AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(7): 13361–13382.
DOI: 10.3934/math.2022738
Received: 29 December 2021
Revised: 31 March 2022
Accepted: 10 April 2022
Published: 17 May 2022

Research article

Picture fuzzy sub-hyperspace of a hyper vector space and its application in decision making problem

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Abstract: In this paper, the notion of picture fuzzy sub-hyperspace of a hyper vector space is introduced and some related results are investigated on the basis of some basic operations (intersection, union, Cartesian product etc.) on picture fuzzy sets. The concept of picture fuzzy linear transformation with respect to some picture fuzzy sub-hyperspace is initiated here and some important results are studied in this regard. It is shown that with respect to some pre-assumed picture fuzzy sub-hyperspace, linear combination of two picture fuzzy linear transformations is a picture fuzzy linear transformation, composition of two picture fuzzy linear transformations is a picture fuzzy linear transformation and inverse of a bijective picture fuzzy linear transformation is a picture fuzzy linear transformation. The effect of good linear transformation on picture fuzzy sub-hyperspaces is discussed here. It is shown that the image of a picture fuzzy sub-hyperspace is a picture fuzzy sub-hyperspace under bijective good linear transformation and the inverse image of a picture fuzzy sub-hyperspace is a picture fuzzy sub-hyperspace under good linear transformation. Some important results on picture fuzzy sub-hyperspaces in the light of (θ, ϕ, ψ)-cut of picture fuzzy set are studied here. Finally, an application of picture fuzzy sub-hyperspace conditions in decision making problem is presented here.

Keywords: picture fuzzy sub-hyperspace; (θ, ϕ, ψ)-cut on picture fuzzy sub-hyperspace; picture fuzzy linear transformation
Mathematics Subject Classification: 08A72

Abbreviations: VS: Vector space; HVS: Hyper vector space; SHS: Sub-hyperspace; LT: Linear transformation; FS: Fuzzy set; IFS: Intuitionistic fuzzy set; PFS: Picture fuzzy set; PFSs: Picture fuzzy sets; MMS: Measure of membership; MNonMS: Measure of non-membership; MPMS: Measure of positive membership; MNeuMS: Measure of neutral membership; MNegMS: Measure of negative membership; MRefMS: Measure of refusal membership; CP: Cartesian product; FSHS: Fuzzy sub-hyperspace; FSS: Fuzzy subspace; PFSHS: Picture fuzzy sub-hyperspace; PFSS: Picture fuzzy subspace; PFLT: Picture fuzzy linear transformation; PFLTs: Picture fuzzy linear transformations

1. Introduction

Uncertainty is a part of our real life. In almost each and every problem, uncertainty arises. Classical set theory is only capable to handle certain cases. Uncertain case can not be handled with classical point view. As an uncertainty handling tool, fuzzy set was initiated by Zadeh [1]. It is necessary to mention that fuzzy set deals only with the grade of membership. It is in fact the extension of classical sense. Certain case can be easily derived from uncertain cases by some particular choice of uncertainty measurement parameters. That is why fuzzy set theory is so more popular than classical set theory. After the introduction of fuzzy set, many researchers worked on fuzzy set based on algebraic structures. Subspace in fuzzy environment was introduced by Katsaras and Liu [2]. Fuzzy vector space under triangular norm was investigated by Das [3]. Idea of Das was enriched by Kumar [4]. Fuzzy set theory was used to solve different types of linear programming problems, transportation problems by the researchers [5–8]. First study on hyperstructure was done by Marty [9] in the form hypergroup. After that several researchers [10–12] worked on different types of hyperstructures. Fuzzy hypergroup and fuzzy hypermodule was studied by Davvaz [13, 14]. Hyper vector space, an important kind of algebraic hyperstructure, was initiated by Tallini [15]. Ameri and Dehghan [16] applied fuzzy set in hyper vector space. In case of clear information, grade of non-membership = 1− grade of membership. But for doubtful information, this rule does not work. In such cases, individual measurements of grade of membership and non-membership become necessary. Implementation of this concept was done in the form of intuitionistic fuzzy set by Atanassov [17]. In the field of medical sciences, sciences and social sciences; it is observed that two components are not sufficient to represent some special type of information. For instance, a voter can give vote in favour of a candidate (positive sense), against a candidate (negative sense) or he/she may remain neutral. Keeping this type of situation in mind, generalization on intuitionistic fuzzy set was done by Cuong [18] in the form of picture fuzzy set (dealing with grade of positive membership, grade of neutral membership and grade of negative membership). With the advancement of time, different kinds of research works in picture fuzzy environment were done by several researchers [19–24]. A lot of research works on picture fuzzy set based on algebraic structures were done by Dogra and Pal [25–29].

Algebraic hyperstructures are the generalizations of algebraic structures. It is well known that algebraic hyperoperation of two elements gives a set whereas algebraic operation of two elements gives a single element. When algebraic hyperoperation of two elements gives a singleton set, then algebraic hyperstructure is reduced to algebraic structure. Algebraic hyperstructure is an important field of study not only in Mathematics, but also in Computer Science. In real life, uncertainty arises in different forms i.e. types of uncertainty are not always same. When fuzzy and intuitionistic fuzzy tools are not capable to handle uncertainty with one or two uncertainty measurement parameters, then more uncertainty measurement parameters are required to introduce. In this purpose, picture fuzzy set was initiated by Cuong [18]. Algebraic hyperstructures become complicated when the number of uncertainty measurement parameters increases. Study of algebraic hyperstructures under such type of complicated environment is a challenging task to the researchers. If this challenge can be handled, then the study of algebraic hyperstructures in less complicated atmosphere will be quite easy as its consequence. Algebraic hyperstructure is an abstract idea and to show the application of this abstract concept in decision making problem under fuzzy/advanced fuzzy environment is a difficult task as this type of application is not available till now in existing literature. If this type of application can be developed then it will add a new dimension not only in the field of algebra but also in the field of decision making. This allows enough motivation to study the properties of algebraic hyperstructures and their applications under advanced fuzzy environment.

In this paper, we introduce the notion of picture fuzzy sub-hyperspaces of a hyper vector space and investigate some results related to these on the basis of some elementary operations (intersection, union, Cartesian product etc.) on picture fuzzy sets. We initiate the concept of picture fuzzy linear transformation and study some important results in this regard. We show that linear combination of two picture fuzzy linear transformations is a picture fuzzy linear transformation, composition of two picture fuzzy linear transformations is a picture fuzzy linear transformation and inverse of bijective picture fuzzy linear transformation is a picture fuzzy linear transformation. We discuss the effect of good linear transformation on picture fuzzy sub-hyperspaces. We show that the image of a picture fuzzy sub-hyperspace is a picture fuzzy sub-hyperspace under bijective good linear transformation and the inverse image of a picture fuzzy sub-hyperspace is a picture fuzzy sub-hyperspace under good linear transformation. We study some important results on picture fuzzy sub-hyperspaces in the light of (θ, ϕ, ψ)-cut of picture fuzzy sets. Finally, we present an application of picture fuzzy sub-hyperspace conditions in an interesting decision making problem. This decision making is presented here from algebraic point of view.

2. Preliminaries

In this section, we recapitulate the concepts of FS, IFS, hyperoperation, HVS, LT in HVS, SHS, FSHS, PFS, CP of PFSs, (θ, ϕ, ψ)-cut of PFS, image of PFS, inverse image of PFS.

As an extension of classical set theoretic concept, FS was invented by Zadeh [1].

Definition 2.1. [1] *Let ξ be a set of universe. Then a FS over ξ is defined as $\tau = \{(a, \tau_1(a)) : a \in \xi\}$, where $\tau_1(a) \in [0, 1]$ is the MMS of a in ξ.*

The concepts of hyperoperation and HVS are defined as follows.

Definition 2.2. [15] *A mapping $\circ : M \times M \to P^*(M)$ is called a hyperoperation, where $P^*(M)$ is the power set of the set M excluding the null set. This operation can be extended in case of subsets of M as follows.*

$A \circ B = \cup\{a \circ b : a \in A, b \in B\}$.

$a \circ A = \{a\} \circ A$.

and $A \circ a = A \circ \{a\}$.

Definition 2.3. [15] *Let $(\xi, +)$ be an abelian group and F be a field. Then $(\xi, +, \circ)$ forms a HVS over F if*
(i) $p \circ (a + b) \subseteq p \circ +p \circ b$
(ii) $(p + q) \circ a \subseteq p \circ a + q \circ a$
(iii) $p \circ (q \circ a) = (pq) \circ a$
(iv) $p \circ (-a) = (-p) \circ a = -(p \circ a)$
(v) $a \in 1 \circ a$ for all $a, b \in \xi$ and for all $p, q \in F$.

LT in HVS is defined as follows.

Definition 2.4. [15] *Let ξ_1 and ξ_2 be two HVSs over the same field F. Then a map $T : \xi_1 \to \xi_2$ is said to be LT on ξ_1 if*
(i) $T(a + b) = T(a) + T(b)$
(ii) $T(p \circ a) \subseteq p \circ T(a)$ for all $a, b \in \xi$ and for all $p \in F$.

Note that T is called good LT if
(i) $T(a + b) = T(a) + T(b)$
(ii) $T(p \circ a) = p \circ T(a)$ for all $a, b \in \xi$ and for all $p \in F$.

SHS of a HVS is defined as follows.

Definition 2.5. [15] *Let ξ be a hyper vector space over a field F and S be a subset of ξ. Then S is called a SHS of ξ if $p \in F$ and $a \in S \Rightarrow p \circ a \subseteq S$.*

The concept of SHS in fuzzy setting is as follows.

Definition 2.6. [16] *Let ξ be a hyper vector space over a field F. A FS $\tau = \{(a, \tau_1(a)) : a \in \xi\}$ over ξ is said to be FSHS of ξ if*
(i) $\tau_1(a - b) \geqslant \tau_1(a) \wedge \tau_1(b)$
(ii) $\inf\limits_{c \in p \circ a} \tau_1(c) \geqslant \tau_1(a)$ for all $a \in \xi$ and for all $p \in F$.

