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Research article

On geometry of focal surfaces due to B-Darboux and type-2 Bishop frames in Euclidean 3-space

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Abstract: In Euclidean 3-space E^3, a canonical subject is the focal surface of such a cliched space curve, which would be a two-dimensional corrosive with Lagrangian discontinuities. The tubular surfaces with respect to the B-Darboux frame and type-2 Bishop frame in E^3 are given. These tubular surfaces' focal surfaces are then defined. For such types of surfaces, we acquire some results becoming Weingarten, flat, linear Weingarten conditions and we demonstrate that in E^3, a tubular surface has no minimal focal surface. We also provide some examples of these types of surfaces.

Keywords: Euclidean geometry; focal surface; type-2 Bishop frame; tubular surface; B-Darboux frame
Mathematics Subject Classification: 53A05, 53A10

1. Introduction

Frame fields are useful technique in differential geometry for assessing curves and surfaces. The most well-known frame field is the Frenet frame, but there are others, such as Darboux frame. In addition to the Frenet frame, which is built on a curve with velocity and acceleration vectors, also the Darboux frame, which is a natural moving frame created on a surface, is another important subject in differential geometry. It is named after Jean Gaston Darboux, a French mathematician who produced a four-volume collection of research between 1887 and 1896. Since then, the Darboux frame has had many important repercussions, which have been studied, see for example [3, 4, 6, 13, 16]. The Bishop frame is another approach for defining a moving frame that is clearly defined even when the second derivative of the curve has vanished, see [1]. Transporting an orthonormal frame parallel along a curve is as simple as parallel transferring each element of the frame.

In E^3, The envelope of the normal planes of φ is the focus surface or caustic of a curve φ in Euclidean 3-space. The study of a curve's focal surface can yield useful geometric data about a certain curve, and conversely. Darboux discovered how to calculate a curve's evolutes, or the curves whose tangents are normals of φ. Furthermore, he demonstrated (confirmed) that the evolutes foliate the focal surface of φ or that all of these lay on the focal surface, see [4]. Let $\mathbb{M} : \Psi(\varsigma, \omega)$ be a surface associated with a single real-valued function, and $N(\varsigma, \omega)$ be Ψ's unit normal vector. The parameterized description of the focal surface is included

$$\Psi^*(\varsigma, \omega) = \Psi(\varsigma, \omega) + \kappa_j^{-1}(\varsigma, \omega) N(\varsigma, \omega),$$

at which κ_1 and κ_2 are the Ψ's principal curvature functions [5]. Some studies on focal surfaces can be found here [7, 11, 12, 14, 17]. Within that work, we investigate the focal surfaces of a tubular surface created by $\{\mu_1, \mu_2, \mathrm{B}\}$ and $\{\mathrm{T}, \mathrm{N}_1, \mathrm{N}_2\}$. The mean and Gaussian curvatures of the focal surfaces are calculated, and the conditions as these surfaces to become minimal and flat are determined.

2. Preliminaries

Let E^3 be a three-dimensional Euclidean space. The metric is provided in within it as

$$\langle , \rangle = du_1^2 + du_2^2 + du_3^2,$$

where $(u_1, u_2, u_3) \in E^3$'s coordinate system.

Symbolize the moving Frenet frame along its regular curve $\varphi = \varphi(\varsigma)$ using $\{\mathrm{T}, \mathrm{N}, \mathrm{B}\}$ and curvature functions κ and τ in E^3, the Frenet formulae is given by [5]:

$$\begin{bmatrix} \mathrm{T}(\varsigma) \\ \mathrm{N}(\varsigma) \\ \mathrm{B}(\varsigma) \end{bmatrix}_\varsigma = \begin{bmatrix} 0 & \kappa(\varsigma) & 0 \\ -\kappa(\varsigma) & 0 & \tau(\varsigma) \\ 0 & -\tau(\varsigma) & 0 \end{bmatrix} \begin{bmatrix} \mathrm{T}(\varsigma) \\ \mathrm{N}(\varsigma) \\ \mathrm{B}(\varsigma) \end{bmatrix}, \tag{2.1}$$

where $\langle \mathrm{B}, \mathrm{B} \rangle = 1$, $\langle \mathrm{T}, \mathrm{T} \rangle = 1$, $\langle \mathrm{N}, \mathrm{N} \rangle = 1$ and $\langle \mathrm{T}, \mathrm{N} \rangle = \langle \mathrm{T}, \mathrm{B} \rangle = \langle \mathrm{N}, \mathrm{B} \rangle = 0$.

For any arbitrary curve $\varphi(\varsigma)$ with $\tau \neq 0$ in E^3, ψ's type-2 Bishop frame is handed by [18]:

$$\begin{bmatrix} \mu_1(\varsigma) \\ \mu_2(\varsigma) \\ \mathrm{B}(\varsigma) \end{bmatrix}_\varsigma = \begin{bmatrix} 0 & 0 & -k_1(\varsigma) \\ 0 & 0 & -k_2(\varsigma) \\ k_1(\varsigma) & k_2(\varsigma) & 0 \end{bmatrix} \begin{bmatrix} \mu_1(\varsigma) \\ \mu_2(\varsigma) \\ \mathrm{B}(\varsigma) \end{bmatrix}, \tag{2.2}$$

where k_1 and k_2 are the type-2 Bishop curvatures and the relation matrix given by

$$\begin{bmatrix} \mathrm{T}(\varsigma) \\ \mathrm{N}(\varsigma) \\ \mathrm{B}(\varsigma) \end{bmatrix} = \begin{bmatrix} \sin \upsilon(\varsigma) & -\cos \upsilon(\varsigma) & 0 \\ \cos \upsilon(\varsigma) & \sin \upsilon(\varsigma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1(\varsigma) \\ \mu_2(\varsigma) \\ \mathrm{B}(\varsigma) \end{bmatrix},$$

where $\upsilon(\varsigma) = \arctan\left(\frac{k_2}{k_1}\right)$ and

$$k_1 = -\tau \cos \upsilon(\varsigma), \ k_2 = -\tau \sin \upsilon(\varsigma).$$

