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Research article

Properties of R_0-algebra based on hesitant fuzzy MP filters and congruence relations

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Abstract: The hesitant fuzzy MP filter and the hesitant fuzzy congruence relation of algebra are introduced in this study, and their properties are investigated. The comparable characterization of a hesitant fuzzy MP filter is then provided. Furthermore, we established that the set of all hesitant fuzzy congruence relations and the set of all hesitant fuzzy MP filters of R_0-algebra are complete lattice isomorphism based on the features of the hesitant fuzzy congruence relation in R_0-algebra.

Keywords: R_0-algebra; hesitant fuzzy set; hesitant fuzzy MP filter; hesitant fuzzy congruence relation
Mathematics Subject Classification: 03G25, 08A72

1. Introduction

Classical logic is the unification of absoluteness and relativity, and it is the foundation of all knowledge and the fundamental portion of human total understanding and the base of all knowledge. Deductive logic reasoning is binary logic, meaning there are only two options: true or false. Uncertainty, on the other hand, isn't just true or false; it might also have multiple outcomes. Uncertainty reasoning is an important aspect of artificial intelligence research, and examining it in the context of logic is a scientific research approach. Wang [1] established the notion of R_0-algebra after explaining the differences between uncertain logic and classical logic. This was specifically for the sake of investigating fuzzy reasoning. R_0-algebra is slightly more powerful than implication lattice algebra. As reasoning criteria in algebraic structure, filters, ideals, and sub algebras, play a crucial part in the study of algebraic structure. There have been numerous studies on these reasoning criteria, such as Cheng's [2] basic structure of R_0-algebras, Li's [3] equivalent characterization of minimal reduction sets, and the necessary and sufficient conditions for the existence of the maximal reduction R_0-algebra. Hua [4] recently presented the concept of derivation in R_0-algebras and demonstrated how to make a filter a perfect derivation filter. Zhang [5] introduced the generalized relative annihilator of R_0-algebras, and studied equivalent characterization of minimal subtractive sets of R_0-algebras. Xin [6] introduced monadic operator, defined and studied monadic R_0-algebra. He explained monadic filtering and monadic congruence, as well as their qualities. Fan [7] investigated the equivalent characterization of Boolean algebra and R_0-algebra by replacing R_0-algebra with Boolean atoms. Zadeh [8] introduced fuzzy sets in 1965. Fuzzy sets and its expansions do well in dealing with uncertainty in a variety of situations. People's interest in the use of fuzzy sets is developing rapidly all over the world, and intuitionistic fuzzy sets [9] and bipolar valued fuzzy sets [10] have both been extensively investigated. Fuzzification principles have also been extended to other algebraic structures, with a series of conclusions emerging one after the other. For example, Liu et al. [11] explored and addressed bipolar fuzzy ideals in negative non-involutive residual lattices, Zhang [12] developed intuitionistic fuzzy filter theory in algebraic structures, and reference [13,14] has further conclusions. There are several physical interpretations of abstract algebras. The physical interpretation of noncommutative algebraic varieties has been introduced, where among other physical properties, the theory of entanglement: A generalization of parameterizing the objects of physics was introduced and discussed in detail [20]. On the other hand, the physical interpretation of bistable unidirectional Ring-Laser operation was discussed. There are many other applications of some kinds of algebra, for example (see [20,21]).

Torra [15] proposed hesitant fuzzy set theory in 2010, which is a great tool for expressing people's indecision in real life and solves the problem of uncertainty. A hesitant fuzzy set is made up of hesitant fuzzy elements, each of which is a collection of probable values from the unit [0,1] closed interval. As a result, as compared to other extended versions of fuzzy sets, hesitant fuzzy sets can more extensively and precisely reflect the hesitant information of decision makers. Hesitant fuzzy sets have also gotten a lot of attention, and they're used in a variety of mathematical models [16–18]. The study of hesitation fuzzy measure, multi-attribute decision-making model, and linguistic decision-making method is the focus of hesitation fuzzy set theory. In real life, the solution to a problem is not unique, however, the uncertain performance of hesitant fuzzy elements lead to a better illustration in such a problem. R_0-algebra is a kind of important logic algebra, where filter is an important reasoning criterion for studying logic algebra and R_0-algebra by studying filters in depth. The main contribution of the present paper highlighted in the following lines:

As a result, studying the filter on R_0-algebra with hesitant fuzzy set is crucial. There are few findings about the algebraic structure of hesitant fuzzy sets available at the moment. As a result, the notions of hesitant fuzzy MP filter and hesitant fuzzy congruence relationship are presented in this study. The relevant properties are also investigated. We also look at the connection between the hesitant fuzzy MP filter and the hesitant fuzzy congruence. The following is how the rest of the article is structured.

The second segment introduces fundamental definitions and knowledge. Section 3 discusses the features and equivalent characterizations of hesitant fuzzy MP filters and hesitant fuzzy congruence relations on R_0-algebras, as well as their lattice structures. This paper comes to an end with Section 4.

2. Preliminaries

In this section, we recollect some basic definitions and knowledge which will be used in the following.

Definition 2.1. [1] By a R_0-algebras, we shall mean an algebra $(R;\wedge,\vee,\to,\neg,0,1)$ satisfying the following axioms:

(R1) $\neg x \to \neg y = y \to x$,

(R2) $1 \to x = x, x \to x = 1$,

(R3) $y \to z \le (x \to y) \to (x \to z)$,

(R4) $x \to (y \to z) = y \to (x \to z)$,

(R5) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, $x \to (y \vee z) = (x \to y) \wedge (x \to z)$,

(R6) $(x \to y) \wedge ((x \to y) \to (\neg x \wedge y)) = 1$, for any $x, y, z \in R$.

