$ as a domain in $\mathbb{R}^m$, that is supposed to be compact, and $\mathcal{B} \ni \zeta = (\zeta^\beta)$, $\beta = \overline{1,m}$, as a multi-variable of evolution. Denote by $\mathcal{B} \supset \mathsf{C} : \zeta = \zeta(\varsigma), \varsigma \in [x, y]$ a piecewise differentiable curve that links the two points $\zeta_1 = (\zeta_1^1, \ldots, \zeta_1^m)$, $\zeta_2 = (\zeta_2^1, \ldots, \zeta_2^m)$ in $\mathcal{B}$. Also, we introduce U as the space of all piecewise differentiable *state* functions $u : \mathcal{B} \to \mathbb{R}^n$ and V as the space of all *control* functions $v : \mathcal{B} \to \mathbb{R}^k$, which are supposed to be piecewise continuous. In addition, on $\mathsf{U} \times \mathsf{V}$ we define the scalar product

$$\langle (u, v), (\pi, x)\rangle = \int_{\mathsf{C}} \Big[u(\zeta) \cdot \pi(\zeta) + v(\zeta) \cdot x(\zeta)\Big] d\zeta^\beta$$

$$= \int_{\mathsf{C}} \Big[\sum_{i=1}^{n} u^i(\zeta)\pi^i(\zeta) + \sum_{j=1}^{k} v^j(\zeta)x^j(\zeta)\Big] d\zeta^1$$

$$+ \cdots + \Big[\sum_{i=1}^{n} u^i(\zeta)\pi^i(\zeta) + \sum_{j=1}^{k} v^j(\zeta)x^j(\zeta)\Big] d\zeta^m, \quad (\forall)(u, v), (\pi, x) \in \mathsf{U} \times \mathsf{V},$$

together with the norm induced by it.

By using the vector-valued C^2-class functions $h_\beta = (h^l_\beta) : \mathcal{B}\times\mathbb{R}^n\times\mathbb{R}^{nm}\times\mathbb{R}^k \to \mathbb{R}^p,\ \beta = \overline{1,m},\ l = \overline{1,p}$, we introduce the following vector functional defined by curvilinear integrals:

$$H : \mathsf{U}\times\mathsf{V} \to \mathbb{R}^p, \quad H(u,v) = \int_{\mathsf{C}} h_\beta\left(\zeta, u(\zeta), u_\alpha(\zeta), v(\zeta)\right) d\zeta^\beta =$$

$$= \left(\int_{\mathsf{C}} h^1_\beta\left(\zeta, u(\zeta), u_\alpha(\zeta), v(\zeta)\right) d\zeta^\beta, \cdots, \int_{\mathsf{C}} h^p_\beta\left(\zeta, u(\zeta), u_\alpha(\zeta), v(\zeta)\right) d\zeta^\beta \right).$$

In the following, $D_\alpha,\ \alpha \in \{1,\ldots,m\}$, denotes the operator of total derivative, and we assume that the 1-form densities of Lagrange type

$$h_\beta = \left(h^1_\beta, \ldots, h^p_\beta\right) : \mathcal{B}\times\mathbb{R}^n\times\mathbb{R}^{nm}\times\mathbb{R}^k \to \mathbb{R}^p, \quad \beta = \overline{1,m},$$

are closed, that is, $D_\alpha h^l_\beta = D_\beta h^l_\alpha,\ \beta,\alpha = \overline{1,m},\ \beta \neq \alpha,\ l = \overline{1,p}$. Also, throughout the paper, we will use the following rules associated with equalities and inequalities:

$$a = b \Leftrightarrow a^l = b^l, \quad a \leq b \Leftrightarrow a^l \leq b^l, \quad a < b \Leftrightarrow a^l < b^l, \quad a \preceq b \Leftrightarrow a \leq b,\ a \neq b, \quad l = \overline{1,p},$$

for any p-tuples $a = \left(a^1, \cdots, a^p\right),\ b = \left(b^1, \cdots, b^p\right)$ in $\mathbb{R}^p$.

