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Research article

Ulam stability for nonlinear implicit differential equations with Hilfer-Katugampola fractional derivative and impulses

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Abstract: In this paper, we investigate the existence, uniqueness and stability results for a class of nonlinear impulsive Hilfer-Katugampola problems. Our reasoning is founded on the Banach contraction principle and Krasnoselskii's fixed point theorem. In addition, an example is provided to demonstrate the effectiveness of the main results.

Keywords: Hilfer-Katugampola fractional derivative; initial value problem; existence; uniqueness; stability; fixed point; impulses
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1. Introduction

Numerous authors have been interested in fractional differential equations throughout the years [1,4, 5,10,12–14,16,18–20,28]. Several natural events are known to be modeled using fractional differential equations, which provides for a better description of the true state of the problem as compared to the problem modeled using differential equations of integer order [8,20,24,27].

Fractional calculus has played an essential role in several domains during the last two decades, including mechanics, chemistry, economics, biology, control theory, and signal and image processing. Furthermore, it has been discovered that fractional order models may accurately capture the dynamical behavior of many complex systems. Such models are appealing not just to engineers and physicists, but also to mathematicians. For further details and applications [2, 3, 17, 20, 21, 23, 29, 31–35].

Several researchers have recently explored impulsive differential equations given their considerable applicability in diverse domains of science and technology. For a detailed study, see for instance [9, 11, 25].

Motivated by the aforementioned works and the paper [26], in this paper, we consider the following impulsive problem:

$$\left({}^{\rho}D_{\alpha^+}^{\zeta_1,\zeta_2}\mathfrak{p}\right)(\vartheta) = \psi\left(\vartheta, \mathfrak{p}(\vartheta), \left({}^{\rho}D_{\alpha^+}^{\zeta_1,\zeta_2}\mathfrak{p}\right)(\vartheta)\right),\ \vartheta \in (\vartheta_\nu, \vartheta_{\nu+1}];\ \nu = 0, \ldots, \varsigma, \tag{1.1}$$

$$\Delta^{\rho} I_{\alpha^+}^{1-\xi}\mathfrak{p}|_{\vartheta=\vartheta_\nu} = \chi_\nu \in \mathbb{R},\ \nu = 1, \ldots, \varsigma, \tag{1.2}$$

$${}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}(\alpha) = \tilde{\alpha} \in \mathbb{R}, \tag{1.3}$$

where ${}^{\rho}D_{\alpha^+}^{\zeta_1,\zeta_2}, {}^{\rho}I_{\alpha^+}^{1-\xi}$ are the Hilfer-Katugampola fractional derivative of order $\zeta_1 \in (0,1)$ and type $\zeta_2 \in [0,1]$ and Katugampola fractional integral of order $1-\xi$, $(\xi = \zeta_1+\zeta_2-\zeta_1\zeta_2)$ respectively, $\psi : \Theta\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, and $0 < \alpha = \vartheta_0 < \vartheta_1 < \cdots < \vartheta_\varsigma < \vartheta_{\varsigma+1} = \mu$, $\Delta^{\rho} I_{\alpha^+}^{1-\xi}\mathfrak{p}|_{\vartheta=\vartheta_\nu} = {}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta_\nu^+) - {}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta_\nu^-)$,where ${}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta_\nu^+) = \lim\limits_{\kappa\to 0^+} {}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta_\nu+\kappa)$ and ${}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta_\nu^-) = \lim\limits_{\kappa\to 0^-} {}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta_\nu+\kappa)$ represent the right and left limits of ${}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta)$ at $\vartheta = \vartheta_\nu$.

The following is the structure of paper. Section 2 presents certain notations and revisits several notions and auxiliary results. Section 3 presents two results for the problems (1.1)–(1.3) by employing suitable fixed point theorems. In Section 4, the Ulam-Hyers stability for the problems (1.1)–(1.3) is given. Finally, we give an example to illustrate the applicability of our theoretical results.

2. Preliminaries

Let $0 < \alpha < \mu$, $\Theta = [\alpha, \mu]$ and $C(\Theta, \mathbb{R})$ denotes a Banach space composed of all continuous functions from Θ into $\mathbb{R}$ with the norm

$$\|\mathfrak{p}\|_\infty = \sup\{|\mathfrak{p}(\vartheta)| : \vartheta \in \Theta\}.$$

Consider the weighted spaces

$$C_{\xi,\rho}(\Theta) = \left\{\mathfrak{p} : (\alpha,\mu] \to \mathbb{R} : \left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi} \mathfrak{p}(\vartheta) \in C(\Theta,\mathbb{R})\right\}, 0 \le \xi < 1,$$

$$C_{\xi,\rho}^{\beta}(\Theta) = \left\{\mathfrak{p} \in C^{\beta-1}(\Theta) : \mathfrak{p}^{(\beta)} \in C_{\xi,\rho}(\Theta)\right\}, \beta \in \mathbb{N},$$
$$C_{\xi,\rho}^{0}(\Theta) = C_{\xi,\rho}(\Theta),$$

with the norms

$$\|\mathfrak{p}\|_{C_{\xi,\rho}} = \sup_{\vartheta\in\Theta}\left|\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi} \mathfrak{p}(\vartheta)\right|$$

and

$$\|\mathfrak{p}\|_{C_{\xi,\rho}^{\beta}} = \sum_{\nu=0}^{\beta-1} \|\mathfrak{p}^{(\nu)}\|_\infty + \|\mathfrak{p}^{(\beta)}\|_{C_{\xi,\rho}}.$$

