Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(8): 14419–14433.
DOI: 10.3934/math.2022794
Received: 02 April 2022
Revised: 19 May 2022
Accepted: 23 May 2022
Published: 06 June 2022

Research article

Existence theory and generalized Mittag-Leffler stability for a nonlinear Caputo-Hadamard FIVP via the Lyapunov method

Hadjer Belbali[1], **Maamar Benbachir**[2], **Sina Etemad**[3], **Choonkil Park**[4,*] **and Shahram Rezapour**[3,5,*]

[1] Laboratory of Mathematics and Applied Sciences, University of Ghardaia, Algeria

[2] Faculty of Sciences, Saad Dahlab University, Blida 1, Algeria

[3] Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran

[4] Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea

[5] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan

* **Correspondence:** Emails: baak@hanyang.ac.kr, sh.rezapour@azaruniv.ac.ir.

Abstract: This paper discusses the existence, uniqueness and stability of solutions for a nonlinear fractional differential system consisting of a nonlinear Caputo-Hadamard fractional initial value problem (FIVP). By using some properties of the modified Laplace transform, we derive an equivalent Hadamard integral equation with respect to one-parametric and two-parametric Mittag-Leffer functions. The Banach contraction principle is used to give the existence of the corresponding solution and its uniqueness. Then, based on a Lyapunov-like function and a $\mathcal{K}$-class function, the generalized Mittag-Leffler stability is discussed to solve a nonlinear Caputo-Hadamard FIVP. The findings are validated by giving an example.

Keywords: Caputo-Hadamard derivative; Lyapunov direct method; $\mathcal{K}$-class function; fixed point; Mittag-Leffler stability
Mathematics Subject Classification: 26A33, 34A08

1. Introduction

Fractional calculus is a well-known theory regarding fractional differential equations (FDEs) which has received much consideration and attention during the past decades and also has became the most important branch in applied analysis because of its extensive applications in a vast range of applied sciences [1–3].

Meanwhile, the variety of fractional operators defined by mathematicians has led researchers to focus on the differences and outputs of mathematical models designed by these operators and to use a wide range of fractional derivation operators in their studies. Some of the prominent works in this field are different types of fractional mathematical models in which the effects of the order of fractional derivatives on the dynamic behavior of the solutions of the assumed systems are carefully simulated. Some examples include the following: In [4, 5], the use of a Caputo derivative; in [6, 7], the use of a Caputo-conformable derivative; in [8–10], the use of a generalized derivative; in [11, 12], the use of a quantum Caputo derivative; in [13,14], the use of a nonsingular Caputo-Fabrizio derivative; in [15–18], the use of a nonsingular Mittag-Leffler kernel-type derivative.

One of the fractional derivatives that is defined by the combination of the properties of the Caputo and Hadamard operators is the Caputo-Hadamard fractional derivative. There are limited fractional models and problems designed by this operator. Examples can be seen in [19–24].

Hence, as we see, the existence and uniqueness problems for FDEs have many forms according to the shape of the differential model and of course the form of the initial or boundary conditions. In the newly published works, the role of fractional calculus in the topics of control theory can be widely observed. In the meantime, the fractional order controller is one of the key concepts in the field of control problems. One of the most important specifications of the control problems is stability analysis which is considered to be a fundamental condition for every control problem. In 1996, Matignon [25] was one of the first mathematicians to conduct research on the stability of linear differential systems using a Caputo operator. Since then, many researchers have implemented further investigations into the stability of such linear fractional systems [26, 27]. In regard to the nonlinear fractional systems, the stability criterion is much more difficult. The direct method attributed to Lyapunov gives a way to study a special type of stability tremed the Mittag-Leffler stability for a given fractional nonlinear system without solving it explicitly [28, 29]. Such a direct method due to Lyapunov is a sufficient condition to confirm the stability of the nonlinear systems; in other words, the given systems may still be stable even if we cannot choose a Lyapunov's mapping to fulfill the stability property for the mentioned system.

In this paper, the main properties such as the existence , uniqueness and different types of stability are studied for the fractional system involving the nonlinear Caputo-Hadamard FIVP as given by

$$\begin{cases} {}_{CH}D_c^{\ell}\phi(t) = A\phi(t) + \psi(t, \phi(t), {}_{CH}D_c^{\beta}\phi(t)), \quad t > c > 0, \\ \Theta^k\phi(t) \mid_{t=c} = \phi_k, k = 0, 1. \end{cases} \tag{1.1}$$

Where $1 < \ell < 2$, $0 < \beta < \ell - 1$, $\phi_0, \phi_1 \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $\Theta = t\frac{d}{dt}$ and $\psi : [c, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function. ${}_{CH}D_c^{\ell}$ and ${}_{CH}D_c^{\beta}$ are the Caputo-Hadamard derivatives of orders ℓ and β, respectively. The basic motivation and novelty of this work is that we attempt to use some specifications of the modified Laplace transform to the Caputo-Hadamard FIVP to derive the corresponding Hadamard integral equation in terms of one-parametric and two-parametric Mittag-Leffler functions. Also, there is no work about the generalized Mittag-Leffler stability for a fractional system designed by using a Caputo-Hadamard operator so far. Thus, with the help of Lyapunov functions and the aid of $\mathcal{K}$-class functions, we will prove this type of stability.

The manuscript is structured as follows. Section 2 is devoted to recalling definitions, theorems, lemmas and remarks that will be applied throughout the next sections. In Section 3, we shall give several sufficient conditions confirming the existence of the solution and its uniqueness for the nonlinear Caputo-Hadamard FIVP given by (1.1) using the Banach contraction principle. In Section 4, by using a Lyapunov-like function and a $\mathcal{K}$-class function, the generalized Mittag-Leffler stability for the Caputo-Hadamard system (1.1) is established. We validate our findings in Section 5 and end the paper in Section 6.

2. Preliminaries

At first, the fundamental notions related to the scope of the present paper are recollected in this section. Let the space

$$AC^n_\Theta = \{\mathfrak{h} : [c, b] \to \mathbb{R} : \Theta^{n-1}\mathfrak{h}(t) \in AC[c, b]\},$$

be so that $\Theta = t\frac{d}{dt}$ stands for the Hadamard derivative, and $AC([c, b], \mathbb{R})$ consists of all functions on $[c, b]$ with the absolute continuity property.

