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Research article

Sum of the triple divisor function of mixed powers

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Abstract: Let $d_3(n)$ denote the 3-th divisor function. In this paper, we study the asymptotic formula of the sum

$$\sum_{\substack{1\leqslant n_1,n_2\leqslant X^{\frac{1}{2}}\\ 1\leqslant n_3\leqslant X^{\frac{1}{k}}}} d_3(n_1^2+n_2^2+n_3^k)$$

with $n_1, n_2, n_3 \in \mathbb{Z}^+$ and $k \geqslant 3$ be an integer. Previously only the case of $k = 2$ is studied.

Keywords: circle method; divisor problem; asymptotic formula; mixed powers
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1. Introduction

Let $X > 1$ be a large integer and $d_t(n)$ denote the t-th divisor function. For $t = 2$, we denote $d(n) := d_2(n)$. In 2012, Guo and Zhai [3] considered the asymptotic formula of the sum

$$\sum_{1\leqslant n_1,n_2,n_3\leqslant X^{\frac{1}{2}}} d(n_1^2+n_2^2+n_3^2) = c_1X^{\frac{3}{2}}(\log X)^2 + c_2X^{\frac{3}{2}} + O(X^{\frac{4}{3}+\epsilon}),$$

where c_1 and c_2 are constants. The error term was refined to $O(X\log^7 X)$ by Zhao [10]. Later, Hu [4] considered the case of $n_1^2+n_2^2+n_3^2+n_4^2$. In 2016, Lü and Mu [7] considered the asymptotic formula of the sum

$$\sum_{\substack{1\leqslant n_1,n_2\leqslant X^{\frac{1}{2}}\\ 1\leqslant n_3\leqslant X^{\frac{1}{k}}}} d(n_1^2+n_2^2+n_3^k) = c_3X^{1+\frac{1}{k}}\log X + c_4X^{1+\frac{1}{k}} + O(X^{1+\frac{1}{k}-\delta(k)+\epsilon}),$$

with $k \geqslant 3$. Here c_3, c_4 are constants and

$$\delta(k)=\begin{cases}\frac{5}{42}, & k=3,\\ \frac{1}{16}, & k=4,\\ \frac{1}{40}, & k=5,\\ \frac{1}{k2^{k-1}}, & 6\leqslant k\leqslant 7,\\ \frac{1}{2k^2(k-1)}, & k\geqslant 8.\end{cases}$$

For $t=3$, recently, Sun and Zhang [8] considered the asymptotic formula of the sum

$$\sum_{1\leqslant n_1,n_2,n_3\leqslant X^{\frac{1}{2}}} d_3(n_1^2+n_2^2+n_3^2)=c_5X^{\frac{3}{2}}\log^2 X+c_6X^{\frac{3}{2}}\log X+c_7X^{\frac{3}{2}}+O(X^{\frac{3}{2}-\frac{1}{8}+\epsilon}).$$

where c_5, c_6 and c_7 are constants. Hu and Yang [5] also considered the case of $n_1^2+n_2^2+n_3^2+n_4^2$.

In this paper, inspired by Lü and Mu [7], we want to consider the asymptotic formula of the sum

$$\sum_{\substack{1\leqslant n_1,n_2\leqslant X^{\frac{1}{2}}\\ 1\leqslant n_3\leqslant X^{\frac{1}{k}}}} d_3(n_1^2+n_2^2+n_3^k)$$

with $k\geqslant 3$. In order to state our result, let $G_k(a,b,q)$ be the Gauss sum

$$G_k(a,b,q)=\sum_{x \bmod q} e\left(\frac{ax^k+bx}{q}\right).$$

and denote $G(a,b,q):=G_2(a,b,q)$. For $0\leqslant j\leqslant 2$, define

$$A_j(q)=\sum_{b=1}^{q} e\left(-\frac{ab}{q}\right)c_{j+1}(b,q),$$

where $a>0$ is an integer. The coefficients $c_j(b,q)$ are sums of terms of the form

$$\sum_{b_1b_2\equiv b(\bmod q)} f(b_1)$$

for some function f. The number of terms in $c_j(b,q)$ depends only on k. More precisely, The coefficients $c_j(b,q)$ are given explicitly in [1, (2.13)]. From [2, (4.8)] we can get

