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Research article

Multiple solutions for a fractional p-Kirchhoff equation with critical growth and low order perturbations

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Abstract: In this article, we deal with the following fractional p-Kirchhoff type equation

$$\begin{cases} M\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)(-\Delta)_p^s u = \frac{|u|^{p_\alpha^*-2}u}{|x|^\alpha}+\frac{\lambda}{|x|^\beta}, & \text{in } \Omega,\\ u>0, & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain containing 0, $(-\Delta)_p^s$ denotes the fractional p-Laplacian, $M(t)=a+bt^{k-1}$ for $t\geq 0$ and $k>1$, $a,b>0$, $\lambda>0$ is a parameter, $0<s<1$, $0\leq\alpha<ps<N$, $\frac{N(p-2)+ps}{p-1}<\beta<\frac{N(p_\alpha^*-1)+\alpha}{p_\alpha^*}$, $1<p<pk<p_\alpha^*=\frac{p(N-\alpha)}{N-ps}$ is the fractional critical Hardy-Sobolev exponent. With aid of the variational method and the concentration compactness principle, we prove the existence of two distinct positive solutions.

Keywords: Kirchhoff type equation; fractional p-Laplacian; critical growth; variational method; positive solution
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1. Introduction and main result

Consider the following fractional p-Kirchhoff type equation with critical growth

$$\begin{cases} M\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)(-\Delta)_p^s u = \frac{|u|^{p_\alpha^*-2}u}{|x|^\alpha}+\frac{\lambda}{|x|^\beta}, & \text{in } \Omega,\\ u>0, & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain containing 0, $(-\Delta)_p^s$ denotes the fractional p-Laplacian, $M(t) = a + bt^{k-1}$ for $t \geq 0$ and $k > 1$, $a, b > 0$, $\lambda > 0$ is a parameter, $0 < s < 1$, $0 \leq \alpha < ps < N$, $\frac{N(p-2)+ps}{p-1} < \beta < \frac{N(p_\alpha^*-1)+\alpha}{p_\alpha^*}$, $1 < p < pk < p_\alpha^* = \frac{p(N-\alpha)}{N-ps}$ is the fractional critical Hardy-Sobolev exponent.

Problem (1.1) reduces to the following stationary analogue of the Kirchhoff equation

$$-\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), \tag{1.2}$$

which was proposed by Kirchhoff in [12] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = f(x,u). \tag{1.3}$$

Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Here, L is the length of the string, $f(x,u)$ is the area of the cross section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. The appearance of nonlocal term $\int_\Omega |\nabla u|^2 dx$ in the equations make its importance in many physical applications. It was pointed out that such nonlocal problems appear in other fields like biological systems, such as population density, where u describes a process which depends on the average of itself (see [1]).

Recently a great attention has been focused on studying the problems involving fractional Sobolev spaces and corresponding nonlocal equations. Indeed, nonlocal fractional problems arise in a quite natural way in many different contexts, such as, optimization, finance, phase transitions, stratified materials, anomalous diffusion, semipermeable membranes and flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, water waves and so on, we refer to [15] for more details.

In particular, Chen et al. in [6] considered the following fractional p-Laplacian equation with subcritical and critical growths

$$\begin{cases}(-\Delta)_p^s u = \lambda|u|^{r-2}u + \mu\frac{|u|^{q-2}u}{|x|^\alpha}, & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N\backslash\Omega,\end{cases} \tag{1.4}$$

where $0 < s < 1, p > 1, \lambda, \mu > 0, 0 \leq \alpha \leq ps < N, p \leq r \leq p_\alpha^*$. They obtained the existence of positive solutions, ground state solutions and sign-changing solutions of the fractional p-Laplacian Eq (1.4) by using the variational method.

In [22], Xiang et al. studied the following fractional p-Laplacian Kirchhoff type equation with critical growth

$$M\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)(-\Delta)_p^s u = |u|^{p_s^*-2}u + \lambda f(x), \text{ in } \mathbb{R}^N, \tag{1.5}$$

where $M(t) = a + bt^{k-1}$ for $t \geq 0$ and $k > 1$, $a \geq 0$, $b > 0$, $0 < s < 1$ and $1 < p < N/s$, $p_s^* = Np/(N-ps)$ is the critical Sobolev exponent, $\lambda \geq 0$ is a parameter, $f \in L^{p_s^*/(p_s^*-1)}(\mathbb{R}^N\backslash\{0\})$ is a nonnegative function. By the variational method, the authors proved that Eq (1.5) admits at least two nonnegative solutions.

Recently, the following fractional p-Laplacian Kirchhoff type equation with critical growth has been well studied by various authors

$$\begin{cases}M\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)(-\Delta)_p^s u = \frac{|u|^{p_\alpha^*-2}u}{|x|^\alpha} + \lambda f(x,u), & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N\backslash\Omega,\end{cases} \tag{1.6}$$

where $M(t) = a + bt^{k-1}$ with $t \geq 0$ and $k > 1$, $0 < s < 1$, $a \geq 0, b > 0$, $0 \leq \alpha < ps < N$, $\lambda > 0$ is a parameter. When $f(x,u) = w(x)|u|^{q-2}u$, Fiscella and Pucci in [10] deal with the existence and the asymptotic behavior of nontrivial solutions for Eq (1.6) with $pk < q < p_\alpha^*$. In [5], Chen and Gui obtained the existence of multiple solutions to Eq (1.6) with $w(x) = 1$ and $1 < q < p < pk$. When $f(x,u) = (I_\mu * F(x,u))f(u)$, $I_\mu = |x|^{-\mu}$ is the Riesz potential of order $\mu \in (0, \min\{N, 2ps\})$, Chen [4] established the existence of positive solutions to Eq (1.6). Chen, Rădulescu and Zhang in [7] obtained the existence of a positive weak solution of Eq (1.6) with $f(x,u) = |u|^{q-2}u$ and $1 < q < \frac{Np}{N-ps}$. Many papers studied the existence of infinitely many weak solutions and nontrivial solutions of Eq (1.6), we refer the readers to [3, 9, 11, 16–21, 23, 24] and the reference therein.

In this paper, we are interested in the existence and multiplicity of positive solutions of Problem (1.1) with critical growth. Our technique based on the Ekland variational principle and the Mountain pass lemma. Since the Problem (1.1) is critical growth, which leads to the cause of the lack of compactness of the embedding $W^{s,p}(\Omega) \hookrightarrow L^{p_\alpha^*}(\Omega)$, we overcome this difficulty by using the concentration compactness principle.

Now we state our main result.