Further, we introduce the following partial differential equation constrained *multiobjective variational control problem*

$$(CP) \quad \min_{(u,v)} \left\{ H(u,v) = \int_{\mathsf{C}} h_\beta\left(\zeta, u(\zeta), u_\alpha(\zeta), v(\zeta)\right) d\zeta^\beta \right\} \text{ subject to } (u,v) \in \mathcal{S},$$

where

$$H(u,v) = \int_{\mathsf{C}} h_\beta\left(\zeta, u(\zeta), u_\alpha(\zeta), v(\zeta)\right) d\zeta^\beta$$

$$= \left(\int_{\mathsf{C}} h^1_\beta\left(\zeta, u(\zeta), u_\alpha(\zeta), v(\zeta)\right) d\zeta^\beta, \cdots, \int_{\mathsf{C}} h^p_\beta\left(\zeta, u(\zeta), u_\alpha(\zeta), v(\zeta)\right) d\zeta^\beta \right)$$

$$= \left(H^1(u,v), \ldots, H^p(u,v)\right)$$

and

$$\mathcal{S} = \Big\{ (u,v) \in \mathsf{U}\times\mathsf{V} \mid Z\left(\zeta, u(\zeta), u_\alpha(\zeta), v(\zeta)\right) = 0,\ Y\left(\zeta, u(\zeta), u_\alpha(\zeta), v(\zeta)\right) \leq 0,$$

$$(u,v)|_{\zeta=\zeta_1,\zeta_2} = \text{given} \Big\}.$$

In the definition of $\mathcal{S}$, we have considered that $Z = (Z^\iota) : \mathcal{B}\times\mathbb{R}^n\times\mathbb{R}^{nm}\times\mathbb{R}^k \to \mathbb{R}^t,\ \iota = \overline{1,t},\ Y = (Y^r) : \mathcal{B}\times\mathbb{R}^n\times\mathbb{R}^{nm}\times\mathbb{R}^k \to \mathbb{R}^q,\ r = \overline{1,q}$, are assumed to be C^2-class functions.

Definition 2.1 (Mititelu and Treanţă [15]) A point $(u^0, v^0) \in \mathcal{S}$ is called an *efficient solution* in (CP) if there exists no other $(u,v) \in \mathcal{S}$ such that $H(u,v) \preceq H(u^0,v^0)$, or, equivalently, $H^l(u,v) - H^l(u^0,v^0) \leq 0,\ (\forall) l = \overline{1,p}$, with strict inequality for at least one l.

Definition 2.2 (Geoffrion [5]) A point $(u^0, v^0) \in \mathcal{S}$ is called a *proper efficient solution* in (CP) if $(u^0, v^0) \in \mathcal{S}$ is an efficient solution in (CP), and there exists a positive real number M such that, for all $l = \overline{1,p}$, we have

$$H^l(u^0,v^0) - H^l(u,v) \leq M\left(H^s(u,v) - H^s(u^0,v^0)\right),$$

for some $s \in \{1, \cdots, p\}$ such that

$$H^s(u, v) > H^s(u^0, v^0),$$

whenever $(u, v) \in \mathcal{S}$ and

$$H^l(u, v) < H^l(u^0, v^0).$$

According to Treanţă [20], for $u \in \mathsf{U}$ and $v \in \mathsf{V}$, we consider the vector functional of curvilinear integral type (which is independent of the path)

$$K : \mathsf{U} \times \mathsf{V} \to \mathbb{R}^p, \quad K(u, v) = \int_{\mathsf{C}} \kappa_\beta (\zeta, u(\zeta), u_\alpha(\zeta), v(\zeta))\, d\zeta^\beta$$

and introduce the concept of invexity associated with K.