Eliminating the limitation of FS, Atanassov [17] defined IFS as an extended version of FS.

Definition 2.7. [17] *An IFS τ over a set of universe ξ is defined as $\tau = \{(a, \tau_1(a), \tau_2(a),) : a \in \xi\}$, where $\tau_1(a) \in [0, 1]$ is the MMS and $\tau_2(a) \in [0, 1]$ is the MNonMS of a satisfying the condition $0 \leqslant \tau_1(a) + \tau_2(a) + \tau_3(a) \leqslant 1$ for all $a \in \xi$.*

Including more possible types of uncertainty, Cuong [18] defined PFS generalizing the concepts of FS and IFS.

Definition 2.8. [18] *A PFS τ over a set of universe ξ is defined as $\tau = \{(a, \tau_1(a), \tau_2(a), \tau_3(a)) : a \in \xi\}$, where $\tau_1(a) \in [0, 1]$ is the MPMS of a in ξ, $\tau_2(a) \in [0, 1]$ is the MNeuMS of a in τ and $\tau_3(a) \in [0, 1]$ is the MNegMS of a in τ with the condition $0 \leqslant \tau_1(a) + \tau_2(a) + \tau_3(a) \leqslant 1$ for all $a \in \xi$. For all $a \in \xi$, $1 - (\tau_1(a) + \tau_2(a) + \tau_3(a))$ is the MRefMS a in τ.*

Some basic operations on PFSs are as follows.

Definition 2.9. [18] *Let $\tau = \{(a, \tau_1(a), \tau_2(a), \tau_3(a)) : a \in \xi\}$ and $\tau' = \{(a, \tau'_1(a), \tau'_2(a), \tau'_3(a)) : a \in \xi\}$ be two PFSs over a set of universe ξ. Then*

(i) $\tau \subseteq \tau'$ *iff* $\tau_1(a) \leqslant \tau'_1(a)$, $\tau_2(a) \leqslant \tau'_2(a)$, $\tau_3(a) \geqslant \tau'_3(a)$ *for all* $a \in \xi$.
(ii) $\tau = \tau'$ *iff* $\tau_1(a) = \tau'_1(a), \tau_2(a) = \tau'_2(a), \tau_3(a) = \tau'_3(a)$ *for all* $a \in \xi$.
(iii) $\tau \cup \tau' = \{(a, \max(\tau_1(a), \tau'_1(a)), \min(\tau_2(a), \tau'_2(a)), \min(\tau_3(a), \tau'_3(a))) : a \in \xi\}$.
(iv) $\tau \cap \tau' = \{(a, \min(\tau_1(a), \tau'_1(a)), \min(\tau_2(a), \tau'_2(a)), \max(\tau_3(a), \tau'_3(a))) : a \in \xi\}$.

The CP of two PFSs is defined below.

Definition 2.10. [18] *Let* $\tau = \{(a, \tau_1(a), \tau_2(a), \tau_3(a)) : a \in \xi_1\}$ *and* $\tau' = \{(b, \tau'_1(b), \tau'_2(b), \tau'_3(b)) : b \in \xi_2\}$ *be two PFSs over two sets of universe* A_1 *and* A_2 *respectively. Then the CP of* τ *and* τ' *is the PFS* $\tau \times \tau'$=$\{((a, b), \xi_1((a, b)), \xi_2((a, b)), \xi_3((a, b))) : (a, b) \in \xi_1 \times \xi_2\}$, *where* $\xi_1((a, b)) = \tau_1(a) \wedge \tau'_1(b)$, $\xi_2((a, b)) = \tau_2(a) \wedge \tau'_2(b)$ *and* $\xi_3((a, b)) = \tau_3(a) \vee \tau'_3(b)$ *for all* $(a, b) \in \xi_1 \times \xi_2$.

The (θ, ϕ, ψ)-cut of a PFS is defined as follows.

Definition 2.11. [18] *Let* $\tau = \{(a, \tau_1(a), \tau_2(a), \tau_3(a)) : a \in \xi\}$ *be a PFS over a set of universe* ξ. *Then* (θ, ϕ, ψ)*-cut of* τ *is the crisp set in* ξ *denoted by* $C_{\theta,\phi,\psi}(\tau)$ *and is defined as* $C_{\theta,\phi,\psi}(\tau) = \{a \in \xi : \tau_1(a) \geqslant \theta, \tau_2(a) \geqslant \phi, \tau_3(a) \leqslant \psi\}$, *where* $\theta, \phi, \psi \in [0, 1]$ *with the condition* $0 \leqslant \theta + \phi + \psi \leqslant 1$.

The image of a PFS is defined as follows.

Definition 2.12. [18] *Let* ξ_1 *and* ξ_2 *be two sets of universe. Let* $h : \xi_1 \to \xi_2$ *be a surjective mapping and* $\tau = \{(a_1, \tau_1(a_1), \tau_2(a_1), \tau_3(a_1)) : a_1 \in \xi_1\}$ *be a PFS over* ξ_1. *Then the image of* τ *under the map* h *is the PFS* $h(\tau) = \{(a_2, \psi_1(a_2), \psi_2(a_2), \psi_3(a_2)) : a_2 \in \xi_2\}$, *where* $\psi_1(a_2) = \bigvee_{a_1 \in h^{-1}(a_2)} \tau_1(a_1)$, $\psi_2(a_2) = \bigwedge_{a_1 \in h^{-1}(a_2)} \tau_2(a_1)$ *and* $\psi_3(a_2) = \bigwedge_{a_1 \in h^{-1}(a_2)} \tau_3(a_1)$ *for all* $a_2 \in \xi_2$.

The inverse image of a PFS is defined as follows.

Definition 2.13. [18] *Let* ξ_1 *and* ξ_2 *be two sets of universe. Let* $h : \xi_1 \to \xi_2$ *be a mapping and* $\tau' = \{(a_2, \tau'_1(a_2), \tau'_2(a_2), \tau'_3(a_2)) : a_2 \in \xi_2\}$ *be a PFS over* ξ_2. *Then the inverse image of* τ' *under the map* h *is the PFS* $h^{-1}(\tau') = \{(a_1, \psi_1(a_1), \psi_2(a_1), \psi_3(a_1)) : a_1 \in \xi_1\}$, *where* $\psi_1(a_1) = \tau'_1(h(a_1))$, $\psi_2(a_1) = \tau'_2(h(a_1))$ *and* $\psi_3(a_1) = \tau'_3(h(a_1))$ *for all* $a_1 \in \xi_1$.

Throughout the paper, we write PFS $\tau = \{(a, \tau_1(a), \tau_2(a), \tau_3(a)) : a \in \xi\}$ as $\tau = (\tau_1, \tau_2, \tau_3)$.

3. Picture fuzzy sub-hyperspace

In this section, the notion of PFSHS of a HVS is introduced as a generalization of FSHS of a HVS and some important results are investigated in this regard. Some properties of PFSHSs in the light of (θ, ϕ, ψ)-cut of PFS are studied here. An application of PFSHS conditionin decision making problem is shown here.

First we are going to define PFSHS as a generalization of FSHS.

Definition 3.1. *Let* ξ *be a HVS over a field* F. *A PFS* $\tau = (\tau_1, \tau_2, \tau_3)$ *over the set of universe* ξ *is said to be PFSHS of* ξ *if*
(i) $\tau_1(a - b) \geqslant \tau_1(a) \wedge \tau_1(b)$, $\tau_2(a - b) \geqslant \tau_2(a) \wedge \tau_2(b)$ *and* $\tau_3(a - b) \leqslant \tau_3(a) \vee \tau_3(b)$
(ii) $\inf_{c \in p \circ a} \tau_1(c) \geqslant \tau_1(a)$, $\inf_{c \in p \circ a} \tau_2(c) \geqslant \tau_2(a)$ *and* $\sup_{c \in p \circ a} \tau_3(c) \leqslant \tau_3(a)$ *for all* $a \in \xi$ *and for all* $p \in F$.

Now, it is the time to establish some elementary results on PFSHS. The following proposition gives a relation between the null vector and any other vector in a HVS over which PFSHS is defined. This relationship is given here in terms of picture fuzzy membership values.

Proposition 3.1. *Let ξ be a HVS over a field F and $\tau = (\tau_1, \tau_2, \tau_3)$ be a PFSHS of ξ. Then*
(i) $\tau_1(\rho) \geqslant \tau_1(a)$, $\tau_2(\rho) \geqslant \tau_2(a)$ and $\tau_3(\rho) \leqslant \tau_3(a)$
(ii) $\inf\limits_{c\in 1\circ a} \tau_1(c) = \tau_1(a)$, $\inf\limits_{c\in 1\circ a} \tau_2(c) = \tau_2(a)$ and $\sup\limits_{c\in 1\circ a} \tau_3(c) = \tau_3(a)$ for all $a, b \in \xi$.

Proof: (i) Since τ is a PFSHS of ξ therefore $\tau_1(\rho) = \tau_1(a - a) \geqslant \tau_1(a)$, $\tau_2(\rho) = \tau_2(a - a) \geqslant \tau_2(a) \wedge \tau_2(a) = \tau_2(a)$ and $\tau_3(\rho) = \tau_3(a - a) \leqslant \tau_3(a) \vee \tau_3(a) = \tau_3(a)$ for all $a \in \xi$.

(ii) Since τ is a PFSHS therefore $\inf\limits_{c\in 1\circ a} \tau_1(c) \geqslant \tau_1(a)$, $\inf\limits_{c\in 1\circ a} \tau_2(c) \geqslant \tau_2(a)$ and $\sup\limits_{c\in 1\circ a} \tau_3(c) \leqslant \tau_3(a)$ for all $a \in \xi$.
Since $a \in 1 \circ a$ therefore $\tau_1(a) \geqslant \inf\limits_{c\in 1\circ a} \tau_1(c)$, $\tau_2(a) \geqslant \inf\limits_{c\in 1\circ a} \tau_2(c)$ and $\tau_3(a) \leqslant \sup\limits_{c\in 1\circ a} \tau_3(c)$ for all $a \in \xi$.
Thus, $\inf\limits_{c\in 1\circ a} \tau_1(c) = \tau_1(a)$, $\inf\limits_{c\in 1\circ a} \tau_2(c) = \tau_2(a)$ and $\sup\limits_{c\in 1\circ a} \tau_3(c) = \tau_3(a)$ for all $a \in \xi$.