Theorem 1.1. *Let* $0 \leq \alpha < ps < N$ *and* $\frac{N(p-2)+ps}{p-1} < \beta < \frac{N(p_\alpha^*-1)+\alpha}{p_\alpha^*}$, *there exists* $\Lambda_* > 0$ *such that for all* $\lambda \in (0, \Lambda_*)$, *problem* (1.1) *has at least two positive solutions.*

2. Preliminary results

Define $W^{s,p}(\Omega)$, the usual fractional Sobolev space endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy\right)^{\frac{1}{p}}.$$

Let $Q = \mathbb{R}^N \times \mathbb{R}^N \setminus (C\Omega \times C\Omega)$ with $C\Omega = \mathbb{R}^N \setminus \Omega$, define

$$X = \left\{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable, } u|_\Omega \in L^p(\Omega) \text{ and } \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy < \infty\right\}.$$

The space X is endowed with the norm

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy\right)^{\frac{1}{p}},$$

where the norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$. The space X_0 is defined as $X_0 = \{u \in X : u = 0 \text{ on } C\Omega\}$ or equivalently the closure of C_0^∞ in X, for all $p > 1$, it is a uniformly convex Banach space endowed with the norm

$$\|u\| := \|u\|_{X_0} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy\right)^{\frac{1}{p}}. \tag{2.1}$$

The dual space of X_0 will be denoted by X_0^*. Since $u = 0$ in $\mathbb{R}^N \setminus \Omega$, the integral in (2.1) can be extended to $\mathbb{R}^N \times \mathbb{R}^N$.

The energy functional $I_\lambda : X_0 \to \mathbb{R}$ associated with Eq (1.1) is

$$I_\lambda(u) = \frac{a}{p}\|u\|^p + \frac{b}{pk}\|u\|^{pk} - \frac{1}{p_\alpha^*}\int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha} dx - \lambda \int_\Omega \frac{u}{|x|^\beta} dx.$$

We say that u is a weak solution of Eq (1.1), if u satisfies

$$(a+b\|u\|^{p(k-1)})\langle u,\varphi\rangle_{s,p}=\int_\Omega \frac{|u|^{p_\alpha^*-2}u\varphi}{|x|^\alpha}dx+\lambda\int_\Omega \frac{\varphi}{|x|^\beta}dx,$$

for all $\varphi\in X_0$, where

$$\langle u,\varphi\rangle_{s,p}=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))[\varphi(x)-\varphi(y)]}{|x-y|^{N+ps}}dxdy.$$

Let S_α be the best fractional critical Hardy-Sobolev constant

$$S_\alpha:=\inf_{u\in W^{s,p}(\Omega)\setminus\{0\}}\frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy}{\left(\displaystyle\int_\Omega\frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx\right)^{p/p_\alpha^*}}.$$

Denote by S_ρ (respectively, B_ρ) the sphere (respectively, the closed ball) of center zero and radius ρ, i.e., $S_\rho=\{u\in X_0:\|u\|=\rho\}$, $B_\rho=\{u\in X_0:\|u\|\le\rho\}$. $C,C_1,C_2,\dots$ denote various positive constants, which may vary from line to line.

Then, we can obtain the following useful Lemma.

Lemma 2.1. *There exist constants $r,\rho,\Lambda_0>0$, such that the functional I_λ satisfies the following conditions for all $\lambda\in(0,\Lambda_0)$:*
(i) *$I_\lambda(u)\ge r$ with $\|u\|=\rho$ and* $\inf I_\lambda(u)<0$ *for $\|u\|\le\rho$;*
(ii) *There exists $e\in X_0$ such that $\|e\|>\rho$ and $I_\lambda(e)<0$.*

Proof. (i) Let R be a constant such that $\Omega\subset B(0,R)=\{x\in\mathbb{R}^N:|x|<R\}$, by the Hölder inequality and the Sobolev inequality, for all $\beta<\frac{N(p_\alpha^*-1)+\alpha}{p_\alpha^*}$, we have

$$\begin{aligned}
\left|\int_\Omega\frac{u}{|x|^\beta}dx\right| &\le \int_\Omega\left(|u|\cdot\frac{|x|^\alpha}{|x|^\beta}\right)\frac{1}{|x|^\alpha}dx\\
&\le\left(\int_\Omega\frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx\right)^{\frac{1}{p_\alpha^*}}\left(\int_\Omega\frac{1}{|x|^\alpha}\left(\frac{|x|^\alpha}{|x|^\beta}\right)^{\frac{p_\alpha^*}{p_\alpha^*-1}}dx\right)^{\frac{p_\alpha^*-1}{p_\alpha^*}}\\
&\le S_\alpha^{-\frac{1}{p}}\|u\|\left(\omega\int_0^R t^{(N-1)+\alpha\frac{p_\alpha^*}{p_\alpha^*-1}-\alpha-\frac{\beta p_\alpha^*}{p_\alpha^*-1}}dt\right)^{\frac{p_\alpha^*-1}{p_\alpha^*}} \qquad (2.2)\\
&= S_\alpha^{-\frac{1}{p}}\|u\|\left(\frac{\omega}{\alpha\left(\frac{p_\alpha^*}{p_\alpha^*-1}-1\right)+N-\frac{\beta p_\alpha^*}{p_\alpha^*-1}}R^{\alpha\left(\frac{p_\alpha^*}{p_\alpha^*-1}-1\right)+N-\frac{\beta p_\alpha^*}{p_\alpha^*-1}}\right)^{\frac{p_\alpha^*-1}{p_\alpha^*}}\\
&= S_\alpha^{-\frac{1}{p}}\|u\|\psi^{\frac{p_\alpha^*-1}{p_\alpha^*}},
\end{aligned}$$

where $\psi=\frac{\omega}{\alpha\left(\frac{p_\alpha^*}{p_\alpha^*-1}-1\right)+N-\frac{\beta p_\alpha^*}{p_\alpha^*-1}}R^{\alpha\left(\frac{p_\alpha^*}{p_\alpha^*-1}-1\right)+N-\frac{\beta p_\alpha^*}{p_\alpha^*-1}}$ and ω denotes the N-dimensional measure of the unit sphere. Combining with (2.2) and the Sobolev inequality, one has

$$\begin{aligned} I_\lambda(u) &= \frac{a}{p}\|u\|^p + \frac{b}{pk}\|u\|^{pk} - \frac{1}{p_\alpha^*}\int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx - \lambda\int_\Omega \frac{u}{|x|^\beta}dx \\ &\geq \frac{b}{pk}\|u\|^{pk} - \frac{1}{p_\alpha^*}S_\alpha^{-\frac{p_\alpha^*}{p}}\|u\|^{p_\alpha^*} - \lambda S_\alpha^{-\frac{1}{p}}\|u\|\psi^{\frac{p_\alpha^*-1}{p_\alpha^*}} \\ &= \|u\|\left(\frac{b}{pk}\|u\|^{pk-1} - \frac{1}{p_\alpha^*}S_\alpha^{-\frac{p_\alpha^*}{p}}\|u\|^{p_\alpha^*-1} - \lambda S_\alpha^{-\frac{1}{p}}\psi^{\frac{p_\alpha^*-1}{p_\alpha^*}}\right). \end{aligned} \tag{2.3}$$

