Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(7): 13313–13324.
DOI: 10.3934/math.2022735
Received: 06 April 2022
Revised: 01 May 2022
Accepted: 05 May 2022
Published: 16 May 2022

Research article

Existence of solutions to elliptic equation with mixed local and nonlocal operators

Xiangrui Li[1], **Shuibo Huang**[1,*], **Meirong Wu**[2] **and Canyun Huang**[3]

[1] School of Mathematics and Computer Science, Northwest Minzu University, Lanzhou, Gansu, 730030, China

[2] The Department of Basic Courses, Army Logistics University, Chongqing, 430030, China

[3] Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, China

* **Correspondence:** Email: huangshuibo2008@163.com.

Abstract: In this paper, making use of a new non-smooth variational approach established by Moameni [13–16], we establish the existence of solutions to the following mixed local and nonlocal elliptic problem

$$\begin{cases} -\Delta u + (-\Delta)^s u = \mu g(x,u) + b(x), & x \in \Omega, \\ u \geq 0, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $(-\Delta)^s$ is the restricted fractional Laplacian, $\mu > 0$, $0 < s < 1$, $N > 2s$, g satisfies some growth condition and $b(x) \in L^m(\Omega)$ for $m \geq 2$. The interesting feature of our work is that we show that the nonlocal operator has an important influence in the existence of solutions to the above equation since g has new growth condition.

Keywords: non-smooth variational approach; mixed local and nonlocal operators
Mathematics Subject Classification: 35J67, 35R11

1. Introduction

In this paper, we consider the existence of solutions to the following mixed local and nonlocal semi-linear non-homogeneous elliptic problem

$$\begin{cases} -\Delta u + (-\Delta)^s u = \mu g(x,u) + b(x), & x \in \Omega, \\ u \geq 0, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\mu > 0, N > 2s, 0 < s < 1$ and $g : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carátheodory function. The fractional Laplacian $(-\Delta)^s$ is defined by

$$\begin{aligned}(-\Delta)^s u(x) =& a_{N,s}\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy \\ =& a_{N,s} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy,\end{aligned}$$

where $a_{N,s}$ is given by

$$a_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1 - \cos \xi_1}{|\xi|^{N+2s}} d\xi \right)^{-1}.$$

The mixed operators of the form $\mathcal{L}_s := -\Delta + (-\Delta)^s (0 < s < 1)$ appears naturally from the superposition of two stochastic processes with different scales, namely, a classical random walk and a Lévy flight. For example, the population with density u can possibly alternate both short and long-range random walks. See [1, 4, 9] for more details.

Recently, an extensive work has been developed for the mixed local and nonlocal operators $\mathcal{L}_s$. Biagi et al. [7] obtained the radial symmetry of the solutions to mixed operators by moving planes method. Furthermore, they also established the one-dimensional symmetry of the global solutions inspired by a classical conjecture of Gibbons. In [4], Biagi et al. established the existence, maximum principles and interior Sobolev regularity of solutions to the following mixed local and nonlocal elliptic problem

$$\begin{cases} -\Delta u + (-\Delta)^s u = f(x), & x \in \Omega, \\ u \geq 0, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.2}$$

Lamao et al. [12] given some summability of solutions to problem (1.2). Some other results of mixed local and nonlocal operators, see [1, 5, 6, 8, 10, 11] and the references therein.

Now, we give a short description on the development of problem (1.1). For the following semi-linear local elliptic problem

$$\begin{cases} -\Delta u + \lambda u = \mu |u|^{p-2} u + b(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $\mu > 0$ and $b \in L^2(\Omega)$. Bahri [2] showed the existence of infinitely many weak solutions to problem (1.3) with $\lambda = 0$ and $\mu = 1$ for $2 < p < 2^*$, where $2^* = 2N/(N-2)$ for $N \geq 3$ and $2^* = \infty$ for $N = 2$. The results of [2] given a partial answer to an open question put forward in [18, p.124]. Barroso [19], by a new fixed point result of the Krasnoselskii type for the sum of two operators, proved the existence of strong solution to problem (1.3) with a more general nonlinearity. Basiri and Moameni [3] applied a new variational principle introduced in [14, 16], showed that problem (1.3) with $\mu > 0$, $\lambda \geq 0$ and $b \in L^m(\Omega)$ has a strong solution for each $2 < p \leq (2N - 2m)/(N - 2m)$ if $N > 2m$ and for $p > 2$ if $N \leq 2m$, where $m \geq 2$. The importance of the result of [3] is that problem (1.3) includes supercritical nonlinearity as well.

