Research article

Contractivity and expansivity of H-Toeplitz operators on the Bergman spaces

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Abstract: In this paper we consider the properties of H-Toeplitz operators $B_\phi$ on the Bergman space $L^2_0(\mathbb{D})$. We present some necessary and sufficient conditions for the contractive and expansive H-Toeplitz operators $B_\phi$ with various symbols $\phi$.

Keywords: H-Toeplitz operators, contractive operators, expansive operators, Bergman space

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1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $dA$ the area measure on the complex plane $\mathbb{C}$. The space $L^2(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z).$$

The Bergman space $L^2_0(\mathbb{D})$ consists of all analytic functions on $\mathbb{D}$ and $L^\infty(\mathbb{D})$ is the space of the essentially bounded measurable function on $\mathbb{D}$. For $\varphi \in L^\infty(\mathbb{D})$, the multiplication operator $M_\varphi$ on $L^2_0(\mathbb{D})$ is defined by $M_\varphi(f) = \varphi \cdot f$ and the Toeplitz operator $T_\varphi$ on $L^2_0(\mathbb{D})$ is defined by

$$T_\varphi(f) = P(\varphi \cdot f),$$

where $P$ denotes the orthogonal projection of $L^2(\mathbb{D})$ onto $L^2_0(\mathbb{D})$ and $f \in L^2_0(\mathbb{D})$. It is clear that those operators are bounded if $\varphi \in L^\infty(\mathbb{D})$.

The harmonic Bergman space $L^2_{\text{harm}}(\mathbb{D})$ denotes the space of all complex-valued harmonic functions in $L^2(\mathbb{D})$. The space $L^2_{\text{harm}}(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D})$ and it is a Hilbert space. Let $P_{\text{harm}}$ be the orthogonal projection from the space $L^2(\mathbb{D})$ onto the space $L^2_{\text{harm}}(\mathbb{D})$. 

Toeplitz operators on the Bergman space were studied by McDonald and Sundberg in [19]. Recently, lots of research about Toeplitz operators has been conducted in the Bergman space (see [2, 11]). In the Hardy space, the hyponormality of Toeplitz operators was studied in [7, 8, 12, 14, 20]; refer to references therein for more details. Recently, many authors characterized the hyponormality of Toeplitz operators on the Bergman space and weighted Bergman space (see [7, 13, 15, 16, 18, 21]). In 2007, Arora and Paliwal [1] have introduced the notion of H-Toeplitz operators on the Hardy space. Recently, in [10], the authors studied H-Toeplitz operators on the Bergman space. The research of H-Toeplitz operators has arisen naturally in several fields of mathematics and in a variety problems. For example, an H-Toeplitz system comprises a matrix equation of the form $Tx = y$ where $T$ is an $n \times n$ H-Toeplitz matrix with $x, y$ in $\mathbb{C}^n$. The $n \times n$ H-Toeplitz matrix $T$ has $2n - 1$ degrees of freedom rather than $n^2$. Thus for a large $n$, it is easier to solve the system of linear equations for an H-Toeplitz matrix(cf. [10]). In this paper we consider the algebraic properties of H-Toeplitz operators $B_\phi$ on the Bergman space $L^2_a(\mathbb{D})$. More concretely, we establish a tractable and explicit criterion for the contractivity and expansivity of H-Toeplitz operators. Several decades ago, many researchers began studying the contractive and expansive operators (see [3, 4, 5, 6]). In [5], the authors considered the invariant subspace problem for contractive operators. Recently, various results have been derived based on the papers (see [9, 17]).

The organization of this paper is as follows. In Section 2, we introduce the notion of H-Toeplitz operators on the Bergman space and provide various well-known properties of these operators. In Section 3, we focus on the contractive and expansive H-Toeplitz operators with analytic, coanalytic and harmonic symbols.

2. Preliminaries and auxiliary lemmas

Let $s, t$ be nonnegative integers and $P$ be the orthogonal projection from $L^2(\mathbb{D})$ to $L^2_a(\mathbb{D})$. Then we have

$$P(z^s z^t) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t} & \text{if } s \geq t \\ 0 & \text{if } s < t. \end{cases}$$

The following lemmas will be used frequently in this paper.

**Lemma 2.1.** ([10]) In the harmonic Bergman space $L^2_{\text{harm}}(\mathbb{D})$, for nonnegative integers $s$ and $t$, the following:

$$P_{\text{harm}}(z^s z^t) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t} & \text{if } s \geq t \\ \frac{t-s+1}{t+1} z^{t-s} & \text{if } s < t. \end{cases}$$

**Lemma 2.2.** ([15]) For $m \geq 0$, we have

\begin{align*}
(i) \quad & \| \sum_{n=0}^{\infty} c_n z^n \|_2^2 = \sum_{n=0}^{\infty} \frac{1}{i + m + 1} |c_i|^2, \\
(ii) \quad & \| P(\sum_{n=0}^{\infty} c_n z^n) \|_2^2 = \sum_{i=m}^{\infty} \frac{i-m+1}{(i+1)^2} |c_i|^2.
\end{align*}

By using Lemmas 2.1 and 2.2, we have the following result.
Remark 2.3. For $m \geq 0$, we have

$$\|P_{\text{harm}}(z^n)\|_2^2 = \sum_{i=0}^{m-1} \frac{m-i+1}{(m+1)^2} |c_i|^2 + \sum_{i=m}^{\infty} \frac{i-m+1}{(i+1)^2} |c_i|^2.$$ 

In order to define the notion of an H-Toeplitz operator on $L^2_{\alpha}(\mathbb{D})$, we first consider the operator $K : L^2_{\alpha}(\mathbb{D}) \to L^2_{\text{harm}}(\mathbb{D})$ defined by

$$K(e_{2n}(z)) = e_{n}(z) = \sqrt{n+1} z^n \quad \text{and} \quad K(e_{2n+1}(z)) = e_{n+1}(z) = \sqrt{n+2} z^{n+1}$$

for all $n \geq 0$ and $z \in \mathbb{D}$. It can be checked that the operator $K$ is bounded linear on $L^2_{\alpha}(\mathbb{D})$ with $\|K\| = 1$. Moreover, the adjoint $K^*$ of the operator $K$ is given by

$$K^*(e_{n}(z)) = e_{2n}(z) \quad \text{and} \quad K^*(e_{n+1}(z)) = e_{2n+1}(z)$$

for all $n \geq 0$. From the definition of $K$ and $K^*$, we have that $KK^* = I_{L^2_{\text{harm}}(\mathbb{D})}$ and $K^*K = I_{L^2_{\alpha}(\mathbb{D})}$.

Remark 2.4. By the definitions of $K$ and $K^*$, we can easily check that $K(z^n) = \frac{\sqrt{n+1}}{\sqrt{2n+1}} z^n$, $K(z^{2n+1}) = \frac{\sqrt{n+2}}{\sqrt{2n+3}} z^{2n+1}$, $K^*(z^n) = \frac{\sqrt{n+1}}{\sqrt{2n+1}} z^n$ and $K^*(z^{2n}) = \frac{\sqrt{n+2}}{\sqrt{2n+3}} z^{2n-1}$.

Next, we define H-Toeplitz operators on the Bergman space $L^2_{\alpha}(\mathbb{D})$ using the definition of the operator $K$.