Let $h(t) = \frac{b}{pk}t^{pk-1} - \frac{1}{p_\alpha^*}S_\alpha^{-\frac{p_\alpha^*}{p}}t^{p_\alpha^*-1}$ for $t > 0$, then there exists

$$\rho = \left[\frac{bp_\alpha^*(p_\alpha^*-1)}{pk(pk-1)}S_\alpha^{\frac{p_\alpha^*}{p}}\right]^{\frac{1}{p_\alpha^*-pk}} > 0$$

such that $\max_{t>0} h(t) = h(\rho) > 0$. Setting $\Lambda_0 = \frac{S_\alpha^{\frac{1}{p}}h(\rho)}{\psi^{\frac{p_\alpha^*-1}{p_\alpha^*}}}$, there exists a constant $r > 0$, for all $\lambda \in (0, \Lambda_0)$, we obtain that $I_\lambda(u) \geq r > 0$ with $\|u\| = \rho$.

For all $u \in X_0\backslash\{0\}$, we get

$$\lim_{t\to 0}\frac{I_\lambda(tu)}{t} = -\lambda\int_\Omega \frac{u}{|x|^\beta}dx < 0. \tag{2.4}$$

Hence, we obtain that $I_\lambda(tu) < 0$ with $t > 0$ small enough, when $\|u\| \leq \rho$, one has

$$m := \inf_{u\in X_0} I_\lambda(u) < 0.$$

(ii) For every $u \in X_0\backslash\{0\}$, we have

$$I_\lambda(tu) = \frac{at^p}{p}\|u\|^p + \frac{bt^{pk}}{pk}\|u\|^{pk} - \frac{t^{p_\alpha^*}}{p_\alpha^*}\int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx - \lambda\int_\Omega \frac{tu}{|x|^\beta}dx \to -\infty$$

as $t \to +\infty$. Consequently, we can find $e \in X_0$ such that $I_\lambda(e) < 0$ provided with $\|e\| > \rho$. The proof is complete. □

Next, we assume that a, b and k satisfy one of the following cases:

Case 2.1. $k = \frac{N-\alpha}{N-ps}, a > 0, 0 < b < S_\alpha^{-k}$,

Case 2.2. $k = \frac{2N-ps-\alpha}{2(N-ps)}, a > 0, b > 0$.

Denote

$$\Lambda = \begin{cases} \left(\frac{1}{p} - \frac{1}{pk}\right)\left(\frac{a^k S_\alpha^k}{1 - bS_\alpha^k}\right)^{\frac{1}{k-1}}, & (Case\ 2.1.1), \\ a\left(\frac{1}{p} - \frac{1}{pk}\right)\left(\frac{bS_\alpha^{2k-1} + S_\alpha^{k-1}\sqrt{b^2S_\alpha^{2k} + 4aS_\alpha}}{2}\right)^{\frac{1}{k-1}} & \\ +\left(\frac{1}{pk} - \frac{1}{p_\alpha^*}\right)\left(\frac{bS_\alpha^k + \sqrt{b^2S_\alpha^{2k} + 4aS_\alpha}}{2}\right)^{\frac{2k-1}{k-1}}, & (Case\ 2.2.1), \end{cases}$$

and

$$D=\left(\frac{aS_\alpha}{p}-\frac{aS_\alpha}{pk}\right)^{-\frac{1}{p-1}}\left(1-\frac{1}{pk}\right)^{\frac{p}{p-1}}\psi^{\frac{p(p_\alpha^*-1)}{p_\alpha^*(p-1)}}.$$

Then, we have the following compactness result.

Lemma 2.2. *Suppose that $a,b>0$ and $1<p<pk<p_\alpha^*$, then the functional I_λ satisfies the $(PS)_{c_\lambda}$ condition for $c_\lambda<c^*=\Lambda-D\lambda^{\frac{p}{p-1}}$.*

Proof. Let $\{u_n\}\subset X_0$ be a $(PS)_{c_\lambda}$ sequence for

$$I_\lambda(u_n)\to c_\lambda,\ \text{and } I'_\lambda(u_n)\to 0 \text{ as } n\to\infty. \tag{2.5}$$

It follows form (2.2) and (2.5) that

$$\begin{aligned}c_\lambda+1+o(\|u_n\|)&\geq I_\lambda(u_n)-\frac{1}{p_\alpha^*}\langle I'_\lambda(u_n),u_n\rangle\\&\geq\left(\frac{1}{p}-\frac{1}{p_\alpha^*}\right)a\|u_n\|^p+\left(\frac{1}{pk}-\frac{1}{p_\alpha^*}\right)b\|u_n\|^{pk}-\left(1-\frac{1}{p_\alpha^*}\right)\lambda S^{-\frac{1}{p}}\|u_n\|\psi^{\frac{p_\alpha^*-1}{p_\alpha^*}}.\end{aligned}$$

This implies that $\{u_n\}$ is bounded in X_0. Up to a subsequence, still denote by $\{u_n\}$, there exists $u\in X_0$ such that

$$\begin{cases}u_n\rightharpoonup u, & \text{weakly in } X_0,\\ u_n\to u, & \text{strongly in } L^q(\Omega)\ (1\leq q<p_\alpha^*),\\ u_n(x)\to u(x), & \text{a.e. in } \Omega.\end{cases} \tag{2.6}$$