Definition 2.1. *[1, 30] The Hadamard integral of a given function $\psi(t) : [c, b] \to \mathbb{R}$ of the order $\ell > 0$ is defined by*

$${}_HD^{-\ell}_{c^+}\psi(t) = \frac{1}{\Gamma(\ell)} \int_c^t \left(\ln\frac{t}{w}\right)^{\ell-1} \psi(w)\frac{dw}{w}, \quad t > c > 0.$$

Definition 2.2. *[1] The Hadamard derivative of a function $\psi(t) : [c, b] \to \mathbb{R}$ belonging to AC^n_Θ of the order ℓ is defined by*

$$\begin{aligned}
{}_HD^{\ell}_{c^+}\psi(t) &= \Theta^n \left[{}_HD^{-(n-\ell)}_{c^+}\psi(t)\right] \\
&= \frac{1}{\Gamma(n-\ell)}\Theta^n \int_c^t \left(\ln\frac{t}{w}\right)^{n-\ell-1} \psi(w)\frac{dw}{w}, \quad t > c > 0,
\end{aligned}$$

where $\Theta = t\frac{d}{dt}$, and $n - 1 < \ell < n \in \mathbb{Z}^+$.

Lemma 2.3. *[31] Let $\ell > 0,\ n = [\ell]+1$. If $\psi(t) \in AC^n_\Theta$, then the Hadamard fractional derivative ${}_HD^{\ell}_{c^+}$ exists almost everywhere on $[c, b]$ and can be represented in the following form:*

$$({}_HD^{\ell}_{c^+}\psi)(t) = \sum_{k=0}^{n-1} \frac{(\Theta^k\psi)(c)}{\Gamma(1 + k - \ell)}(\ln\frac{t}{c})^{k-\ell} + \frac{1}{\Gamma(n-\ell)} \int_c^t (\ln\frac{t}{w})^{n-\ell-1}(\Theta^n\psi)(w)dw.$$

In particular, when $0 < \ell < 1$, then, for $\psi(t) \in AC[c, b]$,

$$({}_HD^{\ell}_{c^+}\psi)(t) = \frac{\psi(c)}{\Gamma(1-\ell)}(\ln\frac{t}{c})^{-\ell} + \frac{1}{\Gamma(1-\ell)} \int_c^t (\ln\frac{t}{w})^{-\ell}\psi'(w)\frac{dw}{w}.$$

Definition 2.4. *[32] The Caputo-Hadamard derivative of the function $\psi(t)$ of the order $\ell(n-1 < \ell < n)$ is defined by*

$$\begin{aligned}
({}_{CH}D^{\ell}_{c^+}\psi)(t) &= {}_HD^{-(n-\ell)}_{c^+}[\Theta^n\psi(t)] \\
&= \frac{1}{\Gamma(n-\ell)} \int_c^t \left(\ln\frac{t}{w}\right)^{n-\ell-1} \Theta^n\psi(w)\frac{dw}{w}, \quad t > c > 0.
\end{aligned}$$

Lemma 2.5. *[32] If $\psi(t) \in AC_{\Theta}^{n}$ is a function such that ${}_{CH}D^{\ell}\psi(t)$ and ${}_{H}D^{\ell}\psi(t)$ exist, then*

$${}_{CH}D_{c}^{\ell}\psi(t) = {}_{H}D_{c}^{\ell}\psi(t) - \sum_{k=0}^{n-1} \frac{(t\frac{d}{dt})^{k}\psi(c)}{\Gamma(k-\ell+1)}(\ln\frac{t}{c})^{k-\ell},$$

and when $0 < \ell < 1$, then

$${}_{CH}D_{c}^{\ell}\psi(t) = {}_{H}D_{c}^{\ell}\psi(t) - \frac{\psi(c)}{\Gamma(1-\ell)}(\ln\frac{t}{c})^{-\ell}.$$

In view of the aforementioned definitions related to the Hadamard operators (integral and derivative operators), we can not obtain the corresponding Laplace transforms due to the initial value starting at the time $t = c > 0$. For this reason, it is necessary that we provide a new type of definition for the case with the starting value at the time $t = c > 0$.

Definition 2.6. *[33, 34] For a mapping $\psi(t)$ given on $[c, \infty)(c > 0)$, the modified Laplace transform of ψ is defined by*

$$\tilde{\psi}(s) = \mathbb{L}_{c}\{\psi(t)\} = \int_{c}^{\infty} \psi(t)e^{-s\ln\frac{t}{c}}\frac{dt}{t}, \ s \in \mathbb{C}.$$

Also, the inverse modified Laplace transform of $\tilde{\psi}(s)$ is defined by

$$\psi(t) = \mathbb{L}_{c}^{-1}\{\tilde{\psi}(s)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{\psi}(s)e^{s\ln\frac{t}{c}}\,ds, \quad c > 0, \ i^{2} = -1.$$

The following properties are fulfilled for these modified transforms.

Proposition 2.7. *[34] If $\mathbb{L}_{c}\{\psi(t)\} = \tilde{\psi}(s)$, then*

$$\mathbb{L}_{c}\{\Theta^{n}\psi(t)\} = s^{n}\tilde{\psi}(s) - \sum_{k=0}^{n-1} s^{n-k-1}\Theta^{k}\psi(c), \quad t > c > 0, \ n \in \mathbb{Z}^{+},$$

where $\Theta = t\frac{d}{dt}$.

Lemma 2.8. *[34] Let $n - 1 < \ell < n$. Then*

$$\begin{aligned} \mathbb{L}_{c}\{{}_{H}D_{c,t}^{-\ell}\psi(t)\} &= s^{-\ell}\mathbb{L}_{c}\{\psi(t)\}, \\ \mathbb{L}_{c}\{{}_{H}D_{c,t}^{\ell}\psi(t)\} &= s^{\ell}\mathbb{L}_{c}\{\psi(t)\} - \sum_{k=0}^{n-1} s^{n-k-1}[\Theta_{H}^{k}D_{c,t}^{-(n-\ell)}\psi(t)]|_{t=c}, \\ \mathbb{L}_{c}\{{}_{CH}D_{c,t}^{\ell}\psi(t)\} &= s^{\ell}\mathbb{L}_{c}\{\psi(t)\} - \sum_{k=0}^{n-1} s^{\ell-k-1}\Theta^{k}\psi(c). \end{aligned}$$

Definition 2.9. *[34] Let ψ and h be defined on $[c, \infty)$. Then the integral $\int_{c}^{t} \psi(c\frac{t}{w})h(w)\frac{dw}{w}$ is termed the convolution of ψ and h, that is,*

$$\psi(t) * h(t) = (\psi * h)(t) = \int_{c}^{t} \psi(c\frac{t}{w})h(w)\frac{dw}{w}. \tag{2.1}$$

Proposition 2.10. *[34] If* $\mathbb{L}_c\{\psi(t)\} = \tilde{\psi}(s)$ *and* $\mathbb{L}_c\{h(t)\} = \tilde{h}(s)$*, then*

$$\mathbb{L}_c\{\psi(t) * h(t)\} = \mathbb{L}_c\{\psi(t)\}\mathbb{L}_c\{h(t)\} = \tilde{\psi}(s)\tilde{h}(s);$$

conversely,

$$\mathbb{L}_c^{-1}\{\tilde{\psi}(s)\tilde{h}(s)\} = \mathbb{L}_c^{-1}\{\tilde{\psi}(s)\} * \mathbb{L}_c^{-1}\{\tilde{h}(s)\} = \psi(t) * h(t).$$