Definition 2.5. ([10]) For $\varphi \in L^\infty(\mathbb{D})$, the H-Toeplitz operator $B_\varphi$ with the symbol $\varphi$ is defined as the operator $B_\varphi : L^2_{\alpha}(\mathbb{D}) \to L^2_{\alpha}(\mathbb{D})$ such that $B_\varphi(f) = PM_\varphi K(f)$ for all $f \in L^2_{\alpha}(\mathbb{D})$.

The next proposition follows from the definition of the H-Toeplitz operators.

Proposition 2.6. ([10]) For $\varphi, \psi \in L^\infty(\mathbb{D})$, the operator $B_\varphi$ satisfies the following:

(i) $B_\varphi$ is a bounded linear operator on $L^2_{\alpha}(\mathbb{D})$ with $\|B_\varphi\| \leq \|\varphi\|_{\infty}$.

(ii) For any scalar $\alpha$ and $\beta$, $B_{\alpha\varphi + \beta\psi} = \alpha B_\varphi + \beta B_\psi$.

(iii) The adjoint of the H-Toeplitz operator $B_\varphi$ is given by $B^*_\varphi = K^* P_{\text{harm}} M_{\varphi}$.

The following remark provides important information for adjoint operators. It shows the difference between adjoint Toeplitz operators and adjoint H-Toeplitz operators.

Remark 2.7. If $f, g$ are in $L^\infty(\mathbb{D})$ then by the definition of Toeplitz operators $T_f$, we have that

$$T_f^* = T_{\overline{f}} \quad \text{and} \quad T_{\overline{f}} T_g = T_{\overline{fg}} \text{ if } f \text{ or } g \text{ is analytic.}$$

But in the case of the H-Toeplitz operator,

$$B^*_\varphi(az) = K^* P_{\text{harm}} M_{\varphi}(az) = K^* P_{\text{harm}}(a\overline{z}z) = K^* \left( \frac{a}{2} \right) = \frac{a}{2}$$

and

$$B_\varphi(az) = PM_\varphi K(az) = PM_\varphi a\overline{z} = P(a\overline{z}^2) = 0.$$ 

Therefore, $B^*_\varphi(az) \neq B_\varphi(az)$. A straightforward calculation shows that $B_\varphi B_\varphi \neq B_\varphi$ (cf. [10]).
3. Main results

A bounded linear operator $T$ on a Hilbert space is said to be expansive if $T^*T \geq I$, contractive if $T^*T \leq I$, and isometric if $T^*T = I$.

For $k \in L^2_\alpha(D)$, let $k(z) = k_\epsilon(z) + k_\sigma(z)$, where

$$k_\epsilon(z) := \sum_{n=0}^{\infty} c_{2n} z^{2n} \quad \text{and} \quad k_\sigma(z) := \sum_{n=0}^{\infty} c_{2n+1} z^{2n+1}.$$

3.1. H-Toeplitz operators with analytic symbols

In this subsection, we consider the properties of H-Toeplitz operators $B_\varphi$ and $B^*_\varphi$ with analytic symbols. First, we study the contractivity and expansivity of $B_\varphi$ and $B^*_\varphi$ with $\varphi = az^N$ for $N \in \mathbb{N}$ and $a \in \mathbb{C}$. Next, we extend the symbol $\varphi$ of the form $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i$ with $a_i \in \mathbb{C}$.

**Theorem 3.1.** Let $\varphi(z) = az^N$ for $N \in \mathbb{N}$ and $a \in \mathbb{C}$. Then $B_\varphi$ is contractive if and only if $|a| \leq 1$.

**Proof.** For any $k \in L^2_\alpha(D)$,

$$B_\varphi k(z) = PM_\varphi K(k(z)) = PM_\varphi K(k_\epsilon(z) + k_\sigma(z))$$

$$= PM_\varphi \sum_{n=0}^{\infty} \left( \frac{\sqrt{n+1}}{\sqrt{2n+1}} c_{2n} z^{n} + \frac{\sqrt{n+2}}{\sqrt{2n+2}} c_{2n+1} z^{n+1} \right)$$

$$= a \sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{\sqrt{2n+1}} c_{2n} z^{n+N} + \varphi \left( az^N \sum_{n=0}^{\infty} \frac{\sqrt{n+2}}{\sqrt{2n+2}} c_{2n+1} z^{n+1} \right)$$

$$= a \sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{\sqrt{2n+1}} c_{2n} z^{n+N} + a \sum_{n=0}^{N-1} \frac{\sqrt{n+2}}{\sqrt{2n+2}} \cdot \frac{N-n}{N+1} \cdot c_{2n+1} z^{n-N-1},$$

and we have that

$$||B_\varphi k(z)||^2 = |a|^2 \left( \sum_{n=0}^{\infty} \frac{n+1}{(2n+1)(n+N+1)} |c_{2n}|^2 + \sum_{n=0}^{N-1} \frac{(n+2)(N-n)}{2(n+1)(N+1)^2} |c_{2n+1}|^2 \right).$$

According to the definition for the contractivity of $B_\varphi$, the inequality $B^*_\varphi B_\varphi \leq I$ is equivalent to $||B_\varphi k(z)||^2 \leq ||k(z)||^2$ for any $k \in L^2_\alpha(D)$. Thus, $B_\varphi$ on $L^2_\alpha(D)$ is contractive if and only if

$$|a|^2 \left( \sum_{n=0}^{\infty} \frac{n+1}{(2n+1)(n+N+1)} |c_{2n}|^2 + \sum_{n=0}^{N-1} \frac{(n+2)(N-n)}{2(n+1)(N+1)^2} |c_{2n+1}|^2 \right) \leq \sum_{j=0}^{\infty} \frac{1}{j+1} |c_j|^2. \quad (3.1)$$

There are two cases to consider. If $c_\ell \neq 0$ for $\ell$ is even and $c_\ell = 0$ for $\ell$ is odd, from (3.1), we have

$$|a|^2 \frac{n+1}{(2n+1)(n+N+1)} |c_{2n}|^2 \leq \frac{1}{2n+1} |c_{2n}|^2.$$
or equivalently,
\[ |a|^2 \leq \frac{n + N + 1}{n + 1} \]
for any nonnegative integer \( n \). Since \( \frac{n + N + 1}{n + 1} \) is decreasing for \( n \), we have
\[ |a|^2 \leq \min_{n \geq 0} \frac{n + N + 1}{n + 1} = \lim_{n \to \infty} \frac{n + N + 1}{n + 1} = 1. \]  
(3.2)

If \( c_\ell \neq 0 \) for \( \ell \) is odd, and \( c_\ell = 0 \) for \( \ell \) is even, from (3.1), we have
\[ |a|^2 \frac{(n + 2)(N - n)}{2(n + 1)(N + 1)^2} |c_{2n+1}|^2 \leq \frac{1}{2(n + 1)} |c_{2n+1}|^2 \]
or equivalently,
\[ |a|^2 \leq \frac{(N + 1)^2}{(n + 2)(N - n)} \]
for any \( 0 \leq n \leq N - 1 \). Put \( f(n) = \frac{(N+1)^2}{(n+2)(N-n)} \), then \( f \) is increasing for \( \frac{N+2}{2} \leq n \leq N - 1 \), and decreasing for \( 0 \leq n < \frac{N+2}{2} \). Moreover, if \( N \) is even, then \( B_\varphi \) is contractive if and only if
\[ |a|^2 \leq \min_{0 \leq n \leq N-1} \frac{(N + 1)^2}{(n + 2)(N - n)} = f \left( N - \frac{1}{2} \right) = \frac{4(N + 1)^2}{(N + 2)^2}. \]  
(3.3)