By using the concentration compactness principle ([6], Lemma 4.5), there exist $u\in X_0$, two Borel regular measures σ and ν, J denumerable, at most countable set $\{x_j\}_{j\in J}\subset\bar{\Omega}$, and non-negative numbers $\{\sigma_j\}_{j\in J},\{\nu_j\}_{j\in J}\subset[0,\infty)$, for all $j\in J$, such that

$$\begin{gathered}\|u_n\|^p\rightharpoonup\sigma,\ \frac{|u_n|^{p_\alpha^*}}{|x|^\alpha}\rightharpoonup\nu,\\ d\sigma\geq\|u\|^p+\sum_{j\in J}\sigma_j\delta_{x_j},\ \sigma_j:=\sigma(\{x_j\}),\\ d\nu=\frac{|u|^{p_\alpha^*}}{|x|^\alpha}+\sum_{j\in J}\nu_j\delta_{x_j},\ \nu_j:=\nu(\{x_j\}),\\ \sigma_j\geq S_\alpha\nu_j^{\frac{p}{p_\alpha^*}},\end{gathered} \tag{2.7}$$

as $n\to\infty$. Fix $\varepsilon>0$, let $\phi_{\varepsilon,j}(x)\in C_0^\infty(B(x_j,2\varepsilon))$ be a smooth cut-off function centered at x_j such that $0\leq\phi_{\varepsilon,j}\leq 1$, $\|\nabla\phi_{\varepsilon,j}\|_\infty\leq\frac{C}{\varepsilon}$, and

$$\phi_{\varepsilon,j}(x)=\begin{cases}1, & \text{in } B(x_j,\varepsilon),\\ 0, & \text{in } \Omega\setminus B(x_j,2\varepsilon).\end{cases}$$

Clear $\{\phi_{\varepsilon,j}u_n\}$ is bounded in X_0, it follows from $\langle I'_\lambda(u_n), \phi_{\varepsilon,j}u_n\rangle \to 0$ as $n \to \infty$ that

$$
\begin{aligned}
&(a + b\|u_n\|^{p(k-1)}) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))[\phi_{\varepsilon,j}(x)u_n(x) - \phi_{\varepsilon,j}(y)u_n(y)]}{|x - y|^{N+ps}}dxdy \\
&= \int_\Omega \frac{|u_n|^{p^*_\alpha}\phi_{\varepsilon,j}(x)}{|x|^\alpha}dx + \lambda \int_\Omega \frac{u_n\phi_{\varepsilon,j}(x)}{|x|^\beta}dx + o(1).
\end{aligned}
\tag{2.8}
$$

From the first term in (2.8), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))[\phi_{\varepsilon,j}(x)u_n(x) - \phi_{\varepsilon,j}(y)u_n(y)]}{|x - y|^{N+ps}}dxdy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))[u_n(x) - u_n(y)]\phi_{\varepsilon,j}(x)}{|x - y|^{N+ps}}dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))u_n(y)[\phi_{\varepsilon,j}(x) - \phi_{\varepsilon,j}(y)]}{|x - y|^{N+ps}}dxdy.
\end{aligned}
\tag{2.9}
$$

Note that $\{u_n\}$ is bounded in X_0, the third term in (2.9), by using the Hölder inequality and Lemma 2.3 in [22], we get

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left| \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))u_n(y)[\phi_{\varepsilon,j}(x) - \phi_{\varepsilon,j}(y)]}{|x - y|^{N+ps}}dxdy\right| \\
&\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+ps}}dxdy\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\phi_{\varepsilon,j}(x) - \phi_{\varepsilon,j}(y))u_n(y)|^p}{|x - y|^{N+ps}}dxdy\right)^{\frac{1}{p}} \\
&\le C\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\phi_{\varepsilon,j}(x) - \phi_{\varepsilon,j}(y))u_n(y)|^p}{|x - y|^{N+ps}}dxdy\right)^{\frac{1}{p}} \\
&= 0,
\end{aligned}
$$

where $C > 0$ is a positive constant. Letting $\varepsilon \to 0$, by (2.2) and (2.7), we get

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \frac{u_n\phi_{\varepsilon,j}(x)}{|x|^\beta}dx = 0, \\
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p\phi_{\varepsilon,j}(x)}{|x - y|^{N+ps}}dxdy \\
&\ge \left(\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p\phi_{\varepsilon,j}(x)}{|x - y|^{N+ps}}dxdy + \sigma_j\right) = \sigma_j, \\
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \frac{|u_n|^{p^*_\alpha}\phi_{\varepsilon,j}(x)}{|x|^\alpha}dx = \left(\lim_{\varepsilon\to 0}\int_\Omega \frac{|u|^{p^*_\alpha}\phi_{\varepsilon,j}(x)}{|x|^\alpha}dx + \nu_j\right) = \nu_j.
\end{aligned}
\tag{2.10}
$$

Combining the above facts with (2.8), we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(a + b\|u_n\|^{p(k-1)}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p\phi_{\varepsilon,j}(x)}{|x - y|^{N+ps}}dxdy \\
&\ge \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left\{a\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p\phi_{\varepsilon,j}(x)}{|x - y|^{N+ps}}dxdy + a\sigma_j\right. \\
&\quad \left. + b\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p\phi_{\varepsilon,j}(x)}{|x - y|^{N+ps}}dxdy + \sigma_j\Big)^k\right\} \\
&= a\sigma_j + b\sigma_j^k.
\end{aligned}
\tag{2.11}
$$

Hence, taking the limit for $n \to \infty$ and $\varepsilon \to 0$ in (2.8), it follows from (2.10) and (2.11) that

$$\nu_j \geq a\sigma_j + b\sigma_j^k.$$

This together with (2.7) implies that either $\nu_j = 0$ or

$$\nu_j \geq \begin{cases} \left(\dfrac{aS_\alpha}{1-bS_\alpha^k}\right)^{\frac{k}{k-1}}, & (Case\ 2.1.2), \\ \left(\dfrac{bS_\alpha^k + \sqrt{b^2S_\alpha^{2k} + 4aS_\alpha}}{2}\right)^{\frac{2k-1}{k-1}}, & (Case\ 2.2.2). \end{cases} \tag{2.12}$$

From (2.7) and (2.12), we obtain that $\sigma_j = 0$ or

$$\sigma_j \geq \begin{cases} \left(\dfrac{aS_\alpha^k}{1-bS_\alpha^k}\right)^{\frac{1}{k-1}}, & (Case\ 2.1.3), \\ \left(\dfrac{bS_\alpha^{2k-1} + S_\alpha^{k-1}\sqrt{b^2S_\alpha^{2k} + 4aS_\alpha}}{2}\right)^{\frac{1}{k-1}}, & (Case\ 2.2.3). \end{cases} \tag{2.13}$$