Definition 2.11. *[35] The one-parametric Mittag–Leffler function is defined as*

$$\mathbb{E}_\ell(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\ell k + 1)}, \quad \ell > 0,\ z \in \mathbb{C}.$$

Clearly, $\mathbb{E}_\ell(z) = e^z$ *for* $\ell = 1$ *The two-parametric Mittag-Leffler function is of the following form*

$$\mathbb{E}_{\ell,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\ell k + \beta)}, \quad \ell > 0,\ \beta > 0.$$

The derivative of the Mittag-Leffler function is given by

$$\frac{d}{dz}\mathbb{E}_{\ell,1}(cz^\ell) = \sum_{k=1}^{\infty} \frac{c^k z^{\ell k-1}}{\Gamma(\ell k)} = cz^{\ell-1}\sum_{k=0}^{\infty} \frac{(cz^\ell)^k}{\Gamma(\ell k + \ell)} = cz^{\ell-1}\mathbb{E}_{\ell,\ell}(cz^\ell), \tag{2.2}$$

and

$$\frac{d}{dz}\left(z^{\beta-1}\mathbb{E}_{\ell,\beta}(cz^\ell)\right) = z^{\beta-2}\mathbb{E}_{\ell\beta-1}(cz^\ell). \tag{2.3}$$

Subsequently, we present the modified Laplace transform of a Mittag-Leffler function. By utilizing the formula [36]

$$\int_0^\infty e^{-st} t^{\ell k+\beta-1}\mathbb{E}^j_{\ell,\beta}(\pm\lambda t^\ell)dt = \frac{j!s^{\ell-\beta}}{(s^\ell \pm \lambda)^{j+1}}, \quad Re(s) > |\lambda|^{\frac{1}{\ell}},$$

and by using the change of the variable $t = \ln\frac{w}{c}$, we get

$$\int_c^\infty e^{-s\ln\frac{w}{c}}(\ln\frac{w}{c})^{\ell k+\beta-1}\mathbb{E}^j_{\ell,\beta}\left(\pm\lambda(\ln\frac{w}{c})^\ell\right)\frac{dw}{w} = \frac{j!s^{\ell-\beta}}{(s^\ell \pm \lambda)^{j+1}}, \quad Re(s) > |\lambda|^{\frac{1}{\ell}}.$$

Definition 2.12. *[37] For a normed space* $\|\mathbb{B}\| = (\mathbb{B}, \|.\|)$*, the operator* $N : \mathbb{B} \to \mathbb{B}$ *satisfies the Lipschitz condition, if there is a positive real constant* K *such that for all* ϕ *and* y *in* $\mathbb{B}$*,*

$$\|N\phi - Ny\| < K\|\phi - y\|.$$

Remark 2.13. *[37] Given Definition 2.12, if* $0 < K < 1$*, the operator* N *is called a contraction mapping on the normed space* $\|\mathbb{B}\| = (\mathbb{B}, \|.\|)$*.*

Theorem 2.14 (Banach fixed point theorem)**.** *[38] Let* $\mathbb{B}$ *be a Banach space and* N *be a contraction mapping with the Lipschitz constant* K*. Then* N *has an unique fixed point.*

3. Existence and uniqueness of solution

For a given $T > c > 0$, let $\mathbb{E} = C([c,T],\mathbb{R}^n)$ be a Banach space consisting of continuous n-vector mappings given on $[c,T]$ furnished with the norm

$$\|\phi\| = \sup_{t\in[c,T]} |\phi(t)|.$$

Notice that the norm of an n-vector $\phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t)) \in \mathbb{R}^n$ is presented as

$$|\phi(t)| = \left(\sum_{k=1}^{n} |\phi_k(t)|^2\right)^{1/2}.$$

Based on the problem given by (1.1), we introduce the Banach space $\mathbb{B} = \{\phi; \phi \in \mathbb{E}, {}_{CH}D_c^{\beta}\phi \in \mathbb{E}\}$ via the norm

$$\|\phi\|_{\mathbb{B}} = \|\phi\| + \|{}_{CH}D_c^{\beta}\phi\|.$$

Now, we first derive the equivalent solution to our system.

Lemma 3.1. *For* $1 < \ell < 2$, $0 < \beta < \ell - 1$ *and invertible matrix* $[Is^{\ell} - A]$*, the solution of the nonlinear Caputo-Hadamard FIVP given by* (1.1) *is given as*

$$\phi(t) = \mathbb{E}_{\ell}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_0 + \left(\ln\frac{t}{c}\right)\mathbb{E}_{\ell,2}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_1$$
$$+ \int_c^t \left(\ln\frac{w}{c}\right)^{\ell-1} \mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{w}{c}\right)^{\ell}\right)\psi(w, \phi(w), D_c^{\beta}\phi(w))\,\frac{dw}{w}.$$

Proof. Let $\Psi(s)$ and $\Phi(s)$ be the modified Laplace transforms of $\psi(t)$ and $\phi(t)$, respectively. Then, by using the modified Laplace transform and its properties for the nonlinear Caputo-Hadamard FIVP given by (1.1), we have

$$\mathbb{L}_c\{{}_{CH}D_c^{\ell}\phi(t)\} = \mathbb{L}_c\{A\phi(t)\} + \mathbb{L}_c\{\psi(t, \phi(t), {}_{CH}D_c^{\beta}\phi(t))\},$$

so

$$\Phi(s) = s^{\ell-1}[Is^{\ell} - A]^{-1}\phi_0 + s^{\ell-2}[Is^{\ell} - A]^{-1}\phi_1 + [Is^{\ell} - A]^{-1}\Psi(s, \Phi(s), {}_{CH}D_c^{\beta}\Phi(s)).$$

By applying the inverse modified Laplace transform to the above relation, we obtain

$$\phi(t) = \mathbb{E}_{\ell}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_0 + \left(\ln\frac{t}{c}\right)\mathbb{E}_{\ell,2}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_1$$
$$+ \int_c^t \left(\ln\frac{w}{c}\right)^{\ell-1} \mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{w}{c}\right)^{\ell}\right)\psi(w, \phi(w), {}_{CH}D_c^{\beta}\phi(w))\,\frac{dw}{w},$$

and this concludes the proof. □

We will use the Banach's contraction principle to prove the existence of a solution of the nonlinear Caputo-Hadamard FIVP given by (1.1).

