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Research article

Comparative analysis for fractional nonlinear Sturm-Liouville equations with singular and non-singular kernels

Ahu Ercan*

Department of Mathematics, Science Faculty, Firat University, Elazig 23119, Turkey

* **Correspondence:** Email: ahuduman24@gmail.com.

Abstract: This article presents the Laplace-Adomian decomposition method (LADM), which produces a fast convergence series solution, for two types of nonlinear fractional Sturm-Liouville (SL) problems. The fractional derivatives are defined in the Caputo, conformable, Caputo-Fabrizio in the sense of Caputo (CFC), Caputo type Atangana-Baleanu (ABC) senses. With the help of this method, approximate solutions of the investigated problems were obtained. The solutions generated from the Caputo and ABC derivatives are represented by the Mittag-Leffler function, which is intrinsic to fractional derivatives, and the solution obtained using the conformable and CFC derivatives generate the hyperbolic sine and cosine functions. Thus, we derive some novel solutions for fractional-order versions of nonlinear SL equations. The fractional calculus provides more data than classical calculus and has been widely used in mathematical modeling with memory effect. Finally, we analyzed and compared these novel solutions of the considered problems by graphs under different values of p, λ and different orders of α.

Keywords: nonlinear Sturm-Liouville equation; Laplace-Adomian decomposition method; Caputo; conformable; Caputo-Fabrizio; Atangana-Baleanu fractional derivatives
Mathematics Subject Classification: 34A08, 34A45, 34L30

1. Introduction

In the last years, fractional calculus has been one of the topics of interest to many scientists due to its usefulness in its application to real-world problems and its comparative results [35]. This situation has led to the acquisition of many new derivatives such as the Riemann-Liouville, Caputo, Grünwald-Letnikov, Hilfer, Hilfer-Prabhakar, conformable, Atangana-Baleanu and Caputo-Fabrizio.

Although these definitions are used by many scientists, there are some limiting deficiencies. Some analysis rules, such as the derivatives of the product of functions and the quotient, the Leibniz rule, and the chain rule, do not work in these derivatives. However, the rules mentioned above work in the conformable derivative due to its definition. On the other hand, the Caputo fractional derivative is convenient because it allows the use of conventional initial and boundary conditions despite having a singular kernel. The singular kernel in the Caputo fractional derivative imposes constraints on some real world problem modeling. Therefore, by replacing the singular kernel in the Caputo derivative definition with exponential and Mittag-Leffler functions, Caputo-Fabrizio and Atangana-Baleanu derivative definitions were given, respectively. This type of change improves the quality of the computation and results [1, 2, 4, 7–9, 13, 16, 23, 27, 33].

The Sturm-Liouville equation is defined by

$$-\frac{d}{dt}\left(w(t)\frac{dy}{dt}\right)+q(t)y=\lambda r(t)y,\quad a\leq t\leq b,$$

where $w(t)$, $w'(t)$, $q(t)$ and $r(t)$ are continuous functions over $[a,b]$, λ is a spectral parameter and $w(t), r(t) > 0$. In nonlinear eigenvalue problems, linearization of the problem around zero, that is, the Fréchet derivative at the origin, plays an important role (see [28]). In terms of this linearizability, a nonlinear version of the classical results for the linear Sturm-Liouville equations is given by Rabinowitz [36]. A nonlinear Sturm-Liouville eigenvalue problem has made important contributions to the modeling of many physical problems [14, 15]. It is very convenient in terms of applicability to various problems, including electromagnetic waves in a resonant cavity, heat conduction, and vibration. The aim of this study is to examine nonlinear SL problems in terms of fractional derivatives and obtain approximate solutions by using LADM. The considered problems need not have any linearization, thanks to the method used. Much work has been done on the theory of the SL problem, and many results have been obtained regarding eigenvalues and corresponding eigenfunctions [30, 31]. It should be noted that many numerical algorithms have been produced to search for approximate solutions, as finding analytical solutions for this problem is a cumbersome job. The fractional SL problem was studied [3, 18, 19, 34]. Many numerical and analytical methods have been proposed to solve nonlinear fractional differential equations: The fractional-order Legendre Tau method [29], homotopy perturbation transform method [24, 28], reproducing kernel method [37], homotopy perturbation method [25], homotopy asymptotic method [38], Adomian descomposition method (ADM) [5, 6, 17], LADM [10–12, 20, 26, 32], extended Laplace transform method (ELTM) [21, 22], etc. The ADM is one of the useful methods that provides efficient algorithms for obtaining approximate or analytical solutions to real world problems. There is also a forceful hybrid method that combines the Laplace transform with the Adomian decomposition method, called the Laplace-Adomian decomposition method (LADM). The most beneficial aspect of this method is its stretch to provide approximate or analytical solutions to non-linear or linear equations and the freedom of small or large parameters. This method generates an analytical solution in the form of a polynomial. This method does not consume much computer time when applied to nonlinear differential equations. The method is very useful for physical problems, as it does not need perturbation, linearization or other constraining methods or assumptions that change the physical state being solved. Also, ELTM, a technique for extending LADM, makes it possible to solve nonlinear differential equations. Unlike LADM, this technique has developed several theorems involving Adomian polynomials and Adomian and Rach theorem of transformation of series, and thus, the Laplace transform of nonlinear expressions is made possible.

The reason for using fractional operators called Caputo, conformable, CFC and ABC in the examined problems is that the derivative of the constant function is zero, and the fractional versions of the operator can be examined with traditional initial conditions. Moreover, CFC and ABC fractional derivatives having non-singular kernel ensure a non-rigid degree to choose appropriate values for the fractional-order parameter and lets us achieve more detailed results than their classical ones.

The improving results of this study can be listed as follows: Approximate solutions of fractional nonlinear SL equations are presented in generalized versions. An effective numerical method is provided to solve fractional nonlinear SL problems. Simulation analysis for considered problems is discussed by means of graphs. LADM is applied to fractional nonlinear SL problems of the types we examined for the first time.

The rest of the article is organized as follows: Section 2 presents some fundamental concepts to shed light on the results obtained in the following parts. Afterwards, by using LADM, approximate solutions of nonlinear SL problems have been obtained by means of Caputo, conformable, CFC and ABC fractional derivatives in Section 3. Section 4 contains two examples of the nonlinear SL problems in the frame of ABC and CFC derivatives for p=2. In Section 5, some important discussions are proposed by means of graphs. Finally, Section 6 summarizes all the important discoveries of this study.