Definition 2.3 If there exist

$$\vartheta : \mathcal{B} \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n,$$

$$\vartheta = \vartheta\left(\zeta, u(\zeta), v(\zeta), u^0(\zeta), v^0(\zeta)\right) = \left(\vartheta^i\left(\zeta, u(\zeta), v(\zeta), u^0(\zeta), v^0(\zeta)\right)\right), \quad i = \overline{1, n},$$

of C^1-class with $\vartheta\left(\zeta, u^0(\zeta), v^0(\zeta), u^0(\zeta), v^0(\zeta)\right) = 0,\ (\forall)\zeta \in \mathcal{B},\ \vartheta(\zeta_1) = \vartheta(\zeta_2) = 0$, and

$$\upsilon : \mathcal{B} \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k,$$

$$\upsilon = \upsilon\left(\zeta, u(\zeta), v(\zeta), u^0(\zeta), v^0(\zeta)\right) = \left(\upsilon^j\left(\zeta, u(\zeta), v(\zeta), u^0(\zeta), v^0(\zeta)\right)\right), \quad j = \overline{1, k},$$

of C^0-class with $\upsilon\left(\zeta, u^0(\zeta), v^0(\zeta), u^0(\zeta), v^0(\zeta)\right) = 0,\ (\forall)\zeta \in \mathcal{B},\ \upsilon(\zeta_1) = \upsilon(\zeta_2) = 0$, such that

$$K(u, v) - K\left(u^0, v^0\right)$$

$$\geq \int_{\mathsf{C}} \left[\frac{\partial \kappa_\beta}{\partial u}\left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta)\right)\vartheta + \frac{\partial \kappa_\beta}{\partial u_\alpha}\left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta)\right) D_\alpha \vartheta\right] d\zeta^\beta$$

$$+ \int_{\mathsf{C}} \left[\frac{\partial \kappa_\beta}{\partial v}\left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta)\right) \upsilon\right] d\zeta^\beta,$$

for any $(u, v) \in \mathsf{U} \times \mathsf{V}$, then K is said to be *invex at* $\left(u^0, v^0\right) \in \mathsf{U} \times \mathsf{V}$ *with respect to* ϑ *and* υ.

Definition 2.4 In the above definition, with $(u, v) \neq \left(u^0, v^0\right)$, if we replace $\geq$ with $>$, we say that K is *strictly invex at* $\left(u^0, v^0\right) \in \mathsf{U} \times \mathsf{V}$ *with respect to* ϑ *and* υ.

Some examples of invex curvilinear integral functionals can be consulted in Treanţă [20].

Definition 2.5 The nonempty subset $\mathsf{X} \times \mathsf{Q} \subset \mathsf{U} \times \mathsf{V}$ is said to be *invex with respect to* ϑ *and* υ if

$$(u^0, v^0) + \lambda\left(\vartheta\left(\zeta, u, v, u^0, v^0\right), \upsilon\left(\zeta, u, v, u^0, v^0\right)\right) \in \mathsf{X} \times \mathsf{Q},$$

for all $(u, v),\ (u^0, v^0) \in \mathsf{X} \times \mathsf{Q}$ and $\lambda \in [0, 1]$.

Now, in order to formulate and prove some results on the existence of solutions for problem (CP), we introduce the following *vector controlled variational inequalities*: *find* $(u^0, v^0) \in \mathcal{S}$ *such that there exists no* $(u, v) \in \mathcal{S}$ *satisfying*

$$(VI) \quad \Big(\int_{\mathsf{C}} \left[\frac{\partial h^1_\beta}{\partial u} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) \vartheta + \frac{\partial h^1_\beta}{\partial v} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) \upsilon \right] d\zeta^\beta$$

$$+ \int_{\mathsf{C}} \left[\frac{\partial h^1_\beta}{\partial u_\alpha} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) D_\alpha \vartheta \right] d\zeta^\beta, \cdots,$$

$$\int_{\mathsf{C}} \left[\frac{\partial h^p_\beta}{\partial u} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) \vartheta + \frac{\partial h^p_\beta}{\partial v} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) \upsilon \right] d\zeta^\beta$$

$$+ \int_{\mathsf{C}} \left[\frac{\partial h^p_\beta}{\partial u_\alpha} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) D_\alpha \vartheta \right] d\zeta^\beta \Big) \leq 0.$$

3. Main results

In this section, we will formulate the characterization results and connections between the solutions of the considered vector controlled variational inequalities and (proper) efficient solutions of the introduced multiobjective variational control problem (CP).