If \( N \) is odd, then \( B_\varphi \) is contractive if and only if
\[ |a|^2 \leq \min_{0 \leq n \leq N-1} \frac{(N + 1)^2}{(n + 2)(N - n)} = f \left( N - \frac{1}{2} \right) = \frac{4(N + 1)}{N + 3}. \]  
(3.4)

Since \( \frac{4(N+1)^2}{(N+2)^2} \geq 1 \) and \( \frac{4(N+1)}{N+3} \geq 1 \) for any \( N \in \mathbb{N} \), from (3.2)–(3.4), \( B_\varphi \) is contractive if and only if \( |a| \leq 1 \). This completes the proof. \( \square \)

**Corollary 3.2.** Let \( \varphi(z) = az^N \) for \( N \in \mathbb{N} \) and \( a \in \mathbb{C} \). Then \( B_\varphi \) is neither expansive nor isometric.

**Proof.** From the proof of Theorem 3.1, \( B_\varphi \) is expansive if and only if
\[ |a|^2 \left( \sum_{n=0}^{\infty} \frac{n + 1}{(2n + 1)(n + N + 1)} |c_{2n}|^2 + \sum_{n=0}^{N-1} \frac{(n + 2)(N - n)}{2(n + 1)(N + 1)^2} |c_{2n+1}|^2 \right) \geq \sum_{j=0}^{\infty} \frac{1}{j + 1} |c_j|^2. \]
(3.5)

Set \( c_{2N+1} \neq 0 \) and \( c_i = 0 \) for \( i \neq 2N + 1 \). Then from (3.5), \( \frac{1}{2(N+1)} \leq 0; \) it is a contradiction. \( \square \)

In the next result, we have the sufficient condition for the contractivity and expansivity of the H-Toeplitz operators \( B_\varphi \) with symbols \( \varphi(z) = \sum_{i=0}^{\infty} a_i z^i \) where \( a_i \in \mathbb{C} \) on \( L_2^2(\mathbb{D}) \).

**Theorem 3.3.** Let \( \varphi(z) = \sum_{i=0}^{\infty} a_i z^i \) and \( a_i \in \mathbb{C} \).
(i) If \( B_\varphi \) is contractive then
\[ \sum_{i=0}^{\infty} \frac{1}{s + i + 1} |a_i|^2 \leq \frac{1}{s + 1} \]
and \( \sum_{i=s+1}^{\infty} \frac{i - s}{(i + 1)^2} |a_i|^2 \leq \frac{1}{s + 2} \).
for any nonnegative integer \( s \).

(ii) If \( B_\varphi \) is expansive then

\[
\sum_{i=0}^{\infty} \frac{1}{s+i+1} |a_i|^2 \geq \frac{1}{s+1} \quad \text{and} \quad \sum_{i=s+1}^{\infty} \frac{i-s}{(i+1)^2} |a_i|^2 \geq \frac{1}{s+2}
\]

(3.6)

for any nonnegative integer \( s \).

Proof. For any \( k \in L^2_\varphi(\mathbb{D}) \),

\[
B_\varphi k(z) = PM_\varphi K(k(z)) = PM_\varphi K(k_0(z) + k_\alpha(z))
\]

\[
= PM_\varphi \left[ \sum_{n=0}^{\infty} \left( \frac{\sqrt{n+1}}{2^{n+1}} c_2 \zeta^n + \frac{\sqrt{n+2}}{2^{n+2}} c_{2n+1} \zeta^{n+1} \right) \right]
\]

\[
= \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{2^{n+1}} a_i c_2 \zeta^{n+i} + \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} \frac{\sqrt{n+2}}{2^{n+2}} \frac{i-n}{i+1} a_i c_{2n+1} \zeta^{n+1-i}
\]

(3.7)

for any \( c_j \in \mathbb{C} (j = 0, 1, 2, \cdots) \). Then on comparing the coefficient of \( \zeta^m \), by the equation (3.7) we have that

\[
a_m c_0 + \frac{\sqrt{2}}{\sqrt{3}} a_{m-1} c_2 + \cdots + \frac{\sqrt{m+1}}{\sqrt{2m+1}} c_2 a_{m} + \sum_{n=0}^{\infty} \frac{\sqrt{n+2}}{\sqrt{2n+2}} \frac{m+1}{n+m+2} a_n c_{2n+1} c_{2n+1}.
\]

Set \( c_\ell \neq 0 \) for some \( \ell \) and \( c_j = 0 \) for any \( j \neq \ell \). Then we consider that the following two cases arise:

**Case 1:** If \( \ell = 2s \) for any nonnegative integer \( s \), then

\[
B_\varphi k(z) = \sum_{i=0}^{\infty} \frac{\sqrt{s+1}}{2s+1} a_i c_2 \zeta^{s+i}.
\]

If \( B_\varphi \) on \( L^2_\varphi(\mathbb{D}) \) is contractive then

\[
\sum_{i=0}^{\infty} \frac{s+1}{(2s+1)(s+i+1)} |a_i|^2 |c_{2s}|^2 \leq \frac{1}{2s+1} |c_{2s}|^2.
\]

Thus, \( \sum_{i=0}^{\infty} \frac{1}{s+i+1} |a_i|^2 \leq \frac{1}{s+1} \) for any nonnegative integer \( s \). Similarly, if \( B_\varphi \) on \( L^2_\varphi(\mathbb{D}) \) is expansive then

\[
\sum_{i=0}^{\infty} \frac{1}{s+i+1} |a_i|^2 \geq \frac{1}{s+1} \quad \text{for any nonnegative integer } s.
\]

**Case 2:** If \( \ell = 2s+1 \) for any nonnegative integer \( s \), then

\[
B_\varphi k(z) = \sum_{i=s+1}^{\infty} \frac{\sqrt{s+2}}{2s+2} \frac{i-s}{i+1} a_i c_{2s+1} \zeta^{i-s-1}.
\]

If \( B_\varphi \) on \( L^2_\varphi(\mathbb{D}) \) is contractive then

\[
\sum_{i=s+1}^{\infty} \frac{(s+2)(i-s)}{2(s+1)(i+1)^2} |a_i|^2 |c_{2s+1}|^2 \leq \frac{1}{2(s+1)} |c_{2s+1}|^2.
\]

Thus, \( \sum_{i=s+1}^{\infty} \frac{i-s}{(i+1)^2} |a_i|^2 \leq \frac{1}{s+2} \) for any nonnegative integer \( s \). Similarly, if \( B_\varphi \) on \( L^2_\varphi(\mathbb{D}) \) is expansive then

\[
\sum_{i=s+1}^{\infty} \frac{i-s}{(i+1)^2} |a_i|^2 \geq \frac{1}{s+2} \quad \text{for any nonnegative integer } s. \]

This completes the proof. \( \square \)

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Example 3.4. Let \( \varphi(z) = \sum_{i=0}^{\infty} \frac{1}{\sqrt{i+1}} \cdot z^i \). Then
\[
\sum_{i=0}^{\infty} \frac{1}{(i+1)^2} = \frac{\pi^2}{6} > 1,
\]
and so, \( B_\varphi \) is not contractive.