Theorem 3.2. *Let $\psi : [c,\infty)\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ be a continuous function that fulfills the following Lipschitz inequality*

$$\|\psi(t,\phi_1(t),y_1(t)) - \psi(t,\phi_2(t),y_2(t))\| \le K(\|\phi_1-\phi_2\| + \|y_1-y_2\|),\quad t\in[c,T],\ K>0.$$

Then the nonlinear Caputo-Hadamard FIVP given by (1.1) *has a solution uniquely on* $[c,T]$ *if*

$$\left[\frac{1}{\ell} + \frac{(T-c)\,\Gamma(\ell)}{Tc\,\Gamma(\ell-\beta+1)}\left(\ln\frac{T}{c}\right)^{-\beta}\right]KM_\ell\left(\ln\frac{T}{c}\right)^{\ell} < 1, \tag{3.1}$$

where $\|\psi(t,0,0)\| \le M_0$ *and* $\left\|\mathbb{E}_{\ell,i}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\right\| \le M_i,\ i\in\{1,2,\ell\}$.

Proof. Consider the operator $N : \mathbb{B}\to\mathbb{B}$ formulated by

$$N\phi(t) = \mathbb{E}_\ell\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_0 + \left(\ln\frac{t}{c}\right)\mathbb{E}_{\ell,2}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_1$$
$$+\int_c^t\left(\ln\frac{w}{c}\right)^{\ell-1}\mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{w}{c}\right)^{\ell}\right)\psi(w,\phi(w),{}_{CH}D_c^{\beta}\phi(w))\,\frac{dw}{w}.$$

We follow the proof in some steps:

Step 1: N is well–defined: Given $\phi\in\mathbb{B}$ and $t\in[c,T]$, we have

$$\|N\phi(t)\| \le \left\|\mathbb{E}_\ell\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\right\|\|\phi_0\| + \left(\ln\frac{t}{c}\right)\left\|\mathbb{E}_{\ell,2}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\right\|\|\phi_1\|$$
$$+\int_c^t\left(\ln\frac{w}{c}\right)^{\ell-1}\left\|\mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{w}{c}\right)^{\ell}\right)\right\|\left\|\psi(w,\phi(w),{}_{CH}D_c^{\beta}\phi(w))\right\|\frac{dw}{w}$$
$$\le M_1\|\phi_0\| + M_2\left(\ln\frac{t}{c}\right)\|\phi_1\| + M_\ell\int_c^t\left(\ln\frac{w}{c}\right)^{\ell-1}\left[K\left(\|\phi(w)\| + \left\|{}_{CH}D_c^{\beta}\phi(w)\right\|\right)\right]\frac{dw}{w}$$
$$+\int_c^t\left(\ln\frac{w}{c}\right)^{\ell-1}\|\psi(s,0,0)\|\frac{dw}{w}$$
$$\le M_1\|\phi_0\| + M_2\left(\ln\frac{t}{c}\right)\|\phi_1\| + \frac{KM_\ell}{\ell}\left(\ln\frac{t}{c}\right)^{\ell}\|\phi\|_{\mathbb{B}} + \frac{M_0M_\ell}{\ell}\left(\ln\frac{t}{c}\right)^{\ell}.$$

Consequently, we obtain

$$\|N\phi\| \le M_1\|\phi_0\| + M_2\left(\ln\frac{T}{c}\right)\|\phi_1\| + \frac{M_0M_\ell}{\ell}\left(\ln\frac{T}{c}\right)^{\ell} + \frac{KM_\ell}{\ell}\left(\ln\frac{T}{c}\right)^{\ell}\|\phi\|_{\mathbb{B}}. \tag{3.2}$$

Applying the first derivative of $N\phi(t)$ and using (2.2) and (2.3), we have

$$N'\phi(t) = A\left(\ln\frac{t}{c}\right)^{\ell-1}\mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_0 + \mathbb{E}_{\ell,1}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_1$$
$$+\frac{1}{t}\left(\ln\frac{t}{c}\right)^{\ell-1}\mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\psi(t,\phi(t),{}_{CH}D_c^{\beta}\phi(t)).$$

Hence,

$$\|N'\phi(t)\| \leq \|A\| \left(\ln\frac{t}{c}\right)^{\ell-1} \left\|\mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\right\| \|\phi_0\| + \left\|\mathbb{E}_{\ell,1}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\right\| \|\phi_1\|$$

$$+ \frac{1}{t}\left(\ln\frac{t}{c}\right)^{\ell-1} \left\|\mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\right\| \left\|\psi(t,\phi(t),{}_{CH}D_c^{\beta}\phi(t))\right\|$$

$$\leq M_\ell\|A\| \left(\ln\frac{t}{c}\right)^{\ell-1} \left\|\phi_0\right\| + M_1\left\|\phi_1\right\| + \frac{KM_\ell}{c}\left(\ln\frac{t}{c}\right)^{\ell-1} \left\|\phi\right\|_{\mathbb{B}} + \frac{M_0M_\ell}{c}\left(\ln\frac{t}{c}\right)^{\ell-1}$$

$$\leq M_\ell\|A\| \left(\ln\frac{t}{c}\right)^{\ell-1} \left\|\phi_0\right\| + M_1\left\|\phi_1\right\| + KM'_\ell\left(\ln\frac{t}{c}\right)^{\ell-1} \left\|\phi\right\|_{\mathbb{B}} + M_0M'_\ell\left(\ln\frac{t}{c}\right)^{\ell-1},$$

where $M'_\ell = \frac{M_\ell}{c}$.

Now, one can estimate that

$$\left\|{}_{CH}D_c^{\beta}N\phi(t)\right\| \leq \frac{1}{\Gamma(1-\beta)}\int_c^t \left(\ln\frac{t}{w}\right)^{-\beta} \left\|N'\phi(w)\right\|\frac{dw}{w}$$

$$\leq \frac{M_1\|\phi_1\|}{\Gamma(1-\beta)}\int_c^t \left(\ln\frac{t}{w}\right)^{-\beta} \frac{dw}{w}$$

$$+ \frac{1}{\Gamma(1-\beta)}\left[M_\ell\|A\|\|\phi_0\| + KM'_\ell\|\phi\|_{\mathbb{B}} + M_0M'_\ell\right]\int_c^t \left(\ln\frac{t}{w}\right)^{-\beta}\left(\ln\frac{w}{c}\right)^{\ell-1} \frac{dw}{w}$$

$$\leq \frac{M_1\|\phi_1\|}{\Gamma(2-\beta)}\left(\ln\frac{t}{c}\right)^{1-\beta} + \frac{\Gamma(\ell)}{\Gamma(\ell-\beta+1)}\left[M_\ell\|A\|\|\phi_0\| + KM'_\ell\|\phi\|_{\mathbb{B}} + M_0M'_\ell\right]\left(\ln\frac{t}{c}\right)^{\ell-\beta}.$$