$$A_j(q)\ll_k q^{-1}. \tag{1.1}$$

For $0\leqslant i\leqslant j\leqslant 2$, define

$$\mathfrak{S}=\frac{1}{j!}\sum_{q=1}^{\infty}\sum_{a \bmod q}{}^{*}\frac{A_j(q)G^2(a,0,q)G_k(a,0,q)}{q^3} \tag{1.2}$$

and

$$\mathfrak{J} = \int_{-\infty}^{\infty} \left(\int_0^3 e(-\beta u)(\log u)^i \, du \right) \left(\int_0^1 e(\beta v^2) \, dv \right)^2 \left(\int_0^1 e(\beta v^k) dv \right) d\beta. \tag{1.3}$$

Our result is the following theorem.

Theorem 1.1. *let $X > 1$ and $k \geqslant 3$ be integers and let $d_3(n)$ denote the 3-th divisor function. We have*

$$\sum_{\substack{1 \leqslant n_1, n_2 \leqslant X^{\frac{1}{2}} \\ 1 \leqslant n_3 \leqslant X^{\frac{1}{k}}}} d_3(n_1^2 + n_2^2 + n_3^k)$$
$$= \sum_{j=0}^{2} \mathfrak{S} \sum_{i=0}^{j} C_j^i \mathfrak{J} X^{1+\frac{1}{k}} (\log X)^{j-i} + O(X^{1+\frac{1}{k}-\delta(k)+\epsilon}),$$

where $\mathfrak{S}$ are defined in (1.2), $\mathfrak{J}$ are defined in (1.3), and

$$\delta(k) = \begin{cases} \dfrac{1}{15}, & k = 3, \\ \dfrac{1}{k2^{k-1}}, & 4 \leqslant k \leqslant 7, \\ \dfrac{1}{2k^2(k-1)}, & k \geqslant 8. \end{cases}$$

To prove our theorem, we use the circle method and some strategies in the work of [7].

2. Outline of the method

Let $X > 1$ and $k \geqslant 3$ be integers. In order to apply the circle method, we choose the parameters P and Q such that

$$P = X^{\theta} \quad \text{and} \quad Q = X^{1-\theta},$$

where

$$\theta = \begin{cases} \dfrac{k+1}{6k+2}, & 3 \leqslant k \leqslant 9, \\ \dfrac{2}{k+2}, & k \geqslant 10. \end{cases}$$

Obviously, we have $Q > X^{1/2}$. By Dirichlet's lemma on rational approximations, each $\alpha \in [Q^{-1}, 1 + Q^{-1}]$ can be written in the form

$$\alpha = \frac{a}{q} + \beta, \quad |\beta| \leqslant \frac{1}{qQ} \tag{2.1}$$

for some integers a, q with $1 \leqslant a \leqslant q \leqslant Q$ and $(a, q) = 1$. We denote by $\mathfrak{M}(q, a)$ the set of α satisfying (2.1) and define the major arcs and the minor arcs as follows.

$$\mathfrak{M} = \bigcup_{q \leqslant P} \bigcup_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \mathfrak{M}(q, a), \quad \mathfrak{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q} \right] \setminus \mathfrak{M}.$$

Let

$$F(\alpha, X) = \sum_{1 \leqslant n \leqslant 3X} d_3(n) e(-n\alpha),$$
$$S_k(\alpha, X) = \sum_{1 \leqslant m \leqslant X^{\frac{1}{k}}} e(m^k \alpha). \tag{2.2}$$

Our Theorem 1.1 is a consequence of the following Proposition 2.1. We will give a proof of Proposition 2.1 in Section 3.

Proposition 2.1. *Let $F(\alpha, X)$ and $S_k(\alpha, X)$ be defined as in (2.2). For $\alpha \in \mathfrak{M}$, we have*

$$\int_{\mathfrak{M}} S_2^2(\alpha, X) S_k(\alpha, X) F(\alpha, X) d\alpha$$
$$= \sum_{j=0}^{2} \mathfrak{S} \sum_{i=0}^{j} C_j^i \mathfrak{I} X^{1+\frac{1}{k}} (\log X)^{j-i} + O(X^{1+\frac{1}{k}-\delta'(k)+\epsilon}),$$

where $\mathfrak{S}$ and $\mathfrak{I}$ are defined as in Theorem 1.1 and

$$\delta'(k) = \begin{cases} \dfrac{k+1}{2k(3k+1)}, & 3 \leqslant k \leqslant 9, \\ \dfrac{2}{k(k+2)}, & k \geqslant 10. \end{cases}$$

Now we can get Theorem 1.1 from Proposition 2.1.