For a curve $\varphi(\varsigma)$ lying on surface $\mathbb{M} = \Psi(\varsigma, \omega)$. Recognize the Darboux frame on the surface $\{\mathrm{T}, \mathrm{G}, \mathbb{N}\}$, where $\mathrm{G} = \mathbb{N} \times \mathrm{T}$ and $\mathbb{N}$ is just surface's normal [6]. Then

$$\begin{bmatrix} \mathrm{T}(\varsigma) \\ \mathrm{G}(\varsigma) \\ \mathbb{N}(\varsigma) \end{bmatrix}_{\varsigma} = \begin{bmatrix} 0 & \kappa_g(\varsigma) & \kappa_n(\varsigma) \\ -\kappa_g(\varsigma) & 0 & \tau_g(\varsigma) \\ -\kappa_n(\varsigma) & -\tau_g(\varsigma) & 0 \end{bmatrix} \begin{bmatrix} \mathrm{T}(\varsigma) \\ \mathrm{G}(\varsigma) \\ \mathbb{N}(\varsigma) \end{bmatrix}, \tag{2.3}$$

where even the geodesic curvature κ_g, normal curvature κ_n, and relative torsion τ_g are defined as:

$$\tau_g = \langle \mathrm{G}', \mathbb{N} \rangle, \ \kappa_n = \langle \mathrm{T}', \mathbb{N} \rangle, \ \kappa_g = \langle \mathrm{T}', \mathrm{G} \rangle.$$

In matrix form, the B-Darboux frame's variation equation $\{\mathrm{T}, \mathrm{N}_1, \mathrm{N}_2\}$ on the surface is as shown below [6]:

$$\begin{bmatrix} \mathrm{T}(\varsigma) \\ \mathrm{N}_1(\varsigma) \\ \mathrm{N}_2(\varsigma) \end{bmatrix}_{\varsigma} = \begin{bmatrix} 0 & \varrho_1(\varsigma) & \varrho_2(\varsigma) \\ -\varrho_1(\varsigma) & 0 & 0 \\ -\varrho_2(\varsigma) & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{T}(\varsigma) \\ \mathrm{N}_1(\varsigma) \\ \mathrm{N}_2(\varsigma) \end{bmatrix}, \tag{2.4}$$

where ϱ_1 and ϱ_2 the B-Darboux curvatures are acquired in the following way:

$$\begin{aligned} \varrho_1 &= \kappa_g \sin \upsilon + \kappa_n \cos \upsilon, \\ \varrho_2 &= \kappa_n \sin \upsilon - \kappa_g \cos \upsilon. \end{aligned}$$

Also, the relation matrix given by

$$\begin{bmatrix} \mathrm{T}(\varsigma) \\ \mathrm{N}_1(\varsigma) \\ \mathrm{N}_2(\varsigma) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin \upsilon & \cos \upsilon \\ 0 & -\cos \upsilon & \sin \upsilon \end{bmatrix} \begin{bmatrix} \mathrm{T}(\varsigma) \\ \mathrm{G}(\varsigma) \\ \mathbb{N}(\varsigma) \end{bmatrix},$$

such that angle υ between $\mathbb{N}$ and N_1 is acquired around $\upsilon - \upsilon_0 = \displaystyle\int \tau_g \, dt$, for any arbitrary constant υ_0.

Let $\mathbb{M} : \Upsilon(\varsigma, \omega)$ be regular surface Υ in E^3, then the Υ's unit normal vector V can be written as

$$V = \frac{\Upsilon_\varsigma \times \Upsilon_\omega}{\|\Upsilon_\varsigma \times \Upsilon_\omega\|},$$

where $\Upsilon_\varsigma = \dfrac{\partial \Upsilon}{\partial \varsigma}$ and $\Upsilon_\omega = \dfrac{\partial \Upsilon}{\partial \omega}$. The Gaussian K and mean H curvature were also provided by [2, 5, 9]:

$$\begin{aligned} K &= \frac{\ell n - m^2}{EG - F^2}, \\ H &= \frac{En + G\ell - 2mF}{2(EG - F^2)}, \end{aligned}$$

where $E = \|\Upsilon_\varsigma\|^2, \ F = \langle \Upsilon_\varsigma, \Upsilon_\omega \rangle, \ G = \|\Upsilon_\omega\|^2, \ \ell = \langle \Upsilon_{\varsigma\varsigma}, V \rangle, \ m = \langle \Upsilon_{\varsigma\omega}, V \rangle$ and $n = \langle \Upsilon_{\omega\omega}, V \rangle$.

3. Obtaining the focal surface of tubular surface due to B-Darboux frame

Let $\varphi(\varsigma)$ be an arc-length-parameterized curve in E^3. Then, the tubular surface due to the B-Darboux frame has the parametrization [8, 10, 15]:

$$\Omega(\varsigma,\omega) = \varphi(\varsigma) + r[\cos\omega\, N_1(\varsigma) + \sin\omega\, N_2(\varsigma)], \tag{3.1}$$

for which $r = const.$ be spheres's radius. The Ω's velocity vectors are

$$\Omega_\varsigma = [1 - rf(\varsigma,\omega)]T,$$
$$\Omega_\omega = (-r\sin\omega)N_1 + (r\cos\omega)N_2,$$

where $f(\varsigma,\omega) = \varrho_1(\varsigma)\cos\omega + \varrho_2(\varsigma)\sin\omega$. As a result, the trying to follow are the features of Ω's first fundamental form: $E = (1 - rf)^2,\ F = 0,\ G = r^2$. The Ω's unit surface normal vector N_Ω, from the other hand, is acquired by

$$N_\Omega = \frac{\Omega_\varsigma \times \Omega_\omega}{\|\Omega_\varsigma \times \Omega_\omega\|} = -\cos\omega N_1 - \sin\omega N_2. \tag{3.2}$$

Ω's second order partial diffrentials are discovered as

$$\Omega_{\varsigma\varsigma} = (-rf_\varsigma)T + \varrho_1(1 - rf)N_1 + \varrho_2(1 - rf)N_2,$$
$$\Omega_{\varsigma\omega} = (-rf_\omega)T,$$
$$\Omega_{\omega\omega} = (-r\cos\omega)N_1 + (-r\sin\omega)N_2.$$

The second fundamental form coefficients are calculated using (3.2) and that the last three equations, as shown below

$$\ell = -f(1 - rf), \quad m = 0, \quad n = r.$$

Thus, the Gaussian K_Ω and mean curvature H_Ω functions are calculated as

$$K_\Omega = -\frac{f}{r(1 - rf)}, \quad H_\Omega = \frac{1 - 2rf}{2r(1 - rf)}. \tag{3.3}$$

Corollary 3.1. *The tubular surface $\mathbb{M} : \Omega(\varsigma,\omega)$ due to the B-Darboux frame defined by (3.1) has a constant Gaussian curvature iff*

$$f = \frac{rc}{1 + r^2c},$$

for some real constant c.