Where 1 is the largest element of R, then R is called a R_0-algebras.

Proposition 2.1. [3] Let R be a R_0-algebras, then for all $x, y \in R$,

(P1) $x \to y = 1$ if and only if $x \le y$,

(P2) $\neg x = x \to 0, x = \neg x \to 0$,

(P3) $(x \to y) \vee (y \to x) = 1$,

(P4) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,

(P5) $(M, \otimes, 1)$ is a commutative semigroup with unit 1,

(P6) $x \otimes y \le x \wedge y$,

(P7) $x \otimes y \le z$ if and only $x \le y \to z$,

(P8) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$,

Where $x \otimes y = \neg(x \to \neg y)$.

Definition 2.2. [2] Let R be a R_0-algebras, $\varnothing \ne A \subseteq R$. Then A is a MP-filter of R if, and only if:

(i) $1 \in A$,

(ii) If $x \in A, x \to y \in A$. Then $y \in A$, for all $x, y \in A$.

Definition 2.3. [8] A fuzzy set in R is a mapping $f : R \to [0,1]$.

Definition 2.4. [19] A fuzzy set A in R is called a fuzzy MP-filter of R if it satisfies the following conditions:

(F1) $A(1) \ge A(x)$,

(F2) $A(y) \ge A(x \to y) \wedge A(x)$, for all $x, y \in R$.

Definition 2.5. [15] Let X be a reference set, then the set η is called a hesitant fuzzy set (briefly, HF set) on X and is expressed as:

$$\eta = \{< x, \eta(x) > \mid \eta(x) \in P[0,1], x \in X\},$$

where $P([0,1])$ is, the power set of $[0,1]$.

If there are hesitant fuzzy sets η and μ on X, we define $\eta \subseteq \mu \Leftrightarrow (x \in X)(\eta(x) \subseteq \mu(x))$. In [15], a hesitant fuzzy set is defined by: Let X be a non-empty set, a hesitant fuzzy set h on X is a function that when applied to X returns a subset of [0,1]. If we consider the case when *h(x)* represents the possible membership values of the set at *x*. Then we have

- Empty set: $h(x) = \{0\}$ for every x in X.
- Full set: $h(x)=\{1\}$ for every x in X.

- Complete ignorance for x in X (all is possible): *h(x)*=[0,1].
- Set for a nonsense x: *h(x)*=∅.

Keeping in view the above points, we can impose the constraint condition on a hesitant fuzzy h: $0 \subseteq h(x) \subseteq \{1\}$ for all x in X.

In other words, since a hesitant fuzzy set h is a collection of elements of the unit interval [0,1], hence, the largest number of this collection, that is, union of this collection should be less than or equal to 1.

There are different types of filters discussed in literature for several kinds of algebras. However, we will collect only literature on (fuzzy) filters of R_0-algebra (see Table 1).

Table 1. literature on (fuzzy) filters of R_0-algebra.

order number #	Author	Type of filter	Type of algebra
1	L. Z. Liu, K. T. Li [23]	Fuzzy implicative and Boolean filters	R_0-algebras
2	J. S. Han, Y. B. Jun, H. S. Kim [24]	Fuzzy Fated-filters	R_0-algebras
3	X. L. Ma, J Zhan, Y. Xu [22]	Generalized fuzzy filters	R_0-algebras
4	J. Zhan, X. L. Ma, Y. B. Jun [25]	$(\in, \in \vee q)$-fuzzy filters	R_0-algebras
5	Y. B. Jun, Y. J. Lee [26]	Redefined fuzzy filters	R_0-algebras
6	J. Zhan, Y. B. Jun, D. W. Pei [27]	Falling fuzzy (implicative) filters	R_0-algebras and application
7	G. J. Wang [28]	MV-algebras, BL-algebras, R_0-algebras and multiple-valued logic	
8	Y. B. Jun, S. Z. Song, J. Zhan [29]	Generalizations of $(\in, \in \vee q)$-fuzzy filters	R_0-algebras
9	Proposed	Hesitant fuzzy MP filters and Congruence relations	R_0-algebras

In literature, a lot of research work is demonstrated for fuzzy filters in several algebraic structures including BCI/BCK-algebras, MTL-algebras, MV-algebras and others. The present work of hesitant fuzzy set in MP filters of R_0-algebra is introduced for the first time. We hope that this work will provide a strong foundation for researchers doing work in R_0-algebras. In future work, we will consider other types of fuzzy MP filters of R_0-algebras, including: intuitionistic MP filters of R_0-algebras, interval-valued fuzzy MP filters of R_0-algebra and others.

3. Hesitant fuzzy MP filters and congruence relation on R_0- algebras

In this section, we give the definitions of hesitant fuzzy MP filters and hesitant fuzzy congruence relations on R_0- algebras. Then, we discuss their properties and equivalent characterizations. Finally, their lattice structures are studied.

Let R be a R_0-algebra unless otherwise specified.

Definition 3.1. An HF set η of R is called a hesitant fuzzy MP filter (briefly, HFMF) if it satisfies the following conditions:

(i) $\eta(x) \subseteq \eta(1)$,

(ii) $\eta(y) \supseteq \eta(x) \cap \eta(x \to y)$ for all $x, y \in R$.