Proof of Theorem 1.1. Applying the circle method, we get

$$\sum_{\substack{1 \leqslant n_1, n_2 \leqslant X^{\frac{1}{2}} \\ 1 \leqslant n_3 \leqslant X^{\frac{1}{k}}}} d_3(n_1^2 + n_2^2 + n_3^k)$$
$$= \int_{\frac{1}{Q}}^{1+\frac{1}{Q}} S^2(\alpha, X) S_k(\alpha, X) F(\alpha, X) d\alpha$$
$$= \int_{\mathfrak{M}} S^2(\alpha, X) S_k(\alpha, X) F(\alpha, X) d\alpha + \int_{\mathfrak{m}} S^2(\alpha, X) S_k(\alpha, X) F(\alpha, X) d\alpha.$$

To estimate the contribution from the minor arcs, we note that each $\alpha \in \mathfrak{m}$ can be written as (2.1) for $P < q \leqslant Q$ and $1 \leqslant a \leqslant q$ with $(q, a) = 1$. In order to make the error term in the asymptotic formula as small as possible, we consider it in two cases.

In the case of $3 \leqslant k \leqslant 7$, for $\alpha \in \mathfrak{m}$, by Weyl's inequality we have

$$S_k(\alpha, X) \ll X^{\frac{1}{k}+\epsilon} (P^{-1} + X^{-\frac{1}{k}} + QX^{-1})^{\frac{1}{2^{k-1}}}.$$

Combining Cauchy's inequality and Hua's inequality, we have

$$\int_{\mathfrak{m}} S^2(\alpha, X) S_k(\alpha, X) F(\alpha, X) d\alpha$$

$$\ll \max_{\alpha\in\mathfrak{m}} |S_k(\alpha, X)| \left(\int_0^1 |S_2(\alpha, X)|^4 \, d\alpha\right)^{\frac{1}{2}} \left(\int_0^1 |F(\alpha, X)|^2 \, d\alpha\right)^{\frac{1}{2}}$$
$$\ll X^{1+\frac{1}{k}+\epsilon} P^{-2^{(1-k)}} + X^{1+\frac{1}{k}(1-2^{(1-k)})+\epsilon} + X^{1+\frac{1}{k}-2^{(1-k)}+\epsilon} Q^{2^{(1-k)}}. \tag{2.3}$$

In the case of $k \geqslant 8$, for $\alpha \in \mathfrak{m}$, we take ([7, Lemma 1.6]) in place of Weyl's inequality and get

$$S_k(\alpha, X) \ll X^{\frac{1}{k}+\epsilon}(P^{-1} + X^{-\frac{1}{k}} + QX^{-1})^{\frac{1}{2k(k-1)}}.$$

Similarly with (2.3), we have

$$\int_{\mathfrak{m}} S^2(\alpha, X) S_k(\alpha, X) F(\alpha, X) d\alpha$$
$$\ll X^{1+\frac{1}{k}+\epsilon} P^{-\frac{1}{2k(k-1)}} + X^{1+\frac{1}{k}(1-\frac{1}{1-2k(k-1)})+\epsilon} + X^{1+\frac{1}{k}-\frac{1}{2k(k-1)}+\epsilon} Q^{\frac{1}{2k(k-1)}}. \tag{2.4}$$

From Propositions 2.1, (2.3) and (2.4), Theorem 1.1 follows by taking $Q = X^{1-\theta}$ with

$$\theta = \begin{cases} \dfrac{k+1}{6k+2}, & 3 \leqslant k \leqslant 9, \\ \dfrac{2}{k+2}, & k \geqslant 10. \end{cases}$$