2. Preliminaries

This section presents some basic definitions and theorems about the Caputo, conformable, ABC and CFC fractional derivatives.

Definition 2.1. *[35] The Liouville-Caputo fractional derivative is defined by*

$$ {}_{0}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \quad n-1<\alpha\leq n. $$

Definition 2.2. *[1] Let $f:[a,\infty)\to\mathbb{R}$ and $t>0$. The left fractional conformable derivative of f of order α is given by*

$$ (D_{\alpha}^{a} f)(t) = \lim_{\varepsilon\to 0} \frac{f\left(t+\varepsilon (t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}, \ 0<\alpha\leq 1. $$

If, in addition, f is differentiable, then $T_{\alpha}^{a}(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Definition 2.3. *[16] The left sided fractional derivative in Caputo sense with exponential kernel is defined by*

$$ {}_{a}^{CFC}D^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(s) \exp\left(\frac{-\alpha}{1-\alpha}(t-s)\right) ds, $$

where $M(\alpha)>0$ is a normalization function with $M(0)=M(1)=1$, $f\in H^1(a,b)$ and $\alpha\in[0,1]$.

Definition 2.4. *[9] The left sided Atangana-Baleanu fractional derivative in the Caputo sense is given by*

$$ {}_{a}^{ABC}D^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t f'(s) E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-s)^{\alpha}\right) ds, $$

where $B(\alpha)>0$ is a normalization function with $B(0)=B(1)=1$, $f\in H^1(a,b)$ and $\alpha\in[0,1]$.

Theorem 2.1. *[16] The Laplace transform (LT) of CFC fractional derivative is defined by*

$$\mathcal{L}\left\{\left({}^{CFC}_{a}D^{\alpha}f\right)(t)\right\}(s)=\frac{M(\alpha)}{1-\alpha}\frac{s\mathcal{L}\{f(t)\}(s)}{s+\frac{\alpha}{1-\alpha}}-\frac{M(\alpha)}{1-\alpha}f(a)e^{-as}\frac{1}{s+\frac{\alpha}{1-\alpha}}.$$

Theorem 2.2. *[8] The LT with Mittag-Leffler kernel is defined by*

$$\mathcal{L}\left\{{}^{ABC}_{a}D^{\alpha}f(t)\right\}(s)=\frac{B(\alpha)}{1-\alpha}\frac{s\mathcal{L}\{f(t)\}(s)-s^{\alpha-1}f(a)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}.$$

Definition 2.5. *[35] The Mittag-Leffler function $E_{\delta}(z)$ is defined by*

$$E_{\delta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\delta k+1)},\ z\in\mathbb{C},\ Re(\delta)>0,$$

and the Mittag-Leffler function with two parameters is defined by

$$E_{\delta,\theta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\delta k+\theta)},\ z,\theta\in\mathbb{C},\ Re(\delta)>0.$$

3. Main results

In this section, LADM is discussed for two types of nonlinear Sturm-Liouville equations by means of the Caputo, conformable, CFC and ABC fractional derivatives. The reason for using fractional operators called Caputo, conformable, CFC and ABC in the examined problems is that the derivative of the constant function is zero, and the fractional versions of the operator can be examined with traditional initial conditions. So, they can be easily applied for real-world problems. Riemann-Liouville (RL) fractional derivative is not well-suited since it requires the initial conditions in the RL sense.

Theorem 3.1. *Consider the nonlinear SL equation with the Caputo derivative:*

$$-{}^{C}D_{0}^{\alpha}{}^{C}D_{0}^{\alpha}y(x)+y^{p}(x)=\lambda y(x),\ x\in(0,1),\ \frac{1}{2}<\alpha\leq 1, \tag{3.1}$$

subject to initial conditions

$$y(0)=0,\ \ D_{0}^{\alpha}y(x)\big|_{x=0}=1, \tag{3.2}$$

where $p>0$ is a real constant, λ is a spectral parameter. Then, the approximate solution is given by

$$\begin{aligned} y(x)=&x^{\alpha}E_{2\alpha,\alpha+1}\left(-\lambda x^{2\alpha}\right)+\frac{x^{\alpha(p+2)}\Gamma(1+\alpha p)}{(\Gamma(1+\alpha))^{p}\Gamma(1+\alpha(p+2))}\\ &+\frac{px^{\alpha(3+2p)}(\Gamma(1+\alpha))^{1-2p}\Gamma(1+\alpha p)\Gamma(1+\alpha+2\alpha p)}{\Gamma(1+\alpha(3+2p))\Gamma(1+\alpha(2+p))}+\dots\,. \end{aligned} \tag{3.3}$$

Proof. Applying the LT to Eq (3.1), we find that

$$\mathcal{L}\{y(x)\}=\frac{s^{2\alpha-1}}{s^{2\alpha}}y(0)+\frac{s^{\alpha-1}}{s^{2\alpha}}D_{0+}^{\alpha}y(x)\big|_{x=0}+\frac{1}{s^{2\alpha}}\mathcal{L}\{y^{p}(x)-\lambda y(x)\}. \tag{3.4}$$

In order to obtain $y(x)$, we apply the Adomian iterative scheme as follows:

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{3.5}$$

Then, by using Eq (3.5), we may rewrite Eq (3.4) as

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} y_n(x)\right\} = \frac{s^{2\alpha-1}}{s^{2\alpha}} y(0) + \frac{s^{\alpha-1}}{s^{2\alpha}} \left. D_{0+}^{\alpha} y(x)\right|_{x=0} + \frac{1}{s^{2\alpha}} \mathcal{L}\left\{\sum_{n=0}^{\infty} A_n - \lambda \sum_{n=0}^{\infty} y_n(x)\right\}, \tag{3.6}$$

where A_n is an Adomian polynomial, which represents the nonlinear term $y^p(x)$ as follows:

$$Ny(x) = \sum_{n=0}^{\infty} A_n,$$

and indicated by the following series [17]:

$$A_n = \frac{1}{n!} \frac{d^n}{d\mu^n} \left[N\left(\sum_{i=0}^{\infty} \mu^i y_i(x) \right) \right]_{\mu=0}. \tag{3.7}$$