It is observed that for $p, q \in F$ and $a, b \in \xi$, $p \circ a + q \circ b \in P^*(\xi)$ ($P^*(\xi) = P(\xi) - \phi$). If $\tau = (\tau_1, \tau_2, \tau_3)$ be a PFS over $p \circ a + q \circ b$ then we define $\inf\limits_{c=c_1+c_2\in p\circ a+q\circ b} \tau_1(c) = \inf\limits_{c_1\in p\circ a} \tau_1(c_1) \wedge \inf\limits_{c_2\in p\circ a} \tau_1(c_2)$, $\inf\limits_{c=c_1+c_2\in p\circ a+q\circ b} \tau_2(c) = \inf\limits_{c_1\in p\circ a} \tau_2(c_1) \wedge \inf\limits_{c_2\in p\circ a} \tau_2(c_2)$ and $\sup\limits_{c=c_1+c_2\in p\circ a+q\circ b} \tau_3(c) = \sup\limits_{c_1\in p\circ a} \tau_3(c_1) \vee \sup\limits_{c_2\in p\circ a} \tau_3(c_2)$. This definition will be useful to establish crucial results on PFSHSs.

Now, we are going to propose a necessary and sufficient condition under which a PFS will be a PFSHS.

Proposition 3.2. *Let ξ be a HVS over a field F and $\tau = (\tau_1, \tau_2, \tau_3)$ be a PFS over ξ. Then τ is a PFSHS of ξ iff $\inf\limits_{c\in p\circ a+q\circ b} \tau_1(c) \geqslant \tau_1(a) \wedge \tau_1(b)$, $\inf\limits_{c\in p\circ a+q\circ b} \tau_2(c) \geqslant \tau_2(a) \wedge \tau_2(b)$ and $\sup\limits_{c\in p\circ a+q\circ b} \tau_3(c) \leqslant \tau_3(a) \vee \tau_3(b)$ for all $a, b \in \xi$ and for all $p, q \in F$.*

Proof: Let τ be a PFSHS of ξ. Therefore,
$\inf\limits_{c=c_1+c_2\in p\circ a+q\circ b} \tau_1(c) = [\inf\limits_{c_1\in p\circ a} \tau_1(c_1)] \wedge [\inf\limits_{c_2\in p\circ a} \tau_1(c_2)] \geqslant \tau_1(a) \wedge \tau_1(b)$,
$\inf\limits_{c=c_1+c_2\in p\circ a+q\circ b} \tau_2(c) = [\inf\limits_{c_1\in p\circ a} \tau_2(c_1)] \wedge [\inf\limits_{c_2\in p\circ a} \tau_2(c_2)] \geqslant \tau_2(a) \wedge \tau_2(b)$
and $\sup\limits_{c=c_1+c_2\in p\circ a+q\circ b} \tau_3(c) = [\sup\limits_{c_1\in p\circ a} \tau_3(c_1)] \vee [\sup\limits_{c_2\in p\circ a} \tau_3(c_3)] \leqslant \tau_3(a) \vee \tau_3(b)$ for all $a, b \in \xi$ and for all $p, q \in F$.

Conversely, let the conditions $\inf\limits_{c\in p\circ a+q\circ b} \tau_1(c) \geqslant \tau_1(a) \wedge \tau_1(b)$, $\inf\limits_{c\in p\circ a+q\circ b} \tau_2(c) \geqslant \tau_2(a) \wedge \tau_2(b)$ and $\sup\limits_{c\in p\circ a+q\circ b} \tau_3(c) \leqslant \tau_3(a) \vee \tau_3(b)$ for all $a, b \in \xi$ and for all $p, q \in F$ be hold. Then putting $p = 1$, $q = -1$ it is obtained that $\inf\limits_{c\in 1\circ a+(-1)\circ b} \tau_1(c) \geqslant \tau_1(a) \wedge \tau_1(b)$, $\inf\limits_{c\in 1\circ a+(-1)\circ b} \tau_2(c) \geqslant \tau_2(a) \wedge \tau_2(b)$ and $\sup\limits_{c\in 1\circ a+(-1)\circ b} \tau_3(c) \leqslant \tau_3(a) \vee \tau_3(b)$, that is, $\inf\limits_{c\in a-b} \tau_1(c) \geqslant \tau_1(a) \wedge \tau_1(b)$, $\inf\limits_{c\in a-b} \tau_2(c) \geqslant \tau_2(a) \wedge \tau_2(b)$ and $\sup\limits_{c\in a-b} \tau_3(c) \leqslant \tau_3(a) \vee \tau_3(b)$ for all $a, b \in \xi$. Now, putting $q = \rho$ in the given conditions it is obtained that $\inf\limits_{c\in p\circ a+q\circ\rho} \tau_1(c) \geqslant \tau_1(a) \wedge \tau_1(\rho) = \tau_1(a)$, $\inf\limits_{c\in p\circ a+q\circ\rho} \tau_2(c) \geqslant \tau_2(a) \wedge \tau_2(\rho) = \tau_2(a)$ and $\sup\limits_{c\in p\circ a+q\circ\rho} \tau_3(c) \leqslant \tau_3(a) \vee \tau_3(\rho) = \tau_3(a)$, that is, $\inf\limits_{c\in p\circ a} \tau_1(c) \geqslant \tau_1(a)$, $\inf\limits_{c\in p\circ a} \tau_2(c) \geqslant \tau_2(a)$ and $\sup\limits_{c\in p\circ a} \tau_3(c) \leqslant \tau_3(a)$

for all $a \in \xi$ and for all $p \in F$. Hence, τ is a PFSHS of ξ.

Proposition 3.3. *Let ξ be a HVS over a field F and $\tau = (\tau_1, \tau_2, \tau_3)$ be a PFSHS of ξ. Then $C_{\theta,\phi,\psi}(\tau)$ is a SHS of ξ.*

Proof: Let $a, b \in C_{\theta,\phi,\psi}(\tau)$. Then $\tau_1(a) \geqslant \theta$, $\tau_2(a) \geqslant \phi$, $\tau_3(a) \leqslant \psi$ and $\tau_1(b) \geqslant \theta$, $\tau_2(b) \geqslant \phi$, $\tau_3(b) \leqslant \psi$.
Now, $\tau_1(a - b) \geqslant \tau_1(a) \wedge \tau_1(b) \geqslant \theta \wedge \theta = \theta$, $\tau_2(a - b) \geqslant \tau_2(a) \wedge \tau_2(b) \geqslant \phi \wedge \phi = \phi$ and $\tau_3(a - b) \leqslant \tau_3(a) \vee \tau_3(b) \leqslant \psi \vee \psi = \psi$.

Also, $\inf\limits_{c \in p \circ a} \tau_1(c) \geqslant \tau_1(a) \geqslant \theta$, $\inf\limits_{c \in p \circ a} \tau_2(c) \geqslant \tau_2(a) \geqslant \phi$ and $\sup\limits_{c \in p \circ a} \tau_3(c) \leqslant \tau_3(a) \leqslant \phi$. Hence, $\tau_1(c) \geqslant \theta$, $\tau_2(c) \geqslant \phi$ and $\tau_3(c) \leqslant \psi$ for all $c \in p \circ a$. Thus, $p \in F$ and $a \in C_{\theta,\phi,\psi}(\tau) \Rightarrow p \circ a \subseteq C_{\theta,\phi,\psi}(\tau)$. This implies $C_{\theta,\phi,\psi}(\tau)$ is a SHS of ξ.

Proposition 3.4. *Let ξ be a HVS over a field F and $\tau = (\tau_1, \tau_2, \tau_3)$ be a PFS over ξ. Then τ is a PFSHS of ξ if all (θ, ϕ, ψ)-cuts of τ are SHSs of ξ.*

Proof: Let $a, b \in \xi$. Also, let $\theta = \tau_1(a) \wedge \tau_1(b)$, $\phi = \tau_2(a) \wedge \tau_2(b)$ and $\psi = \tau_3(a) \vee \tau_3(b)$. Then $\theta \in [0, 1]$, $\phi \in [0, 1]$ and $\psi \in [0, 1]$ with $0 \leqslant \theta + \phi + \psi \leqslant 1$.
Now, $\tau_1(a) \geqslant \tau_1(a) \wedge \tau_2(b) = \theta$, $\tau_2(a) \geqslant \tau_2(a) \wedge \tau_2(b) = \phi$, $\tau_3(a) \leqslant \tau_3(a) \vee \tau_3(b) = \psi$ and $\tau_1(b) \geqslant \tau_1(a) \wedge \tau_1(b) = \theta$, $\tau_2(b) \geqslant \tau_2(a) \wedge \tau_2(b) = \phi$, $\tau_3(b) \leqslant \tau_3(a) \vee \tau_3(b) = \psi$. Therefore, $a, b \in C_{\theta,\phi,\psi}(\tau)$. Since $C_{\theta,\phi,\psi}(\tau)$ is a SHS of ξ therefore $a - b \in C_{\theta,\phi,\psi}(\tau)$. This implies, $\tau_1(a - b) \geqslant \theta = \tau_1(a) \wedge \tau_1(b)$, $\tau_2(a - b) \geqslant \phi = \tau_2(a) \wedge \tau_2(b)$ and $\tau_3(a - b) \leqslant \psi = \tau_3(a) \vee \tau_3(b)$. Since a and b are arbitrary elements of ξ therefore $\tau_1(a - b) \geqslant \tau_1(a) \wedge \tau_1(b)$, $\tau_2(a - b) \geqslant \tau_2(a) \wedge \tau_2(b)$ and $\tau_3(a - b) \leqslant \tau_3(a) \vee \tau_3(b)$ for all $a, b \in \xi$.