3. Proof of Proposition 2.1

For $\alpha \in \mathfrak{M}$, we write $\alpha = a/q + \beta$, $1 \leqslant a \leqslant q \leqslant P$ and $|\beta| \leqslant 1/(qQ)$ with $(a, q) = 1$. Then we have

$$\int_{\mathfrak{M}} S_2^2(\alpha, X) S_k(\alpha, X) F(\alpha, X) d\alpha$$
$$= \sum_{q\leqslant P} \sideset{}{^*}\sum_{a \bmod q} \int_{\mathfrak{M}(q,a)} S_2^2(\alpha, X) S_k(\alpha, X) F(\alpha, X) d\alpha$$
$$= \sum_{q\leqslant P} \int_{|\beta|\leqslant\frac{1}{qQ}} \sideset{}{^*}\sum_{a \bmod q} S_2^2\left(\frac{a}{q}+\beta, X\right) S_k\left(\frac{a}{q}+\beta, X\right) F\left(\frac{a}{q}+\beta, X\right) d\beta$$
$$. \tag{3.1}$$

First we need to estimate $F(a/q + \beta, X)$.

Lemma 3.1. *Suppose that* $(a, q) = 1$, $q \leqslant P \leqslant x^{1/3}$ *and* $|\beta| \leqslant 1/(qQ)$. *We have*

$$F\left(\frac{a}{q}+\beta, X\right) = \sum_{j=0}^{2} A_j(q) I_j(\beta) + O(P^{3+\epsilon} + X^{\eta+\epsilon} P),$$

where $\eta = 2/5$, $A_j(q)$ *are defined in (1.2) and*

$$I_j(\beta) = \int_1^{3X} e(-\beta u) \frac{\log^j u}{j!} \, du. \tag{3.2}$$

Proof. This is lemma 3.2 in [2] for $k = 3$ and $l = 3$. □

Integrating by parts, we have

$$I_j(\beta) \ll_k X^\epsilon \min\left\{X, |\beta|^{-1}\right\}. \tag{3.3}$$

Next we need to estimate $S_2(a/q + \beta, X)$ and $S_k(a/q + \beta, X)$.

Lemma 3.2. *Suppose that* $(a, q) = 1$, $q \leqslant P \leqslant X^{\frac{1}{3}}$ *and* $|\beta| \leqslant 1/(qQ)$. *We have*

$$S_2\left(\frac{a}{q} + \beta, X\right) = \frac{G(a, 0, q)}{q}\Psi_0(\beta) + \sum_{-3q/2<b<3q/2} G(a, b, q)\Psi(b, q, \beta),$$

$$S_k\left(\frac{a}{q} + \beta, X\right) = \frac{G_k(a, 0, q)}{q}\Psi_k(\beta) + O\left(q^{\frac{1}{2}+\epsilon}(1 + X|\beta|)^{\frac{1}{2}}\right)$$

where $G_k(a, b, q)$ *are defined as in (1.1),* $\Psi_0(\beta)$ *and* $\Psi_k(\beta)$ *are the integral*

$$\Psi_0(\beta) = \int_0^{X^{\frac{1}{2}}} e(\beta u^2)\, du,$$

$$\Psi_k(\beta) = \int_0^{X^{\frac{1}{k}}} e(\beta u^k)\, du$$

and $\Psi(b, q, \beta)$ *satisfies*

$$\sum_{-3q/2<b<3q/2} |\Psi(b, q, \beta| \ll \log(q + 2). \tag{3.4}$$

Proof. The first formula is come from [10, Lemma 4.1]. We can also find the second formula in [9, Theorem 4.1]. □

From Lemmas 3.1 and 3.2, we have

$$\begin{aligned} &\sum_{a \bmod q}^{*} S^2\left(\frac{a}{q} + \beta, X\right) S_k\left(\frac{a}{q} + \beta, X\right) F\left(\frac{a}{q} + \beta, X\right) \\ = &\sum_{i=1}^{3} T_i(q, \beta) S_k\left(\frac{a}{q} + \beta, X\right) \\ &+O\left((P^{3+\epsilon} + X^{\eta+\epsilon}) \sum_{a \bmod q}^{*} \left|S\left(\frac{a}{q} + \beta, X\right)\right|^2 \left|S_k\left(\frac{a}{q} + \beta, X\right)\right|\right), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} T_i(q, \beta) &= \sum_{j=0}^{2} \sum_{a \bmod q}^{*} A_j(q) I_j(\beta) C_2^{i-1} \left(\frac{G(a, 0, q)\Psi_0(\beta)}{q}\right)^{3-i} \\ &\quad \times \left(\sum_{-3q/2<b<3q/2} G(a, b, q)\Psi(b, q, \beta)\right)^{i-1}. \end{aligned}$$