The following example shows that the converse of Theorem 3.3 (ii) is not true.

Example 3.5. Consider the polynomial \( \varphi(z) = z + \sqrt{3}z^3 \). Then the conditions in (3.6) hold. Put \( k(z) = -\frac{\sqrt{3}}{3} + \frac{1}{\sqrt{6}} + z^2 \). A straightforward calculation shows that \( B_\varphi k(z) = \frac{1}{2\sqrt{6}} + \sqrt{2}z^3 \). Thus \( \|B_\varphi k(z)\|^2 = \frac{13}{24} \) and \( \|k(z)\|^2 = \frac{23}{36} \). Therefore, \( B_\varphi \) is not expansive.

We obtained the contractivity and expansivity of the adjoint H-Toeplitz operators \( B_\varphi^* \) on \( L_2^2(\mathbb{D}) \).

Theorem 3.6. Let \( \varphi(z) = az^N \) for \( N \in \mathbb{N} \) and \( a \in \mathbb{C} \). Then \( B_\varphi^* \) is contractive if and only if \( |a| \leq 1 \).

Proof. For any \( k \in L_2^2(\mathbb{D}) \),
\[
B_\varphi^* k(z) = K^* P_{\text{harm}} M_\varphi k(z)
\]
\[
= K^* P_{\text{harm}} \left( \sum_{n=0}^{\infty} c_n z^n \right)
\]
\[
= \sum_{n=0}^{N-1} \frac{N-n+1}{N+1} c_n z^{N-n} + \sum_{n=N}^{\infty} \frac{n-N+1}{n+1} c_n z^{n-N}
\]
\[
= \sum_{n=0}^{N-1} \frac{\sqrt{N-n+1} \sqrt{2N-2n}}{N+1} c_n z^{2N-2n-1} + \sum_{n=N}^{\infty} \frac{\sqrt{n-N+1} \sqrt{2n-2N+1}}{n+1} c_n z^{2n-2N}.
\]

Thus
\[
\|B_\varphi^* k(z)\|^2 = |a|^2 \left( \sum_{n=0}^{N-1} \frac{N-n+1}{(N+1)^2} |c_n|^2 + \sum_{n=N}^{\infty} \frac{n-N+1}{(n+1)^2} |c_n|^2 \right).
\]

Thus, \( B_\varphi^* \) on \( L_2^2(\mathbb{D}) \) is contractive if and only if
\[
|a|^2 \left( \sum_{n=0}^{N-1} \frac{N-n+1}{(N+1)^2} |c_n|^2 + \sum_{n=N}^{\infty} \frac{n-N+1}{(n+1)^2} |c_n|^2 \right) \leq \sum_{n=0}^{\infty} \frac{1}{n+1} |c_n|^2.
\]
(3.8)

If \( 0 \leq n \leq N-1 \), then \( |a|^2 \leq \frac{(N+1)^2}{(n+1)(N-n+1)} \); so,
\[
|a|^2 \leq \min_{0 \leq n \leq N-1} \frac{(N+1)^2}{(n+1)(N-n+1)} = \frac{(N+1)^2}{2N},
\]
since \( \frac{(N+1)^2}{(n+1)(N-n+1)} \) is decreasing. If \( n \geq N \), then \( |a|^2 \leq \frac{n+1}{n-N+1} \); so,
\[
|a|^2 \leq \min_{n \geq N} \frac{n+1}{n-N+1} = 1,
\]
since \( \frac{n+1}{n-N+1} \) is decreasing. Hence, for any arbitrary \( c_i \), the inequality given by (3.8) holds if and only if \( |a|^2 \leq \min \left( \frac{(N+1)^2}{2N}, 1 \right) = 1 \). This completes the proof. \( \square \)
From Theorem 3.6, we get the following corollaries and example.

**Corollary 3.7.** Let \( \varphi(z) = az^N \) for \( N \in \mathbb{N} \) and \( a \in \mathbb{C} \). Then \( B_\varphi^* \) is expansive if and only if \( |a|^2 \geq N + 1 \).

**Proof.** From the proof of Theorem 3.6, \( B_\varphi^* \) is expansive if and only if

\[
|a|^2 \left( \sum_{n=0}^{N-1} \frac{N-n+1}{(n+1)^2} |c_n|^2 + \sum_{n=N}^{\infty} \frac{n-N+1}{(n+1)^2} |c_n|^2 \right) \geq \sum_{n=0}^{\infty} \frac{1}{n+1} |c_n|^2.
\]

(3.9)

If \( 0 \leq n \leq N-1 \), then \( |a|^2 \geq \frac{(N+1)^2}{(n+1)^2} \); thus \( |a|^2 \geq N + 1 \) since \( \frac{(N+1)^2}{(n+1)(N-n+1)} \) is decreasing. If \( n \geq N \), then \( |a|^2 \geq \frac{n+1}{n+1-N} \); thus \( |a|^2 \geq N + 1 \) since \( \frac{n+1}{n+1-N} \) is decreasing. Hence, the inequality given by (3.9) holds for any arbitrary \( c_i \) \((i = 0, 1, 2, \cdots)\) if and only if \( |a|^2 \geq N + 1 \). \( \square \)

**Example 3.8.** Let \( \varphi(z) = 2z^4 \). By a direct calculation,

\[
\|B_\varphi^* k(z)\|^2 = 4 \left( \sum_{n=0}^{3} \frac{5-n}{25} |c_n|^2 + \sum_{n=4}^{\infty} \frac{n-3}{(n+1)^2} |c_n|^2 \right)
\]

and

\[
\|k(z)\|^2 = \sum_{n=0}^{\infty} \frac{1}{n+1} |c_n|^2.
\]

Since \( c_i \)'s are arbitrary, set \( c_0 \neq 0 \) and \( c_i = 0 \) for \( i > 0 \); then \( \|B_\varphi^* k(z)\|^2 = \frac{1}{5} |c_0|^2 \) and \( \|k(z)\|^2 = |c_0|^2 \). Thus \( \|B_\varphi^* k(z)\|^2 < \|k(z)\|^2 \). Set \( c_5 \neq 0 \) and \( c_i = 0 \) for \( i \neq 5 \); then \( \|B_\varphi^* k(z)\|^2 = \frac{1}{5} |c_5|^2 \) and \( \|k(z)\|^2 = \frac{1}{5} |c_5|^2 \). Thus, \( \|B_\varphi^* k(z)\|^2 > \|k(z)\|^2 \). Hence \( B_{2z^4} \) is neither contractive nor expansive.

**Corollary 3.9.** Let \( \varphi(z) = az^N \) for \( N \in \mathbb{N} \) and \( a \in \mathbb{C} \). Then \( B_\varphi \) is not self-adjoint.