Consider the weighted Banach space of piecewise continuous functions defined by

$$PC_{1-\xi,\rho}(\Theta,\mathbb{R}) = \{\mathfrak{p}:\Theta\to\mathbb{R} : \mathfrak{p}\in C_{1-\xi,\rho}((\vartheta_\nu,\vartheta_{\nu+1}],\mathbb{R}),\ \nu=0,\ldots,\varsigma \text{ and there exist } {}^\rho I^{1-\xi}_{\alpha^+}\mathfrak{p}(\vartheta_\nu^-) \text{ and } {}^\rho I^{1-\xi}_{\alpha^+}\mathfrak{p}(\vartheta_\nu^+),\ \nu=1,\ldots,\varsigma \text{ with } {}^\rho I^{1-\xi}_{\alpha^+}\mathfrak{p}(\vartheta_\nu^-) = {}^\rho I^{1-\xi}_{\alpha^+}\mathfrak{p}(\vartheta_\nu)\},$$

with the norm

$$\|\mathfrak{p}\|_{PC_{1-\xi,\rho}} = \sup_{\vartheta\in\Theta}\left|\left(\frac{\vartheta^\rho-\vartheta_\nu^\rho}{\rho}\right)^{1-\xi}\mathfrak{p}(\vartheta)\right|.$$

For $\xi=1$, we obtain the space

$$\begin{aligned} PC_{0,\rho}(\Theta,\mathbb{R}) &= PC(\Theta,\mathbb{R}) = \{\mathfrak{p}:\Theta\to\mathbb{R} : \mathfrak{p}\in C((\vartheta_\nu,\vartheta_{\nu+1}],\mathbb{R}),\ \nu=0,\ldots,\varsigma \text{ and} \\ &\quad \text{there exist } \mathfrak{p}(\vartheta_\nu^-) \text{ and } \mathfrak{p}(\vartheta_\nu^+),\ \nu=1,\ldots,\varsigma \text{ with } \mathfrak{p}(\vartheta_\nu^-)=\mathfrak{p}(\vartheta_\nu)\}, \end{aligned}$$

with the norm

$$\|\mathfrak{p}\|_{PC} = \|\mathfrak{p}\|_\infty .$$

Now, we consider the weighted spaces:

$$\begin{aligned} &PC^\beta_{1-\xi,\rho}(\Theta) = \left\{\mathfrak{p}\in PC^{\beta-1}(\Theta) : \mathfrak{p}^{(\beta)}\in PC_{1-\xi,\rho}(\Theta)\right\}, \beta\in\mathbb{N}, \\ &PC^0_{1-\xi,\rho}(\Theta) = PC_{1-\xi,\rho}(\Theta), \end{aligned}$$

with the norms

$$\|\mathfrak{p}\|_{PC^\beta_{1-\xi,\rho}} = \sum_{\nu=0}^{\beta-1}\|\mathfrak{p}^{(\nu)}\|_\infty + \|\mathfrak{p}^{(\beta)}\|_{PC_{1-\xi,\rho}}.$$

By $X^p_{\tilde{\alpha}}(\alpha,\bar{\alpha})$, $(\tilde{\alpha}\in\mathbb{R},\ 1\le p\le\infty)$, we denote the space of those complex-valued Lebesgue measurable functions ψ on $[\alpha,\bar{\alpha}]$ for which $\|\psi\|_{X^p_{\tilde{\alpha}}}<\infty$, with the norm

$$\|\psi\|_{X^p_{\tilde{\alpha}}} = \left(\int_\alpha^{\bar{\alpha}} |\vartheta^{\tilde{\alpha}}\psi(\vartheta)|^p \frac{d\vartheta}{\vartheta}\right)^{\frac{1}{p}},\ \ (1\le p<\infty, \tilde{\alpha}\in\mathbb{R}).$$

Definition 2.1. *[26] Let $\zeta_1\in\mathbb{R}_+, \tilde{\alpha}\in\mathbb{R}$ and $\varkappa\in X^p_{\tilde{\alpha}}(\alpha,\bar{\alpha})$. The Katugampola fractional integral of order ζ_1 is given by*

$$\left({}^\rho I^{\zeta_1}_{\alpha^+}\varkappa\right)(\vartheta) = \int_\alpha^\vartheta \varrho^{\rho-1}\left(\frac{\vartheta^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\frac{\varkappa(\varrho)}{\Gamma(\zeta_1)}d\varrho,\ \vartheta>\alpha, \rho>0.$$