Comparing both sides of Eq (3.6), we can easily write the first term of the series as

$$\mathcal{L}\{y_0(x)\} = \frac{s^{2\alpha-1}}{s^{2\alpha}} y(0) + \frac{s^{\alpha-1}}{s^{2\alpha}} \left. D_{0+}^{\alpha} y(x)\right|_{x=0}, \tag{3.8}$$

and we obtain the general recursive relation as

$$\mathcal{L}\{y_{n+1}(x)\} = \frac{1}{s^{2\alpha}} \mathcal{L}\{A_n - \lambda y_n(x)\}, \quad n \geq 0, \tag{3.9}$$

in which A_n represents the Adomian polynomial given by

$$\begin{aligned} A_0 &= N(y_0), \\ A_1 &= y_1 N'(y_0), \\ A_2 &= y_2 N'(y_0) + \frac{y_1^2}{2} N''(y_0), \\ A_3 &= y_3 N'(y_0) + y_1 y_2 N''(y_0) + \frac{y_1^3}{3!} N'''(y_0), \\ &\;\;. \\ &\;\;. \\ &\;\;. \end{aligned}$$

Then, by applying the inverse LT of Eq (3.8), we find that

$$\begin{aligned} y_0(x) &= \mathcal{L}^{-1}\left\{\frac{s^{2\alpha-1}}{s^{2\alpha}} y(0)\right\} + \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{2\alpha}} \left. D_{0+}^{\alpha} y(x)\right|_{x=0}\right\} \\ &= \frac{x^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned}$$

Let us continue by taking the LT on both sides of Eq (3.9) for $n = 0$ as follow:

$$\mathcal{L}\{y_1(x)\} = \frac{1}{s^{2\alpha}}\mathcal{L}\left\{\frac{x^{\alpha p}}{(\Gamma(\alpha+1))^p}\right\} - \frac{\lambda}{s^{2\alpha}}\mathcal{L}\left\{\frac{x^{\alpha}}{\Gamma(\alpha+1)}\right\}.$$

Afterwards, by applying the inverse LT on both sides of the last equality, we find that

$$y_1(x) = \frac{x^{\alpha(p+2)}\Gamma(1+\alpha p)}{(\Gamma(1+\alpha))^p\Gamma(1+\alpha(p+2))} - \frac{\lambda x^{3\alpha}}{\Gamma(1+3\alpha)}.$$

Then, by using the recursive relation formed in Eq (3.9), we can easily calculate the remaining terms of the function $y(x)$ as follows:

$$\begin{aligned} y_2(x) &= \lambda^2\frac{x^{5\alpha}}{\Gamma(1+5\alpha)} + \frac{p x^{\alpha(3+2p)}(\Gamma(1+\alpha))^{1-2p}\Gamma(1+\alpha p)\Gamma(1+\alpha+2\alpha p)}{\Gamma(1+\alpha(3+2p))\Gamma(1+\alpha(2+p))} \\ &\quad - \frac{\lambda p x^{\alpha(4+p)}(\Gamma(1+\alpha))^{1-p}\Gamma(1+\alpha(2+p))}{\Gamma(1+3\alpha)\Gamma(1+\alpha(4+p))} - \frac{\lambda x^{\alpha(4+p)}\Gamma(1+\alpha p)}{(\Gamma(1+\alpha))^p\Gamma(1+\alpha(4+p))}. \end{aligned}$$

So, the approximate solution of $y(x)$ is given by

$$\begin{aligned} y(x) &= y_0 + y_1 + y_2 + ... \\ &= x^{\alpha}\sum_{k=0}^{\infty}\frac{\left(-\lambda x^{2\alpha}\right)^k}{\Gamma(2\alpha k+\alpha+1)} + \frac{x^{\alpha(p+2)}\Gamma(1+\alpha p)}{(\Gamma(1+\alpha))^p\Gamma(1+\alpha(p+2))} \\ &\quad + \frac{p x^{\alpha(3+2p)}(\Gamma(1+\alpha))^{1-2p}\Gamma(1+\alpha p)\Gamma(1+\alpha+2\alpha p)}{\Gamma(1+\alpha(3+2p))\Gamma(1+\alpha(2+p))} \\ &\quad - \frac{\lambda p x^{\alpha(4+p)}(\Gamma(1+\alpha))^{1-p}\Gamma(1+\alpha(2+p))}{\Gamma(1+3\alpha)\Gamma(1+\alpha(4+p))} + ... \end{aligned}$$

From here, we arrive at Eq (3.3), which is the generalized version of the solution. □

Theorem 3.2. *The following nonlinear SL equation in the Caputo sense is analyzed:*

$$-{}^{C}D_0^{\alpha\,C}D_0^{\alpha}y(x) + e^{y(x)} = \lambda y(x),\ x\in(0,1),\ \frac{1}{2}<\alpha\le 1, \tag{3.10}$$

with the initial conditions

$$y(0) = 1,\ \ D_0^{\alpha}y(x)\Big|_{x=0} = 0. \tag{3.11}$$

So, the approximate solution of $y(x)$ *is given by*

$$y(x) = E_{2\alpha}\left((e-\lambda)x^{2\alpha}\right) + \frac{1}{2}(e-\lambda)^2 e\frac{x^{6\alpha}\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^2\Gamma(6\alpha+1)} + ... \tag{3.12}$$

Proof. Applying the LT to Eq (3.10), we find that

$$\mathcal{L}\{y(x)\} = \frac{s^{2\alpha-1}}{s^{2\alpha}}y(0) + \frac{s^{\alpha-1}}{s^{2\alpha}}D_{0+}^{\alpha}y(x)\Big|_{x=0} + \frac{1}{s^{2\alpha}}\mathcal{L}\left\{e^{y(x)} - \lambda y(x)\right\}. \tag{3.13}$$

In order to obtain $y(x)$, we apply the Adomian iterative scheme as follows:

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \ N = \sum_{n=0}^{\infty} A_n. \tag{3.14}$$

Then, substituting (3.14) into (3.13), we obtain

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} y_n(x)\right\} = \frac{s^{2\alpha-1}}{s^{2\alpha}} y(0) + \frac{s^{\alpha-1}}{s^{2\alpha}} \ D_{0+}^{\alpha} y(x)\Big|_{x=0} + \frac{1}{s^{2\alpha}} \mathcal{L}\left\{\sum_{n=0}^{\infty} A_n - \lambda \sum_{n=0}^{\infty} y_n(x)\right\}. \tag{3.15}$$