Consequently, we obtain

$$\left\|{}_{CH}D_c^{\beta}N\phi\right\| \leq \frac{M_1\|\phi_1\|}{\Gamma(2-\beta)}\left(\ln\frac{T}{c}\right)^{1-\beta} + \frac{\Gamma(\ell)}{\Gamma(\ell-\beta+1)}\Big[M_\ell\|A\|\|\phi_0\|$$

$$+ KM'_\ell\|\phi\|_{\mathbb{B}} + M_0M'_\ell\Big]\left(\ln\frac{T}{c}\right)^{\ell-\beta}. \tag{3.3}$$

From (3.2) and (3.3), we find that

$$\|N\phi\|_{\mathbb{B}} \leq \left[M_1 + \frac{\Gamma(\ell)M_\ell\|A\|}{\Gamma(\ell-\beta+1)}\left(\ln\frac{T}{c}\right)^{\ell-\beta}\right]\|\phi_0\| + \left[M_2\left(\ln\frac{T}{c}\right) + \frac{M_1}{\Gamma(2-\beta)}\left(\ln\frac{T}{c}\right)^{1-\beta}\right]\|\phi_1\|$$

$$+\left[\frac{KM'_\ell}{\ell}\left(\ln\frac{T}{c}\right)^{\ell} + \frac{\Gamma(\ell)KM_\ell}{\Gamma(\ell-\beta+1)}\left(\ln\frac{T}{c}\right)^{\ell-\beta}\right]\|\phi\|_{\mathbb{B}}$$

$$+\left[\frac{M_0M'_\ell}{\ell}\left(\ln\frac{T}{c}\right)^{\ell} + \frac{\Gamma(\ell)M_0M_\ell}{\Gamma(\ell-\beta+1)}\left(\ln\frac{T}{c}\right)^{\ell-\beta}\right].$$

This implies that N is well defined.

Step 2: N is a contraction on $\mathbb{B}$: For $\phi, y \in \mathbb{B}$ and $t \in [c, T]$, we get

$$\|N\phi(t) - Ny(t)\| \le M_\ell \int_c^t \left(\ln \frac{w}{c}\right)^{\ell-1} \|\psi(w, \phi(w), {}_{CH}D_c^\beta \phi(w)) - \psi(w, y(w), {}_{CH}D_c^\beta y(w))\| \frac{dw}{w}$$

$$\le KM_\ell \int_c^t \left(\ln \frac{w}{c}\right)^{\ell-1} \left[\|\phi(w) - y(w)\| + \|{}_{CH}D_c^\beta \phi(w) - {}_{CH}D_c^\beta y(w)\|\right] \frac{dw}{w}$$

$$\le \frac{KM_\ell}{\ell} \left(\ln \frac{t}{c}\right)^{\ell} \|\phi - y\|_{\mathbb{B}}.$$

On the other hand, $\|N'\phi(t) - N'y(t)\| \le \frac{1}{t} KM_\ell \left(\ln \frac{t}{c}\right)^{\ell-1} \|\phi - y\|_{\mathbb{B}}$.

So,

$$\|{}_{CH}D_c^\beta N\phi(t) - {}_{CH}D_c^\beta Ny(t)\| \le \frac{1}{\Gamma(1-\beta)} \int_c^t \left(\ln \frac{t}{w}\right)^{-\beta} \|N'\phi(t) - N'y(t)\| \frac{dw}{w}$$

$$\le \frac{KM_\ell}{\Gamma(1-\beta)} \|\phi - y\|_{\mathbb{B}} \int_c^t \left(\ln \frac{t}{w}\right)^{-\beta} \left(\ln \frac{w}{c}\right)^{\ell-1} \frac{1}{w} \frac{dw}{w}$$

$$\le \frac{(t-c) KM_\ell \Gamma(\ell)}{tc\, \Gamma(\ell - \beta + 1)} \left(\ln \frac{t}{c}\right)^{\ell-\beta} \|\phi - y\|_{\mathbb{B}}.$$

Then,

$$\|N\phi - Ny\|_{\mathbb{B}} \le \left[\frac{1}{\ell} + \frac{(T-c)\,\Gamma(\ell)}{Tc\,\Gamma(\ell - \beta + 1)} \left(\ln \frac{T}{c}\right)^{-\beta}\right] KM_\ell \left(\ln \frac{T}{c}\right)^{\ell} \|\phi - y\|_{\mathbb{B}}.$$

The contractive property for N, thanks to (3.1), is established. As a consequence, Theorem 2.14 confirms the existence of a unique solution for the nonlinear Caputo-Hadamard FIVP given by (1.1) on $[c, T]$. This completes the proof. □

4. Generalized Mittag-Leffler stability

In this section, we follow our study in relation to the stability of the nonlinear Caputo-Hadamard FIVP given by (1.1) by using terms of a Lyapunov-like function and $\mathcal{K}$-class function. For more information, see [39–41].

From now on, we suppose that the Lyapunov function $\mathbb{V} : [c, \infty) \times \mathbb{R}^n \to \mathbb{R}^+$ is continuously differentiable with respect to the time variable t, Lipschtiz with respect to the unknown function ϕ, and also $\mathbb{V}(t, 0) = 0$.

Definition 4.1. *[42] The solution of the nonlinear Caputo-Hadamard FIVP given by* (1.1) *is said to be as follows:*

- *Stable if for all ϕ_0, there exists $\varepsilon > 0$ such that $\|\phi(t)\| \le \varepsilon$ for $t \ge 0$.*
- *Asymptotically stable if $\|\phi(t)\| \to 0$ as $t \to \infty$.*

Definition 4.2. *[42] The solution of the nonlinear Caputo-Hadamard FIVP given by* (1.1) *is Mittag-Leffler stable if*

$$\|\phi(t)\| \le \left[m(\phi(t_0)) \mathbb{E}_\ell \left(-\lambda \left(\ln \frac{t}{c}\right)^{\ell}\right)\right]^{\gamma}, \quad t > c,$$

where $\ell \in (1,2)$, $\lambda \geq 0$, $\gamma > 0$, $m(0) = 0$, $m(\phi) \geq 0$ and $m(\phi)$ is locally Lipschitz on $\phi \in \mathbb{B} \in \mathbb{R}^n$ with a constant m_0.

Definition 4.3. *[42] The solution of the nonlinear Caputo-Hadamard FIVP given by* (1.1) *is generalized Mittag-Leffler stable if*

$$\|\phi(t)\| \leq \left[m(\phi(t_0))\left(\ln\frac{t}{c}\right)^{-\rho}\mathbb{E}_{\ell,1-\rho}\left(-\lambda\left(\ln\frac{t}{c}\right)^{\ell}\right)\right]^{\gamma}, \quad t > c,$$

such that $\ell \in (1,2)$, $-\ell < \rho < 1-\ell$, $\gamma \geq 0$, $\lambda > 0$, $m(0) = 0$, $m(\phi) \geq 0$ and $m(\phi)$ is locally Lipschitz on $\phi \in \mathbb{B} \in \mathbb{R}^n$ with a constant m_0.