Example 3.1. Let $R = <\{0, a, b, c, 1\}, \wedge, \vee, \to>$ be a set with the following tables (see Tables 2 and 3):

Table 2

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	b
b	b	b	1	1	1
c	a	a	b	1	1
1	0	a	b	c	1

Table 3

x	¬x
0	1
a	c
b	b
c	a
1	0

Then $R = <\{0, a, b, c, 1\}, \wedge, \vee, \to>$ is a R_0-algebras. For any $x \in R$, we define: $\eta(x)$ as follows:

$$\eta(0) = \frac{1}{3}, \eta(a) = \frac{1}{2}, \eta(b) = \frac{2}{3}, \eta(c) = \frac{9}{10}, \eta(1) = 1.$$

Then R is hesitant fuzzy MP filter of R .

Definition 3.2. Let R be R_0 -algebras and $\theta = \{<(x, y), \theta(x, y)> | (x, y) \in R \times R\}$ is the hesitant fuzzy equal relation (briefly, HFE) on R , and $\theta : R \times R \to [0,1], 0 \le \theta(x, y) \le 1$. Then, θ satisfy the following conditions: for all $x, y, z \in R$,

(i) $\theta(x, x) \supseteq \theta(x, y)$;

(ii) $\theta(x, y) = \theta(y, x)$;

(iii) $\theta(x, z) \supseteq \theta(x, y) \cap \theta(y, z)$.

Definition 3.3. Let R be a R_0 -algebras and $\theta \in \mathrm{HFE}[R]$, then for all $x, y, z \in R, \theta$ satisfy the following conditions:

(iv) $\theta(x \to z, y \to z) \supseteq \theta(x, y)$, $\theta(z \to x, z \to y) \supseteq \theta(x, y)$.

(v) $\theta(x \vee z, y \vee z) \supseteq \theta(x, y)$.

Then θ is called a hesitant fuzzy congruence relation (briefly, HFC) of R.

Theorem 3.1. Let $\eta \in$ HFMF[R] and $x, y, z \in R$. Then

(i) If $x \le y$, then $\eta(x) \subseteq \eta(y)$,

(ii) $\eta(x \to z) \supseteq \eta(x \to y) \cap \eta(y \to z)$,

(iii) $\eta(x \otimes y) \supseteq \eta(x) \cap \eta(y)$,

(iv) $\eta(x \otimes y) = \eta(x) \cap \eta(y)$,

(v) $\eta(x \wedge y) = \eta(x) \cap \eta(y)$,

(vi) $x \le y \to z \Rightarrow \eta(z) \supseteq \eta(x) \cap \eta(y)$.

Proof. (i). Let $x \le y$. Then $x \to y = 1$, which implies

$\eta(y) \supseteq \eta(x) \cap \eta(x \to y) = \eta(x) \cap \eta(1) = \eta(x)$.

(ii). Since $x \to y \le (y \to z) \to (x \to z)$ and (i), which implies that

$\eta((y \to z) \to (x \to z)) \supseteq \eta(x \to y)$

by using Definition 3.1 (ii), we have

$\eta(x \to z) \supseteq \eta(y \to z) \cap \eta((y \to z) \to (x \to z))$

Therefore, we have $\eta(x \to z) \supseteq \eta(x \to y) \cap \eta(y \to z)$.

(iii). Since $y \le x \to (x \otimes y)$ and using (i), we have $\eta(x \to x \otimes y) \supseteq \eta(y)$.

It follows from Definition 3.1 (ii)

$\eta(x \otimes y) \supseteq \eta(x) \cap \eta(x \to x \otimes y) \supseteq \eta(x) \cap \eta(y)$.

(iv). Since $x \otimes y \le x, x \otimes y \le y$ and using (i), we have $\eta(x \otimes y) \subseteq \eta(x), \eta(x \otimes y) \subseteq \eta(y)$. Hence $\eta(x \otimes y) \subseteq \eta(x) \cap \eta(y)$, and by (iii) we have $\eta(x \otimes y) = \eta(x) \cap \eta(y)$.

(v). By using (i) we have $\eta(x \wedge y) \subseteq \eta(x) \cap \eta(y)$. It from (i), (iv), that $\eta(x \wedge y) \supseteq \eta(x \otimes y) = \eta(x) \cap \eta(y)$. Hence $\eta(x \wedge y) = \eta(x) \cap \eta(y)$.

(vi). Assume that $x \le y \to z$, by using $x \otimes y \le z$, (i) and (iii), we have

$\eta(z) \supseteq \eta(x \otimes y) = \eta(x) \cap \eta(y)$.

Theorem 3.2. Let $\eta \in$ HF[R]. The following are equivalent:

(i) $\eta \in$ HFMP[R],

(ii) $(\forall \gamma \in P([0,1]))$ $R(A, \gamma) \ne \varnothing$ implies $R(A, \gamma) \ne \varnothing$ is a MP filter of R.

Proof. $(i) \Rightarrow (ii)$. Let $x, y \in R$ be such that $x, x \to y \in R(A, \gamma)$, for any $\gamma \in P([0,1])$.

Then $\eta(x) \supseteq \gamma$ and $\eta(x \to y) \supseteq \gamma$. Hence $\eta(y) \supseteq \eta(x) \cap \eta(x \to y) \supseteq \gamma$.