We can easily write the first term of the series in (3.15) as

$$\mathcal{L}\{y_0(x)\} = \frac{s^{2\alpha-1}}{s^{2\alpha}} y(0) + \frac{s^{\alpha-1}}{s^{2\alpha}} \ D_{0+}^{\alpha} y(x)\Big|_{x=0}, \tag{3.16}$$

and the general recursive relation is given by

$$\mathcal{L}\{y_{n+1}(x)\} = \frac{1}{s^{2\alpha}} \mathcal{L}\{A_n - \lambda y_n(x)\}, \quad n \geq 0. \tag{3.17}$$

By applying the inverse LT to both sides of Eq (3.16), we find that

$$y_0(x) = \mathcal{L}^{-1}\left\{\frac{s^{2\alpha-1}}{s^{2\alpha}} y(0)\right\} + \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{2\alpha}} \ D_{0+}^{\alpha} y(x)\Big|_{x=0}\right\} = 1.$$

From the general recursive relation in Eq (3.17) for $n = 0, 1, 2$, we obtain

$$\mathcal{L}\{y_1(x)\} \ = \ \frac{1}{s^{2\alpha}} \mathcal{L}\{e^{y_0}\} - \frac{1}{s^{2\alpha}} \mathcal{L}\{\lambda y_0(x)\}.$$

By applying the inverse LT, we obtain

$$y_1(x) = (e - \lambda) \frac{x^{2\alpha}}{\Gamma(1 + 2\alpha)}.$$

Considering Eq (3.17) for $n = 1$, we get

$$\mathcal{L}\{y_2(x)\} \ = \ \frac{1}{s^{2\alpha}} \mathcal{L}\{e^{y_0} y_1\} - \frac{1}{s^{2\alpha}} \mathcal{L}\{\lambda y_1(x)\}.$$

By applying inverse LT, we obtain

$$y_2(x) = (e - \lambda)^2 \frac{x^{4\alpha}}{\Gamma(1 + 4\alpha)}.$$

Consider the iteration in (3.17) for $n = 2$, then we have

$$\begin{aligned} \mathcal{L}\{y_3(x)\} &= \frac{1}{s^{2\alpha}} \mathcal{L}\left\{e^{y_0} y_2 + \frac{1}{2} e^{y_0} y_1^2\right\} - \frac{1}{s^{2\alpha}} \mathcal{L}\{\lambda y_2(x)\} \\ &= \frac{e(e-\lambda)^2}{s^{1+6\alpha}} + \frac{e(e-\lambda)^2 \Gamma(1+4\alpha)}{2 s^{1+6\alpha} (\Gamma(1+2\alpha))^2} - \frac{\lambda(e-\lambda)^2}{s^{1+6\alpha}}. \end{aligned}$$

By applying the inverse LT, we find that

$$y_3(x) = (e-\lambda)^3 \frac{x^{6\alpha}}{\Gamma(1+6\alpha)} + \frac{1}{2} e (e-\lambda)^2 \frac{x^{6\alpha}\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^2 \Gamma(1+6\alpha)}.$$

Similarly, by applying the scheme, $y(x)$ is obtained as

$$\begin{aligned} y(x) &= y_0 + y_1 + y_2 + y_3 + ... \\ &= \sum_{k=0}^{\infty} \frac{\left((e-\lambda) x^{2\alpha}\right)^k}{\Gamma(2\alpha k+1)} + \frac{1}{2}(e-\lambda)^2 e \frac{x^{6\alpha}\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^2 \Gamma(6\alpha+1)} + ... \end{aligned}$$

From here, we arrive at Eq (3.12), which is the generalized version of the solution. □

Theorem 3.3. *Consider nonlinear Sturm-Liouville equations given by*

$$-D_0^\alpha D_0^\alpha y(x) + y^p(x) = \lambda y(x), \; x \in (0,1), \; \frac{1}{2} < \alpha \leq 1, \tag{3.18}$$

with initial conditions

$$y(0) = 0, \;\; D_{0+}^\alpha y(x)\big|_{x=0} = 1, \tag{3.19}$$

in which D^α is a conformable derivative, $p > 0$ is a real constant, and λ is a spectral parameter. Thus, the approximate solution is given by

$$y(x) = \frac{1}{\sqrt{\lambda}} \sinh\left(\sqrt{\lambda} x\right) + \frac{x^{p+2}\Gamma^2(1+p)}{\Gamma(3+p)} + \frac{p x^{3+2p}\Gamma^2(1+p)\Gamma(2+2p)}{\Gamma(3+p)\Gamma(4+2p)} + ... \tag{3.20}$$

Proof. Applying the LT to Eq (3.18), we have

$$\mathcal{L}_\alpha \{y(x)\} = \frac{1}{s} y(0) + \frac{1}{s^2} D_{0+}^\alpha y(x)\big|_{x=0} + \frac{1}{s^2} \mathcal{L}_\alpha \{y^p(x) - \lambda y(x)\}. \tag{3.21}$$

Now, assume the solution $y(x)$ in the series form as follows:

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \; Ny(x) = \sum_{n=0}^{\infty} A_n. \tag{3.22}$$

Substituting (3.22) to (3.21), we find that

$$\mathcal{L}_\alpha \left\{ \sum_{n=0}^{\infty} y_n(x) \right\} = \frac{1}{s} y(0) + \frac{1}{s^2} D_{0+}^\alpha y(x)\big|_{x=0} + \frac{1}{s^{2\alpha}} \mathcal{L} \left\{ \sum_{n=0}^{\infty} A_n - \lambda \sum_{n=0}^{\infty} y_n(x) \right\}. \tag{3.23}$$

The first term of the series in (3.23) is given by

$$\begin{aligned} y_0(x) &= \mathcal{L}_\alpha^{-1} \left\{ \frac{1}{s} y(0) + \frac{1}{s^2} D_{0+}^\alpha y(x)\big|_{x=0} \right\} \\ &= x, \end{aligned}$$

and the general recursive relation is given by

$$\mathcal{L}_\alpha\left\{y_{n+1}(x)\right\} = \frac{1}{s^2}\mathcal{L}_\alpha\left\{A_n - \lambda y_n(x)\right\}, \quad n \geq 0. \tag{3.24}$$