**Proof.** In the proof of Theorems 3.1 and 3.6,

\[
B_\varphi k(z) = a \sum_{n=0}^{N} \frac{\sqrt{n+1}}{\sqrt{2n+1}} c_{2n} z^{n+N} + a \sum_{n=0}^{N-1} \frac{\sqrt{n+2}}{\sqrt{2n+2}} \frac{N-n}{N+1} c_{2n+1} z^{n+N-1}
\]

and

\[
B_\varphi^* k(z) = \bar{a} \sum_{n=0}^{N-1} \frac{\sqrt{N-n+1} \sqrt{2N-2n}}{N+1} c_{2N-2n} z^{2N-2n-1} + \bar{a} \sum_{n=N}^{\infty} \frac{\sqrt{n-N+1} \sqrt{2n-2N+1}}{n+1} c_{2n} z^{2n-2N}.
\]

Then, on comparing the coefficient of \( z^0 \), we get

\[
\frac{a}{\sqrt{2N(2N+1)}} c_{2N-1} \text{ and } \frac{\bar{a}}{N+1} c_N.
\]

Since \( c_{2N-1} \) and \( c_N \) are arbitrary, \( B_\varphi \) is not self-adjoint. \( \square \)

**Corollary 3.10.** Let \( \varphi(z) = az^N \) for \( N \in \mathbb{N} \) and \( a \in \mathbb{C} \). Then \( B_\varphi \) is not normal.
Proof. For any $k \in L^2_a(\mathbb{D})$ such that $k(z) = \sum_{n=0}^{\infty} c_n z^n$, $B_\varphi$ is normal if and only if $B_\varphi^* B_\varphi k(z) = B_\varphi B_\varphi^* k(z)$ or equivalently, $\|B_\varphi k(z)\| = \|B_\varphi^* k(z)\|$. As in the proof of Theorems 3.1 and 3.6, we have

$$
\|B_\varphi k(z)\|^2 = |a|^2 \left( \sum_{n=0}^{\infty} \frac{n+1}{2(n+1)(n+N+1)} |c_{2n}|^2 + \sum_{n=0}^{N-1} \frac{(n+2)(N-n)}{2(n+1)(N+1)^2} |c_{2n+1}|^2 \right)
$$

and

$$
\|B_\varphi^* k(z)\|^2 = |a|^2 \left( \sum_{n=0}^{\infty} \frac{N-n+1}{(N+1)^2} |c_n|^2 + \sum_{n=N}^{\infty} \frac{n-N+1}{(n+1)^2} |c_n|^2 \right).
$$

Since $c_i$'s are arbitrary, set $c_{2N+1} \neq 0$ and $c_i = 0$ for $i \neq 2N+1$. Then $\|B_\varphi k(z)\|^2 = 0$ and $\|B_\varphi^* k(z)\|^2 = \frac{|a|^2(N+2)}{4(N+1)^2} |c_{2N+1}|^2$; thus, $\|B_\varphi k(z)\|^2 \neq \|B_\varphi^* k(z)\|^2$. \hfill $\Box$

In the next result, we investigated a sufficient condition for the contractivity and expansivity of the adjoint H-Toeplitz operators $B_\varphi^*$ with symbols $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i$ where $a_i \in \mathbb{C}$ on $L^2_a(\mathbb{D})$.

**Theorem 3.11.** Let $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i$ and $a_i \in \mathbb{C}$.

(i) If $B_\varphi^*$ is contractive then

$$
\sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2} |a_i|^2 + \sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2} |a_i|^2 \leq \frac{1}{s+1}
$$

for any nonnegative integer $s$.

(ii) If $B_\varphi^*$ is expansive then

$$
\sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2} |a_i|^2 + \sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2} |a_i|^2 \geq \frac{1}{s+1}
$$

for any nonnegative integer $s$.

**Proof.** For any $k \in L^2_a(\mathbb{D})$,

$$
B_\varphi^* k(z) = K^* P_{\text{harm}} M_{\varphi} k(z)
$$

$$
= K^* P_{\text{harm}} \left( \sum_{n=1}^{i-1} \sum_{i=0}^{\infty} \frac{i-n+1}{i+1} a_i c_n z^{i-n} + \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} \frac{n-i+1}{n+1} a_i c_n z^{n-i} \right)
$$

$$
= K^* \left( \sum_{n=1}^{i-1} \frac{i-n+1}{i+1} \sqrt{\frac{2i-2n}{i-n+1}} a_i c_n z^{2i-2n-1} + \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} \frac{n-i+1}{n+1} \sqrt{\frac{2n-2i+1}{n-i+1}} a_i c_n z^{2n-2i} \right)
$$

Set $c_s \neq 0$ for some $s$ and $c_j = 0$ for any $j \neq s$. Then

$$
B_\varphi^* k(z) = \sum_{i=s+1}^{\infty} \frac{i-s+1}{i+1} \sqrt{\frac{2(i-s)}{i-s+1}} a_i c_s z^{2i-2s-1} + \sum_{i=0}^{s} \frac{s-i+1}{s+1} \sqrt{\frac{2s-2i+1}{s-i+1}} a_i c_i z^{2s-2i}.
$$
If $B^*_\varphi$ on $L^2_\alpha(\mathbb{D})$ is contractive then

$$
\|B^*_\varphi k(z)\|^2 = \sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2} |a_i|^2 |c_i|^2 + \sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2} |a_i|^2 |c_i|^2 \leq \frac{1}{s+1} |c_i|^2.
$$

Thus,

$$
\sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2} |a_i|^2 + \sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2} |a_i|^2 \leq \frac{1}{s+1}.
$$

Similarly, if $B^*_\varphi$ on $L^2_\alpha(\mathbb{D})$ is expansive then

$$
\sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2} |a_i|^2 + \sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2} |a_i|^2 \geq \frac{1}{s+1}.
$$

This completes the proof.

The following example shows that the converse of Theorem 3.11 (ii) is not true.

**Example 3.12.** Consider the polynomial $\varphi(z) = \sqrt{2} z + \sqrt{2} z^{-2}$. Then the condition given by (3.10) holds. Put $k(z) = \frac{z}{9} - \frac{z^2}{3} + z^2$. A straightforward calculation shows that $B^*_\varphi k(z) = \frac{2\sqrt{3}}{3} z + \frac{8\sqrt{3}}{27} z^{-3}$. Then $\|B^*_\varphi k(z)\|^2 = \frac{140}{243}$ and $\|k(z)\|^2 = \frac{183}{243}$. Therefore, $B^*_\varphi$ is not expansive.

**Corollary 3.13.** Let $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ and $a_i \in \mathbb{C}$. If $B^*_\varphi$ is contractive then $\sum_{n=0}^{\infty} \frac{1}{i+1} |a_i|^2 \leq 1$ and if $B^*_\varphi$ is expansive then $\sum_{n=0}^{\infty} \frac{1}{i+1} |a_i|^2 \geq 1$.

**Proof.** We have the result by putting $s = 0$ in Theorem 3.11.

**Example 3.14.** Let $\varphi(z) = \sum_{i=0}^{\infty} \frac{1}{\sqrt{i+1}} z^i$. Then

$$
\sum_{i=0}^{\infty} \frac{1}{(i+1)^2} = \frac{\pi^2}{6} > 1
$$

and by Corollary 3.13, $B^*_\varphi$ is not contractive.