Corollary 3.2. *The tubular surface $\mathbb{M} : \Omega(\varsigma,\omega)$ due to the B-Darboux frame defined by (3.1) has a constant mean curvature iff*

$$f = \frac{2rc - 1}{2r(1 + rc)},$$

for some real constant c.

Corollary 3.3. *The tubular surface $\mathbb{M} : \Omega(\varsigma,\omega)$ due to the B-Darboux frame defined by (3.1) is a (K_Ω, H_Ω)-Weingarten surface.*

Corollary 3.4. *The tubular surface $\mathbb{M} : \Omega(\varsigma, \omega)$ due to the B-Darboux frame defined by (3.1) is a (K_Ω, H_Ω)-linear Weingarten surface iff*

$$f = \frac{rc + c_2}{2(c_1 + c_2 - rc)},$$

where c_1, c_2 and c are not all zero real numbers.

We now concentrate on the parametrization of $\mathbb{M}^*$ focal surface of $\mathbb{M}$ by using (3.3) as well as the equation $\kappa_j = H_\Omega \pm \sqrt{H_\Omega^2 - K_\Omega}$, $j = (1, 2)$, yields the principal curvature performs the following functions

$$\kappa_1 = \frac{1}{r}, \quad \kappa_2 = -\frac{f}{1 - rf}. \tag{3.4}$$

Through using Eq (3.4) we define $\mathbb{M}^*$ as

$$\Omega^*(\varsigma, \omega) = \varphi(\varsigma) + \frac{1}{f}\Big[\cos\omega\, \mathrm{N}_1(\varsigma) + \sin\omega\, \mathrm{N}_2(\varsigma)\Big], \tag{3.5}$$

where $f(\varsigma, \omega) = \varrho_1(\varsigma)\cos\omega + \varrho_2(\varsigma)\sin\omega$. The Ω^*'s velocity vectors are

$$\Omega^*_\varsigma = -\frac{1}{f^2}\Big[f_\varsigma \cos\omega\Big]\mathrm{N}_1 - \frac{1}{f^2}\Big[f_\varsigma \sin\omega\Big]\mathrm{N}_2,$$
$$\Omega^*_\omega = -\frac{1}{f^2}\Big[f_\omega \cos\omega + f\sin\omega\Big]\mathrm{N}_1 - \frac{1}{f^2}\Big[f_\omega \sin\omega - f\cos\omega\Big]\mathrm{N}_2.$$

As a result, the features of Ω^*'s first fundamental form:

$$E^* = \frac{f_\varsigma^2}{f^4}, \quad F^* = \frac{f_\varsigma f_\omega}{f^4}, \quad G^* = \frac{f_\omega^2 + f^2}{f^4}.$$

The Ω^*'s unit surface normal vector N^*_Ω, from the other hand, is acquired by

$$N^*_{\Omega^*} = -T. \tag{3.6}$$

Ω^*'s second order partial diffrentials are discovered as

$$\Omega^*_{\varsigma\varsigma} = -\Big(\frac{f_\varsigma}{f}\Big)\mathrm{T} - \Big(\frac{f_\varsigma}{f^2}\Big)_\varsigma \cos\omega\mathrm{N}_1 - \Big(\frac{f_\varsigma}{f^2}\Big)_\varsigma \sin\omega\mathrm{N}_2,$$
$$\Omega^*_{\varsigma\omega} = \Big[-\Big(\frac{f_\varsigma}{f^2}\Big)_\omega \cos\omega + \Big(\frac{f_\varsigma}{f^2}\Big)\sin\omega\Big]\mathrm{N}_1 - \Big[\Big(\frac{f_\varsigma}{f^2}\Big)_\omega \sin\omega + \Big(\frac{f_\varsigma}{f^2}\Big)\cos\omega\Big]\mathrm{N}_2,$$
$$\Omega^*_{\omega\omega} = -\frac{1}{f^2}\Big[f_\omega\cos\omega + f\sin\omega\Big]_\omega \mathrm{N}_1 - \frac{1}{f^2}\Big[f_\omega\sin\omega - f\cos\omega\Big]_\omega \mathrm{N}_2.$$

The second fundamental form coefficients are calculated using (3.6) and that the last three equations, as shown below

$$\ell^* = \frac{f_\varsigma}{f}, \quad m^* = 0, \quad n^* = 0.$$

Thus, the Gaussian $K^*_{\Omega^*}$ and mean curvature $H^*_{\Omega^*}$ functions are calculated as

$$K^*_{\Omega^*} = 0, \quad H^*_{\Omega^*} = \frac{f(f_\omega^2 + f^2)}{2f_\varsigma}. \tag{3.7}$$

Theorem 3.1. *The focal surface M^* defined by (3.5) via B-Darboux frame is flat surface.*

Theorem 3.2. *The tubular surface $\mathbb{M}$ has no minimal focal surface $\mathbb{M}^*$ defined through it.*

Corollary 3.5. *Let M^* be focal surface (3.5) via B-Darboux frame in* E^3. *Then the following is satisfied:*

(1) $\mathbb{M}^*$'s ς-parameter curves not possible asymptotic curves.

(2) $\mathbb{M}^*$'s ω-parameter curves are asymptotic curves.

Corollary 3.6. *Let M^* be focal surface (3.5) via B-Darboux frame in* E^3. *Then the following holds:*

(1) $\mathbb{M}^*$'s ς-parameter curves are geodesic curves iff $f\, f_{\varsigma\varsigma} - 2(f_\varsigma)^2 = 0$.

(2) $\mathbb{M}^*$'s ω-parameter curves are not geodesic curves.