Definition 2.2. *[26] Let $\zeta_1\in\mathbb{R}_+\setminus\mathbb{N}$ and $\rho>0$. The Katugampola fractional derivative ${}^\rho D^{\zeta_1}_{\alpha^+}$ of order ζ_1 is given by*

$$\begin{aligned} \left({}^\rho D^{\zeta_1}_{\alpha^+}\varkappa\right)(\vartheta) &= \delta^\beta_\rho({}^\rho I^{\beta-\zeta_1}_{\alpha^+}\varkappa)(\vartheta) \\ &= \left(\vartheta^{1-\rho}\frac{d}{d\vartheta}\right)^\beta \int_\alpha^\vartheta \varrho^{\rho-1}\left(\frac{\vartheta^\rho-\varrho^\rho}{\rho}\right)^{\beta-\zeta_1-1}\frac{\varkappa(\varrho)}{\Gamma(\beta-\zeta_1)}d\varrho,\ \vartheta>\alpha,\rho>0, \end{aligned}$$

where $\beta=[\zeta_1]+1$ and $\delta^\beta_\rho = \left(\vartheta^{1-\rho}\frac{d}{d\vartheta}\right)^\beta$.

Theorem 2.3. *[26] Let* $\zeta_1 > 0, \zeta_2 > 0, 1 \le p \le \infty, 0 < \alpha < \bar{\alpha} < \infty$ *and* $\rho, \tilde{\alpha} \in \mathbb{R}, \rho \ge \tilde{\alpha}$. *Then, for* $\varkappa \in X^p_{\tilde{\alpha}}(\alpha, \bar{\alpha})$ *the semigroup property is valid, i.e.*

$$\left({}^{\rho}I^{\zeta_1}_{\alpha^+}\, {}^{\rho}I^{\zeta_2}_{\alpha^+}\varkappa\right)(\vartheta) = \left({}^{\rho}I^{\zeta_1+\zeta_2}_{\alpha^+}\varkappa\right)(\vartheta).$$

Lemma 2.4. *[22] Let* $\zeta_1 > 0$, *and* $0 \le \xi < 1$. *Then,* ${}^{\rho}I^{\zeta_1}_{\alpha^+}$ *is bounded from* $C_{\xi,\rho}(\Theta)$ *into* $C_{\xi,\rho}(\Theta)$.

Lemma 2.5. *[7] Let* $\vartheta > \alpha$. *Then, for* $\zeta_1 \ge 0$ *and* $\zeta_2 > 0$, *we have*

$$\begin{aligned}
\left[{}^{\rho}I^{\zeta_1}_{\alpha^+}\left(\frac{\varrho^{\rho}-\alpha^{\rho}}{\rho}\right)^{\zeta_2-1}\right](\vartheta) &= \frac{\Gamma(\zeta_2)}{\Gamma(\zeta_1+\zeta_2)}\left(\frac{\vartheta^{\rho}-\alpha^{\rho}}{\rho}\right)^{\zeta_1+\zeta_2-1}, \\
\left[{}^{\rho}D^{\zeta_1}_{\alpha^+}\left(\frac{\varrho^{\rho}-\alpha^{\rho}}{\rho}\right)^{\zeta_1-1}\right](\vartheta) &= 0, \quad 0 < \zeta_1 < 1.
\end{aligned}$$

Lemma 2.6. *[22] Let* $\zeta_1 > 0, 0 \le \xi < 1$ *and* $\varkappa \in C_{\xi}[\alpha, \bar{\alpha}]$. *Then,*

$$\left({}^{\rho}D^{\zeta_1}_{\alpha^+}\, {}^{\rho}I^{\zeta_1}_{\alpha^+}\varkappa\right)(\vartheta) = \varkappa(\vartheta), \quad \text{for all} \quad \vartheta \in (\alpha, \bar{\alpha}].$$

Lemma 2.7. *[22] Let* $0 < \zeta_1 < 1, 0 \le \xi < 1$. *If* $\varkappa \in C_{\xi,\rho}[\alpha, \bar{\alpha}]$ *and* ${}^{\rho}I^{1-\zeta_1}_{\alpha^+}\varkappa \in C^1_{\xi,\rho}[\alpha, \bar{\alpha}]$, *then*

$$\left({}^{\rho}I^{\zeta_1}_{\alpha^+}\, {}^{\rho}D^{\zeta_1}_{\alpha^+}\varkappa\right)(\vartheta) = \varkappa(\vartheta) - \frac{\left({}^{\rho}I^{1-\zeta_1}_{\alpha^+}\varkappa\right)(\alpha)}{\Gamma(\zeta_1)}\left(\frac{\vartheta^{\rho}-\alpha^{\rho}}{\rho}\right)^{\zeta_1-1}, \quad \text{for all} \quad \vartheta \in (\alpha, \bar{\alpha}].$$

Definition 2.8. *[22] Let the order* ζ_1 *and the type* ζ_2 *satisfy* $\beta - 1 < \zeta_1 < \beta$ *and* $0 \le \zeta_2 \le 1$, *with* $\beta \in \mathbb{N}$. *The Hilfer-Katugampola fractional derivative of a function* $\varkappa \in C_{1-\xi,\rho}[\alpha, \bar{\alpha}]$, *is defined by*

$$\begin{aligned}
\left({}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\varkappa\right)(\vartheta) &= \left({}^{\rho}I^{\zeta_2(\beta-\zeta_1)}_{\alpha^+}\left(\vartheta^{\rho-1}\frac{d}{d\vartheta}\right)^{\beta}\, {}^{\rho}I^{(1-\zeta_2)(\beta-\zeta_1)}_{\alpha^+}\varkappa\right)(\vartheta) \\
&= \left({}^{\rho}I^{\zeta_2(\beta-\zeta_1)}_{\alpha^+}\delta^{\beta}_{\rho}\, {}^{\rho}I^{(1-\zeta_2)(\beta-\zeta_1)}_{\alpha^+}\varkappa\right)(\vartheta).
\end{aligned}$$

In this paper we consider the case $\beta = 1$ *since* $0 < \zeta_1 < 1$.