Now, let $\tau_1(a) = \theta_1$, $\tau_2(a) = \phi_1$ and $\tau_3(a) = \psi_1$. Then $\theta_1 \in [0, 1]$, $\phi_1 \in [0, 1]$ and $\psi_1 \in [0, 1]$ with $0 \leqslant \theta_1 + \phi_1 + \psi_1 \leqslant 1$. Since $C_{\theta_1,\phi_1,\psi_1}(\tau)$ is a SHS of ξ therefore $p \in F$ and $a \in C_{\theta,\phi,\psi}(\tau)$ implies $p \circ a \subseteq C_{\theta_1,\phi_1,\psi_1}(\tau)$. So, for any $c \in p \circ a$, $\inf\limits_{c \in p \circ a} \tau_1(c) \geqslant \theta_1 = \tau_1(a)$, $\inf\limits_{c \in p \circ a} \tau_2(c) \geqslant \phi_1 = \tau_2(a)$ and $\sup\limits_{c \in p \circ a} \tau_3(c) \leqslant \psi_1 = \tau_3(a)$. Thus, τ is a PFSHS of ξ.

Proposition 3.5. *Let ξ be a HVS over a field F and $\tau = (\tau_1, \tau_2, \tau_3)$, $\tau' = (\tau'_1, \tau'_2, \tau'_3)$ be two PFSHSs of ξ. Then $\tau \cap \tau'$ is a PFSHS of ξ.*

Thus, it is observed that the intersection of two PFSHSs is a PFSHS. But, the union of two PFSHSs is not necessarily a PFSHS which can be shown by the following example.

Example 3.1. *Let $\xi = \mathbb{R}^2$. Then $(\xi, +, \circ)$ forms a HVS under vector addition '$+$' and external composition $\circ$ defined by*

$$p \circ a = \begin{cases} A, & \text{when } a \neq (0,0) \\ \{(0,0)\}, & \text{when } a = (0,0) \end{cases}$$

where A the collection of the points on the line joining the point a and (0,0).
Now, take $S_1 = \{(b, 0) : b \in \mathbb{R}\}$ and $S_2 = \{(0, c) : c \in \mathbb{R}\}$. Then $(S_1, +, \circ)$ and $(S_2, +, \circ)$ are SHSs of ξ where $\circ$ is defined by

$$p \circ (b, 0) = \begin{cases} S_1, & \text{when } b \neq 0 \\ \{(0,0)\}, & \text{otherwise} \end{cases}$$

$$p \circ (0, c) = \begin{cases} S_2, & \text{when } c \neq 0 \\ \{(0, 0)\}, & \text{otherwise} \end{cases}$$

Now, let us consider two PFSs $\tau = (\tau_1, \tau_2, \tau_3)$ and $\tau' = (\tau'_1, \tau'_2, \tau'_3)$ over ξ as follows.

$$\tau_1(a) = \begin{cases} 0.45, & \text{when } a \in S_1 \\ 0.2, & \text{otherwise} \end{cases}$$

$$\tau_2(a) = \begin{cases} 0.4, & \text{when } a \in S_1 \\ 0.25, & \text{otherwise} \end{cases}$$

and

$$\tau_3(a) = \begin{cases} 0.15, & \text{when } a \in S_1 \\ 0.48, & \text{otherwise} \end{cases}$$

$$\tau'_1(a) = \begin{cases} 0.35, & \text{when } a \in S_2 \\ 0.25, & \text{otherwise} \end{cases}$$

$$\tau'_2(a) = \begin{cases} 0.3, & \text{when } a \in S_2 \\ 0.2, & \text{otherwise} \end{cases}$$

and

$$\tau'_3(a) = \begin{cases} 0.27, & \text{when } a \in S_2 \\ 0.39, & \text{otherwise} \end{cases}$$

Thus, $\tau \cup \tau' = \psi = (\psi_1, \psi_2, \psi_3)$ is given by

$$\psi_1(a) = \begin{cases} 0.45, & \text{when } a \in S_1 \\ 0.35, & \text{when } a \in S_2 - \{0\} \\ 0.25, & \text{when } a \in \xi - S_1 \cup S_2 \end{cases}$$

$$\psi_2(a) = \begin{cases} 0.2, & \text{when } a \in S_1 \\ 0.25, & \text{when } a \in S_2 - \{0\} \\ 0.2, & \text{when } a \in \xi - S_1 \cup S_2 \end{cases}$$

$$\psi_3(a) = \begin{cases} 0.15, & \text{when } a \in S_1 \\ 0.27, & \text{when } a \in S_2 - \{0\} \\ 0.39, & \text{when } a \in \xi - S_1 \cup S_2 \end{cases}$$

Here, $C_{0.35,0.2,0.27}(\tau \cup \tau')$
$= \{a : \tau_1(a) \geqslant 0.35, \tau_2(a) \geqslant 0.2, \tau_3(a) \leqslant 0.27\}$
$= \{a \in \xi : \tau_1(a) \geqslant 0.35\} \cap \{a \in \xi : \tau_2(a) \geqslant 0.2\} \cap \{a \in \xi : \tau_3(a) \leqslant 0.27\}$
$= \{a \in \xi : \tau_1(a) = 0.35, 0.45\} \cap \{a \in \xi : \tau_2(a) = 0.2, 0.25\} \cap \{a \in \xi : \tau_3(a) = 0.15, 0.27\}$
$= (S_1 \cup S_2) \cap \xi \cap (S_1 \cup S_2)$
$= S_1 \cup S_2$.
Since $S_1 \cup S_2$ is not a SHS of ξ therefore $\psi = \tau \cup \tau'$ is not a PFSHS of ξ, although τ and τ' are PFSHSs of ξ.

Proposition 3.6. *Let ξ be a HVS over a field F and $\tau = (\tau_1, \tau_2, \tau_3)$, $\tau' = (\tau'_1, \tau'_2, \tau'_3)$ be two PFSHSs of ξ. Then $\tau \cup \tau'$ is a PFSHS of ξ if either $\tau \subseteq \tau'$ or $\tau' \subseteq \tau$.*

From the above proposition, it is observed that the union of two PFSHSs is a PFSHS if one is subset of another. This condition is a sufficient condition for union of two PFSHSs to be a PFSHS. But the condition is not necessary which can be shown by the following example.

Example 3.2. *Let us consider the HVS given in Example 3.1. Now, let us consider PFSHs* $\tau = (\tau_1, \tau_2, \tau_3)$ *and* $\tau' = (\tau'_1, \tau'_2, \tau'_3)$ *over* ξ *as follows.*

$$\tau_1(a) = \begin{cases} 0.45, & \text{when } a = (0,0) \\ 0.2, & \text{otherwise} \end{cases}$$

$$\tau_2(a) = \begin{cases} 0.4, & \text{when } a = (0,0) \\ 0.25, & \text{otherwise} \end{cases}$$

and

$$\tau_3(a) = \begin{cases} 0.15, & \text{when } a = (0,0) \\ 0.48, & \text{otherwise} \end{cases}$$

$$\tau'_1(a) = \begin{cases} 0.35, & \text{when } a = (0,0) \\ 0.25, & \text{otherwise} \end{cases}$$

$$\tau'_2(a) = \begin{cases} 0.3, & \text{when } a = (0,0) \\ 0.2, & \text{otherwise} \end{cases}$$

and

$$\tau'_3(a) = \begin{cases} 0.27, & \text{when } a = (0,0) \\ 0.39, & \text{otherwise} \end{cases}$$

Thus, $\tau \cup \tau' = \psi = (\psi_1, \psi_2, \psi_3)$ *is given by*

$$\psi_1(a) = \begin{cases} 0.45, & \text{when } a = (0,0) \\ 0.25, & \text{otherwise} \end{cases}$$

$$\psi_2(a) = \begin{cases} 0.3, & \text{when } a = (0,0) \\ 0.2, & \text{otherwise} \end{cases}$$

$$\psi_3(a) = \begin{cases} 0.15, & \text{when } a = (0,0) \\ 0.39, & \text{otherwise} \end{cases}$$

Notice that $C_{0.27,0.25,0.37}(\tau \cup \tau')$
$= \{a \in \xi : \tau_1(a) \geqslant 0.27\} \cap \{a \in \xi : \tau_2(a) \geqslant 0.25\} \cap \{a \in \xi : \tau_3(a) \leqslant 0.37\}$
$= \{a \in \xi : \tau_1(a) = 0.45\} \cap \{a \in \xi : \tau_2(a) = 0.3\} \cap \{a \in \xi : \tau_3(a) = 0.15\}$
$= \{(0,0)\}$
and $C_{0.25,0.2,0.39}(\tau \cup \tau')$
$= \{a \in \xi : \tau_1(a) \geqslant 0.25\} \cap \{a \in \xi : \tau_2(a) \geqslant 0.2\} \cap \{a \in \xi : \tau_3(a) \leqslant 0.39\}$
$= \{a \in \xi : \tau_1(a) = 0.25, 0.45\} \cap \{a \in \xi : \tau_2(a) = 0.2, 0.3\} \cap \{a \in \xi : \tau_3(a) = 0.15, 0.39\}$
$= \xi = \mathbb{R}^2$.
Here $C_{0.27,0.25,0.37}(\tau \cup \tau')$ *and* $C_{0.25,0.2,0.39}(\tau \cup \tau')$ *are SHSs of* ξ *although neither* $\tau \subseteq \tau'$ *nor* $\tau' \subseteq \tau$.

4. Picture fuzzy linear transformation

In this section, the notion of PFLT is initiated in a very interesting way which is different from existing literature. Also, some properties of PFLT are investigated.

Definition 4.1. *Let ξ_1, ξ_2 be two HVSs over the same field F and $\tau = (\tau_1, \tau_2, \tau_3), \tau' = (\tau'_1, \tau'_2, \tau'_3)$ be two PFSHSs of ξ_1 and ξ_2 respectively. Also, let $T : \xi_1 \to \xi_2$ be a mapping. Then T is said to be PFLT on ξ_1 if*
(i) T is a linear transformation in crisp sense.
(ii) $\tau'_1(T(a)) \geqslant \tau_1(a)$, $\tau'_2(T(a)) \geqslant \tau_2(a)$ and $\tau'_3(T(a)) \leqslant \tau_3(a)$ for all $a \in \xi_1$.