For the nonlocal case, by a new variational principle, Moameni and Wong [13] proved the existence of a weak solution to the following nonlocal supercritical semi-linear elliptic problem

$$\begin{cases}(-\Delta)^s u = \mu|u|^{p-2}u + b(x), & x \in \Omega,\\ \quad u = 0, & x \in \mathbb{R}^N \setminus \Omega,\end{cases} \tag{1.4}$$

where $0 < s \leq 1$. They extended the corresponding existence results of classical elliptic problem (1.3) to nonlocal elliptic problem (1.4).

In order to establish the existence of solutions to problem (1.1), now we need to recall some definitions and notations. Denote

$$\mathcal{U} = \mathcal{H}_0^1(\Omega) \cap L^p(\Omega), \tag{1.5}$$

where $p \geq 1$ and $\mathcal{H}_0^1(\Omega) = \{u \in H^1(\mathbb{R}^N) : u = 0 \text{ a. e in } \mathbb{R}^N \setminus \Omega\}$. It is obvious that $\mathcal{U}$ is a Banach space equipped with the norm $\|u\|_{\mathcal{U}} = \|u\|_{L^p(\Omega)} + \|u\|_{\mathcal{H}_0^1(\Omega)}$, where $\|u\|_{\mathcal{H}_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$ for any $u \in \mathcal{H}_0^1(\Omega)$. The use of this space in the framework of nonlinear equations and variational methods was put in [17].

Define the convex and weakly closed subset $\mathcal{K}$ of $\mathcal{U}$ in the following two cases:

Case 1: Let r be a positive real number, when $m > \frac{N}{s+1}$, denote

$$\mathcal{K} := \{u \in \mathcal{U} : u \geq 0, \|u\|_{L^\infty(\Omega)} \leq r\}. \tag{1.6}$$

Case 2: When $2 \leq m < \frac{N}{s+1}$, denote

$$\mathcal{K} := \{u \in \mathcal{U} : u \geq 0, \|u\|_{L^{m^*}(\Omega)} \leq r\}, \tag{1.7}$$

where $r > 0$ and

$$m^* = \frac{mN(N-2s)}{(N-2)(N-2ms)}. \tag{1.8}$$

In order to utilize the new principle, by similar arguments as [1, Lemma 3.1] or the proof of Theorems 1 and 2 in [17], we define the energy functional corresponding to problem (1.1) as

$$I_{\mathcal{K}}(u) = \Psi_{\mathcal{K}}(u) - \Phi(u), \tag{1.9}$$

where

$$\Psi_{\mathcal{K}}(u) = \begin{cases}\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\iint_{\mathcal{D}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy, & u \in \mathcal{K},\\ \infty, & \text{otherwise},\end{cases} \tag{1.10}$$

and

$$\Phi(u) = \int_\Omega \mu G(x,u) + ub(x) dx, \tag{1.11}$$

where $G : \Omega \times \mathbb{R} \to \mathbb{R}$ is denoted by $G(x, y) = \int_0^y g(x, s)ds$ and

$$\mathcal{D} := (\mathbb{R}^N \times \mathbb{R}^N) \setminus (\Omega^c \times \Omega^c). \tag{1.12}$$

Motivated by the works of Basiri and Moameni [3], Moameni and Wong [13] and Lamao et al. [12], in this paper, we prove the following theorem by the non-smooth critical point theory and non-smooth variational principle [14, 15].

Theorem 1.1. *Assume that $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carátheodory function which is increasing in u and satisfies the following growth condition*

$$|g(x, u)| \leq a|u|^{p-1} + c, \tag{1.13}$$

where a, c are positive constants, $b(x) \in L^m(\Omega)$ for $m \geq 2$ and

$$\begin{cases} 2 < p \leq \dfrac{N(N-2s) + (N-2)(N-2ms)}{(N-2)(N-2ms)}, & N > m(s+1), \\ p > 2, & N < m(s+1). \end{cases} \tag{1.14}$$

Then there exists $\mu^* > 0$ such that, for any $\mu \in (0, \mu^*)$, problem (1.1) *has at least one positive solution $u(x) \in \mathcal{U}$, where $\mathcal{U}$ is defined by* (1.5).