Property 2.9. *[22] The operator* ${}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}$ *can be written as*

$${}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+} = {}^{\rho}I^{\zeta_2(1-\zeta_1)}_{\alpha^+}\delta_{\rho}\, {}^{\rho}I^{1-\xi}_{\alpha^+} = {}^{\rho}I^{\zeta_2(1-\zeta_1)}_{\alpha^+}\, {}^{\rho}D^{\xi}_{\alpha^+}, \quad \xi = \zeta_1 + \zeta_2 - \zeta_1\zeta_2.$$

Definition 2.10. *We assume that the parameters* ζ_1, ζ_2, ξ *satisfy*

$$\xi = \zeta_1 + \zeta_2 - \zeta_1\zeta_2, \quad 0 < \zeta_1, \zeta_2, \xi < 1.$$

Then, we can define the spaces

$$C^{\zeta_1,\zeta_2}_{1-\xi,\rho}(\Theta) = \left\{\mathfrak{p} \in C_{1-\xi,\rho}(\Theta),\ {}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{p} \in C_{1-\xi,\rho}(\Theta)\right\},$$

$$C^{\xi}_{1-\xi,\rho}(\Theta) = \left\{\mathfrak{p} \in C_{1-\xi,\rho}(\Theta),\ {}^{\rho}D^{\xi}_{\alpha^+}\mathfrak{p} \in C_{1-\xi,\rho}(\Theta)\right\},$$

$$PC^{\zeta_1,\zeta_2}_{1-\xi,\rho}(\Theta) = \left\{\mathfrak{p} \in PC_{1-\xi,\rho}(\Theta),\ {}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{p} \in PC_{1-\xi,\rho}(\Theta)\right\},$$

and

$$PC^{\xi}_{1-\xi,\rho}(\Theta) = \left\{\mathfrak{p} \in PC_{1-\xi,\rho}(\Theta),\ {}^{\rho}D^{\xi}_{\alpha^+}\mathfrak{p} \in PC_{1-\xi,\rho}(\Theta)\right\}.$$

Since ${}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{p} = {}^{\rho}I^{\xi(1-\zeta_1)}_{\alpha^+}\,{}^{\rho}D^{\xi}_{\alpha^+}\mathfrak{p}$, *by Lemma 2.4, we get*

$$C^{\xi}_{1-\xi,\rho}(\Theta) \subset C^{\zeta_1,\zeta_2}_{1-\xi,\rho}(\Theta) \subset C_{1-\xi,\rho}(\Theta),$$

and

$$PC^{\xi}_{1-\xi,\rho}(\Theta) \subset PC^{\zeta_1,\zeta_2}_{1-\xi,\rho}(\Theta) \subset PC_{1-\xi,\rho}(\Theta).$$

Lemma 2.11. *[22] Let* $0 < \zeta_1 < 1, 0 \le \zeta_2 \le 1$ *and* $\xi = \zeta_1 + \zeta_2 - \zeta_1\zeta_2$. *If* $\mathfrak{p} \in C^{\xi}_{1-\xi,\rho}(\Theta)$, *then*

$${}^{\rho}I^{\xi}_{\alpha^+}\,{}^{\rho}D^{\xi}_{\alpha^+}\mathfrak{p} = {}^{\rho}I^{\zeta_1}_{\alpha^+}\,{}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{p}$$

and

$${}^{\rho}D^{\xi}_{\alpha^+}\,{}^{\rho}I^{\zeta_1}_{\alpha^+}\mathfrak{p} = {}^{\rho}D^{\zeta_2(1-\zeta_1)}_{\alpha^+}\mathfrak{p}.$$

Lemma 2.12. *[22] Let* $0 < \zeta_1 < 1, 0 \le \zeta_2 \le 1$ *and* $\xi = \zeta_1 + \zeta_2 - \zeta_1\zeta_2$. *If* $\varphi \in C_{1-\xi}[\alpha,\bar{\alpha}]$ *and* ${}^{\rho}I^{1-\zeta_2(1-\zeta_1)}_{\alpha^+}\varphi \in C^1_{1-\xi,\rho}[\alpha,\bar{\alpha}]$, *then* ${}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\,{}^{\rho}I^{\zeta_1}_{\alpha^+}\varphi$ *exist on* $(\alpha,\bar{\alpha}]$ *and*

$${}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\,{}^{\rho}I^{\zeta_1}_{\alpha^+}\varphi = \varphi.$$