Example 4.1. *Let us consider the Example 3.1. Let us define a map $T : \xi \to \xi$ by $T((a_1, a_2)) = (a_1 + a_2, 0)$ for all $(a_1, a_2) \in \xi$. Clearly, T is a linear map in crisp sense. For any $(a_1, a_2) \in \xi$, it is observed that*

$$\begin{aligned}
\tau_1(T(a_1, a_2)) &= \tau_1((a_1 + a_2, 0)) = 0.45 \geqslant \tau_1((a_1, a_2)),\\
\tau_2(T(a_1, a_2)) &= \tau_2((a_1 + a_2, 0)) = 0.4 \geqslant \tau_2((a_1, a_2))\\
\text{and } \tau_3(T(a_1, a_2)) &= \tau_3((a_1 + a_2, 0) = 0.15 \leqslant \tau_3((a_1, a_2)).
\end{aligned}$$

Thus, T is a PFLT on ξ with respect to PFSHS τ.

The following proposition states that linear combination of two PFLTs is a PFLT.

Proposition 4.1. *Let ξ_1, ξ_2 be two HVSs over the same field F and $\tau = (\tau_1, \tau_2, \tau_3), \tau' = (\tau'_1, \tau'_2, \tau'_3)$ be two PFSHSs of ξ_1, ξ_2 respectively. If T_1 and T_2 be two PFLTs on ξ_1 then so is $aT_1 + bT_2$ for some $a, b \in F$.*

Proof: We have,

$$\begin{aligned}
&\inf_{c \in (aT_1 + bT_2)(d)} \tau'_1(c)\\
&= \inf_{c = c_1 + c_2 \in a \circ T_1(d) + b \circ T_2(d)} \tau'_1(c)\\
&= \inf_{c_1 \in a \circ T_1(d)} \tau'_1(c_1) \wedge \inf_{c_2 \in b \circ T_2(d)} \tau'_1(c_2)\\
&\geqslant \tau'_1(T_1(d)) \wedge \tau'_1(T_2(d)) \text{ [as } \tau' \text{ is a PFSHS of } \xi_2]\\
&\geqslant \tau_1(d) \wedge \tau_1(d) \text{ [as } T_1, T_2 \text{ are PFLTs on } \xi_1]\\
&= \tau_1(d)
\end{aligned}$$

$$\begin{aligned}
&\inf_{c \in (aT_1 + bT_2)(d)} \tau'_2(c)\\
&= \inf_{c = c_1 + c_2 \in a \circ T_1(d) + b \circ T_2(d)} \tau'_2(c)\\
&= \inf_{c_1 \in a \circ T_1(d)} \tau'_2(c_1) \wedge \inf_{c_2 \in b \circ T_2(d)} \tau'_2(c_2)\\
&\geqslant \tau'_2(T_1(d)) \wedge \tau'_2(T_2(d)) \text{ [as } \tau' \text{ is a PFSHS of } \xi_2]\\
&\geqslant \tau_2(d) \wedge \tau_2(d) \text{ [as } T_1, T_2 \text{ are PFLTs on } \xi_1]
\end{aligned}$$

$$= \tau_2(d)$$

$$\begin{aligned}
\text{and} \sup_{c\in(aT_1+bT_2)(d)} \tau_3'(c) &= \sup_{c=c_1+c_2\in a\circ T_1(d)+b\circ T_2(d)} \tau_3'(c) \\
&= \sup_{c_1\in a\circ T_1(d)} \tau_3'(c_1) \vee \sup_{c_2\in b\circ T_2(d)} \tau_3'(c_2) \\
&\leqslant \tau_3'(T_1(d)) \vee \tau_3'(T_2(d)) \text{ [as } \tau' \text{ is a PFSHS of } \xi_2] \\
&\leqslant \tau_3(d) \vee \tau_3(d) \text{ [as } T_1, T_2 \text{ are PFLTs on } \xi_1] \\
&= \tau_3(d)
\end{aligned}$$

Thus, $aT_1 + bT_2$ is a PFLT on ξ_1 for some scalar $a, b \in F$.

The following proposition states that composition of two PFLTs is a PFLT.

Proposition 4.2. *Let ξ_1, ξ_2, ξ_3 be three HVSs over the same field F and $\tau = (\tau_1, \tau_2, \tau_3), \tau' = (\tau_1', \tau_2', \tau_3'), \tau'' = (\tau_1'', \tau_2'', \tau_3'')$ be three PFSHSs of ξ_1, ξ_2, ξ_3 respectively. If $T_1 : \xi_1 \to \xi_2$ and $T_2 : \xi_2 \to \xi_3$ be two PFLTs then so is $T_2 \circ T_1$.*

Proof: Let $a \in \xi_1$.

$$\begin{aligned}
\text{Now, } \tau_1''((T_2T_1)(a)) &= \tau_1''(T_2(T_1(a)) \\
&\geqslant \tau_1'(T_1(a)) \text{ [because } T_2 \text{ is a PFLT]} \\
&\geqslant \tau_1(a) \text{ [because } T_1 \text{ is a PFLT]}
\end{aligned}$$

$$\begin{aligned}
\tau_2''((T_2T_1)(a)) &= \tau_2''(T_2(T_1(a)) \\
&\geqslant \tau_2'(T_1(a)) \text{ [because } T_2 \text{ is a PFLT]} \\
&\geqslant \tau_2(a) \text{ [because } T_1 \text{ is a PFLT]}
\end{aligned}$$

$$\begin{aligned}
\tau_3''((T_2T_1)(a)) &= \tau_3''(T_2(T_1(a)) \\
&\leqslant \tau_3'(T_1(a)) \text{ [because } T_2 \text{ is a PFLT]} \\
&\leqslant \tau_3(a) \text{ [because } T_1 \text{ is a PFLT]}
\end{aligned}$$

Since a is an arbitrary element of ξ_1 therefore $\tau_1''((T_2 \circ T_1)(a)) \geqslant \tau_1(a)$ and $\tau_2''((T_2 \circ T_1)(a)) \geqslant \tau_2(a)$ and $\tau_3''((T_2 \circ T_1)(a)) \leqslant \tau_3(a)$ for all $a \in \xi_1$. Consequently, $T_2 \circ T_1$ is a PFLT on ξ_1.

The following proposition states that the inverse of a PFLT is a PFLT when the PFLT is bijective.

Proposition 4.3. *Let ξ_1 and ξ_2 be two HVSs over the same field F and $\tau = (\tau_1, \tau_2, \tau_3)$ be a PFSHS of ξ_1. If $T : \xi_1 \to \xi_2$ is a bijective good PFLT then T^{-1} is a PFLT on ξ_2.*

Proof: Let $T(\tau) = \psi = (\psi_1, \psi_2, \psi_3)$. For $b \in \xi_2$, we have,

$$\psi_1(b) = \bigvee_{a \in T^{-1}(b)} \tau_1(a)$$
$$\psi_2(b) = \bigwedge_{a \in T^{-1}(b)} \tau_2(a)$$
$$\text{and } \psi_3(b) = \bigwedge_{a \in T^{-1}(b)} \tau_3(a).$$

Since T is bijective therefore $T^{-1}(b)$ must be a singleton set. So, for $b \in \xi_2$, there exists an unique $a \in \xi_1$ such that $a = T^{-1}(b)$ i.e. $T(a) = b$. Thus, in this case, $\psi_1(b) = \tau_1(a)$ and $\psi_2(b) = \tau_2(a)$ and $\psi_3(b) = \tau_3(a)$, that is, $\psi_1(b) = \tau_1(T^{-1}(b))$, $\psi_2(b) = \tau_2(T^{-1}(b))$ and $\psi_3(b) = \tau_3(T^{-1}(b))$.
Thus, it can be written as $\tau_1(T^{-1}(b)) = \psi_1(b) \geqslant \psi_1(b)$, $\tau_2(T^{-1}(b)) = \psi_2(b) \geqslant \psi_2(b)$ and $\tau_3(T^{-1}(b)) = \psi_3(b) \leqslant \psi_3(b)$ for all $b \in \xi_2$. Hence, T^{-1} is a PFLT on ξ_2.

5. Effect of linear transformation on picture fuzzy sub-hyperspaces

In this section, we establish two propositions to discuss the effect of LT on PFSHSs. The first proposition states that the image of a PFSHS under bijective good LT is a PFSHS and the second proposition states that the inverse image of a PFSHS is a PFSHS under good LT.