Remark 1.2. In [3], Basiri and Moameni showed that there exists at least one solution $u \in W^{2,m}(\Omega)$ to the following equation

$$\begin{cases} -\Delta u + \lambda u = \mu g(x, u) + b(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where g satisfies the growth condition (1.13), $a > 0$, $b \in L^m(\Omega)$ for $m \geq 2$ and

$$\begin{cases} 2 < p \leq \dfrac{2N-2m}{N-2m}, & N > 2m, \\ p > 2, & N \leq 2m. \end{cases}$$

It is worth noting that for $0 < s < 1$, we have

$$\frac{N(N-2s) + (N-2)(N-2ms)}{(N-2)(N-2ms)} < \frac{2N-2m}{N-2m},$$

and

$$\lim_{s \to 1^-} \frac{N(N-2s) + (N-2)(N-2ms)}{(N-2)(N-2ms)} = \frac{2N-2m}{N-2m}.$$

Remark 1.3. Note that

$$\frac{N(N-2s) + (N-2)(N-2ms)}{(N-2)(N-2ms)} \geq \frac{2N}{N-2},$$

provided

$$N \geq s + \frac{1}{s}.$$

Therefore, our main results show the existence of solutions to problems (1.1) with supercritical nonlinear term by means of Sobolev spaces.

The main structure of this paper is as follows. Section 2 is specialized in introducing some basic definitions and properties of non-smooth analysis and variational principle, which is the basis for the proof of main theorem. In Section 3, we prove Theorem 1.1 by variational principle.

2. Preliminaries

Let $\mathcal{U}$ be a locally convex space, $\mathcal{U}^*$ is the topological dual of $\mathcal{U}$, $\langle\cdot,\cdot\rangle$ is a dual product of $\mathcal{U}$ and $\mathcal{U}^*$. In order to utilize the variational method and prove the existence of solutions to problem (1.1), it is essential to review some important definitions and results in non-smooth analysis [13–16].

Definition 2.1. A function $\Psi : \mathcal{U} \to \mathbb{R}$ is known as weakly lower semi-continuous if $\Psi(u) \leq \liminf_{n\to\infty} \Psi(u_n)$ for any sequence u_n approaching $u \in \mathcal{U}$ in the weak topology on $\mathcal{U}$.

Definition 2.2. $\Psi : \mathcal{U} \to \mathbb{R} \cup \{\infty\}$ is said to be proper if $\mathrm{Dom}(\Psi) := \{u \in \mathcal{U} : \Psi(u) < \infty\} \neq \emptyset$.

Definition 2.3. The subdifferential $\partial\Psi$ of Ψ is defined to be the following set-valued operator

$$\partial\Psi(u) = \{u^* \in \mathcal{U}^* : \langle u^*, v-u\rangle + \Psi(u) \leq \Psi(v)\},\ \forall v \in \mathcal{U},$$

if $u \in \mathrm{Dom}(\Psi)$ and $\partial\Psi(u) = \emptyset$ if $u \notin \mathrm{Dom}(\Psi)$.

Assume that Ψ is Gâteaux differentiable at u, denote the derivative of Ψ at u by $D\Psi(u)$, in this case $\partial\Psi(u) = \{D\Psi(u)\}$. For a given functional $\Phi \in C^1(\mathcal{U}, \mathbb{R})$ denote by $D\Phi(u) \in \mathcal{U}^*$ its derivative.

The critical point of function is important in this paper.

Definition 2.4. Suppose $\mathcal{U}$ is a real Banach space. Let $I = \Psi - \Phi : \mathcal{U} \to \mathbb{R}\cup\{+\infty\}$, where $\Phi \in C^1(\mathcal{U}, \mathbb{R})$ and $\Psi : \mathcal{U} \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous. A point $u \in \mathcal{U}$ is a critical point of $I = \Psi - \Phi$ if $u \in \mathrm{Dom}(\Psi)$ and for any $v \in \mathcal{U}$,

$$\Psi(v) - \Psi(u) \geq \langle D\Phi(u), v-u\rangle, \tag{2.1}$$

where where $\langle .,.\rangle$ is the duality pairing between $\mathcal{U}$ and its dual $\mathcal{U}^*$.

Definition 2.5. Let $b \in L^2(\Omega)$, we say that a function $u \in \mathcal{H}_0^1(\Omega) \cap L^p(\Omega)$ is a weak solution to problem (1.1) if for every test function $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla\phi dx + \iint_{\mathcal{D}} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}} dxdy = \int_\Omega (\mu g(x,u(x)) + b(x))\phi(x)dx,$$

where $\mathcal{D}$ is defined by (1.12).