So $y \in R(A, \gamma)$, we have $R(A, \gamma) \ne \varnothing$ is a MP filter of R.

$(ii) \Rightarrow (i)$. Let $R(A, \gamma)$ be a MP filter of R, for any $\gamma \in P([0,1])$ with $R(A, \gamma) \ne \varnothing$.

Put $\eta(x) = \gamma_1$, for any $x \in R$, then $x \in R(A, \gamma_1)$. Since $R(A, \gamma_1)$ is a MP filter of R, we have $1 \in R(A, \gamma_1)$ and so $\eta(1) \supseteq \gamma_1 = \eta(x)$.

Now, for any $x, y \in R$, let $\gamma_2 = \eta(x) \cap \eta(x \to y)$. Then $x, x \to y \in R(A, \gamma_2)$, so $R(A, \gamma) \ne \varnothing$.

Hence $R(A, \gamma_1)$ is a MP filter of R, so $y \in R(A, \gamma_2)$. Hence $\eta(y) \supseteq \gamma_2 = \eta(x) \cap \eta(x \to y)$.

Theorem 3.3. Let $\eta_1, \eta_2 \in$ HFMF(R), then $\eta_1 \cap \eta_2 \in$ HFMF(R).

Proof. Let $x, y \in \eta_1 \cap \eta_2$. Then $x \in \eta_1$, $y \in \eta_2$ and $(\eta_1 \cap \eta_2)(y) = \eta_1(y) \cap \eta_2(y)$.

Now put $x \le y \to z$, which implies $\eta_1(z) \supseteq \eta_1(x) \cap \eta_1(y)$, $\eta_2(z) \supseteq \eta_2(x) \cap \eta_2(y)$.

Therefore $(\eta_1 \cap \eta_2)(z) = \eta_1(z) \cap \eta_2(z) \supseteq \eta_1(x) \cap \eta_1(y) \cap \eta_2(x) \cap \eta_2(y)$

$= (\eta_1 \cap \eta_2)(x) \cap (\eta_1 \cap \eta_2)(y)$.

Hence, $\eta_1 \cap \eta_2 \in \text{HFMF}(R)$.

The above theorem can be generalized as follows.

Theorem 3.4. Let $\{\eta_i \mid i \in I\} \in \text{HFMF(R)}$. Then $\cap \eta_i \in \text{HFMF(R)}$ where $\cap \eta_i = \min \eta_i(x)$.

Theorem3.3 shows that, if $\eta_1, \eta_2 \in \text{HFMF(R)}$, then we have $\eta_1 \cap \eta_2 \in \text{HFMF(R)}$. But the following example shows that $\eta_1 \cup \eta_2 \notin \text{HFMF(R)}$.

Example 3.2 $R = \{0, a, b, 1\}$, define two hesitant fuzzy set η, μ in R by

$\eta(0) = \frac{1}{3}, \eta(a) = \frac{1}{2}, \eta(b) = \frac{2}{3}, \eta(1) = 1$, $\mu(0) = \frac{1}{3}, \mu(a) = \frac{1}{4}, \mu(b) = \frac{2}{3}, \mu(1) = 1$;

So, we have $\eta, \mu \in \text{HFMF(R)}$.

Let $\eta \cup \mu = \{< x, \eta(x) \cup \mu(x) > \mid x \in R\}$, in which $(\eta \cup \mu)(0) = \frac{1}{3}, (\eta \cup \mu)(a) = \frac{1}{2}$, $(\eta \cup \mu)(b) = \frac{1}{4}, (\eta \cup \mu)(1) = 1$. But $(\eta \cup \mu)(b) \subseteq (\eta \cup \mu)(0) \cap (\eta \cup \mu)(0 \to b)$, hence $\eta \cup \mu \notin \text{HFMF(R)}$.

After introducing the properties of HFMF, we discuss the properties of HFC on R_0-algebra.

Theorem 3.5. Let R be a R_0-algebra and $\theta \in \text{HFC(R)}$. Then for any $x, y, z \in R$ the following assertions are true:

(i) $\theta(\neg x, \neg y) = \theta(x, y)$;

(ii) $\theta(x \wedge z, y \wedge z) \supseteq \theta(x, y)$, $\theta(x \otimes z, y \otimes z) \supseteq \theta(x, y)$;

(iii) $\theta(x, y) = \theta(y, y)$;

(iv) $\theta(x, y) = \theta(x, x \vee y) \cap \theta(x \vee y, y)$, $\theta(x, y) = \theta(x, x \wedge y) \cap \theta(x \wedge y, y)$;

(v) $\theta(x \to y, y \to x) = \theta(1, x \to y) \cap \theta(1, y \to x)$;

(vi) $\theta(x, y) = \theta(x \to y, y \to x)$;

(vii) If $\eta(x) = \theta(1, x)$, then $\eta \in \text{HFMF(R)}$.

Proof. Let $x, y, z \in R$

(i) We have $\theta(\neg x, \neg y) = \theta(x \to 0, y \to 0) \supseteq \theta(x, y)$.

Because $\neg$ is reverse order of R, then $\theta(x, y) = \theta(\neg\neg x, \neg\neg y) \supseteq \theta(\neg x, \neg y)$,

Hence $\theta(\neg x, \neg y) = \theta(x, y)$.