Applying the inverse LT to the recursive relation formed in Eq (3.24), we can easily calculate the remaining terms of the function $y(x)$ for $n = 0, 1, 2$ as follows:

$$y_1(x) = \frac{x^{p+2}\Gamma^2(1+p)}{\Gamma(3+p)} - \frac{\lambda x^3}{3!},$$

and

$$y_2(x) = \frac{\lambda^2 x^5}{5!} + \frac{p x^{3+2p}\Gamma^2(1+p)\Gamma(2+2p)}{\Gamma(3+p)\Gamma(4+2p)} - \frac{\lambda p x^{4+p}\Gamma(3+p)}{6\Gamma(5+p)} - \frac{\lambda x^{4+p}\Gamma^2(1+p)}{\Gamma(5+p)}.$$

Therefore, the approximate solution of $y(x)$ is given in the following form:

$$\begin{aligned} y(x) &= y_0 + y_1 + y_2 + ... \\ &= x + \frac{\lambda x^3}{3!} + \frac{\lambda^2 x^5}{5!} + ... + \frac{x^{p+2}\Gamma^2(1+p)}{\Gamma(3+p)} + \frac{p x^{3+2p}\Gamma^2(1+p)\Gamma(2+2p)}{\Gamma(3+p)\Gamma(4+2p)} + ... \\ &= \frac{1}{\sqrt{\lambda}}\sum_{k=0}^{\infty}\frac{\left(\sqrt{\lambda}x\right)^{2k+1}}{(2k+1)!} + \frac{x^{p+2}\Gamma^2(1+p)}{\Gamma(3+p)} + \frac{p x^{3+2p}\Gamma^2(1+p)\Gamma(2+2p)}{\Gamma(3+p)\Gamma(4+2p)} \\ &\quad - \frac{\lambda p x^{4+p}\Gamma(3+p)}{6\Gamma(5+p)} - \frac{\lambda x^{4+p}\Gamma^2(1+p)}{\Gamma(5+p)} + ... \end{aligned}$$

From here, we arrive at Eq (3.20), which is the generalized version of the solution. □

Theorem 3.4. *Consider the second type nonlinear SL problem in the conformable sense as follows:*

$$-D_0^\alpha D_0^\alpha y(x) + e^{y(x)} = \lambda y(x), \ x \in (0,1), \ \frac{1}{2} < \alpha \leq 1, \tag{3.25}$$

subject to initial conditions

$$y(0) = 1, \ \ D_{0+}^\alpha y(x)\Big|_{x=0} = 0. \tag{3.26}$$

Then, the approximate solution is given in the form:

$$y(x) = \cosh\left(\left(\sqrt{e-\lambda}\right)x\right) + \frac{3e(e-\lambda)^2 x^6}{6!} + ...$$

Proof. The proof is straightforward from the proof of Theorem 3.3. □

Theorem 3.5. *Consider the nonlinear fractional Sturm-Liouville equations given by*

$$-{}^{CFC}D_0^{\alpha\,CFC}D_0^\alpha y(x) + y^p(x) = \lambda y(x), \ x \in (0,1), \ \frac{1}{2} < \alpha \leq 1, \tag{3.27}$$

with initial conditions

$$y(0) = 0, \ \ D_0^\alpha y(x)\Big|_{x=0} = 1, \tag{3.28}$$

in which ${}^{CFC}D^{\alpha}$ is a differential operator of order α in the Caputo-Fabrizio in the sense of Caputo, $p > 0$ is a real constant, and λ is a spectral parameter. Therefore, we obtain the required solution as

$$y_0(x) = \frac{1-\alpha+\alpha x}{M(\alpha)},$$

and for $n \geq 0$,

$$y_{n+1}(x) = \mathcal{L}^{-1}\left\{\frac{(s(1-\alpha)+\alpha)^2}{s^2M^2(\alpha)}\mathcal{L}\{A_n\}\right\} - \mathcal{L}^{-1}\left\{\frac{\lambda(s(1-\alpha)+\alpha)^2}{s^2M^2(\alpha)}\mathcal{L}\{y_n(x)\}\right\}. \tag{3.29}$$

Proof. Applying the LT to Eq (3.27), we have

$$\begin{aligned}\mathcal{L}\{y(x)\} &= \frac{(s(1-\alpha)+\alpha)}{s^2M(\alpha)}\,{}^{CF}D^{\alpha}_{0+}y(x)\big|_{x=0} + \frac{(s(1-\alpha)+\alpha)^2}{s^2M^2(\alpha)}\mathcal{L}\{y^p(x)\}\\ &\quad -\frac{\lambda(s(1-\alpha)+\alpha)^2}{s^2M^2(\alpha)}\mathcal{L}\{y(x)\}.\end{aligned}$$

Now, assume the solution $y(x)$ in the series form as follows:

$$\begin{aligned}\mathcal{L}\left\{\sum_{n=0}^{\infty}y_n(x)\right\} &= \frac{(s(1-\alpha)+\alpha)}{s^2M(\alpha)}\,{}^{CF}D^{\alpha}_{0+}y(x)\big|_{x=0} + \frac{(s(1-\alpha)+\alpha)^2}{s^2M^2(\alpha)}\mathcal{L}\left\{\sum_{n=0}^{\infty}A_n\right\}\\ &\quad -\frac{\lambda(s(1-\alpha)+\alpha)^2}{s^2M^2(\alpha)}\mathcal{L}\left\{\sum_{n=0}^{\infty}y_n(x)\right\}.\end{aligned}$$

Applying the inverse LT, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}y_n(x) =& \mathcal{L}^{-1}\left\{\frac{(s(1-\alpha)+\alpha)}{s^2M(\alpha)}\right\} + \mathcal{L}^{-1}\left\{\frac{(s(1-\alpha)+\alpha)^2}{s^2M^2(\alpha)}\mathcal{L}\left\{\sum_{n=0}^{\infty}A_n\right\}\right\}\\ &- \mathcal{L}^{-1}\left\{\frac{\lambda(s(1-\alpha)+\alpha)^2}{s^2M^2(\alpha)}\mathcal{L}\left\{\sum_{n=0}^{\infty}y_n(x)\right\}\right\}.\end{aligned} \tag{3.30}$$

From Eq (3.30), the general recursive relation is given by

$$y_{n+1}(x) = \mathcal{L}^{-1}\left\{\frac{(s(1-\alpha)+\alpha)^2}{s^2M^2(\alpha)}\mathcal{L}\{A_n\}\right\} - \mathcal{L}^{-1}\left\{\frac{\lambda(s(1-\alpha)+\alpha)^2}{s^2M^2(\alpha)}\mathcal{L}\{y_n(x)\}\right\}.$$