Let $\Psi_{\mathcal{K}}$ be defined as (1.10), then for any $v \in \mathcal{U}$,

$$\langle D\Psi_{\mathcal{K}}(u), v\rangle = \int_\Omega \nabla u(x)\cdot\nabla v(x)dx + \iint_{\mathcal{D}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy.$$

Lemma 2.6. *The functional $\Psi_{\mathcal{K}}$ is convex and weakly lower semi-continuous.*

Proof. We begin with showing the convexity of $\Psi_{\mathcal{K}}$. Recall that

$$\Psi_{\mathcal{K}}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\iint_{\mathcal{D}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy,\quad u \in \mathcal{K}.$$

For any $u_1, u_2 \in \mathcal{K}$ and $\alpha \in (0,1]$, we have

$$\Psi_{\mathcal{K}}(\alpha u_1 + (1-\alpha)u_2)$$

$$
\begin{aligned}
=&\frac{1}{2}\int_{\Omega}|\nabla(\alpha u_1+(1-\alpha)u_2)(x)|^2dx\\
&+\frac{1}{2}\iint_{\mathcal{D}}\frac{|(\alpha u_1+(1-\alpha)u_2)(x)-(\alpha u_1+(1-\alpha)u_2)(y)|^2}{|x-y|^{N+2s}}dxdy\\
=&\frac{1}{2}\int_{\Omega}\alpha^2|\nabla u_1(x)|^2dx+\frac{1}{2}\int_{\Omega}(1-\alpha)^2|\nabla u_2(x)|^2dx+\int_{\Omega}\alpha(1-\alpha)|\nabla u_1(x)||\nabla u_2(x)|dx\\
&+\frac{1}{2}\iint_{\mathcal{D}}\frac{\alpha^2|u_1(x)-u_1(y)|^2+(1-\alpha)^2|u_2(x)-u_2(y)|^2}{|x-y|^{N+2s}}dxdy\\
&+\frac{1}{2}\iint_{\mathcal{D}}\frac{2\alpha(1-\alpha)|(u_1(x)-u_1(y))(u_2(x)-u_2(y))|}{|x-y|^{N+2s}}dxdy. \qquad (2.2)
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
&\alpha\Psi_{\mathcal{K}}(u_1)+(1-\alpha)\Psi_{\mathcal{K}}(u_2)\\
=&\frac{1}{2}\int_{\Omega}\alpha|\nabla u_1(x)|^2dx+\int_{\Omega}(1-\alpha)|\nabla u_2(x)|^2dx\\
&+\frac{1}{2}\iint_{\mathcal{D}}\frac{\alpha|u_1(x)-u_1(y)|^2+(1-\alpha)|u_2(x)-u_2(y)|^2}{|x-y|^{N+2s}}dxdy. \qquad (2.3)
\end{aligned}
$$

Thus, taking into account $\alpha\in(0,1]$, (2.2) and (2.3), we get

$$
\begin{aligned}
&\Psi_{\mathcal{K}}(\alpha u_1+(1-\alpha)u_2)-[\alpha\Psi_{\mathcal{K}}(u_1)+(1-\alpha)\Psi_{\mathcal{K}}(u_2)]\\
=&\frac{\alpha(\alpha-1)}{2}\int_{\Omega}|\nabla u_1(x)-\nabla u_2(x)|^2dx\\
&+\frac{\alpha(\alpha-1)}{2}\iint_{\mathcal{D}}\frac{|u_1(x)-u_1(y)|^2+|u_2(x)-u_2(y)|^2}{|x-y|^{N+2s}}dxdy\\
&+\frac{\alpha(\alpha-1)}{2}\iint_{\mathcal{D}}\frac{2|(u_1(x)-u_1(y))(u_2(x)-u_2(y))|}{|x-y|^{N+2s}}dxdy\\
\le&0,
\end{aligned}
$$

which follows that $\Psi_{\mathcal{K}}(u)$ is convex.
Now it remains to show that $\Psi_{\mathcal{K}}$ is weakly lower semi-continuous. In order to do this, according to the convexity of $\Psi_{\mathcal{K}}(u)$ and [13, Lemma 3.4], we only need to show that for $u\in\mathcal{K}$,

$$D\Psi(u)\in\partial\Psi_{\mathcal{K}}(u),$$

which evidently holds by the the definition of the restriction of Ψ to $\mathcal{K}$. □

Lemma 2.7. *Assume that $\mathcal{U}=\mathcal{H}_0^1(\Omega)\cap L^p(\Omega)$ and $\mathcal{K}$ is a convex and weakly closed subset of $\mathcal{U}$. Suppose the following two assertions hold:*

1). The functional $I_{\mathcal{K}}$ defined by (1.9) *has a critical point $u_0\in\mathcal{U}$ as in Definition 2.4.*
2). There exists $v_0\in\mathcal{K}$ such that

$$\mathcal{L}_s v_0=\mu g(x,u_0)+b(x),$$

in the weak sense, i.e.,

$$\int_\Omega \nabla v_0 \cdot \nabla \varphi dx + \iint\limits_{\mathcal{D}} \frac{(v_0(x) - v_0(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} dxdy$$
$$= \int_\Omega (\mu g(x, u_0) + b(x))\varphi dx, \ \forall \varphi \in \mathcal{U}. \tag{2.4}$$

Then $u_0 \in \mathcal{K}$ is a weak solution to the equation

$$\mathcal{L}_s u_0 = \mu g(x, u_0) + b(x),$$

and g satisfies the growth condition (1.13).