(ii) By using $x \wedge y = \neg(\neg x \vee \neg y)$ and (i), which implies

$\theta(x \wedge z, y \wedge z) = \theta(\neg(\neg x \vee \neg z), \neg(\neg y \vee \neg z)) = \theta(\neg x \vee \neg z, \neg y \vee \neg z) \supseteq \theta(\neg x, \neg y) = \theta(x, y)$,

Obviously, $x \otimes y = \neg(x \to \neg y)$ implies

$\theta(x \otimes y, y \otimes z) = \theta(\neg(x \to \neg z), \neg(y \to \neg z)) = \theta(x \to \neg z, y \to \neg z) \supseteq \theta(x, y)$.

(iii) Obviously, $\theta(1,1) = \theta(x \vee 1, x \vee 1) \supseteq \theta(x, x)$.

Conversely, by using (ii) we have $\theta(x, x) = \theta(1 \wedge x, 1 \wedge x) \supseteq \theta(1,1)$.

Then $\theta(x, y) = \theta(1,1)$. Hence $\theta(x, y) = \theta(y, y)$.

(iv) Obviously, $\theta(x, y) \supseteq \theta(x, x \vee y) \cap \theta(x \vee y, y)$.

Conversely, $\theta(x, x \vee y) = \theta(x \vee x, y \vee x) \supseteq \theta(x, y)$.

Similarly, we have $\theta(x \vee y, y) \supseteq \theta(x, y)$, $\theta(x, x \vee y) \cap \theta(x \vee y, y) \supseteq \theta(x, y)$.

Hence $\theta(x, y) = \theta(x, x \vee y) \cap \theta(x \vee y, y)$.

Similarly, we have $\theta(x, y) = \theta(x, x \wedge y) \cap \theta(x \wedge y, y)$.

(v) By using $(x \to y) \vee (y \to x) = 1$ and (iv), which implies

$\theta(x \to y, y \to x) = \theta(1, x \to y) \wedge \theta(1, y \to x)$.

(vi) Obviously, $\theta(x\to y, y\to x)\supseteq\theta(x\to y, x\to x)\cap\theta(x\to x, y\to x)$
$\supseteq\theta(y,x)\cap\theta(x,y)=\theta(x,y)$.
Conversely, by using $x\vee y=((x\to y)\to y)\wedge((y\to x)\to x)$ and from (iv) and (v), we have

$$\begin{aligned}\theta(x,y)&=\theta(x,x\vee y)\cap\theta(x\vee y,y)\\&=\theta(x,((x\to y)\to y)\wedge((y\to x)\to x)\cap\theta(((x\to y)\to y)\cap((y\to x)\to x),y)\\&=\theta(x\wedge((x\to y)\to y),((y\to x)\to x)\wedge((x\to y)\to y))\\&\quad\cap\theta(((x\to y)\to y)\wedge((y\to x)\to x),y\wedge((y\to x)\to x)),\\&\supseteq\theta(x,(y\to x)\to x)\cap\theta((x\to y)\to y,y)\\&=\theta(1\to x,(y\to x)\to x)\cap\theta((x\to y)\to y,1\to y)\\&\supseteq\theta(1,y\to x)\cap\theta(x\to y,1)\\&=\theta(1,x\to y)\cap\theta(1,y\to x)=\theta(x\to y,y\to x).\end{aligned}$$

Hence $\theta(x,y)=\theta(x\to y,y\to x)$.
(vii) Obviously, $\eta(1)=\theta(1,1)\supseteq\theta(1,x)=\eta(x)$.
$\eta(y)=\theta(1,y)\supseteq\theta(1,x\to y)\cap\theta(x\to y,y)=\theta(1,x\to y)\cap\theta(x\to y,1\to y)$
$\supseteq\theta(1,x\to y)\cap\theta(1,x)=\eta(x)\cap\eta(x\to y)$. Hence $\eta\in\mathrm{HFMF(R)}$.

The following results are related with the equivalent characterization of hesitant fuzzy congruence relations on R_0-algebra and hesitant fuzzy congruence relations on the direct product of R_0-algebra.

Theorem 3.6. Let R be a R_0-algebra. $\theta\in\mathrm{HFE(R)}$, then $\theta\in\mathrm{HFC(R)}$ if and only if for all $x,y,x_i,y_i\in R(i=1,2)$, it satisfies the following conditions:
(i) $\theta(\neg x,\neg x)\supseteq\theta(x,y)$,
(ii) $\theta(x_1\to x_2,y_1\to y_2)\supseteq\theta(x_1,y_1)\cap\theta(x_2,y_2)$.
(iii) $\theta(x_1\vee x_2,y_1\vee y_2)\supseteq\theta(x_1,y_1)\cap\theta(x_2,y_2)$.
Proof. The proofs are obvious.
Necessity. Let $\theta\in\mathrm{HFE[R]}$ and for all $x,y,x_i,y_i\in R(i=1,2)$,
(i) $\theta(\neg x,\neg y)=\theta(x\to 0,y\to 0)\supseteq\theta(x,y)$.
(ii) $\theta(x_1\to x_2,y_1\to y_2)\supseteq\theta(x_1\to x_2,y_1\to x_2)\cap\theta(y_1\to x_2,y_1\to y_2)$
$\supseteq\theta(x_1,y_1)\cap\theta(x_2,y_2)$.
(iii) $\theta(x_1\vee x_2,y_1\vee y_2)\supseteq\theta(x_1\vee x_2,y_1\vee x_2)\cap\theta(y_1\vee x_2,y_1\vee y_2)$
$\supseteq\theta(x_1,y_1)\cap\theta(x_2,y_2)$.
Therefore, $\theta\in\mathrm{HFC[R]}$.