To proceed further we show that (2.12) and (2.13) are impossible. Indeed, by contradiction, we assume that there exists $j_0 \in J$ such that (2.12) and (2.13) hold. Applying (2.2), (2.7) and the Sobolev inequality, we get

$$\begin{aligned} c_\lambda &= \lim_{n\to\infty}\left\{I_\lambda(u_n) - \frac{1}{pk}\langle I'_\lambda(u_n), u_n\rangle\right\} \\ &= \lim_{n\to\infty}\left\{a\left(\frac{1}{p} - \frac{1}{pk}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dxdy\right. \\ &\quad \left. + \left(\frac{1}{pk} - \frac{1}{p_\alpha^*}\right)\int_\Omega \frac{|u_n|^{p_\alpha^*}}{|x|^\alpha}dx - \left(1 - \frac{1}{pk}\right)\lambda\int_\Omega \frac{u_n}{|x|^\beta}dx\right\} \\ &\geq a\left(\frac{1}{p} - \frac{1}{pk}\right)(\|u\|^p + \sigma_{j_0}) + \left(\frac{1}{pk} - \frac{1}{p_\alpha^*}\right)\left(\int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx + \nu_{j_0}\right) - \left(1 - \frac{1}{pk}\right)\lambda\int_\Omega \frac{u}{|x|^\beta}dx \\ &\geq a\left(\frac{1}{p} - \frac{1}{pk}\right)\left(S_\alpha\|u\|_{p_\alpha^*}^p + \sigma_{j_0}\right) + \left(\frac{1}{pk} - \frac{1}{p_\alpha^*}\right)\left(\|u\|_{p_\alpha^*}^{p_\alpha^*} + \nu_{j_0}\right) - \left(1 - \frac{1}{pk}\right)\lambda\psi^{\frac{p_\alpha^*-1}{p_\alpha^*}}\|u\|_{p_\alpha^*}. \end{aligned}$$

By using the Young inequality, when $a > 0$, we have

$$\left(1 - \frac{1}{pk}\right)\lambda\psi^{\frac{p_\alpha^*-1}{p_\alpha^*}}\|u\|_{p_\alpha^*} \leq \left(\frac{a}{p} - \frac{a}{pk}\right)S_\alpha\|u\|_{p_\alpha^*}^p + \left(\frac{aS_\alpha}{p} - \frac{aS_\alpha}{pk}\right)^{-\frac{1}{p-1}}\left(1 - \frac{1}{pk}\right)^{\frac{p}{p-1}}\psi^{\frac{p(p_\alpha^*-1)}{p_\alpha^*(p-1)}}\lambda^{\frac{p}{p-1}}.$$

Consequently, we deduce that $c_\lambda \geq c^* = \Lambda - D\lambda^{\frac{p}{p-1}}$. This is a contradiction. Hence $\sigma_j = \nu_j = 0$ for all $j \in J$, which implies that

$$\int_\Omega \frac{|u_n|^{p_\alpha^*}}{|x|^\alpha}dx \to \int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx, \tag{2.14}$$

as $n \to \infty$. Now, we prove that $u_n \to u$ in X_0, let $\varphi \in X_0$ be fixed and B_φ be the linear functional on X_0 defined by

$$B_\varphi(v) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}(v(x)-v(y))dxdy,$$

for every $v \in X_0$. By using the Hölder inequality, one has

$$|B_\varphi(v)| \le \|\varphi\|^{p-1}\|v\|.$$

According to $I'_\lambda(u_n) \to 0$ in X_0^* and $u_n \rightharpoonup u$ in X_0, we have

$$\begin{aligned} o(1) &= \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle \\ &= (a + b\|u_n\|^{p(k-1)})B_{u_n}(u_n - u) - (a + b\|u\|^{p(k-1)})B_u(u_n - u) \\ &\quad - \int_\Omega \frac{|u_n|^{p_\alpha^*-2}u_n - |u|^{p_\alpha^*-2}u}{|x|^\alpha}(u_n - u)dx - \lambda\int_\Omega \frac{u_n - u}{|x|^\beta}dx \\ &= (a + b\|u_n\|^{p(k-1)})\left[B_{u_n}(u_n - u) - B_u(u_n - u)\right] \\ &\quad + \left[(a + b\|u_n\|^{p(k-1)}) - (a + b\|u\|^{p(k-1)})\right]B_u(u_n - u) \\ &\quad - \int_\Omega \frac{|u_n|^{p_\alpha^*-2}u_n - |u|^{p_\alpha^*-2}u}{|x|^\alpha}(u_n - u)dx - \lambda\int_\Omega \frac{u_n - u}{|x|^\beta}dx. \end{aligned} \tag{2.15}$$

Since $\{u_n\}$ is bounded in X_0, by (2.6), one has

$$\begin{aligned} &\lim_{n\to\infty}(a + b\|u_n\|^{p(k-1)})B_{u_n}(u_n - u) = 0, \\ &\lim_{n\to\infty}(a + b\|u_n\|^{p(k-1)})B_u(u_n - u) = 0, \\ &\lim_{n\to\infty}\int_\Omega \frac{u_n - u}{|x|^\beta}dx = 0, \end{aligned} \tag{2.16}$$

then

$$\lim_{n\to\infty}(a + b\|u_n\|^{p(k-1)})\left[B_{u_n}(u_n - u) - B_u(u_n - u)\right] = 0.$$

By $a, b > 0$, we get

$$\lim_{n\to\infty}\left[B_{u_n}(u_n - u) - B_u(u_n - u)\right] = 0. \tag{2.17}$$

Moreover, it follows from the Brezis-Lieb Lemma that

$$\int_\Omega \frac{|u_n - u|^{p_\alpha^*}}{|x|^\alpha}dx = \int_\Omega \frac{|u_n|^{p_\alpha^*}}{|x|^\alpha}dx - \int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx + o(1) \to 0, \text{ as } n \to \infty.$$

This together with (2.14) and the Hölder inequality, one has

$$\int_\Omega \frac{|u_n|^{p_\alpha^*-2}u_n - |u|^{p_\alpha^*-2}u}{|x|^\alpha}(u_n - u)dx \to 0, \text{ as } n \to \infty.$$

Let us now recall the well-known Simon inequalities. That is, for every $\xi, \eta \in \mathbb{R}^N$

$$|\xi - \eta|^p \le \begin{cases} c_p(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta), & \text{for } p \ge 2, \\ C_p[(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta)]^{\frac{p}{2}}(|\xi|^p + |\eta|^p)^{\frac{2-p}{2}}, & \text{for } 1 < p < 2, \end{cases} \tag{2.18}$$

where $c_p, C_p > 0$ depending only on p. According to (2.18), we distinguish two cases:

Case 2.3. *if $p \geq 2$, it follows from (2.17) and (2.18) as $n \to \infty$ that*

$$\begin{aligned}\|u_n - u\|^p &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u(x) - u_n(y) + u(y)|^p}{|x-y|^{N+ps}}dxdy \\ &\leq c_p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+ps}} \\ &\quad \times (u_n(x) - u(x) - u_n(y) + u(y))dxdy \\ &= c_p[B_{u_n}(u_n - u) - B_u(u_n - u)] = o(1).\end{aligned}$$