Definition 2.13. *[20] A two-parameter Mittag-Leffler function* $E_{\zeta_1,\zeta_2}(\mathfrak{p}), \zeta_1, \zeta_2, \mathfrak{p} \in \mathbb{R}$ *with* $\zeta_1 > 0$ *and* $\zeta_2 > 0$, *is defined by*

$$E_{\zeta_1,\zeta_2}(\mathfrak{p}) = \sum_{\nu=0}^{\infty} \frac{\mathfrak{p}^{\nu}}{\Gamma(\zeta_1\nu + \zeta_2)}.$$

If $\zeta_2 = 1$, *we obtain:*

$$E_{\zeta_1}(\mathfrak{p}) = \sum_{\nu=0}^{\infty} \frac{\mathfrak{p}^{\nu}}{\Gamma(\zeta_1\nu + 1)}.$$

Lemma 2.14. *[6] Let* $\zeta_1 > 0, \mathfrak{p}(\vartheta), \varpi_1(\vartheta)$ *be nonnegative functions and* $\varpi_2(\vartheta)$ *be nonnegative and nondecreasing function for* $\vartheta \in [\vartheta_0,\mu), \mu > 0, \varpi_2(\vartheta) \le \theta$ *where* θ *is a constant. If*

$$\mathfrak{p}(\vartheta) \le \varpi_1(\vartheta) + \varpi_2(\vartheta)\int_{\vartheta_0}^{\vartheta}\left(\frac{\vartheta^{\rho}-\varrho^{\rho}}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}\mathfrak{p}(\varrho)d\varrho, \quad \vartheta \in [\vartheta_0,\mu).$$

Then

$$\mathfrak{p}(\vartheta) \le \varpi_1(\vartheta) + \int_{\vartheta_0}^{\vartheta}\left[\sum_{\beta=1}^{\infty}\frac{(\varpi_2(\vartheta)\Gamma(\zeta_1))^{\beta}}{\Gamma(\beta\zeta_1)}\left(\frac{\vartheta^{\rho}-\varrho^{\rho}}{\rho}\right)^{\beta\zeta_1-1}\varrho^{\rho-1}\right]\varpi_1(\varrho)d\varrho,\ \vartheta \in [\vartheta_0,\mu).$$

Corollary 2.15. *[6] Assume that the requirements of Lemma 2.14 are met, and that* $\varpi_1(\vartheta)$ *is a nondecreasing function for* $\vartheta \in [\vartheta_0,\mu)$. *Then*

$$\mathfrak{p}(\vartheta) \le \varpi_1(\vartheta)E_{\zeta_1}\left(\varpi_2(\vartheta)\Gamma(\zeta_1)\left(\frac{\vartheta^{\rho}-\vartheta_0^{\rho}}{\rho}\right)^{\zeta_1}\right), \quad \vartheta \in [\vartheta_0,\mu).$$

Definition 2.16. *The Eq* (1.1) *is Ulam-Hyers stable if there exists a real number $c_{\psi,\varsigma} > 0$ such that for each $\kappa > 0$ and for each solution $\mathfrak{w} \in PC^{\xi}_{1-\xi,\rho}(\Theta)$ of the inequality*

$$\begin{cases} \left|{}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{w}(\vartheta) - \psi(\vartheta, \mathfrak{w}(\vartheta), {}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{w}(\vartheta))\right| \leq \kappa, \ \vartheta \in (\vartheta_\nu, \vartheta_{\nu+1}], \nu = 1, \ldots, \varsigma \\ \left|\Delta^{\rho} I^{1-\xi}_{\alpha^+}\mathfrak{w}|_{\vartheta=\vartheta_\nu} - \chi_\nu\right| \leq \kappa, \ \nu = 1, \ldots, \varsigma; \end{cases} \tag{2.1}$$

there exists a solution $\mathfrak{p} \in PC^{\xi}_{1-\xi,\rho}(\Theta)$ of Eq (1.1) *with*

$$|\mathfrak{w}(\vartheta) - \mathfrak{p}(\vartheta)| \leq c_{\psi,\varsigma}\kappa, \ \vartheta \in \Theta.$$

Definition 2.17. *The Eq* (1.1) *is generalized Ulam-Hyers stable if there exists $\varpi_{\psi,\varsigma} \in PC_{1-\xi}(\mathbb{R}_+, \mathbb{R}_+)$, $\varpi_{\psi,\varsigma}(0) = 0$, such that for each solution $\mathfrak{w} \in PC^{\xi}_{1-\xi,\rho}(\Theta)$ of the inequality* (2.1), *there exists a solution $\mathfrak{p} \in PC^{\xi}_{1-\xi,\rho}(\Theta)$ of the Eq* (1.1) *with*

$$|\mathfrak{w}(\vartheta) - \mathfrak{p}(\vartheta)| \leq \varpi_{\psi,\varsigma}(\kappa), \vartheta \in \Theta.$$