### 3.2. H-Toeplitz operators with coanalytic symbols

In this subsection, we consider the properties of H-Toeplitz operators $B_\varphi$ and $B^*_\varphi$ with coanalytic, or antianalytic symbols. First, we study the contractivity and expansivity of $B_\varphi$ and $B^*_\varphi$ with $\varphi = b \bar{z}^N$ for $N \in \mathbb{N}$ and $b \in \mathbb{C}$. Next, we extend the symbol $\varphi$ of the form $\varphi(z) = \sum_{i=1}^{\infty} b_i \bar{z}^i$ with $b_i \in \mathbb{C}$.

**Theorem 3.15.** Let $\varphi(z) = b \bar{z}^N$ for $N \in \mathbb{N}$ and $b \in \mathbb{C}$. Then $B_\varphi$ is contractive if and only if $|b| \leq 1$.

**Proof.** For any $k \in L^2_\alpha(\mathbb{D})$,

$$
B_\varphi k(z) = PM \left[ \sum_{n=0}^{\infty} \left( \frac{\sqrt{n+1}}{\sqrt{2n+1}} c_{2n} \bar{z}^n + \frac{\sqrt{n+2}}{\sqrt{2n+2}} c_{2n+1} \bar{z}^{n+1} \right) \right]
$$

$$
= b \sum_{n=N}^{\infty} \frac{c_{2n-N}}{\sqrt{2n+1}} \frac{n-N+1}{\sqrt{n+1}} \bar{z}^{n-N}
$$

$$
= b \sum_{n=0}^{\infty} \frac{c_{2n+2N}}{\sqrt{2n+2N+1}} \frac{n+1}{\sqrt{n+N+1}} \bar{z}^{n}.
$$
Thus

\[ ||B_\varphi k(z)||^2 = |b|^2 \sum_{n=0}^{\infty} \frac{n + 1}{(2n + 2N + 1)(n + N + 1)} |c_{2n+2N}|^2. \]

Hence \( B_\varphi \) is contractive if and only if

\[ |b|^2 \sum_{n=0}^{\infty} \frac{n + 1}{(2n + 2N + 1)(n + N + 1)} |c_{2n+2N}|^2 \leq \sum_{n=0}^{\infty} \frac{1}{n + 1} |c_n|^2. \]

If we compare the coefficients of \( c_{2n+2N} \), we have

\[ \frac{|b|^2}{(2n + 2N + 1)(n + N + 1)} |c_{2n+2N}|^2 \leq \frac{1}{2n + 2N + 1} |c_{2n+2N}|^2 \]

for any \( n \geq 0 \); thus,

\[ |b|^2 \leq \frac{n + N + 1}{n + 1} \]

for any \( n \geq 0 \). Since \( \frac{n + N + 1}{n + 1} \) is decreasing for \( n \), \( B_\varphi \) is contractive if and only if

\[ |b|^2 \leq \min_{n \geq 0} \frac{n + N + 1}{n + 1} = \lim_{n \to \infty} \frac{n + N + 1}{n + 1} = 1. \]

This completes the proof. \( \square \)

**Corollary 3.16.** Let \( \varphi(z) = b z^N \) for \( N \in \mathbb{N} \) and \( b \in \mathbb{C} \). Then \( B_\varphi \) is neither expansive nor isometric.

**Proof.** From the proof of Theorem 3.15, \( B_\varphi \) is expansive if and only if

\[ |b|^2 \sum_{n=0}^{\infty} \frac{n + 1}{(2n + 2N + 1)(n + N + 1)} |c_{2n+2N}|^2 \geq \sum_{n=0}^{\infty} \frac{1}{n + 1} |c_n|^2. \]

Since \( c_i \)'s (0 \leq i < 2N) are arbitrary, we put \( c_i \neq 0 \) if i is odd and \( c_i = 0 \) if i is even; then, \( 0 \geq \frac{1}{i+1} \); it is a contradiction. \( \square \)

In the next result, we get a sufficient condition for the contractivity of \( \text{H-Toeplitz operators} \) \( B_\varphi \) with symbols \( \varphi(z) = \sum_{i=1}^{\infty} b_i z^i \), where \( b_i \in \mathbb{C} \) on \( L^2_\varphi(\mathbb{D}) \).

**Theorem 3.17.** Let \( \varphi(z) = \sum_{i=1}^{\infty} b_i z^i \) and \( b_i \in \mathbb{C} \). If \( B_\varphi \) is contractive then

\[ \sum_{i=1}^{s} (s - i + 1)|b_i|^2 \leq s + 1 \]

for any \( s \in \mathbb{N} \).

**Proof.** For any \( k \in L^2_\varphi(\mathbb{D}) \),

\[ B_\varphi k(z) = P \left( \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} \frac{\sqrt{n + 1}}{\sqrt{2n + 1}} b_i c_{2n+i} z^n \right) = \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{1}{\sqrt{2n + 1}} \cdot \frac{n - i + 1}{\sqrt{n + 1}} b_i c_{2n+i} z^{n-i}. \]
Then on comparing the coefficient of $z^m$, by the equation (3.11), we have that

$$
\sum_{n=m+1}^{\infty} \frac{m + 1}{\sqrt{2n + 1}} \sqrt{n + 1} b_{n-m} c_{2n}.
$$

We set $c_\ell \neq 0$ if $\ell = 2s$ and $c_\ell = 0$ if $\ell \neq 2s$ for some $s \in \mathbb{N}$. Thus $B_\varphi$ on $L^2_a(\mathbb{D})$ is contractive then

$$
\sum_{i=1}^{s} \frac{(s - i + 1)|b_i|^2}{(2s + 1)(s + 1)} \leq \frac{1}{2s + 1} |c_{2s}|^2.
$$

Therefore, $\sum_{i=1}^{s} (s - i + 1)|b_i|^2 \leq s + 1$. This completes the proof. □

On the other hand, we have that

**Corollary 3.18.** Let $\varphi(z) = \sum_{i=1}^{\infty} b_i z^i$ and $b_i \in \mathbb{C}$. Then $B_\varphi$ is not expansive.

**Proof.** Using the equation (3.11), we set $c_\ell = 0$ if $\ell$ is even and $c_\ell \neq 0$ if $\ell$ is odd; then, $B_\varphi k(z) = 0$. Thus, $B_\varphi$ on $L^2_a(\mathbb{D})$ is not expansive. □

The following theorem is purposed to find the necessary and sufficient conditions for the contrac-
tivity of the adjoint H-Toeplitz operator $B_\varphi^*$ with coanalytic symbols $\varphi$.

**Theorem 3.19.** Let $\varphi(z) = b z^N$ for $N \in \mathbb{N}$ and $b \in \mathbb{C}$. Then $B_\varphi^*$ is contractive if and only if $|b| \leq 1$.