Therefore, the proof is over. □

Theorem 3.6. *Consider the second type nonlinear fractional SL problem in the Caputo-Fabrizio sense as follows:*

$$-{}^{CFC}D^{\alpha}_{0}{}^{CFC}D^{\alpha}_{0}y(x) + e^{y(x)} = \lambda y(x),\ x \in (0,1),\ \frac{1}{2} < \alpha \leq 1, \tag{3.31}$$

subject to initial conditions

$$y(0) = 1,\quad D^{\alpha}_{0}y(x)\big|_{x=0} = 0. \tag{3.32}$$

Then, the approximate solution is given in the following form:

$$y(x) = \cosh\left(\frac{\alpha x\sqrt{e-\lambda}}{M(\alpha)}\right) + \frac{(e-\lambda)}{M^2(\alpha)}\left((1-\alpha)^2 - 2\left(-\alpha+\alpha^2\right)x\right) + \ldots.$$

Proof. The proof is straightforward from the proof of Theorem 3.5. □

Theorem 3.7. *Consider the nonlinear fractional Sturm-Liouville equations given by*

$$-{}^{ABC}D_0^{\alpha}{}^{ABC}D_0^{\alpha}y(x)+y^{p}(x)=\lambda y(x),\ x\in(0,1),\ \frac{1}{2}<\alpha\leq 1, \tag{3.33}$$

with initial conditions

$$y(0)=0,\quad D_0^{\alpha}y(x)\Big|_{x=0}=1, \tag{3.34}$$

in which ${}^{ABC}D^{\alpha}$ is an Atangana-Baleanu derivative in the sense of Caputo, $p>0$ is a real constant, and λ is a spectral parameter. Thus, we find the related solution as

$$y_0(x)=\frac{1}{B(\alpha)}\left(1-\alpha+\frac{\alpha x^{\alpha}}{\Gamma(1+\alpha)}\right),$$

and for $n\geq 0$,

$$y_{n+1}(x)=\mathcal{L}^{-1}\left\{\frac{(s^{\alpha}(1-\alpha)+\alpha)^2}{s^{2\alpha}B^2(\alpha)}\mathcal{L}\{A_n\}\right\}-\mathcal{L}^{-1}\left\{\frac{\lambda(s^{\alpha}(1-\alpha)+\alpha)^2}{s^{2\alpha}B^2(\alpha)}\mathcal{L}\{y_n(x)\}\right\}.$$

Proof. The proof is straightforward from the proof of Theorem 3.5. □

Theorem 3.8. *Consider the second type nonlinear SL problem in the frame of the Atangana-Baleanu derivative as follows:*

$$-{}^{ABC}D_0^{\alpha}{}^{ABC}D_0^{\alpha}y(x)+e^{y(x)}=\lambda y(x),\ x\in(0,1),\ \frac{1}{2}<\alpha\leq 1, \tag{3.35}$$

subject to initial conditions

$$y(0)=1,\quad D_0^{\alpha}y(x)\Big|_{x=0}=0. \tag{3.36}$$

Then, the approximate solution is given by

$$y(x)=E_{2\alpha}\left(\frac{(e-\lambda)\alpha^2x^{2\alpha}}{B^2(\alpha)}\right)+\frac{(e-\lambda)(1-\alpha)^2}{B^2(\alpha)}+\frac{2(e-\lambda)(1-\alpha)\alpha x^{\alpha}}{B^2(\alpha)\Gamma(1+\alpha)}+\ldots.$$

Proof. The proof is straightforward from the proof of Theorem 3.5. □

4. Examples

Example 1. Let us examine the following nonlinear SL problem with the Atangana-Baleanu derivative in case of $p=2$:

$$-{}^{ABC}D_0^{\alpha}{}^{ABC}D_0^{\alpha}y(x)+y^{2}(x)=\lambda y(x),\ x\in(0,1),\ \frac{1}{2}<\alpha\leq 1, \tag{3.37}$$

with initial conditions

$$y(0)=0,\quad D_0^{\alpha}y(x)\Big|_{x=0}=1, \tag{3.38}$$

where λ is a spectral parameter. Thus, we find the approximate solution as

$$\begin{aligned} y(x) &= \frac{\alpha x^{\alpha}}{B(\alpha)} E_{2\alpha,\alpha+1}\left(\frac{\alpha^2 x^{2\alpha}\lambda}{B^2(\alpha)}\right) + \frac{1-\alpha}{B(\alpha)} - \frac{\lambda(1-\alpha)^3}{B^3(\alpha)} \\ &\quad + \frac{4(1-\alpha)^3 \alpha x^{\alpha}}{B^4(\alpha)\Gamma(1+\alpha)} - \frac{3\lambda\alpha(1-\alpha)^2 x^{\alpha}}{B^3(\alpha)\Gamma(1+\alpha)} + \dots . \end{aligned}$$

Example 2. Let us examine the following nonlinear SL problem with the Caputo-Fabrizio derivative in case of $p = 2$:

$$-{}^{CFC}D_0^{\alpha\,CFC}D_0^{\alpha} y(x) + y^2(x) = \lambda y(x), \; x \in (0,1), \; \frac{1}{2} < \alpha \le 1, \tag{3.39}$$

with initial conditions

$$y(0) = 0, \;\; D_0^{\alpha} y(x)\big|_{x=0} = 1, \tag{3.40}$$

where λ is a spectral parameter. Thus, we find the approximate solution as

$$\begin{aligned} y(x) &= \frac{1}{\sqrt{\lambda}} \sin\left(\frac{\alpha x\sqrt{\lambda}}{M(\alpha)}\right) + \frac{1-\alpha}{M(\alpha)} + \frac{(1-\alpha)^4}{M^4(\alpha)} \\ &\quad - \frac{\lambda(1-\alpha)^3}{M^3(\alpha)} + \frac{4(1-\alpha)^3 \alpha x}{M^4(\alpha)} - \frac{3\lambda\alpha(1-\alpha)^2 x}{M^3(\alpha)} + \dots . \end{aligned}$$