Proof. Note that u_0 is the critical point of $I_{\mathcal{K}}$, according to Definition 2.4, we get

$$\Psi_{\mathcal{K}}(u) - \Psi_{\mathcal{K}}(u_0) \geq \langle D\Phi(u_0), u - u_0 \rangle, \ \forall u \in \mathcal{K},$$

which, together with the definition of $\Psi_{\mathcal{K}}$, see (1.10), leads to

$$\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\iint_{\mathcal{D}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy - \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \frac{1}{2}\iint_{\mathcal{D}} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{N+2s}} dxdy$$
$$\geq \int_\Omega D\Phi(u_0)(u - u_0) dx, \quad \forall u \in \mathcal{K}. \tag{2.5}$$

Taking $\varphi = v_0 - u_0$ in (2.4), we have

$$\int_\Omega \nabla v_0 \cdot \nabla (v_0 - u_0) dx + \iint\limits_{\mathcal{D}} \frac{(v_0(x) - v_0(y))((v_0 - u_0)(x) - (v_0 - u_0)(y))}{|x-y|^{N+2s}} dxdy$$
$$= \int_\Omega D\Phi(u_0)(v_0 - u_0) dx. \tag{2.6}$$

Substituting $u = v_0$ in (2.5) and taking into account (2.6), we find

$$\frac{1}{2}\int_\Omega |\nabla v_0|^2 dx + \frac{1}{2}\iint_{\mathcal{D}} \frac{|v_0(x) - v_0(y)|^2}{|x-y|^{N+2s}} dxdy - \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \frac{1}{2}\iint_{\mathcal{D}} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{N+2s}} dxdy$$
$$\geq \int_\Omega \nabla v_0 \cdot \nabla (v_0 - u_0) dx + \iint\limits_{\mathcal{D}} \frac{(v_0(x) - v_0(y))((v_0 - u_0)(x) - (v_0 - u_0)(y))}{|x-y|^{N+2s}} dxdy. \tag{2.7}$$

Making use of the convexity of $\Psi_{\mathcal{K}}$, we get

$$\int_\Omega |\nabla v_0|^2 dx + \iint_{\mathcal{D}} \frac{|v_0(x) - v_0(y)|^2}{|x-y|^{N+2s}} dxdy - \int_\Omega |\nabla u_0|^2 dx - \iint_{\mathcal{D}} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{N+2s}} dxdy$$
$$\geq 2\int_\Omega \nabla v_0 \cdot \nabla (u_0 - v_0) dx + 2\iint_{\mathcal{D}} \frac{(v_0(x) - v_0(y))((u_0 - v_0)(x) - (u_0 - v_0)(y))}{|x-y|^{N+2s}} dxdy,$$

which means that

$$\int_\Omega |\nabla v_0|^2 dx + \iint_{\mathcal{D}} \frac{|v_0(x) - v_0(y)|^2}{|x-y|^{N+2s}} dxdy - \int_\Omega |\nabla u_0|^2 dx - \iint_{\mathcal{D}} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{N+2s}} dxdy$$

$$\leq 2\int_{\Omega}\nabla v_0\cdot\nabla(v_0-u_0)dx+2\iint_{\mathcal{D}}\frac{(v_0(x)-v_0(y))((v_0-u_0)(x)-(v_0-u_0)(y))}{|x-y|^{N+2s}}dxdy. \tag{2.8}$$

Thus, taking into account (2.7) and (2.8), we obtain

$$\begin{aligned}&\int_{\Omega}|\nabla v_0|^2dx+\iint_{\mathcal{D}}\frac{|v_0(x)-v_0(y)|^2}{|x-y|^{N+2s}}dxdy-\int_{\Omega}|\nabla u_0|^2dx-\iint_{\mathcal{D}}\frac{|u_0(x)-u_0(y)|^2}{|x-y|^{N+2s}}dxdy\\ =&2\int_{\Omega}\nabla v_0\cdot\nabla(u_0-v_0)dx+2\iint_{\mathcal{D}}\frac{(v_0(x)-v_0(y))((u_0-v_0)(x)-(u_0-v_0)(y))}{|x-y|^{N+2s}}dxdy.\end{aligned}$$