Example 3.1. *Let φ be a circular helix parameterized as*

$$\varphi(\varsigma) = \left(\cos\left(\frac{\varsigma}{\sqrt{2}}\right), \quad \sin\left(\frac{\varsigma}{\sqrt{2}}\right), \frac{\varsigma}{\sqrt{2}}\right).$$

Then, the curve's Darboux frame and curvatures κ_g, κ_n and τ_g are dictated by

$$\begin{aligned}
\mathrm{T}(\varsigma) &= \frac{1}{\sqrt{2}}\left(-\sin\left(\frac{\varsigma}{\sqrt{2}}\right), \cos\left(\frac{\varsigma}{\sqrt{2}}\right), 1\right),\\
\mathrm{G}(\varsigma) &= \frac{1}{\sqrt{2}}\left(-\cos\left(\frac{\varsigma}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\sin\left(\frac{\varsigma}{\sqrt{2}}\right), \sin\left(\frac{\varsigma}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\cos\left(\frac{\varsigma}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right),\\
\mathbb{N}(\varsigma) &= \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\varsigma}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\sin\left(\frac{\varsigma}{\sqrt{2}}\right), -\sin\left(\frac{\varsigma}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\cos\left(\frac{\varsigma}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right),\\
\kappa_g &= \frac{1}{2\sqrt{2}}, \quad \kappa_n = \frac{1}{2\sqrt{2}}, \quad \tau_g = \frac{1}{2}.
\end{aligned}$$

Now,

$$\upsilon = \int_0^\varsigma \tau_g\, dt = \int_0^\varsigma \frac{1}{2}\, dt = \frac{\varsigma}{2}.$$

So, the B-Darboux curvatures are calculated as

$$\begin{aligned}
\varrho_1 &= \frac{1}{2\sqrt{2}}\Big[\sin\left(\frac{\varsigma}{2}\right) + \cos\left(\frac{\varsigma}{2}\right)\Big],\\
\varrho_2 &= \frac{1}{2\sqrt{2}}\Big[\sin\left(\frac{\varsigma}{2}\right) - \cos\left(\frac{\varsigma}{2}\right)\Big].
\end{aligned}$$

Then the B-Darboux frame are given as

$$\begin{aligned}
\mathrm{T}(\varsigma) &= \frac{1}{\sqrt{2}}\Big(-\sin\big(\frac{\varsigma}{\sqrt{2}}\big), \cos\big(\frac{\varsigma}{\sqrt{2}}\big), 1\Big),\\
\mathrm{N}_1(\varsigma) &= \frac{1}{\sqrt{2}}\Big[\cos\big(\frac{\varsigma}{\sqrt{2}}\big)\big[\cos\big(\frac{\varsigma}{2}\big) - \sin\big(\frac{\varsigma}{2}\big)\big] + \frac{1}{\sqrt{2}}\sin\big(\frac{\varsigma}{\sqrt{2}}\big)\big[\cos\big(\frac{\varsigma}{2}\big) + \sin\big(\frac{\varsigma}{2}\big)\big],\\
&\quad \sin\big(\frac{\varsigma}{\sqrt{2}}\big)\big[\cos\big(\frac{\varsigma}{2}\big) - \sin\big(\frac{\varsigma}{2}\big)\big] - \frac{1}{\sqrt{2}}\cos\big(\frac{\varsigma}{\sqrt{2}}\big)\big[\cos\big(\frac{\varsigma}{2}\big) + \sin\big(\frac{\varsigma}{2}\big)\big], \frac{1}{\sqrt{2}}\big[\cos\big(\frac{\varsigma}{2}\big) + \sin\big(\frac{\varsigma}{2}\big)\big]\Big]\\
\mathrm{N}_2(\varsigma) &= \frac{1}{\sqrt{2}}\Big[\cos\big(\frac{\varsigma}{\sqrt{2}}\big)\big[\cos\big(\frac{\varsigma}{2}\big) + \sin\big(\frac{\varsigma}{2}\big)\big] - \frac{1}{\sqrt{2}}\sin\big(\frac{\varsigma}{\sqrt{2}}\big)\big[\cos\big(\frac{\varsigma}{2}\big) - \sin\big(\frac{\varsigma}{2}\big)\big]\Big],\\
&\quad \sin\big(\frac{\varsigma}{\sqrt{2}}\big)\big[\cos\big(\frac{\varsigma}{2}\big) + \sin\big(\frac{\varsigma}{2}\big)\big] - \frac{1}{\sqrt{2}}\cos\big(\frac{\varsigma}{\sqrt{2}}\big)\big[\cos\big(\frac{\varsigma}{2}\big) - \sin\big(\frac{\varsigma}{2}\big)\big], -\frac{1}{\sqrt{2}}\big[\cos\big(\frac{\varsigma}{2}\big) - \sin\big(\frac{\varsigma}{2}\big)\big]\Big].
\end{aligned}$$

As a result and taking $r = \sqrt{2}$, the parameterization of the tubular surface $\mathbb{M}_1$ over the curve φ can be compiled in Darboux frame as (see Figure 1)

$$\begin{aligned}
\Omega_1(\varsigma,\omega) = \Big[&\cos\big(\frac{\varsigma}{\sqrt{2}}\big)\big[1 - \cos\omega + \sin\omega\big] + \frac{1}{\sqrt{2}}\sin\big(\frac{\varsigma}{\sqrt{2}}\big)\big[\cos\omega + \sin\omega\big]\Big],\\
&\sin\big(\frac{\varsigma}{\sqrt{2}}\big)\big[1 - \cos\omega + \sin\omega\big] - \frac{1}{\sqrt{2}}\cos\big(\frac{\varsigma}{\sqrt{2}}\big)\big[\cos\omega + \sin\omega\big], \frac{1}{\sqrt{2}}\big[\varsigma + \cos\omega + \sin\omega\big]\Big].
\end{aligned}$$