On inserting (3.5) into (3.1), we have

$$\begin{aligned}
&\int_{\mathfrak{M}} S_2^2(\alpha,X)S_k(\alpha,X)F(\alpha,X)d\alpha \\
= &\sum_{i=1}^{3}\sum_{q\leqslant P}\int_{|\beta|\leqslant\frac{1}{qQ}} T_i(q,\beta)S_k\left(\frac{a}{q}+\beta,X\right)d\beta \\
&+O\left((P^{3+\epsilon}+X^{\eta+\epsilon})\int_{\mathfrak{M}}|S_2(\alpha,X)|^2|S_k(\alpha,X)|d\alpha\right). \qquad (3.6)
\end{aligned}$$

It follows from integration by parts together with trivial bounds that

$$\Psi_0(\beta)\ll\left(\frac{X}{1+|\beta|X}\right)^{\frac{1}{2}},\quad \Psi_k(\beta)\ll\left(\frac{X}{1+|\beta|X}\right)^{\frac{1}{k}}. \qquad (3.7)$$

3.1. Estimate of $T_1(q,\beta)S_k(a/q+\beta,X)$

By the definition of $T_1(q,\beta)$ and Lemma 3.2, we have

$$\begin{aligned}
&\sum_{q\leqslant P}\int_{|\beta|\leqslant\frac{1}{qQ}} T_i(q,\beta)S_k\left(\frac{a}{q}+\beta,X\right)d\beta \\
= &\sum_{q\leqslant P}\int_{|\beta|\leqslant\frac{1}{qQ}} T_1(q,\beta)\frac{G_k(a,0,q)}{q}\Psi_k(\beta)d\beta \\
&+\sum_{q\leqslant P}\int_{|\beta|\leqslant\frac{1}{qQ}} T_i(q,\beta)O\left(q^{\frac{1}{2}+\epsilon}(1+X|\beta|)^{\frac{1}{2}}\right)d\beta \\
= &\sum_{j=0}^{2}\sum_{q\leqslant P}\sum_{a \bmod q}^{*}\frac{A_j(q)G^2(a,0,q)G_k(a,0,q)}{q^3}\int_{|\beta|\leqslant\frac{1}{qQ}} I_j(\beta)\Psi_0^2(\beta)\Psi_k(\beta)d\beta \\
&+\sum_{j=0}^{2}\sum_{q\leqslant P}\sum_{a \bmod q}^{*}\frac{A_j(q)G^2(a,0,q)q^{\frac{1}{2}+\epsilon}}{q^2}\int_{|\beta|\leqslant\frac{1}{qQ}} I_j(\beta)\Psi_0^2(\beta)(1+X|\beta|)^{\frac{1}{2}}d\beta \\
:= &\sum_1+\sum_2.
\end{aligned}$$

By Lemma 3.4 in [6] and Theorem 4.2 in [9], we have

$$G(a,b,q)\ll q^{\frac{1}{2}},\quad G_k(a,0,q)\ll q^{1-\frac{1}{k}}. \qquad (3.8)$$

To $\sum_1$, we extend the integration over $|\beta|\leqslant 1/(qQ)$ to $(-\infty,\infty)$ and the error term is

$$\sum_{j=0}^{2}\sum_{q\leqslant P}\sum_{a \bmod q}^{*}\frac{A_j(q)G^2(a,0,q)G_k(a,0,q)}{q^3}\int_{|\beta|>\frac{1}{qQ}} I_j(\beta)\Psi_0^2(\beta)\Psi_k(\beta)d\beta$$

$$
\begin{aligned}
&\ll \sum_{q\leqslant P} q^{-1-\frac{1}{k}} \int\limits_{|\beta|>\frac{1}{qQ}} |\beta|^{-2-\frac{1}{k}} d\beta \\
&\ll \sum_{q\leqslant P} q^{-1-\frac{1}{k}} (qQ)^{1+\frac{1}{k}} \\
&\ll XQ^{\frac{1}{k}}.
\end{aligned}
$$