Thus,

$$\int_{\Omega}|\nabla v_0-\nabla u_0|^2dx+\iint_{\mathcal{D}}\frac{((v_0(x)-v_0(y))-(u_0(x)-u_0(y)))^2}{|x-y|^{N+2s}}dxdy=0.$$

It is clear that we can obtain $u_0=v_0$. The proof of this lemma is complete. □

Lemma 2.8. *[12, Theorem 1.1] For any solution $u\in\mathcal{H}_0^1(\Omega)$ to problem*

$$\begin{cases}-\Delta u+(-\Delta)^s u=f(x), & x\in\Omega,\\ u\geq 0, & x\in\Omega,\\ u=0, & x\in\mathbb{R}^N\setminus\Omega,\end{cases} \tag{2.9}$$

where $f\in L^m(\Omega)$ with $m>\frac{N}{s+1}$. Then there exists a constant $C>0$, depending on N, Ω, s and $\|u\|_{\mathcal{H}_0^1(\Omega)}$, such that

$$\|u\|_{L^\infty(\Omega)}\leq C. \tag{2.10}$$

Lemma 2.9. *[12, Theorem 1.3] Suppose that $f\in L^m(\Omega)$ with*

$$1<m<\frac{N}{s+1}. \tag{2.11}$$

Then, there exists a constant $c=c(N,m,s)>0$, such that any solutions to problem (2.9) *satisfy*

$$\|u\|_{L^{m^*}(\Omega)}\leq c\|f\|_{L^m(\Omega)}, \tag{2.12}$$

where m^* is given by (1.8).

3. Existence result

In this section, we prove the existence of solutions to problem (1.1) by Lemma (2.7).

For $r>0$, define $\mathcal{K}(r)$ as (1.6) and (1.7). According to [20, Lemma 3.5], we know that $\mathcal{K}$ is a weakly closed subset of $\mathcal{U}$. The following lemma shows that $I_{\mathcal{K}}(u)$ has a minimum point $u_0\in\mathcal{K}$.

Lemma 3.1. *Suppose that $\mathcal{K}$ is defined by* (1.6) *and* (1.7). *Then there exists $u_0\in\mathcal{K}$ such that $I_{\mathcal{K}}(u_0)=\min_{u\in\mathcal{K}}I_{\mathcal{K}}(u)$.*

Proof. By the definition of $I_{\mathcal{K}}$ (see(1.9) for more details), we know that $I_{\mathcal{K}}(u) = +\infty$ provided $u \notin \mathcal{K}$. Therefore, $\beta := \min_{u\in\mathcal{K}} I_{\mathcal{K}}(u) = \min_{u\in\mathcal{U}} I_{\mathcal{K}}(u)$. Considering the growth condition (1.13), we obtain

$$G(x,u) \le \frac{a}{p}|u|^p + cu,$$

which, together with Hölder inequality, leads to

$$\Phi(u) \le \int_\Omega \frac{\mu a}{p}|u|^p + (b(x)+\mu c)udx \le \frac{\mu a}{p}\|u\|^p_{L^p(\Omega)} + \|b(x)+\mu c\|_{L^m(\Omega)}\|u\|_{L^{m'}(\Omega)}, \tag{3.1}$$

where m' is the conjugate exponent of m.

Case 1: For $m > \frac{N}{s+1}$, we have $\|u\|_{L^\infty(\Omega)} \le r$. Then,

$$\begin{aligned}\Phi(u) &\le \int_\Omega \frac{\mu a}{p}|u|^p + (b(x)+\mu c)udx \\ &\le C(\|u\|^p_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}) \\ &\le C_1 r^p + C_2 r < \infty,\end{aligned} \tag{3.2}$$

where $C(c,\mu,|\Omega|,\|b(x)\|_{L^m(\Omega)}) > 0$ is a constant.

Case 2: For $2 \le m < \frac{N}{s+1}$, we know that $\|u\|_{L^{m^*}(\Omega)} \le r$, where m^* is defined by (1.8). Note that $m^* > m(p-1)$. Therefore, $L^{m^*}(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \le p < m(p-1)$. Thus,

$$\begin{aligned}\Phi(u) &\le \int_\Omega \frac{\mu a}{p}|u|^p + (b(x)+\mu c)udx \\ &\le C\|u\|^p_{L^{m^*}(\Omega)} + C\|u\|_{L^{m^*}(\Omega)} \\ &\le C(r^p + r) < \infty,\end{aligned} \tag{3.3}$$

where $C > 0$ depends on $\mu, a, p, |\Omega|, \|b(x)\|_{L^m(\Omega)}$.