Proposition 5.1. *Let ξ_1 and ξ_2 be two HVSs over the same field F and $\tau = (\tau_1, \tau_2, \tau_3)$ be a PFSHS of ξ_1. Then for a bijective good LT $T : \xi_1 \to \xi_2$, $T(\tau)$ is a PFSHS of ξ_2.*

Proof: Let $T(\tau) = \psi = (\psi_1, \psi_2, \psi_3)$. For $c \in \xi_2$, we have,

$$\psi_1(c) = \bigvee_{a \in T^{-1}(c)} \tau_1(a)$$
$$\psi_2(c) = \bigwedge_{a \in T^{-1}(c)} \tau_2(a)$$
$$\text{and } \psi_3(c) = \bigwedge_{a \in T^{-1}(b)} \tau_3(a).$$

Since T is bijective therefore $T^{-1}(c)$ must be a singleton set. So, for $c \in \xi_2$, there exists an unique $a \in \xi_1$ such that $a = T^{-1}(c)$ i.e. $T(a) = c$. Thus, in this case, $\psi_1(c) = \psi_1(T(a)) = \tau_1(a)$, $\psi_2(c) = \psi_2(T(a)) = \tau_2(a)$ and $\psi_3(c) = \psi_3(T(a)) = \tau_3(a)$. Similarly, for $d \in \xi_2$, $\psi_1(d) = \psi_1(T(b)) = \tau_1(b)$, $\psi_2(d) = \psi_2(T(b)) = \tau_2(b)$ and $\psi_3(d) = \psi_3(T(b)) = \tau_3(b)$.

$$\begin{aligned}
&\text{Now, } \inf_{z \in p \circ c + q \circ d} \psi_1(z) \\
&= \inf_{z \in p \circ T(a) + q \circ T(b)} \psi_1(z) \text{ [where } c = T(a) \text{ and } d = T(b)] \\
&= \inf_{z \in T(p \circ a + q \circ b)} \psi_1(z) \text{ [as } T \text{ is a good LT]} \\
&= \inf_{s = s_1 + s_2 \in p \circ a + q \circ b} \tau_1(s) \text{ [as } T \text{ is a bijective LT and } z = T(s) \text{ for unique } s \in \xi_1] \\
&= \inf_{s_1 \in p \circ a} \tau_1(s_1) \wedge \inf_{s_2 \in q \circ b} \tau_1(s_2) \\
&\geqslant \tau_1(a) \wedge \tau_1(b) \\
&= \psi_1(T(a)) \wedge \psi_1(T(b))
\end{aligned}$$

$= \psi_1(c) \wedge \psi_1(d)$

$$\begin{aligned}
&\inf_{z\in p\circ c+q\circ d} \psi_2(z) \\
&= \inf_{z\in p\circ T(a)+q\circ T(b)} \psi_2(z) \text{ [where } c = T(a) \text{ and } d = T(b)] \\
&= \inf_{z\in T(p\circ a+q\circ b)} \psi_2(z) \text{ [as } T \text{ is a good LT]} \\
&= \inf_{s=s_1+s_2\in p\circ a+q\circ b} \tau_2(s) \text{ [as } T \text{ is a bijective LT and } z = T(s) \text{ for unique } s \in \xi_1] \\
&= \inf_{s_1\in p\circ a} \tau_2(s_1) \wedge \inf_{s_2\in q\circ b} \tau_2(s_2) \\
&\geqslant \tau_2(a) \wedge \tau_2(b) \\
&= \psi_2(T(a)) \wedge \psi_2(T(b)) \\
&= \psi_2(c) \wedge \psi_2(d)
\end{aligned}$$

$$\begin{aligned}
&\sup_{z\in p\circ c+q\circ d} \psi_3(z) \\
&= \sup_{z\in p\circ T(a)+q\circ T(b)} \psi_3(z) \text{ [where } c = T(a) \text{ and } d = T(b)] \\
&= \sup_{z\in T(p\circ a+q\circ b)} \psi_3(z) \text{ [as } T \text{ is a good LT]} \\
&= \sup_{s=s_1+s_2\in p\circ a+q\circ b} \tau_3(s) \text{ [as } T \text{ is a bijective LT and } z = T(s) \text{ for unique } s \in \xi_1] \\
&= \sup_{s_1\in p\circ a} \tau_3(s_1) \vee \sup_{s_2\in q\circ b} \tau_3(s_2) \\
&\leqslant \tau_3(a) \vee \tau_3(b) \\
&= \psi_3(T(a)) \vee \psi_3(T(b)) \\
&= \psi_3(c) \vee \psi_3(d)
\end{aligned}$$

Since, c, d are arbitrary elements of ξ_2 therefore $\inf_{z\in p\circ c+q\circ d} \psi_1(z) \geqslant \psi_1(c) \wedge \psi_1(d)$, $\inf_{z\in p\circ c+q\circ d} \psi_2(z) \geqslant \psi_2(c) \wedge \psi_2(d)$ and $\sup_{z\in p\circ c+q\circ d} \psi_3(z) \leqslant \psi_3(c) \vee \psi_3(d)$ for all $c, d \in \xi_2$ and for all $p, q \in F$. Consequently, $T(\tau)$ is a PFSHS of ξ_2.

Proposition 5.2. *Let ξ_1 and ξ_2 be two HVSs over the same field F and $\tau' = (\tau'_1, \tau'_2, \tau'_3)$ be a PFSHS of ξ_2. Also, let $T : \xi_1 \to \xi_2$ be a good LT. Then $T^{-1}(\tau')$ is a PFSHS of ξ_1.*

Proof: Let $T^{-1}(\tau') = \psi = (\psi_1, \psi_2, \psi_3)$, where $\psi_1(a) = \tau'_1(T(a))$ and $\psi_2(a) = \tau'_2(T(a))$ and $\psi_3(a) = \tau'_3(T(a))$ for all $a \in \xi_1$. Now we have,

$$\begin{aligned}
\text{Now, } \inf_{c\in p\circ a+q\circ b} \psi_1(c) &= \inf_{d\in T(p\circ a+q\circ b)} \tau'_1(d) \\
&= \inf_{d=d_1+d_2\in p\circ T(a)+q\circ T(b)} \tau'_1(d) \text{ [as } T \text{ is a good LT]}
\end{aligned}$$

$$
\begin{aligned}
&= \inf_{d_1 \in p \circ T(a)} \tau_1'(d_1) \wedge \inf_{d_2 \in q \circ T(b)} \tau_1'(d_2) \\
&\geqslant \tau_1'(T(a)) \wedge \tau_1'(T(b)) \\
&= \psi_1(a) \wedge \psi_1(b)
\end{aligned}
$$

$$
\begin{aligned}
\inf_{c \in p \circ a + q \circ b} \psi_2(c) &= \inf_{d \in T(p \circ a + q \circ b)} \tau_2'(d) \\
&= \inf_{d = d_1 + d_2 \in p \circ T(a) + q \circ T(b)} \tau_2'(d) \text{ [as } T \text{ is a good LT]} \\
&= \inf_{d_1 \in p \circ T(a)} \tau_2'(d_1) \wedge \inf_{d_2 \in q \circ T(b)} \tau_2'(d_2) \\
&\geqslant \tau_2'(T(a)) \wedge \tau_2'(T(b)) \\
&= \psi_2(a) \wedge \psi_2(b)
\end{aligned}
$$

$$
\begin{aligned}
\text{and } \sup_{c \in p \circ a + q \circ b} \psi_3(c) &= \sup_{d \in T(p \circ a + q \circ b)} \tau_3'(d) \\
&= \sup_{d = d_1 + d_2 \in p \circ T(a) + q \circ T(b)} \tau_3'(d) \text{ [as } T \text{ is a good LT]} \\
&= \sup_{d_1 \in p \circ T(a)} \tau_3'(d_1) \vee \sup_{d_2 \in q \circ T(b)} \tau_3'(d_2) \\
&\leqslant \tau_3'(T(a)) \vee \tau_3'(T(b)) \\
&= \psi_3(a) \vee \psi_3(b)
\end{aligned}
$$

Thus, $T^{-1}(\tau')$ is a PFSHS of ξ_1.

6. Comparative study

In algebraic structure, the algebraic operation of two elements gives a single element whereas in algebraic hyperstructure, algebraic hyperoperation of two elements gives a set. This is the main difference between algebraic operation and algebraic hyperoperation. When for $a \in \xi$, $p \in F$; $p \circ a$ gives a singleton set $\{p \cdot a\}$, then hyper vector space is reduced to vector space. In this case, hyperoperative condition of picture fuzzy sub-hyperspace is reduced to the following. $\inf\limits_{c \in p \circ a} \tau_1(c) \geqslant \tau_1(a)$, $\inf\limits_{c \in p \circ a} \tau_2(c) \geqslant \tau_2(a)$ and $\sup\limits_{c \in p \circ a} \tau_3(c) \leqslant \tau_3(a)$ for all $a \in \xi$ and for all $p \in F$.

This implies, $\inf\limits_{c \in \{p \cdot a\}} \tau_1(c) \geqslant \tau_1(a)$, $\inf\limits_{c \in \{p \cdot a\}} \tau_2(c) \geqslant \tau_2(a)$ and $\sup\limits_{c \in \{p \cdot a\}} \tau_3(c) \leqslant \tau_3(a)$ for all $a \in \xi$ and for all $p \in F$.

That is, $\tau_1(p \cdot a) \geqslant \tau_1(a)$, $\tau_2(p \cdot a) \geqslant \tau_2(a)$ and $\tau_3(p \cdot a) \leqslant \tau_3(a)$ for all $a \in \xi$ and for all $p \in F$.

Thus, the conditions of PFSHS are reduced to the conditions of PFSS. Hence, the study of subspace under picture fuzzy environment is a particular case of the study of SHS under picture fuzzy environment i.e. in other words, the study of SHS under picture fuzzy environment is a generalization of the study of subspace under picture fuzzy environment. This study can be extended in other types of more complicated uncertain atmosphere (picture fuzzy interval valued atmosphere, spherical fuzzy atmosphere etc.) or to other types of algebraic hyperstructures (hypergroup, hypermodules etc.).

7. Application

Vector space/hyper vector space is an important type of algebraic structure/hyperstructure. Algebraic structure/hyper structure has a lot of applications in different areas of Computer Science such as error correction, coding theory etc. In our daily life, uncertainty occurs in every now and then based on human opinions. As an uncertainty handling tool, fuzzy set was invented by Zadeh [1]. Later on to handle higher level of uncertainties, intuitionistic fuzzy set was introduced by Atanassov [17] as an extension of fuzzy set and later on picture fuzzy set was initiated by Cuong [18] as an extension of intuitionistic fuzzy set. These provide enough motivation among the researchers to study algebraic structures/hyperstructures in fuzzy and advanced fuzzy environment. Vector space deals with two compositions namely internal composition and external composition. By internal composition, we simply mean addition of vectors and by external composition we simply mean scalar multiplication with vectors. In vector space, addition of two vectors produces a vector and scalar multiplication with vector produces vector. In case of hyper vector space, scalar composition with vector produces set of vectors. This is the main key operation in hyper vector space called hyperoperation. This hyperoperation can be linked with human life incident. Brides and grooms are human beings. As a result they are of same category. So brides together with grooms can be taken as field of scalars. Successors may come as a result of composition of bride and groom. Successors may have different genetic characteristics such as body colour, eye colour, height, blood type, intelligence level etc. inherited from their parents. For a finite hyper vector space, using hyperoperative conditions of PFSHS, a bride can choose a suitable groom for marriage out of a finite number of available grooms in order to produce genetically superior successors compared with him. A finite field over that field forms a finite hyper vector space, provided that hyperoperative composition is allowed in the field. In this case, there is no distinction between scalars and vectors i.e. in this case, scalars and vectors are same. It is necessary to mention that when hyperoperative conditions are applied as PFSHS conditions then the successors are genetically superior than their parents. Now our target is to select the best couple in order to produce genetically most superior successors. This selection of best couple with respect to a particular bride/groom follows a sequence of steps. Below we give an algorithm of best couple selection for a particular bride.