Theorem 3.1 *Let* $\mathcal{S} \subset \mathsf{U} \times \mathsf{V}$ *be an invex set with respect to* ϑ *and* υ, *and let* $(u^0, v^0) \in \mathcal{S}$ *be a proper efficient solution of* (CP). *If each curvilinear integral*

$$\int_{\mathsf{C}} h^l_\beta \left(\zeta, u(\zeta), u_\alpha(\zeta), v(\zeta) \right) d\zeta^\beta, \quad l = \overline{1, p},$$

is Fréchet differentiable at $(u^0, v^0) \in \mathcal{S}$, *then the pair* (u^0, v^0) *solves* (VI).

Proof. By reductio ad absurdum, consider that $(u^0, v^0) \in \mathcal{S}$ is a proper efficient solution of (CP), but it does not satisfy (VI). In consequence, there exists $(u, v) \in \mathcal{S}$ such that, for all $l = \overline{1, p}$, we have

$$\int_{\mathsf{C}} \left[\frac{\partial h^l_\beta}{\partial u} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) \vartheta + \frac{\partial h^l_\beta}{\partial v} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) \upsilon \right] d\zeta^\beta$$

$$+ \int_{\mathsf{C}} \left[\frac{\partial h^l_\beta}{\partial u_\alpha} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) D_\alpha \vartheta \right] d\zeta^\beta < 0, \tag{1}$$

and, for $s \neq l$,

$$\int_{\mathsf{C}} \left[\frac{\partial h^s_\beta}{\partial u} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) \vartheta + \frac{\partial h^s_\beta}{\partial v} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) \upsilon \right] d\zeta^\beta$$

$$+ \int_{\mathsf{C}} \left[\frac{\partial h^s_\beta}{\partial u_\alpha} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) D_\alpha \vartheta \right] d\zeta^\beta \leq 0. \tag{2}$$

By hypothesis, we have that $\mathcal{S} \subset \mathsf{U}\times\mathsf{V}$ is an invex set with respect to ϑ and υ. Thus, we can consider the pair $(z, w) = (u^0, v^0) + \lambda_n \left(\vartheta\left(\zeta, u, v, u^0, v^0\right), \upsilon\left(\zeta, u, v, u^0, v^0\right)\right) \in \mathcal{S},\ (\forall)n$, for some sequence $\{\lambda_n\}$ of positive real numbers satisfying $\lambda_n \to 0$ as $n \to \infty$.

Further, we establish that each curvilinear integral $\int_{\mathsf{C}} h^l_\beta\left(\zeta, u(\zeta), u_\alpha(\zeta), v(\zeta)\right) d\zeta^\beta,\ l = \overline{1, p}$, is Fréchet differentiable at $(u^0, v^0) \in \mathcal{S}$ and obtain the following equality:

$$
\begin{aligned}
&H^l(z, w) - H^l(u^0, v^0)\\
&= \int_{\mathsf{C}} \lambda_n \left[\frac{\partial h^l_\beta}{\partial u}\left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta)\right)\vartheta + \frac{\partial h^l_\beta}{\partial v}\left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta)\right)\upsilon\right] d\zeta^\beta\\
&+ \int_{\mathsf{C}} \lambda_n \left[\frac{\partial h^l_\beta}{\partial u_\alpha}\left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta)\right) D_\alpha\vartheta\right] d\zeta^\beta\\
&+ \parallel \lambda_n \left(\vartheta\left(\zeta, u, v, u^0, v^0\right), \upsilon\left(\zeta, u, v, u^0, v^0\right)\right) \parallel \cdot G^l(z, w),
\end{aligned}
\tag{3}
$$

where $G^l : V_{(u^0,v^0)} \to \mathbb{R}$ is a continuous function defined on a neighborhood of (u^0, v^0), denoted by $V_{(u^0,v^0)}$, with $\lim_{n\to\infty} G^l(z, w) = 0$. By dividing (3) by λ_n and taking the limit, we get

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\lambda_n}\left[H^l(z, w) - H^l(u^0, v^0)\right]\\
&= \int_{\mathsf{C}} \left[\frac{\partial h^l_\beta}{\partial u}\left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta)\right)\vartheta + \frac{\partial h^l_\beta}{\partial v}\left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta)\right)\upsilon\right] d\zeta^\beta\\
&+ \int_{\mathsf{C}} \lambda_n \left[\frac{\partial h^l_\beta}{\partial u_\alpha}\left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta)\right) D_\alpha\vartheta\right] d\zeta^\beta.
\end{aligned}
\tag{4}
$$

Combining relations (1) and (4), it results that

$$
H^l(z, w) - H^l(u^0, v^0) < 0,
$$

for some $n \geq N$, with N being a natural number.