**Proof.** For any $k \in L^2_a(\mathbb{D})$,

$$
B_\varphi^* k(z) = K^* P_{\text{harm}} \left( b z^N \sum_{n=0}^{\infty} c_n z^n \right) = \bar{b} \sum_{n=0}^{\infty} \frac{\sqrt{2n + 2N + 1}}{\sqrt{n + N + 1}} c_n z^{2n + 2N};
$$

then,

$$
||B_\varphi^* k(z)||^2 = |b|^2 \sum_{n=0}^{\infty} \frac{1}{n + N + 1} |c_n|^2.
$$

Thus, $B_\varphi^*$ on $L^2_a(\mathbb{D})$ is contractive if and only if

$$
|b|^2 \sum_{n=0}^{\infty} \frac{1}{n + N + 1} |c_n|^2 \leq \sum_{n=0}^{\infty} \frac{1}{n + 1} |c_n|^2.
$$

Since $\frac{n+N+1}{n+1}$ is decreaing, $B_\varphi^*$ on $L^2_a(\mathbb{D})$ is contractive if and only if

$$
|b|^2 \leq \min_{n \geq 0} \frac{n + N + 1}{n + 1} = 1.
$$

This completes the proof. □

From Theorem 3.19, we get the following corollary and example.

**Corollary 3.20.** Let $\varphi(z) = b z^N$ for $N \in \mathbb{N}$ and $b \in \mathbb{C}$. Then, $B_\varphi^*$ is expansive if and only if $|b|^2 \geq N + 1$.
Proof. From the proof of Theorem 3.19, $B^\ast_\varphi$ is expansive if and only if

$$|b|^2 \sum_{n=0}^{\infty} \frac{1}{n + N + 1} |c_n|^2 \geq \sum_{n=0}^{\infty} \frac{1}{n + 1} |c_n|^2$$

or equivalently,

$$|b|^2 \geq \frac{n + N + 1}{n + 1}$$

for any $n \geq 0$. Hence $B^\ast_\varphi$ is expansive if and only if

$$|b|^2 \geq \max_{n \geq 0} \frac{n + N + 1}{n + 1} = N + 1.$$  

□

Example 3.21. Let $\varphi(z) = \frac{3}{2}z^2$. By direct calculations,

$$||B^\ast_\varphi k(z)||^2 = \frac{9}{4} \sum_{n=0}^{\infty} \frac{1}{n + 3} |c_n|^2$$

and

$$||k(z)||^2 = \sum_{n=0}^{\infty} \frac{1}{n + 1} |c_n|^2.$$ 

Since $c_i$'s are arbitrary, we set $c_0 \neq 0$ and $c_i = 0$ for $i > 0$; then, $||B^\ast_\varphi k(z)||^2 = \frac{3}{4} |c_0|^2$ and $||k(z)||^2 = |c_0|^2$. Thus, $||B^\ast_\varphi k(z)||^2 < ||k(z)||^2$. Set $c_1 \neq 0$ and $c_i = 0$ for $i 
eq 1$; then, $||B^\ast_\varphi k(z)||^2 = \frac{9}{16} |c_1|^2$ and $||k(z)||^2 = \frac{1}{2} |c_1|^2$. Thus, $||B^\ast_\varphi k(z)||^2 > ||k(z)||^2$. Hence, $B^\ast_\varphi$ is neither contractive nor expansive.

In view of Corollaries 3.9 and 3.10, we have the following result.

Corollary 3.22. Let $\varphi(z) = \frac{1}{2}z^N$ for $N \in \mathbb{N}$ and $b \in \mathbb{C}$. Then $B^\ast_\varphi$ is neither self-adjoint nor normal.

In the next theorem, we have the necessary and sufficient condition for the contractivity and expansivity of adjoint H-Toeplitz operators $B^\ast_\varphi$ with symbols $\varphi(z) = \sum_{i=1}^{\infty} b_i \bar{z}^i$ where $b_i \in \mathbb{C}$ on $L^2_\mathbb{D}(\mathbb{D})$.

Theorem 3.23. Let $\varphi(z) = \sum_{i=1}^{\infty} b_i \bar{z}^i$ and $b_i \in \mathbb{C}$.

(i) $B^\ast_\varphi$ is contractive if and only if

$$\sum_{m=1}^{\infty} \sum_{i=1}^{m} \frac{1}{m + 1} |b_i|^2 |c_{m-i}|^2 \leq \sum_{j=0}^{\infty} \frac{1}{j + 1} |c_j|^2.$$ 

(ii) $B^\ast_\varphi$ is expansive if and only if

$$\sum_{m=1}^{\infty} \sum_{i=1}^{m} \frac{1}{m + 1} |b_i|^2 |c_{m-i}|^2 \geq \sum_{j=0}^{\infty} \frac{1}{j + 1} |c_j|^2.$$ 

Aims Mathematics
Proof. For any \( k \in L^2_0(\mathbb{D}) \),

\[
B^*_\varphi k(z) = \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} \frac{\sqrt{2n+2i+1}}{\sqrt{n+i+1}} b_n c_{n+2i} z^{2n+2i}.
\]

Then \( B^*_\varphi \) is contractive if and only if

\[
||B^*_\varphi k(z)||^2 = \sum_{m=1}^{\infty} \sum_{i=1}^{m} \frac{1}{m+1} |b_i|^2 |c_{m-i}|^2 \leq \sum_{j=0}^{\infty} \frac{1}{j+1} |c_j|^2.
\]

Similarly, \( B^*_\varphi \) is expansive if and only if

\[
||B^*_\varphi k(z)||^2 = \sum_{m=1}^{\infty} \sum_{i=1}^{m} \frac{1}{m+1} |b_i|^2 |c_{m-i}|^2 \geq \sum_{j=0}^{\infty} \frac{1}{j+1} |c_j|^2.
\]

This completes the proof. \( \square \)

**Corollary 3.24.** Let \( \varphi(z) = b_1 z + b_2 z^2 \) and \( b_1, b_2 \in \mathbb{C} \). Then \( B^*_\varphi \) is contractive if and only if

\[
\frac{1}{s+2} |b_1|^2 + \frac{1}{s+3} |b_2|^2 \leq \frac{1}{s+1}
\]

for any nonnegative integer \( s \).

### 3.3. H-Toeplitz operators with harmonic symbols

Finally, we study the properties of H-Toeplitz operators \( B_\varphi \) with harmonic symbols of the form \( \varphi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{i=1}^{\infty} b_i z^i \) with \( a_i, b_i \in \mathbb{C} \). Specifically, we focus on the necessary and sufficient conditions of contractivity and expansivity for \( B_\varphi \) and \( B^*_\varphi \), respectively.

**Theorem 3.25.** Let \( \varphi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{i=1}^{\infty} b_i z^i \) and \( a_i, b_i \in \mathbb{C} \).

(i) If \( B_\varphi \) is contractive then

\[
\sum_{i=1}^{\infty} \frac{s+1}{(2s+1)(s+i+1)} |a_i|^2 + \sum_{i=1}^{s} \frac{s-i+1}{(2s+1)(s+1)} |b_i|^2 \leq 1
\]

and

\[
\sum_{i=s+1}^{\infty} \frac{i-s}{(i+1)^2} |a_i|^2 \leq \frac{1}{s+2}
\]

for any nonnegative integer \( s \).