Theorem 3.7. Let R_1,R_2 be two R_0-algebra and $\theta_1\in\mathrm{HFC[R_1]}$, $\theta_2\in\mathrm{HFC[R_2]}$, we define
$\theta_1\times\theta_2:(R_1\times R_2)\times(R_1\times R_2)\to[0,1]$: $(x_1,y_1),(x_2,y_2)\in R_1\times R_2$,
$(\theta_1\times\theta_2)((x_1,y_1),(x_2,y_2))=\theta_1(x_1,x_2)\cap\theta_2(y_1,y_2)$.
Then $\theta_1\times\theta_2\in\mathrm{HFC[R_1\times R_2]}$.
And any hesitant fuzzy congruence relation on $\theta_1\times\theta_2$ has this representation.
Proof. First, we will prove $\theta_1\times\theta_2\in\mathrm{HFE[R_1\times R_2]}$ in the following three folds. For any $(x_i,y_i)\in R_1\times R_2(i=1,2,3)$,
(i) $(\theta_1\times\theta_2)((x_1,y_1),(x_1,y_1))=\theta_1(x_1,x_1)\cap\theta_2(y_1,y_1)\supseteq\theta_1(x_1,x_2)\cap\theta_2(y_1,y_2)$
$=(\theta_1\times\theta_2)((x_1,y_1),(x_2,y_2))$. The reflexivity is established.

(ii) $(\theta_1\times\theta_2)((x_1,y_1),(x_2,y_2))=\theta_1(x_1,x_2)\cap\theta_2(y_1,y_2)=\theta_1(x_2,x_1)\cap\theta_2(y_2,y_1)$
$=(\theta_1\times\theta_2)((x_2,y_2),(x_1,y_1))$. The symmetry is established.
(iii) $(\theta_1\times\theta_2)((x_1,y_1),(x_3,y_3))=\theta_1(x_1,x_3)\wedge\theta_2(y_1,y_3)$
$\supseteq\theta_1(x_1,x_2)\cap\theta_1(x_2,x_3)\cap\theta_2(y_1,y_2)\cap\theta_2(y_2,y_3)$
$=(\theta_1(x_1,x_2)\cap\theta_2(y_1,y_2))\cap(\theta_1(x_2,x_3)\cap\theta_2(y_2,y_3))$
$=(\theta_1\times\theta_2)((x_1,y_1),(x_2,y_2))\cap(\theta_1\times\theta_2)((x_2,y_2),(x_3,y_3))$. The transitivity is established.
Then we prove $\theta_1\times\theta_2\in\mathrm{HFC}(\mathrm{R}_1\times\mathrm{R}_2)$. For any $(x_i,y_i)\in R_1\times R_2(i=1,2,3)$,
(i) $(\theta_1\times\theta_2)((x_1,y_1)\to(x_3,y_3),(x_2,y_2)\to(x_3,y_3))$
$=(\theta_1\times\theta_2)((x_1\to x_3,y_1\to y_3),(x_2\to x_3,y_2\to y_3))$
$=\theta_1(x_1\to x_3,x_2\to x_3)\cap\theta_2(y_1\to y_3,y_2\to y_3)$
$\supseteq\theta_1(x_1,x_2)\cap\theta_2(y_1,y_2)=(\theta_1\times\theta_2)((x_1,y_1),(x_2,y_2))$.
Similarly, we have
$(\theta_1\times\theta_2)((x_3,y_3)\to(x_1,y_1),(x_3,y_3)\to(x_2,y_2))\supseteq(\theta_1\times\theta_2)((x_1,y_1),(x_2,y_2))$.
(ii) $(\theta_1\times\theta_2)((x_1,y_1)\vee(x_3,y_3),(x_2,y_2)\vee(x_3,y_3))$
$=(\theta_1\times\theta_2)((x_1\vee x_3,y_1\vee y_3),(x_2\vee x_3,y_2\vee y_3))$
$=\theta_1(x_1\vee x_3,x_2\vee x_3)\cap\theta_2(y_1\vee y_3,y_2\vee y_3)\supseteq\theta_1(x_1,x_2)\cap\theta_2(y_1,y_2)$
$=(\theta_1\times\theta_2)((x_1,y_1),(x_2,y_2))$, hence, we have $\theta_1\times\theta_2\in\mathrm{HFC}(\mathrm{R}_1\times\mathrm{R}_2)$.
Let $\theta=\{<(x,y),\theta(x,y)>|(x,y)\in R\times R\}\in\mathrm{HFC}(\mathrm{R}_1\times\mathrm{R}_2)$ and
$\theta_1:R_1\times R_2\to[0,1]$, $\theta_2:R_1\times R_2\to[0,1]$, as follows: For any $x_1,x_2\in R_1,y_1,y_2\in R_2$,
$\theta_1(x_1,x_2)=\bigcup_{y\in R_2}\theta((x_1,y),(x_2,y))$, $\theta_2(x_1,x_2)=\bigcup_{y\in R_2}\theta((x_1,y),(x_2,y))$.
Then we prove $\theta_1\in\mathrm{HFC}(\mathrm{R}_1)$, $\theta_2\in\mathrm{HFC}(\mathrm{R}_2)$, and $\theta=\theta_1\times\theta_2$.
First, we will take θ_1 as an example to simplify the above equation. For any $y,z\in R_2$,
$\theta((x_1,y\wedge z),(x_2,y\wedge z))=\theta((x_1,y)\wedge(x_1\vee x_2,z),(x_2,y)\wedge(x_1\vee x_2,z))$
$\supseteq\theta((x_1,y),(x_2,y))$.
And $\theta((x_1,z),(x_2,z))=\theta((x_1,y\wedge z)\vee(x_1\wedge x_2,z),(x_2,y\wedge z)\vee(x_1\vee x_2,z))$
$\supseteq\theta((x_1,y\wedge z),(x_2,y\wedge z))$,
Hence, $\theta((x_1,z),(x_2,z))\supseteq\theta((x_1,y),(x_2,y))$.
Similarly, we can prove $\theta((x_1,y),(x_2,y))\supseteq\theta((x_1,z),(x_2,z))$.
Hence $\theta((x_1,y),(x_2,y))=\theta((x_1,z),(x_2,z))$.
If the maximum element in R_1 is 1, then we have:
$\theta_1(x_1,x_2)=\theta((x_1,1),(x_2,1))$, for any $x_1,x_2\in R_1$.
$\theta_2(y_1,y_2)=\theta((1,y_1),(1,y_2))$, for any $y_1,y_2\in R_2$.
Take θ_1 for example, to get $\theta_1\in\mathrm{HFC}(\mathrm{R}_1)$.
Firstly, we prove $\theta_1\in\mathrm{HFE}(\mathrm{R}_1)$,for any $x_1,x_2\in R_1$, we prove it in the following three aspects.
Case 1. $\theta_1(x_1,x_1)=\theta((x_1,1),(x_1,1))\supseteq\theta((x_1,1),(x_2,1))=\theta_1(x_1,x_2)$,
Case 2. $\theta(x_1,x_2)=\theta((x_1,1),(x_2,1))=\theta((x_2,1),(x_1,1))=\theta(x_2,x_1)$,
Case 3. $\theta_1(x_1,x_3)=\theta((x_1,1),(x_3,1))\supseteq\theta((x_1,1),(x_2,1))\cap\theta((x_2,1),(x_3,1))=\theta_1(x_1,x_2)\cap\theta_1(x_2,x_3)$.