Definition 2.18. *The Eq* (1.1) *is Ulam-Hyers-Rassias stable with respect to (ω, ϖ) if there exists a real number $c_{\psi,\varsigma,\omega} > 0$ such that for each $\kappa > 0$ and for each solution $\mathfrak{w} \in PC^{\xi}_{1-\xi,\rho}(\Theta)$ of the inequality*

$$\begin{cases} \left|{}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{w}(\vartheta) - \psi(\vartheta, \mathfrak{w}(\vartheta), {}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{w}(\vartheta))\right| \leq \kappa\omega(\vartheta), \ \vartheta \in (\vartheta_\nu, \vartheta_{\nu+1}], \nu = 1, \ldots, \varsigma \\ \left|\Delta^{\rho} I^{1-\xi}_{\alpha^+}\mathfrak{w}|_{\vartheta=\vartheta_\nu} - \chi_\nu\right| \leq \kappa\varpi, \ \nu = 1, ..., \varsigma; \end{cases} \tag{2.2}$$

there exists a solution $\mathfrak{p} \in PC^{\xi}_{1-\xi,\rho}(\Theta)$ of Eq (1.1) *with*

$$|\mathfrak{w}(\vartheta) - \mathfrak{p}(\vartheta)| \leq c_{\psi,\varsigma,\omega}\kappa(\omega(\vartheta) + \varpi), \ \vartheta \in \Theta.$$

Definition 2.19. *The Eq* (1.1) *is generalized Ulam-Hyers-Rassias stable with respect to (ω, ϖ) if there exists a real number $c_{\psi,\varsigma,\omega} > 0$ such that for each solution $\mathfrak{w} \in PC^{\xi}_{1-\xi,\rho}(\Theta)$ of the inequality*

$$\begin{cases} \left|{}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{w}(\vartheta) - \psi(\vartheta, \mathfrak{w}(\vartheta), {}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{w}(\vartheta))\right| \leq \omega(\vartheta), \ \vartheta \in (\vartheta_\nu, \vartheta_{\nu+1}], \nu = 1, \ldots, \varsigma \\ \left|\Delta^{\rho} I^{1-\xi}_{\alpha^+}\mathfrak{w}|_{\vartheta=\vartheta_\nu} - \chi_\nu\right| \leq \varpi, \ \nu = 1, \ldots, \varsigma; \end{cases} \tag{2.3}$$

there exists a solution $\mathfrak{p} \in PC^{\xi}_{1-\xi,\rho}(\Theta)$ of Eq (1.1) *with*

$$|\mathfrak{w}(\vartheta) - \mathfrak{p}(\vartheta)| \leq c_{\psi,\varsigma,\omega}(\omega(\vartheta) + \varpi(\vartheta)), \ \vartheta \in \Theta.$$

Remark 2.20. *A function $\mathfrak{w} \in PC^{\xi}_{1-\xi,\rho}(\Theta)$ is a solution of the inequality (2.2) if and only if there is $\eta \in PC^{\xi}_{1-\xi,\rho}(\Theta)$ and a sequence $\eta_\nu, \nu = 1, \ldots, \varsigma$, where*

i) $|\eta(\vartheta)| \leq \kappa\omega(\vartheta), \ \vartheta \in (\vartheta_\nu, \vartheta_{\nu+1}], \nu = 1, \ldots, \varsigma$ *and* $|\eta_\nu| \leq \kappa\varpi, \nu = 1, \ldots, \varsigma$;

ii) ${}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{w}(\vartheta) = \psi(\vartheta, \mathfrak{w}(\vartheta), {}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{w}(\vartheta)) + \eta(\vartheta), \ \vartheta \in (\vartheta_\nu, \vartheta_{\nu+1}], \nu = 1, \ldots, \varsigma;$

iii) $\Delta^{\rho} I^{1-\xi}_{\alpha^+}\mathfrak{w}|_{\vartheta=\vartheta_\nu} = \chi_\nu + \eta_\nu, \ \nu = 1, \ldots, \varsigma.$

Theorem 2.21. *[30] Let $\mathfrak{D} \subset PC(\Theta, \mathbb{R})$. $\mathfrak{D}$ is relatively compact (i.e $\overline{\mathfrak{D}}$ is compact) if:*

(1) $\mathfrak{D}$ *is uniformly bounded i.e, there exists* $\bar{\theta} > 0$ *where*

$$|\psi(\mathfrak{p})| < \bar{\theta} \text{ for every } \psi \in \mathfrak{D} \text{ and } \mathfrak{p} \in (\vartheta_\nu, \vartheta_{\nu+1}], \nu = 1, \ldots, \varsigma.$$

(2) $\mathfrak{D}$ *is equicontinuous on* $(\vartheta_\nu, \vartheta_{\nu+1}]$ *i.e, for every* $\kappa > 0$, *there exists* $\delta > 0$ *such that for each* $\mathfrak{p}, \overline{\mathfrak{p}} \in (\vartheta_\nu, \vartheta_{\nu+1}]$, $\left|\mathfrak{p} - \overline{\mathfrak{p}}\right| \leq \delta$ *implies* $\left|\psi(\mathfrak{p}) - \psi(\overline{\mathfrak{p}})\right| \leq \kappa$, *for every* $\psi \in \mathfrak{D}$.

With appropriate modifications, the preceding theorem may be extended to the weighted space $PC_{1-\xi}(\Theta, \mathbb{R})$.