(ii) If \( B_\varphi \) is expansive then

\[
\sum_{i=1}^{\infty} \frac{s+1}{(2s+1)(s+i+1)} |a_i|^2 + \sum_{i=1}^{s} \frac{s-i+1}{(2s+1)(s+1)} |b_i|^2 \geq 1
\]

and

\[
\sum_{i=s+1}^{\infty} \frac{i-s}{(i+1)^2} |a_i|^2 \geq \frac{1}{s+2}
\]

for any nonnegative integer \( s \).
Proof. By a similar argument as in the proof of Theorems 3.3 and 3.17, for any $k \in L^2_a(\mathbb{D})$,

$$B_\varphi k(z) = \sum_{i=1}^\infty \sum_{n=0}^\infty \frac{\sqrt{n+1}}{\sqrt{2n+1}} a_i c_{2n} z^{n+i} + \sum_{i=0}^\infty \sum_{n=0}^\infty \frac{\sqrt{n+2}}{\sqrt{2n+2}} \cdot \frac{i-n}{i+1} a_i c_{2n+1} z^{i-n-1}$$

$$+ \sum_{n=1}^\infty \sum_{i=1}^n \frac{1}{\sqrt{2n+1}} \cdot \frac{n-i+1}{\sqrt{n+1}} b_i c_{2n} z^{-i}$$

for any $c_j \in \mathbb{C}$ $(j = 0, 1, 2, \cdots)$. Set $c_\ell \neq 0$ for some $\ell$ and $c_j = 0$ for any $j \neq \ell$. Then we consider the following two cases:

Case 1: If $\ell = 2s$ for any nonnegative integer $s$ and $c_{2s} \neq 0$ then

$$B_\varphi k(z) = \sum_{i=1}^\infty \frac{\sqrt{s+1}}{\sqrt{2s+1}} a_i c_{2s+1} z^{s+i} + \sum_{i=1}^s \frac{s-i+1}{(2s+1)(s+1)} b_i c_{2s} z^{-s-i}.$$}

If $B_\varphi$ on $L^2_a(\mathbb{D})$ is contractive then

$$\sum_{i=1}^\infty \frac{s+1}{(2s+1)(s+i+1)} |a_i|^2 + \sum_{i=1}^s \frac{s-i+1}{(2s+1)(s+1)} |b_i|^2 \leq 1.$$}

Similarly, if $B_\varphi$ on $L^2_a(\mathbb{D})$ is expansive then

$$\sum_{i=1}^\infty \frac{s+1}{(2s+1)(s+i+1)} |a_i|^2 + \sum_{i=1}^s \frac{s-i+1}{(2s+1)(s+1)} |b_i|^2 \geq 1.$$}

Case 2: If $\ell = 2s + 1$ for any nonnegative integer $s$ and $c_{2s+1} \neq 0$, then it follows from Case 2 of Theorem 3.3. This completes the proof. \hfill \Box

Theorem 3.26. Let $\varphi(z) = \sum_{i=0}^\infty a_i z^i + \sum_{i=1}^\infty b_i z^i$ and $a_i, b_i \in \mathbb{C}$.

(i) If $B_\varphi^s$ is contractive, then

$$\sum_{i=s+1}^\infty \frac{i-s+1}{(i+1)^2} |a_i|^2 + \sum_{i=0}^s \frac{s-i+1}{(s+1)^2} |a_i|^2 + \sum_{i=1}^\infty \frac{1}{s+i+1} |b_i|^2 \leq \frac{1}{s+1},$$

for any nonnegative integer $s$.

(ii) If $B_\varphi^s$ is expansive, then

$$\sum_{i=s+1}^\infty \frac{i-s+1}{(i+1)^2} |a_i|^2 + \sum_{i=0}^s \frac{s-i+1}{(s+1)^2} |a_i|^2 + \sum_{i=1}^\infty \frac{1}{s+i+1} |b_i|^2 \geq \frac{1}{s+1},$$

for any nonnegative integer $s$.

Proof. By a similar argument as in the proof of Theorems 3.11 and 3.23, for any $k \in L^2_a(\mathbb{D})$,

$$B_\varphi^s k(z) = \sum_{i=0}^{i-1} \sum_{n=1}^\infty \frac{i-n+1}{i+1} \cdot \frac{\sqrt{2i-2n}}{|i-n+1|} a_i c_{2i-2n-1} z^{2i-2n-1}$$

$$+ \sum_{n=1}^\infty \sum_{i=0}^{n-i+1} \frac{n-i+1}{n+1} \cdot \frac{\sqrt{2n-2i+1}}{|n-i+1|} a_i c_{2n} z^{2n-2i} + \sum_{i=1}^\infty \sum_{n=0}^{\sqrt{2n+2i}} \frac{\sqrt{2n+2i+1}}{\sqrt{n+i+1}} b_i c_{2n+2i} z^{2n+2i}.$$
Corollary 3.27. Let 

\[ B_\varphi^* k(z) = \sum_{i=s+1}^{\infty} \frac{\sqrt{i-s+1} \sqrt{2i-2s} a_i c_{iz}^{2i-2s-1}}{i+1} + \sum_{i=0}^{s} \frac{\sqrt{i} \sqrt{2i+1} b_i c_{iz}^{2s+2i}}{i+1}, \]

for any nonnegative integer \( s \). If \( B_\varphi^* \) on \( L^2_n(\mathbb{D}) \) is contractive, then

\[ \sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2} |a_i|^2 + \sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2} |a_i|^2 + \sum_{i=1}^{\infty} \frac{1}{s+i+1} |b_i|^2 \leq \frac{1}{s+1}. \]

Similarly, if \( B_\varphi^* \) on \( L^2_n(\mathbb{D}) \) is expansive, then

\[ \sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2} |a_i|^2 + \sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2} |a_i|^2 + \sum_{i=1}^{\infty} \frac{1}{s+i+1} |b_i|^2 \geq \frac{1}{s+1}, \]

for any nonnegative integer \( s \). This completes the proof. \qed

The following results are immediate from Theorem 3.26.

Corollary 3.27. Let \( \varphi(z) = a_1 z + a_2 z^2 + b_1 \bar{z} + b_2 \bar{z}^2 \) and \( a_i, b_i \in \mathbb{C} \) where \( i = 1, 2 \). Then, \( B_\varphi^* \) is contractive

\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{s}{(s+1)^2} |a_1|^2 + \frac{s-1}{(s+1)^2} |a_2|^2 + \frac{1}{s+3} |b_1|^2 + \frac{1}{s+3} |b_2|^2 \leq \frac{1}{s+1} & \text{if } s = 0, 1, \\
\frac{s}{(s+1)^2} |a_1|^2 + \frac{s+1}{(s+1)^2} |a_2|^2 + \frac{1}{s+3} |b_1|^2 + \frac{1}{s+3} |b_2|^2 \leq \frac{1}{s+1} & \text{if } s \geq 2.
\end{array} \right.
\end{align*}
\]

Corollary 3.28. Let \( \varphi(z) = a_1 z + b_1 \bar{z} \) and \( a_1, b_1 \in \mathbb{C} \). Then, \( B_\varphi^* \) is contractive; then,

\[
\frac{s}{(s+1)^2} |a_1|^2 + \frac{1}{s+2} |b_1|^2 \leq \frac{1}{s+1},
\]

for any \( s \in \mathbb{N} \).

4. Conclusions

We characterized the necessary or sufficient conditions for the contractive and expansive H-Toeplitz operators \( B_\varphi \) with various symbols \( \varphi \) on the Bergman space \( L^2_n(\mathbb{D}) \). By these results, we expect to provide the properties of these operators on the Bergman space.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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