Here we use the upper bounds (1.1), (3.3), (3.7) and (3.8). Thus we have

$$
\begin{aligned}
&\sum_{q\leqslant P} \int\limits_{|\beta|\leqslant\frac{1}{qQ}} T_1(q,\beta)\frac{G_k(a,0,q)}{q}\Psi_k(\beta)d\beta \\
= &\sum_{j=0}^{2}\sum_{q\leqslant P}\sum_{a \bmod q}{}^{*} \frac{A_j(q)G^2(a,0,q)G_k(a,0,q)}{q^3}\int_{-\infty}^{\infty} I_j(\beta)\Psi_0^2(\beta)\Psi_k(\beta)d\beta \\
&+O(XQ^{\frac{1}{k}}).
\end{aligned}
$$

Next, we estimate the integral on the right hand side. From the definitions of $I_j(\beta)$, $\Psi_0(\beta)$ and $\Psi_k(\beta)$, we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty} I_j(\beta)\Psi_0^2(\beta)\Psi_k(\beta)d\beta \\
= &\int_{-\infty}^{\infty}\left(\int_1^{3X} e(-\beta u)\frac{\log^j u}{j!}du\right)\left(\int_0^{X^{\frac{1}{2}}} e(\beta v^2)dv\right)^2\left(\int_0^{X^{\frac{1}{k}}} e(\beta v^k)dv\right)d\beta \\
= &\frac{X^{1+\frac{1}{k}}}{j!}\int_{-\infty}^{\infty}\left(\int_1^{3X} e(-\beta u)\log^j u\, du\right)\left(\int_0^1 e(\beta X v^2)\, dv\right)^2\left(\int_0^1 e(\beta X v^k)\, dv\right)d\beta \\
= &\frac{X^{1+\frac{1}{k}}}{j!}\int_{-\infty}^{\infty}\left(\int_{\frac{1}{X}}^{3} e(-\beta u)(\log(Xu))^j du\right)\left(\int_0^1 e(\beta v^2)dv\right)^2\left(\int_0^1 e(\beta v^k)\, dv\right)d\beta \\
= &\frac{X^{1+\frac{1}{k}}}{j!}\int_{-\infty}^{\infty}\left(\int_0^{3} e(-\beta u)(\log(Xu))^j du\right)\left(\int_0^1 e(\beta v^2)dv\right)^2\left(\int_0^1 e(\beta v^k)\, dv\right)d\beta \\
&+O(X^{\frac{1}{k}+\epsilon}).
\end{aligned}
$$

Here

$$
\begin{aligned}
&\frac{X^{1+\frac{1}{k}}}{j!}\int_{-\infty}^{\infty}\left(\int_0^{\frac{1}{X}} e(-\beta u)(\log(Xu))^j du\right)\left(\int_0^1 e(\beta v^2)dv\right)^2\left(\int_0^1 e(\beta v^k)dv\right)d\beta \\
\ll &X^{\frac{1}{k}+\epsilon}.
\end{aligned}
$$

By (1.1) and (3.8), we have

$$
\sum_{j=0}^{2}\sum_{q\leqslant P}\sum_{a \bmod q}{}^{*} \frac{A_j(q)G^2(a,0,q)G_k(a,0,q)}{q^3} \ll 1.
$$

Then we split $\log(Xu)$ and obtain

$$\begin{aligned}
&\sum_{q\leqslant P}\int_{|\beta|\leqslant\frac{1}{qQ}} T_1(q,\beta)\frac{G_k(a,0,q)}{q}\Psi_k(\beta)\,d\beta\\
=\;&\sum_{j=0}^{2}\frac{1}{j!}\sum_{q\leqslant P}\sum_{a \bmod q}^{*}\frac{A_j(q)G^2(a,0,q)G_k(a,0,q)}{q^3}\sum_{i=0}^{j}C_j^i\Im X^{1+\frac{1}{k}}(\log X)^{j-i}\\
&+O(XQ^{\frac{1}{k}}+X^{\frac{1}{k}+\epsilon}),
\end{aligned}$$

where $\Im$ are defined as in (1.3). Further, we extend the summation over $q\leqslant P$ to all positive integers and the estimate of the error term is

$$\begin{aligned}
&\sum_{j=0}^{2}\frac{1}{j!}\sum_{q>P}\sum_{a \bmod q}^{*}\frac{A_j(q)G^2(a,0,q)G_k(a,0,q)}{q^3}\sum_{i=0}^{j}C_j^i\Im X^{1+\frac{1}{k}}(\log X)^{j-i}\\
\ll\;& XQ^{\frac{1}{k}}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\sum_1 &= \sum_{j=0}^{2}\sum_{q\leqslant P}\int_{|\beta|\leqslant\frac{1}{qQ}} T_1(q,\beta)\frac{G_k(a,0,q)}{q}\Psi_k(\beta)d\beta\\
&= \sum_{j=0}^{2}\mathfrak{S}\sum_{i=0}^{j}C_j^i\Im X^{1+\frac{1}{k}}(\log X)^{j-i}+O(XQ^{\frac{1}{k}}+X^{\frac{1}{k}+\epsilon}),
\end{aligned}$$

where $\mathfrak{S}$ and $\Im$ defined in (1.2) and (1.3).