5. Visual results

Our main purpose in this section is to observe the behaviors of the approximate solution curves for different values of α, p and λ. We compare the solutions for fractional-order versions of nonlinear SL equations studied in different types by simulation under different orders of α, and different values of p and λ. It is observed that the solution curves make a right-sided translation as the value of α approaches 1 in Figures 1, 2, 5 and 6. The fractional simulations within the CFC and ABC derivatives are illustrated in Figures 7 and 8 for some values of α. Here, we see that the solution curves intersect at certain intervals and continue their movements with decreasing and increasing slopes. As can be clearly seen from all these figures, the solution curves perform a right-sided translation as α gets closer to 1. The comparisons between the classical and fractional versions of the approximate solutions of the Caputo, conformable, ABC and CFC fractional derivatives are shown in Figures 1, 2 and 5–8, respectively. The fractional simulation shown in Figure 3 shows that the increase in the value of p has little effect on the solution curves. The fractional simulations shown in Figure 4 exhibit the results for $\lambda = 1, 1.5, 2, 2.5, 3$. From this figure, it is clear that the increase in the value of λ promotes a right-sided translation. In Figure 9, comparisons of the first type of nonlinear SL equation are presented under all studied derivatives. Finally, Figure 10 shows the comparisons of the exponential type of nonlinear SL equation in terms of all derivatives.

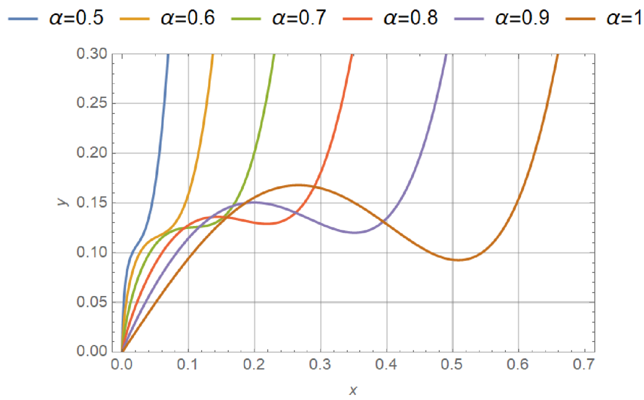

Figure 1. Comparative analysis for the solutions of the problem (3.1), (3.2) for different orders of α, $\lambda = 36$, and $p(x) = 2$.

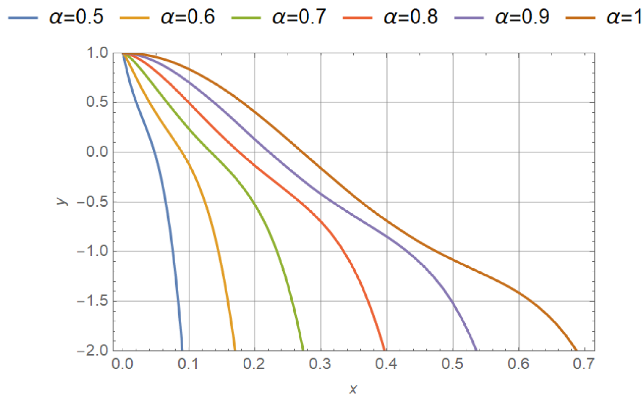

Figure 2. Comparative analysis for the solutions of the problem (3.10), (3.11) for different orders of α, $\lambda = 36$, and $e = 2.73$.

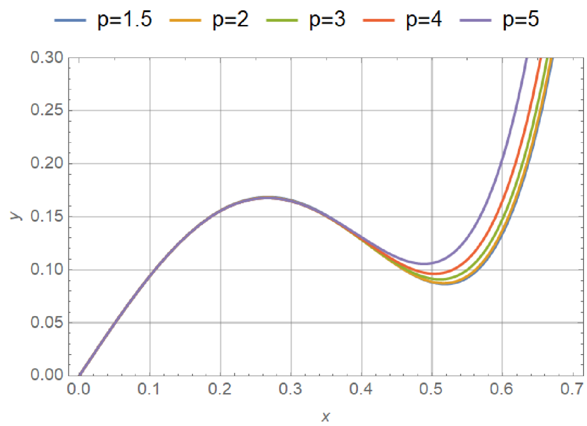

Figure 3. Comparative analysis for the solutions of the problem (3.18), (3.19) for different values of p, and $\lambda = 36$.

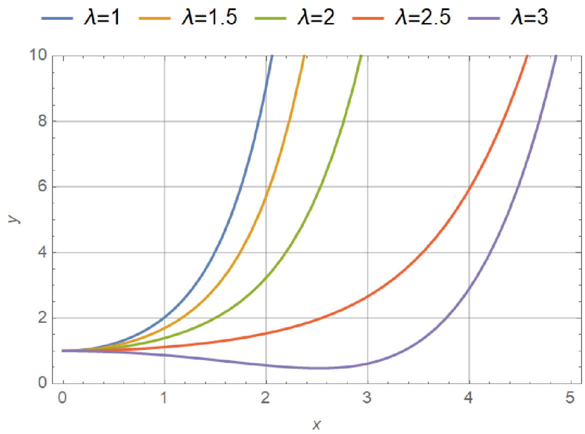

Figure 4. Comparative analysis for the solutions of the problem (3.25), (3.26) for different values of λ.

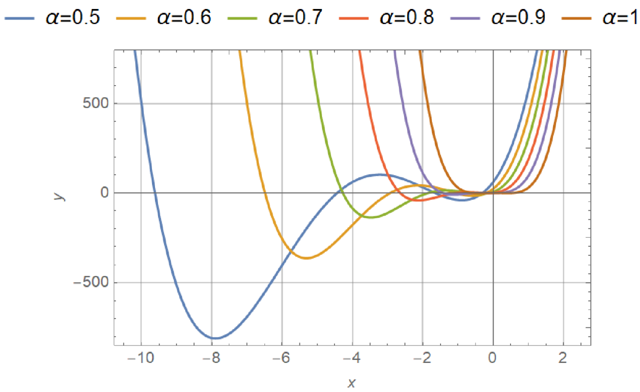

Figure 5. Comparative analysis for the solutions of the problem (3.31), (3.32) for different orders of α, $M = 1$, and $\lambda = 36$.