The tubular surface $\mathbb{M}_2$ over the curve φ via B-Darboux frame can be given as (see Figure 2)

$$\begin{aligned}
\Omega_2(\varsigma,\omega) = \Big[&\cos\big(\frac{\varsigma}{\sqrt{2}}\big) + \big[\cos\big(\frac{\varsigma}{2}\big) - \sin\big(\frac{\varsigma}{2}\big)\big]\big[\cos\omega\cos\big(\frac{\varsigma}{\sqrt{2}}\big) - \frac{1}{\sqrt{2}}\sin\omega\sin\big(\frac{\varsigma}{\sqrt{2}}\big)\big]\\
&+ \big[\cos\big(\frac{\varsigma}{2}\big) + \sin\big(\frac{\varsigma}{2}\big)\big]\big[\sin\omega\cos\big(\frac{\varsigma}{\sqrt{2}}\big) + \frac{1}{\sqrt{2}}\cos\omega\sin\big(\frac{\varsigma}{\sqrt{2}}\big)\big],\\
&\sin\big(\frac{\varsigma}{\sqrt{2}}\big) + \big[\cos\big(\frac{\varsigma}{2}\big) - \sin\big(\frac{\varsigma}{2}\big)\big]\big[\cos\omega\sin\big(\frac{\varsigma}{\sqrt{2}}\big) - \frac{1}{\sqrt{2}}\sin\omega\cos\big(\frac{\varsigma}{\sqrt{2}}\big)\big]\\
&- \big[\cos\big(\frac{\varsigma}{2}\big) + \sin\big(\frac{\varsigma}{2}\big)\big]\big[\sin\omega\sin\big(\frac{\varsigma}{\sqrt{2}}\big) + \frac{1}{\sqrt{2}}\cos\omega\cos\big(\frac{\varsigma}{\sqrt{2}}\big)\big],\\
&\frac{1}{\sqrt{2}}\big[\varsigma + \cos\big(\frac{\sqrt{2}\omega + \varsigma}{\sqrt{2}}\big) - \sin\big(\frac{\sqrt{2}\omega - \varsigma}{\sqrt{2}}\big)\big]\Big].
\end{aligned}$$

The focal surface $\mathbb{M}_1^*$ of $\mathbb{M}_1$ via Darboux frame can be given as (see Figure 3)

$$\begin{aligned}
\Omega_1^*(\varsigma,\omega) = \Big[&\sqrt{2}\sin\big(\frac{\varsigma}{\sqrt{2}}\big) + \cos\big(\frac{\varsigma}{\sqrt{2}}\big)\big[\frac{3\sin\omega - \cos\omega}{\cos\omega + \sin\omega}\big], -\sqrt{2}\cos\big(\frac{\varsigma}{\sqrt{2}}\big)\\
&+ \sin\big(\frac{\varsigma}{\sqrt{2}}\big)\big[\frac{3\sin\omega - \cos\omega}{\cos\omega + \sin\omega}\big], \frac{\varsigma + 2}{\sqrt{2}}\Big].
\end{aligned}$$

The focal surface $\mathbb{M}_2^*$ of $\mathbb{M}_2$ via B-Darboux frame can be given as (see Figure 4)

$$\Omega_2^*(\varsigma,\omega) = \cos\left(\frac{\varsigma}{\sqrt{2}}\right) + \frac{1}{\sin\left(\frac{\varsigma}{2}\right)\left[\cos\omega + \sin\omega\right] + \cos\left(\frac{\varsigma}{2}\right)\left[\cos\omega - \sin\omega\right]}\left[\left[\cos\left(\frac{\varsigma}{2}\right)\right.\right.$$
$$\left. - \sin\left(\frac{\varsigma}{2}\right)\right]\left[\cos\omega\cos\left(\frac{\varsigma}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\sin\omega\sin\left(\frac{\varsigma}{\sqrt{2}}\right)\right] + \left[\cos\left(\frac{\varsigma}{2}\right)\right.$$
$$\left.\left. + \sin\left(\frac{\varsigma}{2}\right)\right]\left[\sin\omega\cos\left(\frac{\varsigma}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\cos\omega\sin\left(\frac{\varsigma}{\sqrt{2}}\right)\right]\right],$$
$$\sin\left(\frac{\varsigma}{\sqrt{2}}\right) + \frac{1}{\sin\left(\frac{\varsigma}{2}\right)\left[\cos\omega + \sin\omega\right] + \cos\left(\frac{\varsigma}{2}\right)\left[\cos\omega - \sin\omega\right]}\left[\left[\cos\left(\frac{\varsigma}{2}\right)\right.\right.$$
$$\left. - \sin\left(\frac{\varsigma}{2}\right)\right]\left[\cos\omega\sin\left(\frac{\varsigma}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\sin\omega\cos\left(\frac{\varsigma}{\sqrt{2}}\right)\right]$$
$$\left. - \left[\cos\left(\frac{\varsigma}{2}\right) + \sin\left(\frac{\varsigma}{2}\right)\right]\left[\sin\omega\sin\left(\frac{\varsigma}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\cos\omega\cos\left(\frac{\varsigma}{\sqrt{2}}\right)\right]\right],$$
$$\left.\frac{\varsigma}{\sqrt{2}} + \frac{2\left[\cos\left(\frac{\sqrt{2}\omega+\varsigma}{\sqrt{2}}\right) - \sin\left(\frac{\sqrt{2}\omega-\varsigma}{\sqrt{2}}\right)\right]}{\sin\left(\frac{\varsigma}{2}\right)\left[\cos\omega + \sin\omega\right] + \cos\left(\frac{\varsigma}{2}\right)\left[\cos\omega - \sin\omega\right]}\right].$$

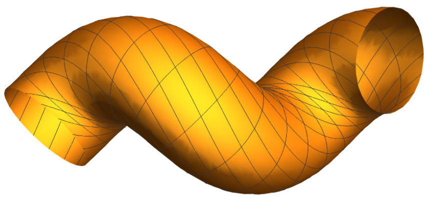

Figure 1. $\mathbb{M}_1$ due to Darboux frame.

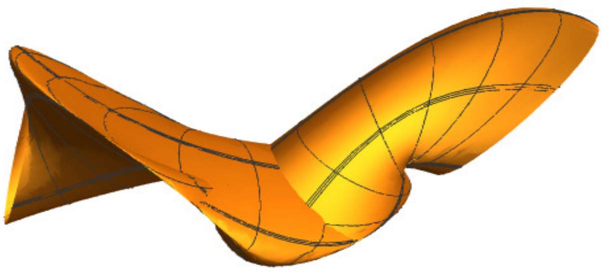

Figure 2. $\mathbb{M}_2$ via B-Darboux frame.