Next, since $(u^0, v^0) \in \mathcal{S}$ is a proper efficient solution of (CP), we consider the nonempty set

$$
\mathcal{M} = \left\{s \in \{1, \cdots, p\} \mid H^s(u^0, v^0) - H^s(z, w) \leq 0,\ (\forall)n \geq N\right\}.
$$

For $s \in \mathcal{M}$, by considering the Fréchet differentiability of $\int_{\mathsf{C}} h^s_\beta\left(\zeta, u(\zeta), u_\alpha(\zeta), v(\zeta)\right) d\zeta^\beta$ at $(u^0, v^0) \in \mathcal{S}$, we obtain

$$
\begin{aligned}
&H^s(z, w) - H^s(u^0, v^0)\\
&= \int_{\mathsf{C}} \lambda_n \left[\frac{\partial h^s_\beta}{\partial u}\left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta)\right)\vartheta + \frac{\partial h^s_\beta}{\partial v}\left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta)\right)\upsilon\right] d\zeta^\beta\\
&+ \int_{\mathsf{C}} \lambda_n \left[\frac{\partial h^s_\beta}{\partial u_\alpha}\left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta)\right) D_\alpha\vartheta\right] d\zeta^\beta
\end{aligned}
$$

$$+ \parallel \lambda_n \left(\vartheta \left(\zeta, u, v, u^0, v^0 \right), \upsilon \left(\zeta, u, v, u^0, v^0 \right) \right) \parallel \cdot G^s(z, w), \tag{5}$$

where $G^s : V_{(u^0,v^0)} \to \mathbb{R}$ is a continuous function defined on a neighborhood of (u^0, v^0), denoted by $V_{(u^0,v^0)}$, with $\lim_{n\to\infty} G^s(z, w) = 0$. By dividing (5) by λ_n and taking the limit, we get

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left[H^s(z, w) - H^s(u^0, v^0) \right]$$

$$= \int_C \left[\frac{\partial h^s_\beta}{\partial u} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) \vartheta + \frac{\partial h^s_\beta}{\partial v} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) \upsilon \right] d\zeta^\beta$$

$$+ \int_C \left[\frac{\partial h^s_\beta}{\partial u_\alpha} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) D_\alpha \vartheta \right] d\zeta^\beta.$$

By using the property of the set $\mathcal{M}$, for $n \geq N$, we get

$$\int_C \left[\frac{\partial h^s_\beta}{\partial u} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) \vartheta + \frac{\partial h^s_\beta}{\partial v} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) \upsilon \right] d\zeta^\beta \tag{6}$$

$$+ \int_C \left[\frac{\partial h^s_\beta}{\partial u_\alpha} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) D_\alpha \vartheta \right] d\zeta^\beta \geq 0.$$

Combining relations (2) and (6), it results that

$$\int_C \left[\frac{\partial h^s_\beta}{\partial u} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) \vartheta + \frac{\partial h^s_\beta}{\partial v} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) \upsilon \right] d\zeta^\beta$$

$$+ \int_C \left[\frac{\partial h^s_\beta}{\partial u_\alpha} \left(\zeta, u^0(\zeta), u^0_\alpha(\zeta), v^0(\zeta) \right) D_\alpha \vartheta \right] d\zeta^\beta = 0,$$

for some $n \geq N$, with N being a natural number, and $s \neq l,\ s \in \mathcal{M}$.