Theorem 2.22. *[26] ($PC_{1-\xi}$ type Arzela-Ascoli theorem). Let* $\mathfrak{D} \subset PC_{1-\xi}(\Theta, \mathbb{R})$. $\mathfrak{D}$ *is relatively compact (i.e* $\overline{\mathfrak{D}}$ *is compact) if:*

(1) $\mathfrak{D}$ *is uniformly bounded i.e, there exists* $\bar{\theta} > 0$ *such that*

$$|\psi(\mathfrak{p})| < \bar{\theta} \text{ for every } \psi \in \mathfrak{D} \text{ and } \mathfrak{p} \in (\vartheta_\nu, \vartheta_{\nu+1}], \nu = 1, \ldots, \varsigma.$$

(2) $\mathfrak{D}$ *is equicontinuous on* $(\vartheta_\nu, \vartheta_{\nu+1}]$ *i.e, for every* $\kappa > 0$, *there exists* $\delta > 0$ *such that for each* $\mathfrak{p}, \overline{\mathfrak{p}} \in (\vartheta_\nu, \vartheta_{\nu+1}]$, $\left|\mathfrak{p} - \overline{\mathfrak{p}}\right| \leq \delta$ *implies* $\left|\psi(\mathfrak{p}) - \psi(\overline{\mathfrak{p}})\right| \leq \kappa$, *for every* $\psi \in \mathfrak{D}$.

3. Existence of solutions

In this section, we study the existence of solution for the problems (1.1)–(1.3). We employ the following lemma to establish our main results.

Lemma 3.1. *Let* $0 < \zeta_1 < 1$, *and* $0 \leq \zeta_2 \leq 1, \xi = \zeta_1 + \zeta_2 - \zeta_1\zeta_2$ *and* $\gamma : \Theta \to \mathbb{R}$ *be continuous function. For any* $\bar{\alpha} \in \Theta$, *a function* $\mathfrak{p} \in C^{\xi}_{1-\xi,\rho}(\Theta, \mathbb{R})$ *is a solution of the equation:*

$$\mathfrak{p}(\vartheta) = \frac{1}{\Gamma(\xi)} \left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi-1} \left\{\left({}^{\rho}I^{1-\xi}_{\alpha^+}\right) \mathfrak{p}(\bar{\alpha}) - \left({}^{\rho}I^{1-\xi+\zeta_1}_{\alpha^+}\right) \gamma(\vartheta)|_{\vartheta=\bar{\alpha}}\right\} + \left({}^{\rho}I^{\zeta_1}_{\alpha^+}\gamma\right)(\vartheta), \tag{3.1}$$

if and only if, $\mathfrak{p}$ *is solution of the Hilfer-Katugampola fractional differential equation*

$${}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{p}(\vartheta) = \gamma(\vartheta); \quad \vartheta \in \Theta. \tag{3.2}$$

Proof. ($\Rightarrow$) Let $\mathfrak{p} \in C^{\xi}_{1-\xi,\rho}(\Theta)$ satisfying (3.1). We demonstrate that $\mathfrak{p}$ also verifies the fractional differential equation (3.2). Applying ${}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}$ on both sides of the Eq (3.1), we get

$$\begin{aligned} {}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{p}(\vartheta) &= \frac{1}{\Gamma(\xi)} \left\{\left({}^{\rho}I^{1-\xi}_{\alpha^+}\right) \mathfrak{p}(\bar{\alpha}) - \left({}^{\rho}I^{1-\xi+\zeta_1}_{\alpha^+}\right) \gamma(\vartheta)|_{\vartheta=\bar{\alpha}}\right\} {}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+} \left(\frac{\varrho^\rho - \alpha^\rho}{\rho}\right)^{\xi-1} (\vartheta) \\ &+ {}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\, {}^{\rho}I^{\zeta_1}_{\alpha^+}\gamma(\vartheta); \quad \vartheta \in \Theta. \end{aligned}$$

From the Lemma 2.4 and by the definition of the space $C^1_{1-\xi,\rho}(\Theta)$, we get

$${}^{\rho}I^{1-\zeta_2(1-\zeta_1)}_{\alpha^+}\gamma \in C^1_{1-\xi,\rho}(\Theta).$$

By Lemma 2.12 and Lemma 2.5 we obtain

$${}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{p}(\vartheta) = \gamma(\vartheta); \quad \vartheta \in \Theta.$$

($\Leftarrow$) Let $\mathfrak{p} \in C^{\xi}_{1-\xi,\rho}(\Theta)$ by a solution of the fractional differential equation (3.2). We prove that $\mathfrak{p}$ is also a solution of (3.1). Then