Remark 4.4. *[42] Mittag-Leffler stability and generalized Mittag-Leffler stability imply asymptotic stability.*

4.1. Lyapunov method

Theorem 4.5. *Let $\phi = 0$ be an equilibrium point of the nonlinear Caputo-Hadamard FIVP given by* (1.1)*, and assume that $\mathbb{V}$ satisfies*

$$c\|\phi\|^b \leq \mathbb{V}(t,\phi(t)), \tag{4.1}$$

$$_{CH}D_c^{\ell}\mathbb{V}(t,\phi(t)) \leq -q\mathbb{V}(t,\phi(t)) \tag{4.2}$$

such that $\phi \in \mathbb{R}^n$, $c, b, q > 0$. Then, the zero solution is Mittag-Leffler stable if $\mathbb{V}(c,\phi(c)) \geq 0$ and $\Theta\mathbb{V}(c,\phi(c)) = 0$, where $\Theta = \frac{d}{dt}$.

Proof. Using the inequality given by (4.2), a nonnegative function $M(t)$ exists and satisfies

$$_{CH}D_c^{\ell}\mathbb{V}(t,\phi(t)) + M(t) = -q\mathbb{V}(t,\phi(t)). \tag{4.3}$$

Let $\mathbb{L}_c\{\mathbb{V}(t,\phi(t))\} = \mathbb{V}(s)$. Then, the application of the Laplace transform given by (4.3) gives

$$s^{\ell}\mathbb{V}(s) - s^{\ell-1}\mathbb{V}_0 - s^{\ell-2}\mathbb{V}_1 + M(s) = -q\mathbb{V}(s). \tag{4.4}$$

By applying the inverse modified Laplace transform to (4.4), we obtain

$$\mathbb{V}(t,\phi(t)) = \mathbb{V}_0\mathbb{E}_{\ell}\left(-q\left(\ln\frac{t}{c}\right)^{\ell}\right) + \mathbb{V}_1\left(\ln\frac{t}{c}\right)\mathbb{E}_{\ell,2}\left(-q\left(\ln\frac{t}{c}\right)^{\ell}\right) - M(t) * \left[\left(\ln\frac{t}{c}\right)^{\ell-1}\mathbb{E}_{\ell,\ell}\left(-q\left(\ln\frac{t}{c}\right)^{\ell}\right)\right].$$

Since both $\left(\ln\frac{t}{c}\right)^{\ell-1}$ and $\mathbb{E}_{\ell,\ell}\left(-q\left(\ln\frac{t}{c}\right)^{\ell}\right)$ are nonnegative functions and $\mathbb{V}_1 = \Theta\mathbb{V}(c,\phi(c)) = 0$, we deduce that

$$\mathbb{V}(t,\phi(t)) \leq \mathbb{V}_0\mathbb{E}_{\ell}\left(-q\left(\ln\frac{t}{c}\right)^{\ell}\right).$$

In accordance with (4.1), we obtain

$$\|\phi(t)\| \leq \left[\frac{\mathbb{V}_0}{c}\mathbb{E}_{\ell}\left(-q\left(\ln\frac{t}{c}\right)^{\ell}\right)\right]^{\frac{1}{b}},$$

for $m = \dfrac{\mathbb{V}_0}{c} \geq 0$. In this case, the zero solution of the nonlinear Caputo-Hadamard FIVP given by (1.1) is Mittag-Leffler stable. □

4.2. Stability via $\mathcal{K}$-class functions

Definition 4.6. *[43] If $\varphi \in C([0,\infty),[0,\infty))$ is strictly increasing, and $\varphi(c) = 0,\ c > 0$, then φ is termed a $\mathcal{K}$-class function, as illustrated by $\varphi \in \mathcal{K}$.*

Theorem 4.7. *Let $\phi = 0$ be an equilibrium point of the nonlinear Caputo-Hadamard FIVP given by* (1.1)*. Suppose that there exists a $\mathcal{K}$-class function φ that satisfies*

$$\mathbb{V}(t,\phi(t)) \geq \varphi^{-1}(\|\phi(t)\|), \tag{4.5}$$

$$_{CH}D_c^{\ell}\mathbb{V}(t,\phi(t)) \leq 0, \tag{4.6}$$

$$\sup_{t\geq c}\varphi\left(\mathbb{V}(c,\phi(c)) + \Theta\mathbb{V}(c,\phi(c))\ln\frac{t}{c}\right) \leq M \tag{4.7}$$

for $M \geq 0$. Then, the zero solution is stable.

Proof. By applying (4.6), there exists some $M \geq 0$ so that

$$_{CH}D_c^{\ell}\mathbb{V}(t,\phi(t)) = -M(t).$$

By using the Laplace transform and its inverse, we obtain

$$\mathbb{V}(t,\phi(t)) = \mathbb{V}_0 + \left(\ln\frac{t}{c}\right)\mathbb{V}_1 - M(t) * \left[\frac{1}{\Gamma(\ell)}\left(\ln\frac{t}{c}\right)^{\ell-1}\right], \tag{4.8}$$

where $\mathbb{V}_0 = \mathbb{V}(c,\phi(c))$, and $\mathbb{V}_1 = \Theta\mathbb{V}(c,\phi(c))$.

Substituting (4.8) into (4.5) yields

$$\varphi^{-1}(\|\phi(t)\|) \leq \mathbb{V}_0 + \left(\ln\frac{t}{c}\right)\mathbb{V}_1 - M(t) * \left[\frac{1}{\Gamma(\ell)}\left(\ln\frac{t}{c}\right)^{\ell-1}\right] \leq \mathbb{V}_0 + \left(\ln\frac{t}{c}\right)\mathbb{V}_1.$$

Therefore

$$\|\phi(t)\| \leq \varphi\left(\mathbb{V}_0 + \left(\ln\frac{t}{c}\right)\mathbb{V}_1\right).$$

Then, by Eq (4.7), we get $\|\phi(t)\| \leq M,\ t > c$, which confirms that the zero solution of the nonlinear Caputo-Hadamard FIVP given by (1.1) is stable. □

5. Example

Here, we validate our results by providing the following example.