To $\sum_2$, by (1.1), (3.3), (3.7) and (3.8) we have

$$\sum_2 \ll \sum_{q\leqslant P} q^{-\frac{1}{2}+\epsilon}\int\limits_{|\beta|\leqslant\frac{1}{qQ}} |\beta|^{-1}X(1+X|\beta|)^{-\frac{1}{2}}d\beta \ll X^{\frac{3}{2}+\epsilon}Q^{-\frac{1}{2}}.$$

To sum up, we have

$$\begin{aligned}
&\sum_{q\leqslant P}\int\limits_{|\beta|\leqslant\frac{1}{qQ}} T_i(q,\beta)S_k\left(\frac{a}{q}+\beta,X\right)d\beta\\
=\;&\sum_{j=0}^{2}\mathfrak{S}\sum_{i=0}^{j}C_j^i\Im X^{1+\frac{1}{k}}(\log X)^{j-i}+O(XQ^{\frac{1}{k}}+X^{\frac{3}{2}+\epsilon}Q^{-\frac{1}{2}}+X^{\frac{1}{k}+\epsilon}),
\end{aligned}$$

where $\mathfrak{S}$ and $\Im$ are defined in (1.2) and (1.3).

3.2. *Estimate of $T_i(q,\beta)S_k(a/q+\beta,X)$ with $i=2,3$*

Now by the definition of $T_i(q,\beta)$ and Lemma 3.2, we have

$$\sum_{q\leqslant P}\int\limits_{|\beta|\leqslant\frac{1}{qQ}} T_i(q,\beta)S_k(a/q+\beta,X)d\beta$$

$$
\begin{aligned}
&= C_2^{i-1}\sum_{j=0}^{2}\sum_{q\leqslant P}\sum_{a \bmod q}{}^{*}\frac{A_j(q)G^{3-i}(a,0,q)G_k(a,0,q)}{q^{4-i}}\\
&\quad\times\int\limits_{|\beta|\leqslant\frac{1}{qQ}} I_j(\beta)\Psi_0^{3-i}(\beta)\Psi_k(\beta)\left(\sum_{-3q/2<b<3q/2}G(a,b,q)\Psi(b,q,\beta)\right)^{i-1}d\beta\\
&\quad+C_2^{i-1}\sum_{j=0}^{2}\sum_{q\leqslant P}\sum_{a \bmod q}{}^{*}\frac{A_j(q)G^{3-i}(a,0,q)}{q^{3-i}}\int\limits_{|\beta|\leqslant\frac{1}{qQ}} I_j(\beta)\Psi_0^{3-i}(\beta)\\
&\quad\times\left(\sum_{-3q/2<b<3q/2}G(a,b,q)\Psi(b,q,\beta)\right)^{i-1} O\left(q^{\frac{1}{2}+\epsilon}(1+X|\beta|)^{\frac{1}{2}}\right)d\beta\\
&:= \sum_3+\sum_4.
\end{aligned}
$$

Combining Lemma 3.3 with (1.1), (3.3), (3.4), (3.7) and (3.8), for $i = 2, 3$ we have

$$\sum_3 \ll \sum_{q\leqslant P} q^{\frac{3-i}{2}}q^{i-4}q^{1-\frac{1}{k}}q^{\frac{i-1}{2}}X^{\frac{3}{2}+\frac{1}{k}-\frac{i}{2}} \ll X^{\frac{i}{2}+\frac{1}{2}}Q^{1+\frac{1}{k}-i}.$$

For $i = 2$,

$$\sum_4 \ll \sum_{q\leqslant P} q^{\frac{1}{2}}q^{-1}q^{\frac{1}{2}+\epsilon}q^{\frac{1}{2}}X^{\frac{1}{2}+\epsilon} \ll X^{2+\epsilon}Q^{-\frac{3}{2}-\epsilon}.$$