Algorithm: Aim: To select the best couple for marriage with respect to a particular bride to produce genetically most superior successors.

Input: A finite set of brides and grooms as field of scalars such that PFSHS conditions are satisfied.

Output: The best couple for marriage with respect to a particular bride.

The following steps are to be followed to reach the desired goal.

Step 1: Collect data about hyperoperative composition of grooms with a particular bride from composition table.

Step 2: Calculate the least picture fuzzy measurement for each set of successors using the data collected in picture fuzzy sense based on human opinions about a specific genetic characteristics of the possible successors.

Step 3: Find the degrees of superiority of the successors.

Step 4: Arrange the degrees of superiority in descending order.

Step 5: Conclude about the selection of the best couple.

It is to be noted that one can use the same algorithm pattern for best couple selection with respect to a particular groom.

The proposed algorithm for best couple selection with respect to a particular bride/ groom can be depicted as a flowchart. Here flowchart for best couple selection with respect to a particular bride is shown in Figure 1.

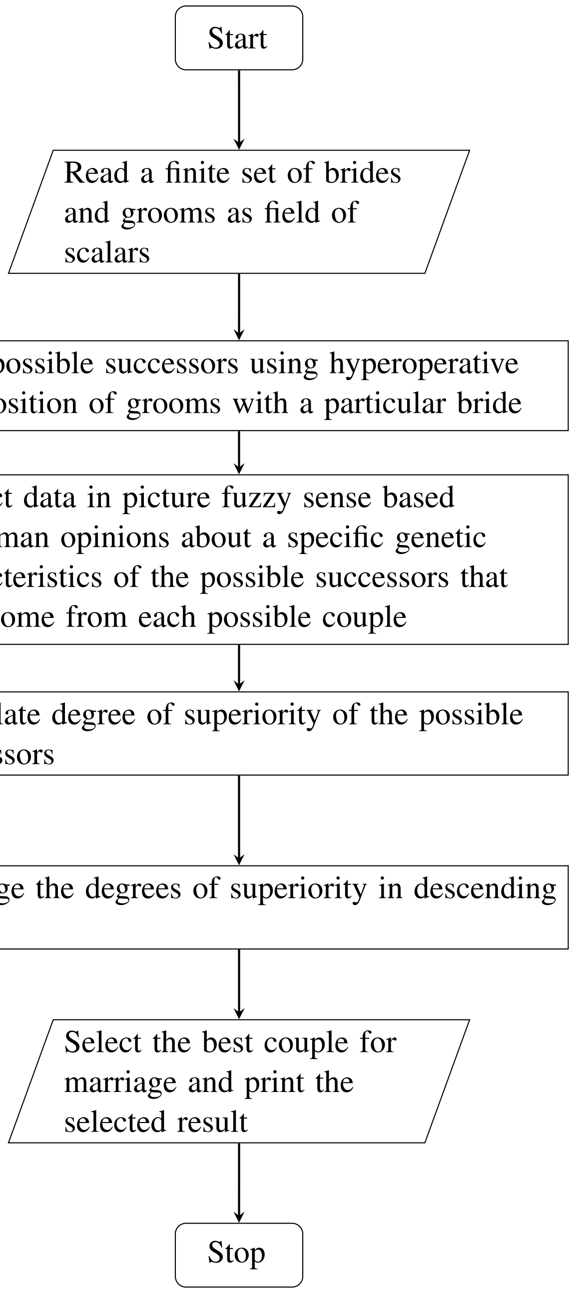

Figure 1. Flowchart of groom selection for a particular bride.

Illustration of Algorithm: Let us consider a finite field $F = \{b_1, b_2, ..., b_n,\ g_1, g_2, ..., g_n\}$, where b_i be the brides, g_i be the grooms for $i = 1, 2, ..., n$. Here $g_i \circ b_k$ (for some $k \in \{1, 2, ..., n\}$) is the set of possible successors that may come from the groom g_i (for $i = 1, 2, ..., n$) and the bride b_k. If $z \in g_i \circ b_k$ then $\tau_1(z)$ is the MPMS, $\tau_2(z)$ is the MNeuMS and $\tau_3(z)$ is the MNegMS of the successor z in connection with a specific genetic characteristics of z. Here the data about genetic characteristics is collected based on human opinions. Say $l_i^{(k)} = \inf_{z \in g_i \circ b_k} \tau_1(z) - \tau_1(b_k)$, $m_i^{(k)} = \inf_{z \in g_i \circ b_k} \tau_2(z) - \tau_2(b_k)$ and $n_i^{(k)} = \tau_3(b_k) - \sup_{z \in g_i \circ b_k} \tau_3(z)$ for $i = 1, 2, 3, ..., n$. Here $l_i^{(k)}, m_i^{(k)}, n_i^{(k)}$ are all non-negative as PFSHS conditions are satisfied. Now degree of superiority of the possible successors that may come from the groom g_i and the bride b_k is $s_i^{(k)} = l_i^{(k)} + m_i^{(k)} - n_i^{(k)}$ for $i = 1, 2, ..., n$. Arranging $s_i^{(k)}$ in descending order, we can conclude about groom selection for the bride b_k. In the similar fashion, one can conclude about bride selection for a particular groom when selection is on the hand of groom.

Now let us illustrate this method for $n = 2$.

Let us consider a finite set $F = \{b_1, b_2, g_1, g_2\}$ as a field, where b_1, b_2 be the brides ; g_1, g_2 be the grooms. Then F over F forms a finite hyper vector space under the '$+$' and '$\circ$' defined by

$+$	b_1	b_2	g_1	g_2
b_1	b_1	b_2	g_1	g_2
b_2	b_2	b_2	g_1	g_2
g_1	g_1	g_1	g_2	b_2
g_2	g_2	g_2	b_2	b_2

$\circ$	b_1	b_2	g_1	g_2
b_1	$\{b_1\}$	$\{b_1\}$	$\{b_1\}$	$\{b_1\}$
b_2	$\{b_1\}$	$\{b_2\}$	$\{b_2\}$	$\{b_2\}$
g_1	$\{b_1\}$	$\{b_2\}$	$\{g_1\}$	$\{g_2\}$
g_2	$\{b_1\}$	$\{b_2\}$	$\{g_1\}$	$\{g_2\}$

Let us consider a PFS $\tau = (\tau_1, \tau_2, \tau_3)$ over F defined by $\tau = \{(b_1, 0.62, 0.18, 0.22),$
$(b_2, 0.59, 0.16, 0.25), (g_1, 0.44, 0.11, 0.36), (g_2, 0.50, 0.14, 0.28)\}$.
Clearly τ forms a PFSHS of ξ.

Here the decision making problem can be viewed from four different angles.

Case 1: When selection is on the hand of the bride b_1.

Since PFSHS conditions are satisfied therefore it can be written as
$\Box \inf_{z \in g_1 \circ b_1} \tau_1(z) \geqslant \tau_1(b_1)$, $\inf_{z \in g_1 \circ b_1} \tau_2(z) \geqslant \tau_2(b_1)$ and $\sup_{z \in g_1 \circ b_1} \tau_3(z) \leqslant \tau_3(b_1)$.
That is, $\tau_1(b_1) \geqslant \tau_1(b_1)$, $\tau_2(b_1) \geqslant \tau_2(b_1)$ and $\tau_3(b_1) \leqslant \tau_3(b_1)$.
So $l_1^{(1)} = m_1^{(1)} = n_1^{(1)} = 0$. Thus the degree of superiority of the possible successors/next generation as a result of hyperoperative composition of the groom g_1 and the bride b_1 (when selection is on the hand of the bride b_1) is $s_1^{(1)} = l_1^{(1)} + m_1^{(1)} - n_1^{(1)} = 0$.

□ $\inf\limits_{z\in g_2\circ b_1}\tau_1(z)\geqslant\tau_1(b_1)$, $\inf\limits_{z\in g_2\circ b_1}\tau_2(z)\geqslant\tau_2(b_1)$ and $\sup\limits_{z\in g_2\circ b_1}\tau_3(z)\leqslant\tau_2(b_1)$.
That is, $\tau_1(b_1)\geqslant\tau_1(b_1)$, $\tau_2(b_1)\geqslant\tau_2(b_1)$ and $\tau_3(b_1)\leqslant\tau_3(b_1)$.
So $l_2^{(1)}=m_2^{(1)}=n_2^{(1)}=0$. Thus the degree of superiority of the possible successors/next generation as a result of hyperoperative composition of the groom g_2 and the bride b_1 (when selection is on the hand of the bride b_1) is $s_2^{(1)}=l_2^{(1)}+m_2^{(1)}-n_2^{(1)}=0$.

Here the possible successors of b_1 are genetically equivalent to b_1. Since b_1 is genetically improved (which is clear from picture fuzzy information of b_1 related to some specific genetic characteristics), so b_1 can choose any one groom for marriage out of the grooms g_1 and g_2.

Case 2: When selection is on the hand of the bride b_2.