Case 2.4. *if $1 < p < 2$, letting $\xi = u_n(x) - u_n(y)$ and $\eta = u(x) - u(y)$ in (2.18) as $n \to \infty$, we get*

$$\begin{aligned}\|u_n - u\|^p &\leq C_p[B_{u_n}(u_n - u) - B_u(u_n - u)]^{\frac{p}{2}}(\|u_n\|^p + \|u\|^p)^{\frac{2-p}{2}} \\ &\leq C_p[B_{u_n}(u_n - u) - B_u(u_n - u)]^{\frac{p}{2}}(\|u_n\|^{\frac{p(2-p)}{2}} + \|u\|^{\frac{p(2-p)}{2}}) \\ &\leq C_p[B_{u_n}(u_n - u) - B_u(u_n - u)]^{\frac{p}{2}} = o(1).\end{aligned}$$

Indeed, since $\|u_n\|^p$ and $\|u\|^p$ are bounded in X_0, by the subadditivity inequality, for all $\xi, \eta \geq 0$ and $1 < p < 2$, one has

$$(\xi + \eta)^{\frac{2-p}{2}} \leq \xi^{\frac{2-p}{2}} + \eta^{\frac{2-p}{2}}.$$

Thus, we obtain that $u_n \to u$ in X_0. The proof is complete. □

From [13], let $0 \leq \alpha < ps < N$, for all minimizer U_α for S_α, there exist $x_0 \in \mathbb{R}^N$ and a non-increasing $u : \mathbb{R}^+ \to \mathbb{R}$ such that $U_\alpha = u(|x - x_0|)$. Next, we fix a positive radially symmetric decreasing minimizer $U_\alpha = U_\alpha(r)$ for S_α, multiplying U_α by a positive constant, we assume that

$$(-\Delta)_p^s U_\alpha = \frac{U_\alpha^{p_\alpha^*-1}}{|x|^\alpha}, \quad \text{in } \mathbb{R}^N.$$

Then, we have the following Lemma.

Lemma 2.3. *([13]) There exist $c_1, c_2 > 0$ and $\theta > 1$ such that*

$$\frac{c_1}{r^{\frac{n-ps}{p-1}}} \leq U_\alpha(r) \leq \frac{c_2}{r^{\frac{n-ps}{p-1}}}, \quad \frac{U_\alpha(\theta r)}{U_\alpha(r)} \leq \frac{1}{2}, \text{ for every } r \geq 1.$$

For any $\varepsilon > 0$, the function

$$U_{\alpha,\varepsilon}(x) = \varepsilon^{-\frac{N-ps}{p}} U_\alpha\left(\frac{x}{\varepsilon}\right)$$

is also a minimizer for S_α. For all $\delta \geq \varepsilon > 0$, let

$$m_{\varepsilon,\delta} = \frac{U_{\alpha,\varepsilon}(\delta)}{U_{\alpha,\varepsilon}(\delta) - U_{\alpha,\varepsilon}(\theta\delta)},$$

and

$$g_{\varepsilon,\delta}(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq U_{\alpha,\varepsilon}(\theta\delta), \\ m_{\varepsilon,\delta}^p(t - U_{\alpha,\varepsilon}(\theta\delta)), & \text{if } U_{\alpha,\varepsilon}(\theta\delta) \leq t \leq U_{\alpha,\varepsilon}(\delta), \\ t + U_{\alpha,\varepsilon}(\delta)(m_{\varepsilon,\delta}^{p-1} - 1), & \text{if } t \geq U_{\alpha,\varepsilon}(\delta), \end{cases}$$

as well as

$$G_{\varepsilon,\delta}(t)=\int_0^t g'_{\varepsilon,\delta}(\tau)^{\frac{1}{p}}d\tau=\begin{cases}0, & \text{if } 0\le t\le U_{\alpha,\varepsilon}(\theta\delta),\\ m_{\varepsilon,\delta}(t-U_{\alpha,\varepsilon}(\theta\delta)), & \text{if } U_{\alpha,\varepsilon}(\theta\delta)\le t\le U_{\alpha,\varepsilon}(\delta),\\ t, & \text{if } t\ge U_{\alpha,\varepsilon}(\delta).\end{cases}$$

The functions $g_{\varepsilon,\delta}$ and $G_{\varepsilon,\delta}$ are nondecreasing and absolutely continuous. Consider now the radially symmetric non-increasing function $u_{\varepsilon,\delta}(r)=G_{\varepsilon,\delta}(U_{\alpha,\varepsilon}(r))$, which satisfies

$$u_{\varepsilon,\delta}(r)=\begin{cases}U_{\alpha,\varepsilon}(r), & \text{if } r\le\delta,\\ 0, & \text{if } r\ge\theta\delta.\end{cases}$$

Moreover, from [6, 13], there exists $C>0$ such that for all $0<2\varepsilon\le\delta<\theta^{-1}\delta_\Omega$, we have

$$\begin{cases}\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_{\varepsilon,\delta}(x)-u_{\varepsilon,\delta}(y)|^p}{|x-y|^{N+ps}}dxdy\le S_\alpha^{\frac{N-\alpha}{ps-\alpha}}+C\left(\frac{\varepsilon}{\delta}\right)^{\frac{N-ps}{p-1}},\\ \displaystyle\int_{\mathbb{R}^N}\frac{|u_{\varepsilon,\delta}|^{p_\alpha^*}}{|x|^\alpha}dx\ge S_\alpha^{\frac{N-\alpha}{ps-\alpha}}-C\left(\frac{\varepsilon}{\delta}\right)^{\frac{N-\alpha}{p-1}}.\end{cases}\tag{2.19}$$

Then we have the following Lemma.