For $i = 3$,

$$\sum_4 \ll \sum_{q\leqslant P} q^{\frac{1}{2}+\epsilon}qX^{\frac{1}{2}}(qQ)^{-\frac{1}{2}} \ll X^{\frac{5}{2}+\epsilon}Q^{-\frac{5}{2}-\epsilon}.$$

3.3. Estimate of the O-term

By Hua's inequality, the contribution of O-term is bounded by

$$
\begin{aligned}
&(P^{3+\epsilon}+X^{\eta+\epsilon}P)\int\limits_{\mathfrak{M}}|S_2(\alpha,X)|^2|S_k(\alpha,X)|\,d\alpha\\
\ll\ &(P^{3+\epsilon}+X^{\eta+\epsilon}P)\left(\int\limits_{\mathfrak{M}}|S_2(\alpha,X)|^4\,d\alpha\right)^{\frac{1}{2}}\left(\int\limits_{\mathfrak{M}}|S_k(\alpha,X)|^2\,d\alpha\right)^{\frac{1}{2}}\\
\ll\ &X^{\frac{1}{2}+\frac{1}{2k}}P^{3+\epsilon}+X^{\frac{1}{2}+\frac{1}{2k}+\eta+\epsilon}P\\
=\ &X^{\frac{7}{2}+\frac{1}{2k}+\epsilon}Q^{-3-\epsilon}+X^{\frac{3}{2}+\frac{1}{2k}+\eta+\epsilon}Q^{-1}.
\end{aligned}
$$

3.4. Completion of the proof

From above, we have

$$\int\limits_{\mathfrak{M}} S_2^2(\alpha,X)S_k(\alpha,X)F(\alpha,X)d\alpha$$

$$
\begin{aligned}
= \;& \sum_{j=0}^{2} \mathfrak{S} \sum_{i=0}^{j} C_j^i \mathfrak{I} X^{1+\frac{1}{k}} (\log X)^{j-i} + O(XQ^{\frac{1}{k}} + X^{\frac{3}{2}+\epsilon} Q^{-\frac{1}{2}} + X^{\frac{1}{k}+\epsilon} \\
& + X^{\frac{3}{2}} Q^{-1+\frac{1}{k}} + X^2 Q^{-2+\frac{1}{k}} + X^{2+\epsilon} Q^{-\frac{3}{2}-\epsilon} + X^{\frac{5}{2}+\epsilon} Q^{-\frac{5}{2}-\epsilon} \\
& + X^{\frac{7}{2}+\frac{1}{2k}+\epsilon} Q^{-3-\epsilon} + X^{\frac{3}{2}+\frac{1}{2k}+\eta+\epsilon} Q^{-1}).
\end{aligned}
$$

By taking $Q = X^{1-\theta}$, where

$$
\theta = \begin{cases} \dfrac{k+1}{6k+2}, & 3 \leqslant k \leqslant 9, \\ \dfrac{2}{k+2}, & k \geqslant 10. \end{cases}
$$

Comparing each item in the error term, we have for $3 \leqslant k \leqslant 9$,

$$
\begin{aligned}
& \int_{\mathfrak{M}} F(\alpha, X) S_2^2(\alpha, X) S_k(\alpha, X) d\alpha \\
= \;& \sum_{j=0}^{2} \mathfrak{S} \sum_{i=0}^{j} C_j^i \mathfrak{I} X^{1+\frac{1}{k}} (\log X)^{j-i} + O\left(X^{1+\frac{1}{k}-\frac{k+1}{2k(3k+1)}+\epsilon} \right).
\end{aligned}
$$

For $k \geqslant 10$,

$$
\begin{aligned}
& \int_{\mathfrak{M}} F(\alpha, X) S_2^2(\alpha, X) S_k(\alpha, X) d\alpha \\
= \;& \sum_{j=0}^{2} \mathfrak{S} \sum_{i=0}^{j} C_j^i \mathfrak{I} X^{1+\frac{1}{k}} (\log X)^{j-i} + O\left(X^{1+\frac{1}{k}-\frac{2}{k(k+2)}+\epsilon} \right).
\end{aligned}
$$

Then we finish the proof of Proposition 2.1.

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Conflict of interest

The authors declare there is no conflict of interests.

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