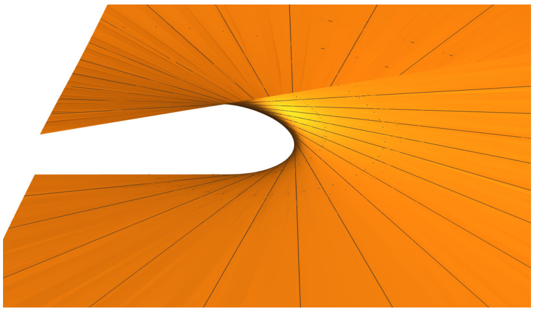

Figure 3. Focal surface $\mathbb{M}_1^*$ via Darboux frame.

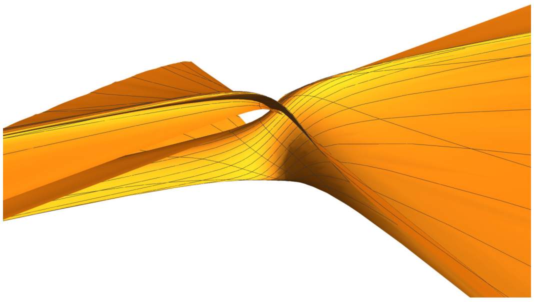

Figure 4. Focal surface $\mathbb{M}_2^*$ via B-Darboux frame.

4. Obtaining the focal surface of tubular surface due to type-2 Bishop frame

Let $\varphi : (x, y) \to \mathrm{E}^3$ be a finite-length smooth unit speed curve that is embedded in E^3 topologically. The way to allocate N_φ of the $\varphi(x, y)$'s normal bundle in E^3 is innately diffeomorphic to the direct product $(x, y) \times \mathrm{E}^2$ due to transcription together with regard to the resulting connection normally. For a relatively tiny $r > 0$, the tube of radius r about the curve φ is the set (see [10]):

$$\mathrm{T}_{r(\varphi)} = \{exp_{\varphi(\varsigma)}\, u | u \in N_{\varphi(\varsigma)},\ ||u|| = r,\ x < \varsigma < y\}.$$

In E^3, the tube $\mathrm{T}_{r(\varphi)}$ is a smooth surface for sufficiently small $r > 0$. Using $\{\mu_1, \mu_2, \mathrm{B}\}$, we can write the tube surface as shown below.

$$\mathbb{M} : \Phi(\varsigma, \omega) = \varphi(\varsigma) + r[\cos\omega\, \mu_2(\varsigma) + \sin\omega\, \mathrm{B}(\varsigma)]. \tag{4.1}$$

The following are the Φ's derivative formulations for type-2 Bishop frame partial differentiation with respect to ς and ω.

$$\begin{aligned} \Phi_\varsigma &= [1 - rk_1 \sin\omega]\mu_1 + (rk_2 \sin\omega)\mu_2 - (rk_2 \cos\omega)\mathrm{B}, \\ \Phi_\omega &= -(r \sin\omega)\mu_2 + (r \cos\omega)\mathrm{B}. \end{aligned}$$

As a consequence, the features of Φ's first fundamental form

$$E = (1 - r f)^2, \quad F = 0, \quad G = r^2.$$

From the other hand, the Φ's unit surface normal vector N_Φ is acquired by

$$N_\Phi = -\cos\omega\,\mu_2 - \sin\omega\,\mathrm{B}. \tag{4.2}$$

Φ's second order partial diffrentials are discovered as

$$\begin{aligned}
\Phi_{\varsigma\varsigma} &= [rk_1' \sin\omega - rk_1k_2\cos\omega]\mu_1 + [rk_2'\sin\omega - rk_2^2\cos\omega]\mu_2 - [k_1 + r\tau^2\sin\omega + rk_2'\cos\omega]\mathrm{B},\\
\Phi_{\varsigma\omega} &= (rk_1\cos\omega)\mu_1 + (rk_2\cos\omega)\mu_2 + (rk_2\sin\omega)\mathrm{B},\\
\Phi_{\omega\omega} &= -(r\cos\omega)\mu_2 - (r\sin\omega)\mathrm{B}.
\end{aligned}$$

The explanatory variables of the second fundamental form were also calculated using (4.2) and the last three equations, which are shown below

$$\ell = rk_2^2 + rk_1^2\sin\omega + k_1\sin\omega, \quad m = -rk_2, \quad n = r.$$

Thus, the Gaussian K_Φ and mean curvature H_Φ are calculated as

$$K_\Phi = \frac{k_1\sin\omega}{r(1+rk_1\sin\omega)}, \quad H_\Phi = \frac{1-2rk_1\sin\omega}{2r(1+rk_1\sin\omega)}. \tag{4.3}$$

The principal curvatures κ_1 and κ_2 of Φ are the root of

$$det(W_{II} - \kappa W_I) = 0,$$

where $W_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$, $W_{II} = \begin{pmatrix} \ell & m \\ m & n \end{pmatrix}$. Then the Φ's principal curvatures are

$$\kappa_1 = \frac{1}{r}, \quad \kappa_2 = \frac{k_1\sin\omega}{1+rk_1\sin\omega}. \tag{4.4}$$

Corollary 4.1. *The tubular surface* $\mathbb{M} : \Phi(\varsigma,\omega)$ *due to the type-2 Bishop frame defined by (4.1) has a constant Gaussian curvature iff*

$$\kappa_1 = \frac{rc}{(1-r^2c)\sin\omega},$$

for some real constant c.

Corollary 4.2. *The tubular surface* $\mathbb{M} : \Phi(\varsigma,\omega)$ *due to the type-2 Bishop frame defined by (4.1) has a constant mean curvature iff*

$$\kappa_1 = \frac{2rc-1}{2r(1-rc)\sin\omega},$$

for some some real constant c.

Corollary 4.3. *The tubular surface* $\mathbb{M} : \Phi(\varsigma,\omega)$ *due to the type-2 Bishop frame defined by (4.1) is a* (K_Φ, H_Φ)*-Weingarten surface.*

Corollary 4.4. *The tubular surface* $\mathbb{M} : \Phi(\varsigma,\omega)$ *due to the type-2 Bishop frame defined by (4.1) is a* (K_Φ, H_Φ)*-linear Weingarten surface iff*

$$\kappa_1 = \frac{2rc-b}{2(a+rb+rc)\sin\omega},$$

where a, b and c are not all zero real numbers.