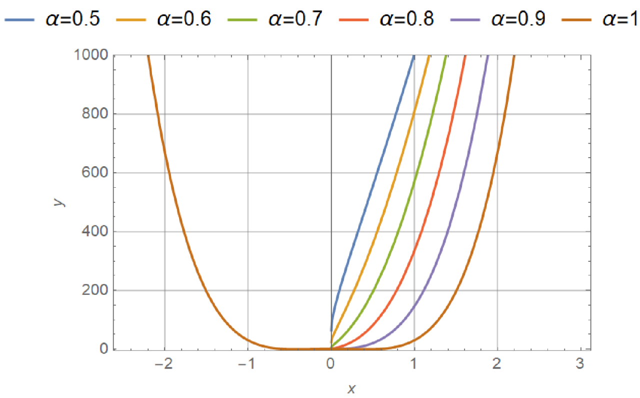

Figure 6. Comparative analysis for the solutions of the problem (3.35), (3.36) for different orders of α, $B = 1$, and $\lambda = 36$.

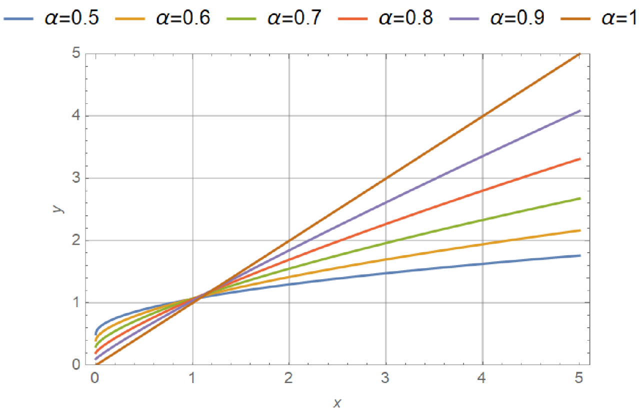

Figure 7. Comparative analysis for the solutions of the problem (3.37), (3.38) for different orders of α, $B = 1$, and $\lambda = 36$.

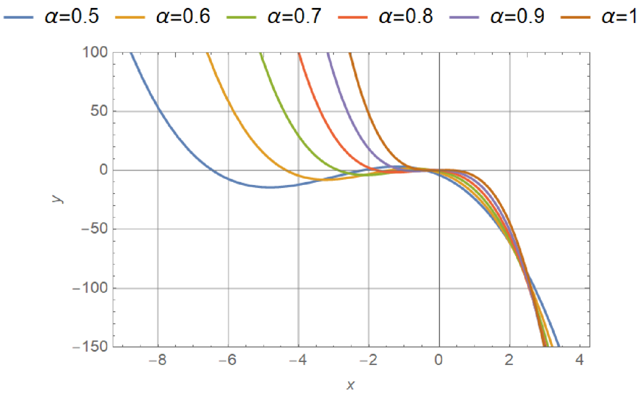

Figure 8. Comparative analysis for the solutions of the problem (3.39), (3.40) for different orders of α, $M = 1$, and $\lambda = 36$.

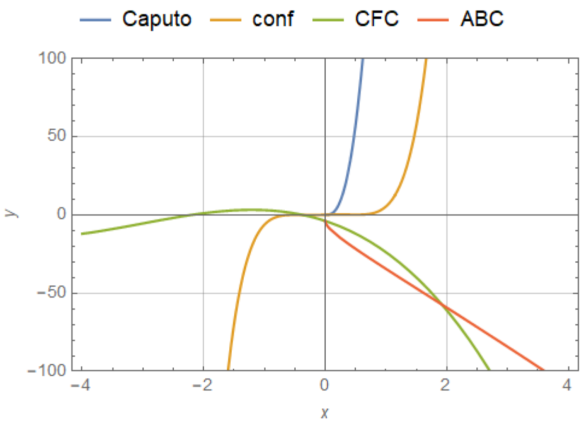

Figure 9. Comparative analysis for the solutions of the first type of nonlinear SL problem for $\alpha = 0.5$, $p = 2$, $M = 1$, $B = 1$, and $\lambda = 36$.

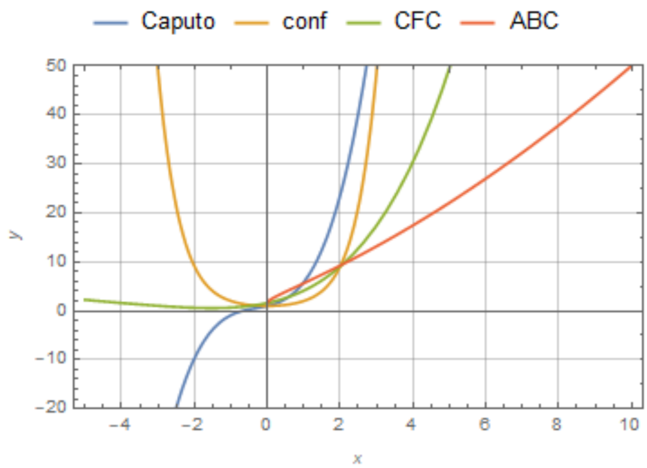

Figure 10. Comparative analysis for the solutions of the second type of nonlinear SL problem for $\alpha = 0.5$, $e = 2.73$, $M = 1$, $B = 1$, and $\lambda = 1$.

6. Conclusions

In this research paper, we have considered two different types of fractional-order versions of nonlinear SL problems involving Caputo, conformable, CFC and ABC derivatives. Also, a powerful analytical technique, called LADM, was applied to find approximate solutions of fractional nonlinear SL problems. The solutions derived from the Caputo and ABC derivatives are represented by the Mittag-Leffler function, and the solutions obtained using the conformable and CFC derivatives generate the hyperbolic sine and cosine functions, that is, the solutions are obtained in generalized form. The main purpose of the article is to obtain approximate solutions of fractional nonlinear SL problems by means of LADM. These solutions can be easily obtained without discretization, linearization or perturbation, thanks to LADM. This method can solve nonlinear situations without difficulty. This indicates that LADM is a convenient and effective method for use in solving nonlinear systems. Also, we analyzed and compared the solutions of these different versions and displayed them by simulation under different orders of α and different values of p and λ. The comparisons we made with the help of graphs are among the Caputo, conformable, ABC, and CFC derivatives, and their classical versions corresponding to $\alpha = 1$. In addition, the approximate solutions obtained from two different types of nonlinearity cases were compared for all studied derivatives. Visual results give better understanding of the solutions obtained.

Conflict of interest

The authors declare that they have no conflicts of interest.

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