Hence $\theta_1 \in \mathrm{HFE(R_1)}$.

Then we prove $\theta_1 \in \mathrm{HFC(R_1)}$,for any $x_i \in R_1 (i=1,2,3)$, we prove it in the following two steps.

Case1. $\theta_1(x_1 \to x_3, x_2 \to x_3) = \theta((x_1 \to x_3, 1), (x_2 \to x_3, 1))$

$= \theta((x_1,1) \to (x_3,1), (x_2,1) \to (x_3,1))$

$\supseteq \theta((x_1,1),(x_2,1)) = \theta_1(x_1, x_2)$,

Similarly, $\theta_1(x_1 \to x_3, x_2 \to x_3) \supseteq \theta_1(x_1, x_2)$.

Case2. $\theta_1(x_1 \vee x_3, x_2 \vee x_3) = \theta((x_1 \vee x_3, 1), (x_2 \vee x_3, 1))$

$= \theta((x_1,1) \vee (x_3,1), (x_2,1) \vee (x_3,1))$

$\supseteq \theta((x_1,1),(x_2,1)) = \theta_1(x_1, x_2)$, Hence $\theta_1 \in \mathrm{HFC(R_1)}$.

Finally, we prove $\theta = \theta_1 \times \theta_2$,for any $(x_1, y_1), (x_2, y_2) \in R_1 \times R_2$.

$(\theta_1 \times \theta_2)((x_1, y_1), (x_2, y_2)) = \theta_1(x_1, x_2) \cap \theta_2(y_1, y_2)$

$= \theta((x_1,1),(x_2,1)) \cap \theta((1, y_1),(1, y_2))$

$= \theta((x_1, y_1) \vee (x_1 \wedge x_2, 1), (x_2, y_2) \vee (x_1 \wedge x_2, 1) \cap \theta((x_1, y_1) \vee (1, y_1 \wedge y_2), (x_2, y_2) \vee (1, y_1 \wedge y_2))$

$\cap \theta((x_1, y_1), (x_2, y_2)) = \theta((x_1, y_1), (x_2, y_2))$. Hence $\theta \subseteq \theta_1 \times \theta_2$.

Conversely,

$\theta((x_1, y_1), (x_2, y_2) \supseteq \theta((x_1, y_1), (x_2, y_1)) \cap \theta((x_2, y_1), (x_2, y_2)) = \theta_1(x_1, x_2) \cap \theta_2(y_1, y_2)$

$= (\theta_1 \times \theta_2)((x_1, y_1), (x_2, y_2))$.

Then we have $\theta \supseteq \theta_1 \times \theta_2$. Hence $\theta = \theta_1 \times \theta_2$.

Theorem 3.8. Let $\eta \in \mathrm{HFMF(R)}$, θ is the hesitation fuzzy relationship defined below. For any $x, y \in R$, $\theta = \{< (x,y), \theta(x,y) > | (x,y) \in R\}$, such that $\theta(x,y) = \eta(x \to y) \cap \eta(y \to x)$ .Then we have $\theta \in \mathrm{HFC(R)}$.