Now, by using (4.4) we can derive the focal surface $\mathbb{M}^*$ of $\mathbb{M}$ from the definition of the focal surface of a given surface

$$\Phi^*(\varsigma,\omega) = \varphi(\varsigma) - \frac{1}{k_1 \sin\omega}\Big[\cos\omega\,\mu_2(\varsigma) + \sin\omega\,\mathrm{B}(\varsigma)\Big]. \tag{4.5}$$

The Φ^*'s velocity vectors are

$$\Phi^*_\varsigma = -\frac{1}{k_1^2 \sin\omega}\Big[k_1' \cos\omega - k_1 k_2 \sin\omega\Big]\mu_2 + \frac{1}{k_1^2 \sin\omega}\Big[k_1' \sin\omega + k_1 k_2 \cos\omega\Big]\mathrm{B},$$
$$\Phi^*_\omega = \Big[\frac{1}{k_1 \sin^2\omega}\Big]\mu_2.$$

As a consequence, the following are the parts of Φ's first fundamental form:

$$E^* = \frac{{k_1'}^2 + k_1^2 k_2^2}{k_1^4 \sin^2\omega}, \quad F^* = \frac{k_1' \cos\omega - k_1 k_2 \sin\omega}{k_1^3 \sin^3\omega}, \quad G^* = \frac{1}{k_1^2 \sin^4\omega}.$$

The Φ^*'s unit surface normal vector N^*_Φ, from the other hand, is acquired by

$$N^*_{\Phi^*} = -\mu_1. \tag{4.6}$$

Φ^*'s second order partial diffrentials are discovered as

$$\Phi^*_{\varsigma\varsigma} = \Big[\frac{k_1' \sin\omega + k_1 k_2 \cos\omega}{k_1^2 \sin\omega}\Big]\mu_1 + \Big[\Big[\frac{k_1' \cos\omega - k_1 k_2 \sin\omega}{k_1^2 \sin\omega}\Big]_\varsigma + k_2\Big[\frac{k_1' \sin\omega + k_1 k_2 \cos\omega}{k_1^2 \sin\omega}\Big]\Big]\mu_2$$
$$- \Big[\Big[\frac{k_1' \sin\omega + k_1 k_2 \sin\omega}{k_1^2 \cos\omega}\Big]_\varsigma - k_2\Big[\frac{k_1' \cos\omega - k_1 k_2 \sin\omega}{k_1^2 \sin\omega}\Big]\Big]\mathrm{B},$$
$$\Phi^*_{\varsigma\omega} = \Big[\frac{-k_1'}{k_1^2 \sin^2\omega}\Big]\mu_2 + \Big[\frac{-k_2}{k_1 \sin^2\omega}\Big]\mathrm{B},$$
$$\Phi^*_{\omega\omega} = \Big[\frac{-2\cos\omega}{k_1 \sin\omega}\Big]\mu_2.$$

The second fundamental form coefficients are calculated using (4.6) as shown below

$$\ell^* = -\Big[\frac{k_1' \sin\omega + k_1 k_2 \cos\omega}{k_1^2 \sin\omega}\Big], \quad m^* = 0, \quad n^* = 0.$$

Thus, the Gaussian $K^*_{\Phi^*}$ and mean curvature $H^*_{\Phi^*}$ functions are calculated as

$$K^*_{\Phi^*} = 0, \quad H^*_{\Phi^*} = -\frac{k_1^3 \sin\omega(k_1' \sin\omega + k_1 k_2 \cos\omega)}{2\Big[{k_1'}^2 + k_1^2 k_2^2 - (k_1' \cos\omega - k_1 k_2 \sin\omega)^2\Big]}. \tag{4.7}$$

Theorem 4.1. *The focal surface M^* defined by (4.1) via type-2 Bishop frame is flat surface.*

Theorem 4.2. The tubular surface $\mathbb{M}$ has no minimal focal surface $\mathbb{M}^*$ defined through it.

Corollary 4.5. *Let M^* be focal surface (4.5) via type-2 Bishop frame in E^3. Then the following is satisfied:*

(1) $\mathbb{M}^*$'s ς-parameter curves are not asymptotic curves.

(2) $\mathbb{M}^*$' ω-parameter curves are asymptotic curves.

Corollary 4.6. *Let M^* be a focal surface (4.5) via type-2 Bishop frame in* E^3*. Then the following holds:*

(1) $\mathbb{M}^*$'s ς-parameter curves are geodesic curves iff $k_1' + k_1 k_2 \cot\omega = 0$.

(2) $\mathbb{M}^*$'s ω-parameter curves are not geodesic curves.

Example 4.1. *Let φ be a curve parameterized as $\varphi(\varsigma) = \left(\cos\left(\frac{\varsigma}{2}\right), \sin\left(\frac{\varsigma}{2}\right), \frac{\sqrt{3}\varsigma}{2}\right)$. Then, the curve's Frenet invariants are dictated by*

$$\begin{aligned}
\mathrm{T}(\varsigma) &= \frac{1}{2}\left(-\sin\left(\frac{\varsigma}{2}\right), \cos\left(\frac{\varsigma}{2}\right), \sqrt{3}\right),\\
\mathrm{N}(\varsigma) &= \left(-\cos\left(\frac{\varsigma}{2}\right), -\sin\left(\frac{\varsigma}{2}\right), 0\right),\\
\mathrm{B}(\varsigma) &= \left(\frac{\sqrt{3}}{2}\sin\left(\frac{\varsigma}{2}\right), -\frac{\sqrt{3}}{2}\cos\left(\frac{\varsigma}{2}\right), \frac{1}{2}\right),\\
\kappa &= \frac{1}{4}, \quad \tau = \frac{\sqrt{3}}{4}.
\end{aligned}$$

Now, $\upsilon = \int_0^{\varsigma} \kappa\, dt = \int_0^{\varsigma} \frac{1}{4}\, dt = \frac{\varsigma}{4}$. So, the type-2 Bishop curvatures are calculated as

$$k_1 = -\frac{\sqrt{3}}{4}\cos\left(\frac{\varsigma}{4}\right), \qquad k_2 = -\frac{\sqrt{3}}{4}\sin\left(\frac{\varsigma}{4}\right).$$