Example 5.1. *According to* (1.1)*, consider the nonlinear Caputo-Hadamard FIVP*

$$\begin{cases} {}_{CH}D_{1,t}^{3/2}\phi(t) = \frac{1}{10}\left(-|\phi(t)| - {}_{CH}D_{1,t}^{1/2}|\phi(t)|\right), & t \in [1,e], \\ \Theta^k\phi(t)\mid_{t=1} = 0, k = 0,1. \end{cases} \tag{5.1}$$

Here, we have $A = 0$ and $\psi\left(t,\phi(t),{}_{CH}D_{1,t}^{1/2}\phi(t)\right) = \frac{1}{10}\left(-|\phi(t)| - {}_{CH}D_{1,t}^{1/2}|\phi(t)|\right)$, where $\psi : [1,e]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$. In order to show that (5.1) *has a unique solution, we simply check that*

$$\left\|\psi\left(t,\phi(t),{}_{CH}D_{1,t}^{1/2}\phi(t)\right) - \psi\left(t,y(t),{}_{CH}D_{1,t}^{1/2}y(t)\right)\right\|_{\mathbb{B}} \leq \frac{1}{10}\left\|\phi(t) - y(t)\right\|_{\mathbb{B}},$$

which is satisfying the Lipschitz condition with $K = \frac{1}{10}$*. Since* $|\mathbb{E}_{\ell,\ell}\left(A(\ln \frac{t}{c})^{\ell}\right)| \leq M_{\ell}$*, for* $A = 0$*, we have* $\mathbb{E}_{\frac{3}{2},\frac{3}{2}}(0) = \frac{2}{\sqrt{\pi}}$ *and*

$$\left[\frac{1}{\frac{3}{2}} + \frac{\Gamma(\frac{3}{2})}{\Gamma(2)}\left(\ln\frac{e}{1}\right)^{-1/2}\right]\frac{1}{10}\frac{2}{\sqrt{\pi}}\left(\ln\frac{e}{1}\right)^{3/2} = 0.26 < 1.$$

From Theorem 3.2, the nonlinear Caputo-Hadamard FIVP given by (5.1) *has a unique solution.*

On the other hand, consider the Lyapunov function $\mathbb{V}(t, \phi(t)) = |\phi(t)|$*. In this case,*

$${}_{CH}D^{\ell}_{1,t}\mathbb{V}(t, \phi(t)) = \frac{1}{10}\left(-\mathbb{V}(t, \phi(t)) - {}_{CH}D^{\beta}_{1,t}\mathbb{V}(t, \phi(t))\right) \leq -\frac{1}{10}\mathbb{V}(t, \phi(t)).$$

Hence, the hypotheses of Theorem 4.5 hold with $c = 0, b = 1$ *and* $q = \dfrac{1}{10}$*. Accordingly, the zero solution of the given nonlinear Caputo-Hadamard FIVP given by* (5.1) *is Mittag-Leffler stable.*

6. Conclusions

In this paper, we provided several hypotheses that demonstrate the existence of a solution and its uniqueness for the nonlinear Caputo-Hadamard FIVP given by (1.1) by using the Banach contraction principle. To do this, the modified Laplace transform played a main role in finding the corresponding integral equation by using Mittag-Leffler functions with one and two parameters to derive the Hadamard integrals. Subsequently, we used a Lyapunov-like function and $\mathcal{K}$-class function to prove the generalized Mittag-Leffler stability for the Caputo-Hadamard system given by (1.1). Further, we examined the theoretical results by designing an illustrative example. In subsequent works, the notion of generalized Mittag-Leffler stability can be discussed for different nonlinear systems furnished with non-singular derivation operators. Also, one can focus on the generalized Mittag-Leffler stability problem of q-FDEs in a variety of different forms.

Acknowledgments

The third and fifth authors would like to thank Azarbaijan Shahid Madani University. Also, the authors would like to thank the dear reviewers for their valuable and constructive comments to improve the quality of the paper.

Conflict of interest

The authors declare no conflicts of interest.

References

1. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Vol. 204, North-Holland Mathematics Studies, 2006.
2. V. E. Tarasov, *Fractional dynamics: Applications of fractional calculus to dynamics of particles, fields and media*, Springer, Higher Education Press, 2011.

3. Y. Zhou, J. Wang, L. Zhang, *Basic theory of fractional differential equations*, Singapore: World Scientific Publishing Company, 2016. https://doi.org/10.1142/10238
4. D. Baleanu, S. Etemad, S. Rezapour, A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions, *Bound. Value Probl.*, **2020** (2020), 64. https://doi.org/10.1186/s13661-020-01361-0
5. D. Baleanu, S. Etemad, S. Rezapour, On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators, *Alex. Eng. J.*, **59** (2020), 3019–3027. https://doi.org/10.1016/j.aej.2020.04.053
6. A. Aphithana, S. K. Ntouyas, J. Tariboon, Existence and Ulam-Hyers stability for Caputo conformable differential equations with four-point integral conditions, *Adv. Differ. Equ.*, **2019** (2019), 139. https://doi.org/10.1186/s13662-019-2077-5
7. S. T. M. Thabet, S. Etemad, S. Rezapour, On a coupled Caputo conformable system of pantograph problems, *Turk. J. Math.*, **45** (2021), 496–519. https://doi.org/10.3906/mat-2010-70
8. S. Rezapour, S. K. Ntouyas, M. Q. Iqbal, A. Hussain, S. Etemad, J. Tariboon, An analytical survey on the solutions of the generalized double-order φ-integrodifferential equation, *J. Funct. Spaces*, **2021** (2021), 6667757. https://doi.org/10.1155/2021/6667757
9. S. P. Bhairat, D. B. Dhaigude, Existence of solutions of generalized fractional differential equation with nonlocal initial condition, *Math. Bohemica*, **144** (2019), 203–220. https://doi.org/10.21136/MB.2018.0135-17
10. M. M. Matar, M. I. Abbas, J. Alzabut, M. K. A. Kaabar, S. Etemad, S. Rezapour, Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives, *Adv. Differ. Equ.*, **2021** (2021), 68. https://doi.org/10.1186/s13662-021-03228-9
11. S. Rezapour, A. Imran, A. Hussain, F. Martinez, S. Etemad, M. K. A. Kaabar, Condensing functions and approximate endpoint criterion for the existence analysis of quantum integro-difference FBVPs, *Symmetry*, **13** (2021), 469. https://doi.org/10.3390/sym13030469
12. M. E. Samei, A. Ahmadi, S. N. Hajiseyedazizi, S. K. Mishra, B. Ram, The existence of nonnegative solutions for a nonlinear fractional q-differential problem via a different numerical approach, *J. Inequal. Appl.*, **2021** (2021), 75. https://doi.org/10.1186/s13660-021-02612-z
13. H. Mohammadi, S. Kumar, S. Rezapour, S. Etemad, A theoretical study of the Caputo-Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control, *Chaos, Solitons Fract.*, **144** (2021), 110668. https://doi.org/10.1016/j.chaos.2021.110668
14. M. I. Abbas, On the initial value problems for the Caputo-Fabrizio impulsive fractional differential equations, *Asian-Eur. J. Math.*, **14** (2021), 2150073. https://doi.org/10.1142/S179355712150073X
15. M. ur Rahman, M. Arfan, Z. Shah, E. Alzahrani, Evolution of fractional mathematical model for drinking under Atangana-Baleanu Caputo derivatives, *Phys. Scripta*, **96** (2021), 115203. https://doi.org/10.1088/1402-4896/ac1218
16. M. ur Rahman, S. Ahmad, M. Arfan, A. Akgul, F. Jarad, Fractional order mathematical model of serial killing with different choices of control strategy, *Fractal Fract.*, **6** (2022), 162. https://doi.org/10.3390/fractalfract6030162