Finally, for $s \neq l,\ s \in \mathcal{M}$, by computing the limit

$$\frac{\frac{1}{\lambda_n} \left[H^l(u^0, v^0) - H^l(z, w) \right]}{\frac{1}{\lambda_n} \left[H^s(z, w) - H^s(u^0, v^0) \right]},$$

we find that it is ∞ as $n \to \infty$, which contradicts the proper efficiency of (u^0, v^0) for (CP), and the proof is now complete. □

The next theorem provides a charaterization of the efficient solutions for (CP) by using the vector controlled variational inequality (VI).

Theorem 3.2 *Let* $(u^0, v^0) \in \mathcal{S}$ *be a solution of* (VI). *If each curvilinear integral* $\int_C h^l_\beta (\zeta, u(\zeta), u_\alpha(\zeta), v(\zeta))\, d\zeta^\beta,\ l = \overline{1, p}$, *is Fréchet differentiable and invex at* $(u^0, v^0) \in \mathcal{S}$ *with respect to* ϑ *and* υ, *then the pair* (u^0, v^0) *is an efficient solution of* (CP).

Proof. By reductio ad absurdum, consider that $(u^0, v^0) \in \mathcal{S}$ is a solution of (VI), but it is not an efficient solution of (CP). In consequence, there exists $(u, v) \in \mathcal{S}$ such that, for all $l = \overline{1, p}$,

$$H^l(u, v) - H^l(u^0, v^0) \leq 0, \tag{7}$$

with strict inequality for at least one l.

By hypothesis, each curvilinear integral $\int_{C} h_{\beta}^{l}(\zeta, u(\zeta), u_{\alpha}(\zeta), v(\zeta))\, d\zeta^{\beta}$, $l = \overline{1,p}$, is Fréchet differentiable and invex at $(u^0, v^0) \in \mathcal{S}$ with respect to ϑ and υ. In consequence, we have

$$H^l(u,v) - H^l(u^0,v^0)$$

$$\geq \int_C \left[\frac{\partial h_{\beta}^{l}}{\partial u}\left(\zeta, u^0(\zeta), u_{\alpha}^0(\zeta), v^0(\zeta)\right)\vartheta + \frac{\partial h_{\beta}^{l}}{\partial v}\left(\zeta, u^0(\zeta), u_{\alpha}^0(\zeta), v^0(\zeta)\right)\upsilon \right] d\zeta^{\beta}$$

$$+ \int_C \left[\frac{\partial h_{\beta}^{l}}{\partial u_{\alpha}}\left(\zeta, u^0(\zeta), u_{\alpha}^0(\zeta), v^0(\zeta)\right) D_{\alpha}\vartheta \right] d\zeta^{\beta}, \tag{8}$$

for any $(u,v) \in \mathcal{S}$ and $l = \overline{1,p}$.

On combining inequalities (7) and (8), we find that, for all $l = \overline{1,p}$, there exists $(u,v) \in \mathcal{S}$ such that

$$\int_C \left[\frac{\partial h_{\beta}^{l}}{\partial u}\left(\zeta, u^0(\zeta), u_{\alpha}^0(\zeta), v^0(\zeta)\right)\vartheta + \frac{\partial h_{\beta}^{l}}{\partial v}\left(\zeta, u^0(\zeta), u_{\alpha}^0(\zeta), v^0(\zeta)\right)\upsilon \right] d\zeta^{\beta}$$

$$+ \int_C \left[\frac{\partial h_{\beta}^{l}}{\partial u_{\alpha}}\left(\zeta, u^0(\zeta), u_{\alpha}^0(\zeta), v^0(\zeta)\right) D_{\alpha}\vartheta \right] d\zeta^{\beta} \leq 0,$$

with strict inequality for at least one l, which contradicts that $(u^0, v^0) \in \mathcal{S}$ is a solution of (VI). The proof is now complete. □

4. Conclusions

In the current paper, by using the invexity and Fréchet differentiability of the involved curvilinear integral functionals (which are independent of the path), we have formulated and proved a connection between the solutions of some vector controlled variational inequalities and (proper) efficient solutions of a multiobjective controlled variational problem. Also, the notion of an invex set with respect to some given functions played an important role in our arguments. The theory developed in this paper can be converted by considering the concept of the variational derivative associated with multiple/curvilinear integral functionals (see Treanţă [19]).

Conflict of interest

The author declares no conflict of interest.

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