Then the type-2 Bishop frame are given as

$$\begin{aligned}
\mu_1(\varsigma) = \Big[&-\cos\left(\frac{\varsigma}{2}\right)\cos\left(\frac{\varsigma}{4}\right) - \frac{1}{2}\sin\left(\frac{\varsigma}{2}\right)\sin\left(\frac{\varsigma}{4}\right), -\sin\left(\frac{\varsigma}{2}\right)\cos\left(\frac{\varsigma}{4}\right)\\
&+ \frac{1}{2}\cos\left(\frac{\varsigma}{2}\right)\sin\left(\frac{\varsigma}{4}\right), \frac{\sqrt{3}}{2}\sin\left(\frac{\varsigma}{4}\right)\Big],\\
\mu_2(\varsigma) = \Big[&-\cos\left(\frac{\varsigma}{2}\right)\sin\left(\frac{\varsigma}{4}\right) + \frac{1}{2}\sin\left(\frac{\varsigma}{2}\right)\cos\left(\frac{\varsigma}{4}\right), -\sin\left(\frac{\varsigma}{2}\right)\sin\left(\frac{\varsigma}{4}\right)\\
&- \frac{1}{2}\cos\left(\frac{\varsigma}{2}\right)\cos\left(\frac{\varsigma}{4}\right), -\frac{\sqrt{3}}{2}\cos\left(\frac{\varsigma}{4}\right)\Big],\\
\mathrm{B}(\varsigma) = \Big(&\frac{\sqrt{3}}{2}\sin\left(\frac{\varsigma}{2}\right), -\frac{\sqrt{3}}{2}\cos\left(\frac{\varsigma}{2}\right), \frac{1}{2}\Big).
\end{aligned}$$

Taking $r = 2$, the parameterization of the tubular surface $\mathbb{M}$ over φ can be compiled in type-2 Bishop frame as (see Figure 5)

$$\begin{aligned}
\Phi(\varsigma, \omega) = \Big\{ &\sin\left(\frac{\varsigma}{2}\right)\left[\frac{\sqrt{3}}{2} + \cos\omega\cos\left(\frac{\varsigma}{4}\right)\right] + \cos\left(\frac{\varsigma}{2}\right)\left[1 - \cos\omega\sin\left(\frac{\varsigma}{4}\right)\right],\\
&\sin\left(\frac{\varsigma}{2}\right)\left[1 - \cos\omega\sin\left(\frac{\varsigma}{4}\right)\right] - \cos\left(\frac{\varsigma}{2}\right)\left[\frac{\sqrt{3}}{2} + \cos\omega\cos\left(\frac{\varsigma}{4}\right)\right],\\
&\frac{1}{2}\left[\sqrt{3}\,\varsigma - 2\sqrt{3}\cos\omega\cos\left(\frac{\varsigma}{4}\right) + 1\right]\Big\}.
\end{aligned}$$

The focal surface $\mathbb{M}^*$ of $\mathbb{M}$ via type-2 Bishop frame can be given as (see Figure 6)

$$\Phi^*(\varsigma,\omega)=\Big\{\cos\left(\frac{\varsigma}{2}\right)\left[1+\frac{4}{\sqrt{3}}\cot\omega\tan\left(\frac{\varsigma}{4}\right)\right]-2\sin\left(\frac{\varsigma}{2}\right)\left[\csc\omega\sec\left(\frac{\varsigma}{4}\right)+\frac{1}{\sqrt{3}}\cot\omega\right],$$
$$\sin\left(\frac{\varsigma}{2}\right)\left[1+\frac{4}{\sqrt{3}}\cot\omega\tan\left(\frac{\varsigma}{4}\right)\right]+2\cos\left(\frac{\varsigma}{2}\right)\left[\csc\omega\sec\left(\frac{\varsigma}{4}\right)+\frac{1}{\sqrt{3}}\cot\omega\right],$$
$$\frac{\sqrt{3}\varsigma}{2}+2\csc\omega\left[1-\frac{2}{\sqrt{3}}\cos\left(\frac{\varsigma}{4}\right)\right]\Big\}.$$

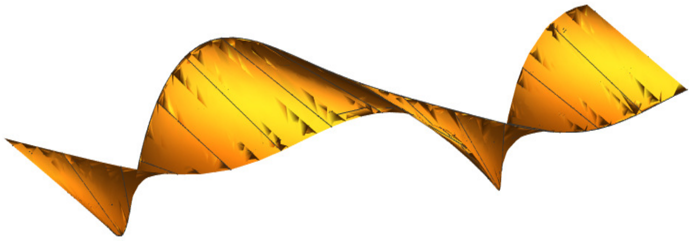

Figure 5. $\mathbb{M}$ due to type-2 Bishop frame.

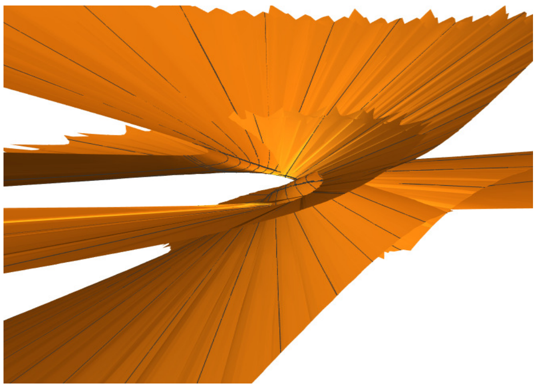

Figure 6. $\mathbb{M}^*$ via type-2 Bishop frame.

5. Conclusions

The focal surface of a space curve in Euclidean space is the equivalent of the evolute of some well plane curve that is a smooth curve far from the plane curve's inflection points. It is the critical value of a Lagrangian map and the local bifurcation set of the family of distance squared functions on a planar curve. As a result, it solely possesses Lagrangian singularities. We may deduce that for places corresponding to ordinary vertices of a plane curve, the evolute possesses an ordinary cusp singularity. The tubular surfaces in E^3 are supplied with regard to the B-Darboux frame and type-2 Bishop frame. The focal surfaces of these tubular surfaces are then specified. We obtain some results for these kinds of surfaces as Weingarten, flat, linear Weingarten conditions, and we show that in E^3, a tubular surface has no minimal focal surface.

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Conflicts of interest

The authors declare no competing interest.

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