17. H. Qu, M. ur Rahman, M. Arfan, M. Salimi, S. Salahshour, A. Ahmadian, Fractal-fractional dynamical system of Typhoid disease including protection from infection, *Eng. Comput.*, 2021. https://doi.org/10.1007/s00366-021-01536-y

18. X. Liu, M. Arfan, M. ur Rahman, B. Fatima, Analysis of SIQR type mathematical model under Atangana-Baleanu fractional differential operator, *Comput. Methods Biomech. Biomed. Eng.*, 2022. https://doi.org/10.1080/10255842.2022.2047954

19. B. Ahmad, S. K. Ntouyas, An existence theorem for fractional hybrid differential inclusions of Hadamard type with Dirichlet boundary conditions, *Abstr. Appl. Anal.*, **2014** (2014), 705809. https://doi.org/10.1155/2014/705809

20. B. Ahmad, S. K. Ntouyas, J. Tariboon, Existence results for mixed Hadamard and Riemann-Liouville fractional integro-differential inclusions, *Adv. Differ. Equ.*, **2015** (2015), 293. https://doi.org/10.1186/s13662-015-0625-1

21. K. Pei, G. Wang, Y. Sun, Successive iterations and positive extremal solutions for a Hadamard type fractional integro-differential equations on infinite domain, *Appl. Math. Comput.*, **312** (2017), 158–168. https://doi.org/10.1016/j.amc.2017.05.056

22. C. Derbazi, H. Hammouche, Caputo-Hadamard fractional differential equations with nonlocal fractional integro-differential boundary conditions via topological degree theory, *AIMS Math.*, **5** (2020), 2694–2709. https://doi.org/10.3934/math.2020174

23. S. Belmor, F. Jarad, T. Abdeljawad, On Caputo-Hadamard type coupled systems of nonconvex fractional differential inclusions, *Adv. Differ. Equ.*, **2021** (2021), 377. https://doi.org/10.1186/s13662-021-03534-2

24. S. Etemad, S. Rezapour, M. E. Samei, On a fractional Caputo-Hadamard inclusion problem with sum boundary value conditions by using approximate endpoint property, *Math. Methods Appl. Sci.*, **43** (2020), 9719–9734. https://doi.org/10.1002/mma.6644

25. D. Matignon, Stability results for fractional differential equations with applications to control processing, *Comput. Eng. Syst. Appl.*, **2** (1996), 963–968.

26. W. Deng, C. Li, Q. Guo, Analysis of fractional differential equations with multi-orders, *Fractals*, **15** (2007), 173–182. https://doi.org/10.1142/S0218348X07003472

27. W. Deng, C. Li, J. Lu, Stability analysis of linear fractional differential system with multiple time delays, *Nonlinear Dyn.*, **48** (2007), 409–416. https://doi.org/10.1007/s11071-006-9094-0

28. A. Bayati Eshkaftaki, J. Alidousti, R. Khoshsiar Ghaziani, Stability analysis of fractional-order nonlinear systems via Lyapunov method, *J. Mahani Math. Res. Center*, **3** (2014), 61–73.

29. L. G. Zhang, J. M. Li, G. P. Chen, Extension of Lyapunov second method by fractional calculus, *Pure Appl. Math.*, **3** (2005), 1008–5513.

30. H. Belbali, M. Benbachir, Existence results and Ulam-Hyers stability to impulsive coupled system fractional differential equations, *Turk. J. Math.*, **45** (2021), 1368–1385. https://doi.org/10.3906/mat-2011-85

31. A. K. Anatoly, Hadamard-type fractional calculus, *J. Korean Math. Soc.*, **38** (2001), 1191–1204.

32. F. Jarad, T. Abdeljawad, D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives, *Adv. Differ. Equ.*, **2012** (2012), 142. https://doi.org/10.1186/1687-1847-2012-142

33. C. Li, Z. Li, Asymptotic behaviours of solution to Caputo-Hadamard fractional partial differential equation with fractional Laplacian, *Int. J. Comput. Math.*, **98** (2021), 305–339. https://doi.org/10.1080/00207160.2020.1744574

34. C. Li, Z. Li, Z. Wang, Mathematical analysis and the local discontinuous Galerkin method for Caputo-Hadamard fractional partial differential equation, *J. Sci. Comput.*, **85** (2020), 41. https://doi.org/10.1007/s10915-020-01353-3

35. H. J. Haubold, A. M. Mathai, R. K. Saxena, Mittag-Leffler functions and their applications, *J. Appl. Math.*, **2011** (2011), 298628. https://doi.org/10.1155/2011/298628

36. I. Podlubny, *Fractional differential equations, mathematics in science and engineering*, San Diego, Calif, USA: Academic Press, 1999.

37. V. Daftardar-Gejji, H. Jafari, Analysis of a system of nonautonomous fractional differential equations involving Caputo derivatives, *J. Math. Anal. Appl.*, **328** (2007), 1026–1033. https://doi.org/10.1016/j.jmaa.2006.06.007

38. D. R. Smart, *Fixed point theorems*, Cambridge: Cambridge University Press, 1974.

39. S. J. Sadati, D. Baleanu, A. Ranjbar, R. Ghaderi, T. Abdeljawad, Mittag-Leffler stability theorem for fractional nonlinear systems with delay, *Abstr. Appl. Anal.*, **2010** (2010), 108651. https://doi.org/10.1155/2010/108651

40. K. Liu, J. R. Wang, D. O'Regan, Ulam-Hyers-Mittag-Leffler stability for ψ-Hilfer fractional-order delay differential equations, *Adv. Differ. Equ.*, **2019** (2019), 50. https://doi.org/10.1186/s13662-019-1997-4

41. X. Li, S. Liu, W. Jiang, q-Mittag-Leffler stability and Lyapunov direct method for differential systems with q-fractional order, *Adv. Differ. Equ.*, **2018** (2018), 78. https://doi.org/10.1186/s13662-018-1502-5

42. Y. Li, Y. Q. Chen, I. Podlubny, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability, *Comput. Math. Appl.*, **59** (2010), 1810–1821. https://doi.org/10.1016/j.camwa.2009.08.019

43. H. Belbali, M. Benbachir, Stability for coupled systems on networks with Caputo-Hadamard fractional derivative, *J. Math. Model.*, **9** (2021), 107–118. https://doi.org/10.22124/JMM.2020.17303.1500