In both cases, we derive that $\Phi(u) < \infty$, which, combined with $\Psi_{\mathcal{K}}(u) \ge 0$, implies that $\beta = \min_{u\in\mathcal{K}} I_{\mathcal{K}}(u) \ge -\infty$ for any $u \in \mathcal{K}$.

Consider the minimizing sequence $\{u_n\} \subset \mathcal{K}$ such that $I_{\mathcal{K}}(u_n) \to \beta$. Therefore, up to a subsequence, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup u_0$ weakly in $\mathcal{K}$. Furthermore, $u_0 \in \mathcal{K}$ since $\mathcal{K}$ is weakly closed. It is easily seen that $\Phi(u_n) \to \Phi(u_0)$ as $n \to +\infty$. On the other hand, by Lemma 2.6, we know that $\Psi_{\mathcal{K}}(u_0) \le \liminf_{n\to+\infty} \Psi_{\mathcal{K}}(u_n)$, which implies that $I_{\mathcal{K}}(u_0) \le \liminf_{n\to+\infty} \Psi_{\mathcal{K}}(u_n) = \beta$. Therefore, $\beta = I_{\mathcal{K}}(u_0)$. □

Now we give the proof of the main theorem.

Proof of Theorem 1.1. Using Lemma 3.1, we know that there exists $u_0 \in \mathcal{K}$ such that $I_{\mathcal{K}}(u_0) = \min_{u\in\mathcal{K}} I_{\mathcal{K}}(u)$, where $I_{\mathcal{K}}$ is given by (1.9).
Now we shall prove there exists $v_0 \in \mathcal{K}$ such that

$$\begin{cases} -\Delta v_0 + (-\Delta)^s v_0 = \mu g(x,u_0) + b(x), & x \in \Omega, \\ v_0 \ge 0, & x \in \Omega, \\ v_0 = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.4}$$

Let $f(x) = \mu g(x, u_0) + b(x)$. Recall that $\Psi_{\mathcal{K}}$ and Φ are defined as (1.10) and (1.11), respectively. In order to get the existence of solution u_0 to problem (3.4), it is necessary to show that $\|v_0\|_{L^\infty(\Omega)} \leq r$ if $m > \frac{N}{s+1}$ and $\|v_0\|_{L^{m^*}(\Omega)} \leq r$ if $2 \leq m < \frac{N}{s+1}$.
Let $m > \frac{N}{s+1}$, choose $r > 0$ such that $C\|b(x)\|_{L^m(\Omega)} < r$, where C appears in the following inequality. Then by Lemma 2.8, we get

$$\begin{aligned}\|v_0\|_{L^\infty(\Omega)} &\leq C\|\mu g(x, u_0) + b(x)\|_{L^m(\Omega)}\\ &\leq \mu C\|g(x, u_0)\|_{L^m(\Omega)} + C\|b(x)\|_{L^m(\Omega)}\\ &\leq \mu C\|u_0\|^{p-1}_{L^{m(p-1)}(\Omega)} + C\|b(x)\|_{L^m(\Omega)} + \mu C\\ &\leq \mu C r^{p-1} + C\|b(x)\|_{L^m(\Omega)} + \mu C,\end{aligned}$$

where we have use the fact that $\|u_0\|_{L^\infty(\Omega)} \leq r$. Choose $\mu^* > 0$ small enough, such that $\mu C r^{p-1} + C\|b(x)\|_{L^m(\Omega)} + \mu C \leq r$ for each $\mu \in (0, \mu^*)$.
Similarly, when $2 \leq m < \frac{N}{s+1}$, then by Lemma 2.9, we have

$$\begin{aligned}\|v_0\|_{L^{m^*}(\Omega)} &\leq C\|\mu g(x, u_0) + b(x)\|_{L^m(\Omega)}\\ &\leq C\mu\|g(x, u_0)\|_{L^m(\Omega)} + C\|b(x)\|_{L^m(\Omega)}\\ &\leq C\left(\mu\|u_0\|^{p-1}_{L^{m(p-1)}(\Omega)} + \|b(x)\|_{L^m(\Omega)} + \mu\right)\\ &\leq C\left(\mu r^{p-1} + \|b(x)\|_{L^m(\Omega)} + \mu\right).\end{aligned}$$

Choose $\mu^* > 0$ small enough, such that $C\left(\mu r^{p-1} + \|b(x)\|_{L^m(\Omega)} + \mu\right) \leq r$ for each $\mu \in (0, \mu^*)$.
According to Lemma 2.7, we conclude problem (1.1) has at least one positive solution $u_0 \in \mathcal{K}$. □