Lemma 2.4. *Suppose that $0\le\alpha<ps<N$ and $\frac{N(p-2)+ps}{p-1}<\beta<\frac{N(p_\alpha^*-1)+\alpha}{p_\alpha^*}$. Then there exists $\Lambda_*>0$, for all $\lambda\in(0,\Lambda_*)$ such that $\sup_{t\ge0}I_\lambda(tu_\varepsilon)<c^*=\Lambda-D\lambda^{\frac{p}{p-1}}$(where c^* is the constant given in Lemma 2.2).*

Proof. By Lemma 2.1, we obtain that $I_\lambda(tu)\to-\infty$ as $t\to\infty$ and $I_\lambda(tu)<0$ as $t\to0$, then there exists $t_\varepsilon>0$ such that $I_\lambda(t_\varepsilon u)=\sup_{t>0}I_\lambda(tu)\ge r>0$. Assume that there exist positive constants $t_1,t_2>0$ such that $0<t_1<t_\varepsilon<t_2<+\infty$. Without loss of generality, we take $\delta=1$ in the definition of $u_{\varepsilon,\delta}\in W^{s,p}(\Omega)$ given in Lemma 2.3, for any sufficiently small $0<\varepsilon<1$, set $u_\varepsilon=u_{\varepsilon,1}$ and $I_\lambda(tu_\varepsilon)=J(t)-\lambda\int_\Omega\frac{u_\varepsilon}{|x|^\beta}dx$, where

$$J(t)=\frac{at^p}{p}\|u_\varepsilon\|^p+\frac{bt^{pk}}{pk}\|u_\varepsilon\|^{pk}-\frac{t^{p_\alpha^*}}{p_\alpha^*}\int_\Omega\frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx.$$

It is easy to see that $\lim_{t\to0}J(t)=0$ and $\lim_{t\to\infty}J(t)=-\infty$. Hence, there exists $t_\varepsilon>0$ such that $J(t_\varepsilon)=\max_{t\ge0}J(t)$, that is

$$J'(t)|_{t_\varepsilon}=at_\varepsilon^{p-1}\|u_\varepsilon\|^p+bt_\varepsilon^{pk-1}\|u_\varepsilon\|^{pk}-t_\varepsilon^{p_\alpha^*-1}\int_\Omega\frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx=0.\tag{2.20}$$

By (2.20), we have

$$t_\varepsilon=\begin{cases}\left(\dfrac{a\|u_\varepsilon\|^p}{\int_\Omega\frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx-b\|u_\varepsilon\|^{pk}}\right)^{\frac{1}{p(k-1)}}, & (Case\ 2.1.4),\\ \left[\dfrac{b\|u_\varepsilon\|^{pk}+\sqrt{b^2\|u_\varepsilon\|^{2pk}+4a\|u_\varepsilon\|^p\int_\Omega\frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx}}{2\int_\Omega\frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx}\right]^{\frac{1}{p(k-1)}}, & (Case\ 2.2.4).\end{cases}\tag{2.21}$$

In addition, by the definition of u_ε, we have

$$\begin{aligned}\int_\Omega \frac{u_\varepsilon}{|x|^\beta}dx &\geq \int_{B_\delta} \frac{u_\varepsilon}{|x|^\beta}dx\\ &= \int_{B_\delta} \frac{\varepsilon^{-\frac{N-ps}{p}}U(\frac{x}{\varepsilon})}{|x|^\beta}dx\\ &= \varepsilon^{N-\beta-\frac{N-ps}{p}}\int_{B_{\frac{\delta}{\varepsilon}}} \frac{U(x)}{|x|^\beta}dx\\ &\geq C\varepsilon^{N-\beta-\frac{N-ps}{p}}\int_{B_{\frac{\delta}{\varepsilon}}\setminus B_1} \frac{1}{|x|^{\frac{N-ps}{p-1}}|x|^\beta}dx\\ &\geq C_1\varepsilon^{N-\beta-\frac{N-ps}{p}},\end{aligned} \tag{2.22}$$

where $\beta < \frac{N(p_\alpha^*-1)+\alpha}{p_\alpha^*} = N - \frac{N-ps}{p}$ and $C_1 > 0$ is a positive constant. Now, we consider the following two cases:

For Case 2.1, if $k = \frac{N-\alpha}{N-ps}, a > 0, 0 < b < S_\alpha^{-k}$ and $p_\alpha^* = pk$. It follows from (2.19), (2.20) and (2.22) that

$$\begin{aligned}\sup_{t\geq 0} J(t) = J(t_\varepsilon) &= t_\varepsilon^p\left(\frac{a}{p}||u_\varepsilon||^p + \frac{b}{pk}t_\varepsilon^{p(k-1)}||u_\varepsilon||^{pk} - \frac{1}{p_\alpha^*}t_\varepsilon^{p_\alpha^*-p}\int_\Omega \frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx\right)\\ &= t_\varepsilon^p\left(\frac{a}{p}-\frac{a}{pk}\right)||u_\varepsilon||^p + \left(\frac{1}{pk}-\frac{1}{p_\alpha^*}\right)t_\varepsilon^{p_\alpha^*}\int_\Omega \frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx\\ &= \left(\frac{a}{p}-\frac{a}{pk}\right)||u_\varepsilon||^p\left(\frac{a||u_\varepsilon||^p}{\int_\Omega \frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx - b||u_\varepsilon||^{pk}}\right)^{\frac{1}{k-1}}\\ &\leq \left(\frac{a}{p}-\frac{a}{pk}\right)\left(S_\alpha^{\frac{N-\alpha}{ps-\alpha}} + C\varepsilon^{\frac{N-ps}{p-1}}\right)\\ &\quad\left(\frac{a\left(S_\alpha^{\frac{N-\alpha}{ps-\alpha}} + C\varepsilon^{\frac{N-ps}{p-1}}\right)}{\left(S_\alpha^{\frac{N-\alpha}{ps-\alpha}} - C\varepsilon^{\frac{N-\alpha}{p-1}}\right) - b\left(S_\alpha^{\frac{N-\alpha}{ps-\alpha}} + C\varepsilon^{\frac{N-ps}{p-1}}\right)^k}\right)^{\frac{1}{k-1}}\\ &\leq \left(\frac{1}{p}-\frac{1}{pk}\right)\left(\frac{a^k S_\alpha^k}{1-bS_\alpha^k}\right)^{\frac{1}{k-1}} + C_2\varepsilon^{\frac{N-ps}{p-1}}.\end{aligned}$$

Consequently, from the above information, we obtain

$$\begin{aligned}\sup_{t\geq 0} I_\lambda(tu_\varepsilon) &\leq J(t_\varepsilon) - \lambda\int_\Omega \frac{u_\varepsilon}{|x|^\beta}dx\\ &\leq \Lambda + C_2\varepsilon^{\frac{N-ps}{p-1}} - C_1\lambda\varepsilon^{N-\beta-\frac{N-ps}{p}}\\ &< \Lambda - D\lambda^{\frac{p}{p-1}},\end{aligned}$$

where $C_1, C_2 > 0$ (independent of ε, λ). Here we have used the fact that $\frac{N(p-2)+ps}{p-1} < \beta < \frac{N(p_\alpha^*-1)+\alpha}{p_\alpha^*}$, and let $\varepsilon = \lambda^{\frac{p}{N-ps}}$, $0 < \lambda < \Lambda_1 = \min\{1, (\frac{C_2+D}{C_1})^{\frac{(N-ps)(p-1)}{p[(N-\beta)(p-1)-(N-ps)]}}\}$, then

$$\begin{aligned} C_2\lambda^{\frac{p}{p-1}} - C_1\lambda\left(\lambda^{\frac{p}{N-ps}}\right)^{N-\beta-\frac{N-ps}{p}} &= \lambda^{\frac{p}{p-1}}(C_2 - C_1\lambda^{\frac{p[(N-\beta)(p-1)-(N-ps)]}{(N-ps)(p-1)}}) \\ &< -D\lambda^{\frac{p}{p-1}}. \end{aligned} \tag{2.23}$$