Proof. For any $x, y, z \in R$,

$\theta(x,z) = \{< (x,z), \theta(x,z) > | (x,z) \in R\}$. By using $\eta \in \mathrm{HFMF(R)}$, we have

$\eta(x \to z) \supseteq \eta(x \to y) \cap \eta(y \to z), \eta(z \to x) \supseteq \eta(z \to y) \cap \eta(y \to x)$.

Hence, $\theta(x,z) = \eta(x \to z) \cap \eta(z \to x)$

$\supseteq (\eta(x \to y) \cap \eta(y \to z)) \cap (\eta(z \to y) \cap \eta(y \to x))$

$= (\eta(x \to y) \wedge \eta(y \to x)) \cap (\eta(z \to y) \cap \eta(y \to z))$

$= \theta(x,y) \cap \theta(y,z)$,

For any $x, y, z \in R$, we have $\theta(x,z) \supseteq \theta(x,y) \cap \theta(y,z)$.

Hence $\theta \in \mathrm{HFC(R)}$.

Finally, we discuss the relationship between HFMF and HFC in R_0 -algebra.

Theorem 3.9. $(\mathrm{HFC(R)}, \vee, \wedge)$ and $(\mathrm{HFMF(R)}, \vee, \wedge)$ is a complete lattice isomorphism.

Proof. θ is a hesitant fuzzy congruence relation of R_0 -algebra.

Let $\theta = \{< (x,y), \theta(x,y) > | (x,y) \in R \times R\}$ and $\eta = \{< x, \eta(x) > | x \in R\}$ is the hesitation fuzzy *MP* filter.

We defined $f : \mathrm{HFC(R)} \to \mathrm{HFMF(R)}$.

and $f(\theta) = \{< x, \eta(x) > | x \in R\}, \eta(x) = \theta(1, x)$, for all $x \in R$.

First, according to Theorem 3.5 (vii) and $f(\theta) \in \mathrm{HFMF(M)}$, and so the definition of f is reasonable.

Second, it is proved that f is a one-to-one mapping. If $\eta_1(x) = \eta_2(x)$, then we have for any $x \in R$, $\theta_1(1,x)=\theta_2(1,x)$.

By using Theorem 3.5 (v) and (vi), for any $x, y \in R$, we have

$\theta_1(x,y)=\theta_1(x \to y, y \to x)=\theta_1(1, x \to y) \cap \theta_1(1, y \to x)$

$=\theta_2(1, x \to y) \cap \theta_2(1, y \to x)=\theta_2(x \to y, y \to x)=\theta_2(x,y)$,

That is, f is monotonic.

For any $x, y \in R$, define $\theta=\{<(x,y),\theta(x,y)>|(x,y) \in R \times R\}$ as a hesitant fuzzy relation of R and $\theta(x,y)=\eta(x \to y) \cap \eta(y \to x)$.

It can be verified that θ meets all the conditions in Definitions 3.2 and 3.3, then $\theta \in$ HFC(R) and $f(\theta)=\eta$. Therefore, f is full. So f is a one-to-one mapping.

Finally, ensure the arbitrary union and arbitrary intersection of f.

Let $\{\theta_i\}_{i \in I} \subseteq$ HFC(R), define $\underset{i\in I}{\cup}\theta, \underset{i\in I}{\cup}\theta : R \times R \to [0,1]$ as follows:

$(\underset{i\in I}{\cup}\theta_i)(x,y)=(\underset{i\in I}{\cup}\eta_i(x \to y)) \cap (\underset{i\in I}{\cup}\eta_i(y \to x))$.

Then obviously, we have $f(\underset{i\in I}{\cup}\theta_i)(1,x)=\underset{i\in I}{\cup} f(\theta_i)(1,x)=\underset{i\in I}{\cup}\eta_i(x)$.

Put $f(\underset{i\in I}{\cap}\theta_i)(1,x)=\underset{i\in I}{\cap}\eta_i(x)=\eta(x)$,

Because it can be verified that $\underset{i\in I}{\cap}\theta_i \in$ HFC(R), we just verify that for any $x \in R$, $\eta(x)=\underset{i\in I}{\cap}\eta_i(x)$ is established and $\theta_i(1,x)=\eta_i(x)$.

Because $\eta(x)=\underset{i\in I}{\cap}\theta_i(1,x)=\underset{i\in I}{\cap}(\theta_i(1,x))=\underset{i\in I}{\cap}\eta_i(x)$. Thus, f is lattice isomorphism.

4. Conclusions

The notions of hesitant fuzzy MP filter and hesitant fuzzy congruence relation of *R0*-algebras were explored in this article. The attributes of many equivalent characterizations and characterizations are next investigated. The relationships between the hesitant fuzzy MP filter and the hesitant fuzzy congruence relation have been discovered. The results obtained in this study, in our opinion, can be applied to expanding other algebraic systems, such as BF-algebras and MV-algebras. We hope that this publication has paved the way for future research into the theory of other logical algebras.

Summary of the manuscript

Properties of *R0*-algebra based on hesitant fuzzy MP filters and Congruence relations	The definitions of hesitant fuzzy MP filters and hesitant fuzzy congruence relations on *R0*-algebras	The properties and equivalent characterizations of hesitant fuzzy MP filters and hesitant fuzzy congruence relations on *R0*-algebras	Hesitant fuzzy congruence relations on the direct product of R_0 -algebra.

Conflict of interest

The author declares no conflict of interests.

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