Since PFSHS conditions are satisfied therefore it can be written as
□ $\inf\limits_{z\in g_1\circ b_2}\tau_1(z)\geqslant\tau_1(b_2)$, $\inf\limits_{z\in g_1\circ b_2}\tau_2(z)\geqslant\tau_2(b_2)$ and $\sup\limits_{z\in g_1\circ b_2}\tau_3(z)\leqslant\tau_3(b_2)$.
That is, $\tau_1(b_2)\geqslant\tau_1(b_2)$, $\tau_2(b_2)\geqslant\tau_2(b_2)$ and $\tau_3(b_2)\leqslant\tau_3(b_2)$.
So $l_1^{(2)}=m_1^{(2)}=n_1^{(2)}=0$. Thus the degree of superiority of the possible successors/next generation as a result of hyperoperative composition of the groom g_1 and the bride b_2 (when selection is on the hand of bride b_2) is $s_1^{(2)}=l_1^{(2)}+m_1^{(2)}-n_1^{(2)}=0$.

□ $\inf\limits_{z\in g_2\circ b_2}\tau_1(z)\geqslant\tau_1(b_2)$, $\inf\limits_{z\in g_2\circ b_2}\tau_2(z)\geqslant\tau_2(b_2)$ and $\sup\limits_{z\in g_2\circ b_2}\tau_3(z)\leqslant\tau_2(b_2)$.
That is, $\tau_1(b_2)\geqslant\tau_1(b_2)$, $\tau_2(b_2)\geqslant\tau_2(b_2)$ and $\tau_3(b_2)\leqslant\tau_3(b_2)$.
So $l_2^{(2)}=m_2^{(2)}=n_2^{(2)}=0$. Thus the degree of superiority of the possible successors/next generation as a result of hyperoperative composition of the groom g_2 and the bride b_2 (when selection is on the hand of bride b_2) is $s_2^{(2)}=l_2^{(2)}+m_2^{(2)}-n_2^{(2)}=0$.

Here the possible successors of b_2 are genetically equivalent to b_2. Since b_2 is genetically improved (which is clear from picture fuzzy information of b_2 related to some specific genetic characteristics), so b_2 can choose any one groom for marriage out of the grooms g_1 and g_2.

Case 3: When selection is on the hand of the groom g_1.

Since PFSHS conditions are satisfied therefore it can be written as
□ $\inf\limits_{z\in b_1\circ g_1}\tau_1(z)\geqslant\tau_1(g_1)$, $\inf\limits_{z\in b_1\circ g_1}\tau_2(z)\geqslant\tau_2(g_1)$ and $\sup\limits_{z\in b_1\circ g_1}\tau_3(z)\leqslant\tau_3(g_1)$.
That is, $\tau_1(b_1)\geqslant\tau_1(g_1)$, $\tau_2(b_1)\geqslant\tau_2(g_1)$ and $\tau_3(b_1)\leqslant\tau_3(g_1)$.
That is, $0.62\geqslant0.44$, $0.18\geqslant0.11$ and $0.22\leqslant0.36$.
So $l_1^{(1)}=0.62-0.44=0.18$, $m_1^{(1)}=0.18-0.11=0.07$ and $n_1^{(1)}=0.36-0.22=0.14$. Thus the degree of superiority of the possible successors/next generation as a result of hyperoperative composition of the bride b_1 and the groom g_1 (when selection is on the hand of the groom g_1) is $s_1^{(1)}=l_1+m_1-n_1=0.18+0.07-0.14=0.11$.

□ $\inf\limits_{z\in b_2\circ g_1}\tau_1(z)\geqslant\tau_1(g_1)$, $\inf\limits_{z\in b_2\circ g_1}\tau_2(z)\geqslant\tau_2(g_1)$ and $\sup\limits_{z\in b_2\circ g_1}\tau_3(z)\leqslant\tau_2(g_1)$.

That is, $\tau_1(b_2) \geqslant \tau_1(g_1)$, $\tau_2(b_2) \geqslant \tau_2(g_1)$ and $\tau_3(b_2) \leqslant \tau_3(g_1)$.
That is, $0.59 \geqslant 0.44$, $0.16 \geqslant 0.11$ and $0.25 \leqslant 0.36$.
So $l_2^{(1)} = 0.59 - 0.44 = 0.15$, $m_2^{(1)} = 0.16 - 0.11 = 0.05$ and $n_2^{(1)} = 0.36 - 0.25 = 0.11$. Thus the degree of superiority of the possible successors/next generation as a result of hyperoperative composition of the bride b_2 and the groom g_1 (when selection is on the hand of the groom g_1) is $s_2^{(1)} = l_2^{(1)} + m_2^{(1)} - n_2^{(1)} = 0.15 + 0.05 - 0.11 = 0.09$.

Since $s_1^{(1)} > s_2^{(1)}$, therefore the groom g_1 can select the bride b_1 for marriage out of the brides b_1 and b_2.

Case 4: When selection is on the hand of the groom g_2.

Since PFSHS conditions are satisfied therefore it can be written as
□ $\inf\limits_{z \in b_1 \circ g_2} \tau_1(z) \geqslant \tau_1(g_2)$, $\inf\limits_{z \in b_1 \circ g_2} \tau_2(z) \geqslant \tau_2(g_2)$ and $\sup\limits_{z \in b_1 \circ g_2} \tau_3(z) \leqslant \tau_3(g_2)$.
That is, $\tau_1(b_1) \geqslant \tau_1(g_2)$, $\tau_2(b_1) \geqslant \tau_2(g_2)$ and $\tau_3(b_1) \leqslant \tau_3(g_2)$.
That is, $0.62 \geqslant 0.50$, $0.18 \geqslant 0.14$ and $0.22 \leqslant 0.28$.
So $l_1^{(2)} = 0.62 - 0.50 = 0.12$, $m_1^{(2)} = 0.18 - 0.14 = 0.04$ and $n_1^{(2)} = 0.28 - 0.22 = 0.06$. Thus the degree of superiority of the possible successors/next generation as a result of hyperoperative composition of the bride b_1 and the groom g_2 (when selection is on the hand of groom g_2) is $s_1^{(2)} = l_1^{(2)} + m_1^{(2)} - n_1^{(2)} = 0.12 + 0.04 - 0.06 = 0.10$.

□ $\inf\limits_{z \in b_2 \circ g_2} \tau_1(z) \geqslant \tau_1(g_2)$, $\inf\limits_{z \in b_2 \circ g_2} \tau_2(z) \geqslant \tau_2(g_2)$ and $\sup\limits_{z \in b_2 \circ g_2} \tau_3(z) \leqslant \tau_3(g_2)$.
That is, $\tau_1(b_2) \geqslant \tau_1(g_2)$, $\tau_2(b_2) \geqslant \tau_2(g_2)$ and $\tau_3(b_2) \leqslant \tau_3(g_2)$.
That is, $0.59 \geqslant 0.50$, $0.16 \geqslant 0.14$ and $0.25 \leqslant 0.28$.
So $l_2^{(2)} = 0.59 - 0.50 = 0.09$, $m_2^{(2)} = 0.16 - 0.14 = 0.02$ and $n_2^{(2)} = 0.28 - 0.25 = 0.03$. Thus the degree of superiority of the possible successors/next generation as a result of hyperoperative composition of the groom b_2 and the bride g_2 (when selection is on the hand of groom g_2) is $s_2^{(2)} = l_2^{(2)} + m_2^{(2)} - n_2^{(2)} = 0.09 + 0.02 - 0.03 = 0.08$.

Since $s_1^{(2)} > s_2^{(2)}$, therefore the groom g_2 can select the bride b_1 for marriage out of the brides b_1 and b_2.

So the best couple for marriage with respect to a particular bride/groom is listed below.

When selection is on the hand of	Selected best couple
b_1	any of $b_1 - g_1$ or $b_1 - g_2$
b_2	any of $b_2 - g_1$ or $b_2 - g_2$
g_1	$g_1 - b_1$
g_2	$g_2 - b_1$

Limitations of the Proposed Method: Note that when for a particular bride b_k (for some $k \in \{1, 2, ..., n\}$), all $s_i^{(k)} = 0$ for $i = 1, 2, ..., n$; then possible successors are genetically equivalent to b_k. In this case, our algorithm does not work. For decision making, we have to look at the picture fuzzy information of b_k related to some specific genetic characteristics of b_k. Moreover, for large domain, n

becomes large and the composition table becomes little difficult. So careful data handling is to be done in order to satisfy PFSHS conditions. This method can only be used in real life applications where composition concept arises in practical situation. These are some of the limitations of our proposed method. Despite these limitations, we think that our method will add a new dimension not only in the field of decision making but also in the field of algebra because decision making has been done here from algebraic point of view.

8. Conclusions

In this paper, we have introduced PFSHS of a HVS and studied some elementary properties in connection with some basic operations (intersection, union, Cartesian product etc.) on picture fuzzy sets. We have discussed the effect of good LT on PFSHS. We have proved that image of a PFSHS is a PFSHS under bijective good LT and inverse image of a PFSHS is a PFSHS under good LT. We have initiated the notion of PFLT with respect to some pre-assumed PFSHS. We have shown that linear combination of two PFLTs is a PFLT. This proposition merges two results: (1) The sum of two PFLTs is a PFLT and (ii) scalar multiplication with PFLT is a PFLT. Finally, we have presented an application of PFSHS condition in decision making problem. As a result of our study, the researchers will able to justify the validity of the results on sub-hyperspace as a certain case of our study. This study can be treated as the study of a special type of advanced fuzzy hyper algebraic structure. This study opens a new window for the researchers who are interested to study more about sub-hyperspace in picture fuzzy setting. Moreover, investigation about sub-hyperspace under some other types of uncertain environment will be easy for the researchers who will go through this research work. Researchers will be able to solve different types of transportation problems, linear programming problems using picture fuzzy set algorithm in those cases where fuzzy set algorithm is unable to provide fruitful solution. Our method is so much interesting due to its application in decision making field from algebraic point of view.

Acknowledgements

This work is supported by Research Council Faroe Islands and University of the Faroe Islands for the third author. The authors are grateful to the anonymous referees for careful checking of the details and for giving helpful comments towards the improvement of the overall presentation of the paper.

Conflict of interest

The authors declare that they have no conflict of interest.

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