For Case 2.2, if $k = \frac{2N-ps-\alpha}{2(N-ps)}$, $a > 0, b > 0$. According to (2.19), (2.20) and (2.22), we have

$$\begin{aligned} \sup_{t\geq 0} J(t) &= J(t_\varepsilon) \\ &= t_\varepsilon^p\left(\frac{a}{p}\|u_\varepsilon\|^p + \frac{b}{pk}t_\varepsilon^{p(k-1)}\|u_\varepsilon\|^{pk} - \frac{1}{p_\alpha^*}t_\varepsilon^{p_\alpha^*-p}\int_\Omega \frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx\right) \\ &= \left(\frac{a}{p} - \frac{a}{pk}\right)t_\varepsilon^p\|u_\varepsilon\|^p + \left(\frac{1}{pk} - \frac{1}{p_\alpha^*}\right)t_\varepsilon^{p_\alpha^*}\int_\Omega \frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx \\ &= a\left(\frac{1}{p} - \frac{1}{pk}\right)\left[\frac{b\|u_\varepsilon\|^{pk} + \sqrt{b^2\|u_\varepsilon\|^{2pk} + 4a\|u_\varepsilon\|^p\int_\Omega \frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx}}{2\int_\Omega \frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx}\right]^{\frac{1}{k-1}}\|u_\varepsilon\|^p \\ &\quad + \left(\frac{1}{pk} - \frac{1}{p_\alpha^*}\right)\left[\frac{b\|u_\varepsilon\|^{pk} + \sqrt{b^2\|u_\varepsilon\|^{2pk} + 4a\|u_\varepsilon\|^p\int_\Omega \frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx}}{2\int_\Omega \frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx}\right]^{\frac{p_\alpha^*}{p(k-1)}}\int_\Omega \frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx \\ &\leq a\left(\frac{1}{p} - \frac{1}{pk}\right)\left(\frac{bS_\alpha^{2k-1} + S_\alpha^{k-1}\sqrt{b^2S_\alpha^{2k} + 4aS_\alpha}}{2}\right)^{\frac{1}{k-1}} \\ &\quad + \left(\frac{1}{pk} - \frac{1}{p_\alpha^*}\right)\left(\frac{bS_\alpha^k + \sqrt{b^2S_\alpha^{2k} + 4aS_\alpha}}{2}\right)^{\frac{2k-1}{k-1}} + C_3\varepsilon^{\frac{N-ps}{p-1}}, \end{aligned}$$

where $C_3 > 0$ (independent of ε, λ). Consequently, it is similar to Case 2.1, by (2.23), there exists $\Lambda_2 > 0$ such that $0 < \lambda < \Lambda_2$, we get

$$\begin{aligned} \sup_{t\geq 0} I_\lambda(tu_\varepsilon) &\leq J(t_\varepsilon) - \lambda\int_\Omega \frac{u_\varepsilon}{|x|^\beta}dx \\ &\leq \Lambda + C_3\varepsilon^{\frac{N-ps}{p-1}} - C_1\varepsilon^{N-\beta-\frac{N-ps}{p}} \\ &< \Lambda - D\lambda^{\frac{p}{p-1}}. \end{aligned}$$

The proof is complete. □

3. Proof of Theorem 1.1

Theorem 3.1. *Suppose that* $0 < \lambda < \Lambda_*$ ($\Lambda_* = \min\{\Lambda_0, \Lambda_1, \Lambda_2, 1\}$). *Then the Eq* (1.1) *has two positive solutions.*

Proof. It follows from Lemma 2.1 that

$$m := \inf_{u\in B\rho(0)} I_\lambda(u) < 0.$$

By the Ekland variational principle [8], there exists a minimizing sequence $\{u_n\} \subset \overline{B\rho(0)}$ such that

$$I_\lambda(u_n) \le \inf_{u \in \overline{B\rho(0)}} I_\lambda(u) + \frac{1}{n},\ I_\lambda(v) \ge I_\lambda(u_n) - \frac{1}{n}\|v - u_n\|,\ v \in \overline{B\rho(0)}.$$

Hence, we obtain that $I_\lambda(u_n) \to m$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. By Lemma 2.2, we have $u_n \to u_\lambda$ in X_0 with $I_\lambda(u_n) \to m < 0$, which implies that $u_\lambda \not\equiv 0$. Note that $I_\lambda(u_n) = I_\lambda(|u_n|)$, we have $u_\lambda \ge 0$. Thus, by using the strong maximum principle ([14], Lemma 2.3), we obtain that u_λ is a positive solution of Eq (1.1) such that $I_\lambda(u_\lambda) < 0$.

Applying the Mountain pass Lemma [2], Lemmas 2.1 and 2.2, there exists a sequence $\{u_n\} \subset X_0$ such that

$$I_\lambda(u_n) \to c_\lambda,\ \text{and } I'_\lambda(u_n) \to 0 \text{ as } n \to \infty,$$

where

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

and

$$\Gamma = \{\gamma \in C([0,1], X_0) : \gamma(0) = 0, \gamma(1) = e\}.$$

According to Lemma 2.2, we know that $\{u_n\} \subset X_0$ has a convergent subsequence, still denoted by $\{u_n\}$, we may assume that $u_n \to u_*$ in X_0 as $n \to \infty$.

$$I_\lambda(u_*) = \lim_{n\to\infty} I_\lambda(u_n) > r > 0,$$

which implies that $u_* \not\equiv 0$. Similarly, we can obtain that u_* is a positive solution of Eq (1.1) with $I_\lambda(u_*) > 0$. That is, the proof of Theorem 1.1 is complete. □

4. Conclusions

In this paper, we consider a class of fractional p-Kirchhoff type equations with critical growth. Under some suitable assumptions, by using the variational method and the concentration compactness principle, we obtain the existence and multiplicity of positive solutions.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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