Acknowledgements

This works was partially supported by Fundamental Research Funds for the Central Universities (No.31920220067), Innovation Team Project of Northwest Minzu University (No.1110130131) and Leading Talents Project of State Ethnic Affairs Commission of China.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

References

1. R. Arora, V. D. Rădulescu, Combined effects in mixed local-nonlocal stationary problems, 2021, arXiv: 2111.06701.
2. A. Bahri, Topological results on a certain class of functional and application, *J Funct. Anal.*, **41** (1981), 397–427. http://doi.org/10.1016/0022-1236(81)90083-5
3. M. Basiri, A. Moameni, Solutions of supercritical semilinear non-homogeneous elliptic problems, *Nonlinear Anal.*, **165** (2017), 121–142. http://doi.org/10.1016/j.na.2017.09.014
4. S. Biagi, S. Dipierro, E. Valdinoci, E. Vecchi, Mixed local and nonlocal elliptic operators: regularity and maximum principles, *Commun. Part. Diff. Eq.*, **47** (2022), 585–629. https://doi.org/10.1080/03605302.2021.1998908
5. S. Biagi, S. Dipierro, E. Valdinoci, E. Vecchi, A Faber-Krahn inequality for mixed local and nonlocal operators, 2021, arXiv: 2104.00830.
6. S. Biagi, S. Dipierro, E. Valdinoci, E. Vecchi, A Hong-Krahn-Szegö inequality for mixed local and nonlocal operators, *Math. Eng.-US*, **5** (2023), 1–25. https://doi.org/10.3934/mine.2023014
7. S. Biagi, E. Vecchi, S. Dipierro, E. Valdinoci, Semilinear elliptic equations involving mixed local and nonlocal operators, *Proc. R. Soc. Edinburgh, Sect. A: Math.*, **151** (2021), 1611–1641. https://doi.org/10.1017/prm.2020.75
8. S. Dipierro, E. Lippi, E. Valdinoci, Linear theory for a mixed operator with Neumann conditions, *Asymptotic Anal.*, 2021, 1–24, pre–press. https://doi.org/10.3233/ASY-211718
9. S. Dipierro, E. Lippi, E. Valdinoci, (Non)local logistic equations with Neumann conditions, 2021, arXiv: 2101.02315.
10. P. Garain, A. Ukhlov, Mixed local and nonlocal Sobolev inequalities with extremal and associated quasilinear singular elliptic problems, 2021, arXiv: 2106.04458.
11. B. Z. Hu, Y. Yang, A note on the combination between local and nonlocal p-Laplacian operators, *Complex Var. Elliptic.*, **65** (2020), 1763–1776. https://doi.org/10.1080/17476933.2019.1701450
12. C. D. Lamao, S. B. Huang, Q. Y. Tian, C. B. Huang, Regularity results of solutions to elliptic equations involving mixed local and nonlocal operators, *AIMS Mathematics*, **7** (2022), 4199–4210. https://doi.org/10.3934/math.2022233
13. A. Moameni, K. Wong, Existence of solutions for nonlocal supercritical elliptic problems, *J. Geom. Anal.*, **31** (2021), 164–186. https://doi.org/10.1007/s12220-019-00254-8
14. A. Moameni, A variational principle for problems with a hint of convexity, *Cr. Math*, **355** (2017), 1236–1241. https://doi.org/10.1016/j.crma.2017.11.003
15. A. Moameni, Critical point theory on convex subsets with applications in differential equations and analysis, *J. Math. Pure. Appl.*, **141** (2020), 266–315. https://doi.org/10.1016/j.matpur.2020.05.005
16. A. Moameni, Non-convex self-dual Lagrangians: new variational principles of symmetric boundary value problems, *J. Funct. Anal.*, **260** (2011), 2674–2715. https://doi.org/10.1016/j.jfa.2011.01.010

17. R. Servadei, E. Valdinoci, Variational methods for nonlocal operators of elliptic type, *Discrete Cont. Dyn.-A*, **33** (2013), 2105–2137. https://doi.org/10.3934/dcds.2013.33.2105

18. M. Struwe, *Variational methods*, Berlin: Springer, 1990.

19. C. Barroso, Semilinear elliptic equations and fixed points, *P. Am. Math. Soc.*, **133** (2005), 745–749. https://doi.org/10.1090/S0002-9939-04-07718-4

20. N. Kouhestani, H. Mahyar, A. Moameni, Multiplicity results for a non-local problem with concave and convex nonlinearities, *Nonlinear Anal.*, **182** (2019), 263–279. https://doi.org/10.1016/j.na.2018.12.006

AIMS Press

© 2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)