$$\begin{aligned} & \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}\left\{\left({}^{\rho}I^{1-\xi}_{\alpha^{+}}\right)\mathfrak{p}(\bar{\alpha})-\left({}^{\rho}I^{1-\xi+\zeta_{1}}_{\alpha^{+}}\right)\gamma(\vartheta)|_{\vartheta=\bar{\alpha}}\right\}+\left({}^{\rho}I^{\zeta_{1}}_{\alpha^{+}}\gamma\right)(\vartheta) \\ = & \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}\left\{\left({}^{\rho}I^{1-\xi}_{\alpha^{+}}\right)\mathfrak{p}(\bar{\alpha})-\left({}^{\rho}I^{1-\xi+\zeta_{1}}_{\alpha^{+}}\right)^{\rho}D^{\zeta_{1},\zeta_{2}}_{\alpha^{+}}\mathfrak{p}(\vartheta)|_{\vartheta=\bar{\alpha}}\right\} \\ + & {}^{\rho}I^{\zeta_{1}}_{\alpha^{+}}\,{}^{\rho}D^{\zeta_{1},\zeta_{2}}_{\alpha^{+}}\mathfrak{p}(\vartheta) \\ = & \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}\left\{\left({}^{\rho}I^{1-\xi}_{\alpha^{+}}\right)\mathfrak{p}(\bar{\alpha})-{}^{\rho}I^{1-\xi}_{\alpha^{+}}\left({}^{\rho}I^{\zeta_{1}}_{\alpha^{+}}\,{}^{\rho}D^{\zeta_{1},\zeta_{2}}_{\alpha^{+}}\right)\mathfrak{p}(\vartheta)|_{\vartheta=\bar{\alpha}}\right\} \\ + & {}^{\rho}I^{\zeta_{1}}_{\alpha^{+}}\,{}^{\rho}D^{\zeta_{1},\zeta_{2}}_{\alpha^{+}}\mathfrak{p}(\vartheta). \end{aligned} \tag{3.3}$$

By Lemma 2.7, with $\zeta_1 = \xi$, we obtain

$$\left({}^{\rho}I^{\xi}_{\alpha^{+}}\,{}^{\rho}D^{\xi}_{\alpha^{+}}\mathfrak{p}\right)(\vartheta)=\mathfrak{p}(\vartheta)-\frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}\,{}^{\rho}I^{1-\xi}_{\alpha^{+}}\mathfrak{p}(\alpha),$$

where $\vartheta \in (\alpha, \mu]$. By hypothesis, $\mathfrak{p} \in C^{\xi}_{1-\xi,\rho}(\Theta)$, using Lemma 2.11, we have

$$\left({}^{\rho}I^{\xi}_{\alpha^{+}}\,{}^{\rho}D^{\xi}_{\alpha^{+}}\mathfrak{p}\right)(\vartheta)=\left({}^{\rho}I^{\zeta_{1}}_{\alpha^{+}}\,{}^{\rho}D^{\zeta_{1},\zeta_{2}}_{\alpha^{+}}\mathfrak{p}\right)(\vartheta)=\mathfrak{p}(\vartheta)-\frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}\,{}^{\rho}I^{1-\xi}_{\alpha^{+}}\mathfrak{p}(\alpha), \tag{3.4}$$

which implies that

$${}^{\rho}I^{1-\xi}_{\alpha^{+}}\left({}^{\rho}I^{\zeta_{1}}_{\alpha^{+}}\,{}^{\rho}D^{\zeta_{1},\zeta_{2}}_{\alpha^{+}}\mathfrak{p}\right)(\vartheta)=({}^{\rho}I^{1-\xi}_{\alpha^{+}}\mathfrak{p})(\vartheta)-\frac{1}{\Gamma(\xi)}\,{}^{\rho}I^{1-\xi}_{\alpha^{+}}\mathfrak{p}(\alpha){}^{\rho}I^{1-\xi}_{\alpha^{+}}\left(\frac{\varrho^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}(\vartheta).$$

By applying Lemma 2.5 we get

$$\begin{aligned} {}^{\rho}I^{1-\xi}_{\alpha^{+}}\left({}^{\rho}I^{\zeta_{1}}_{\alpha^{+}}\,{}^{\rho}D^{\zeta_{1},\zeta_{2}}_{\alpha^{+}}\mathfrak{p}\right)(\vartheta) &= ({}^{\rho}I^{1-\xi}_{\alpha^{+}}\mathfrak{p})(\vartheta)-\frac{1}{\Gamma(\xi)}\,{}^{\rho}I^{1-\xi}_{\alpha^{+}}\mathfrak{p}(\alpha)\Gamma(\xi) \\ &= ({}^{\rho}I^{1-\xi}_{\alpha^{+}}\mathfrak{p})(\vartheta)-{}^{\rho}I^{1-\xi}_{\alpha^{+}}\mathfrak{p}(\alpha). \end{aligned} \tag{3.5}$$

By replacing (3.4) and (3.5) in (3.3) we have

$$\begin{aligned} & \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}\left\{\left({}^{\rho}I^{1-\xi}_{\alpha^{+}}\mathfrak{p}\right)(\bar{\alpha})-\left({}^{\rho}I^{1-\xi+\zeta_{1}}_{\alpha^{+}}\gamma\right)(\vartheta)|_{\vartheta=\bar{\alpha}}\right\}+{}^{\rho}I^{\zeta_{1}}_{\alpha^{+}}\gamma(\vartheta) \\ = & \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}\left\{\left({}^{\rho}I^{1-\xi}_{\alpha^{+}}\mathfrak{p}\right)(\bar{\alpha})-{}^{\rho}I^{1-\xi}_{\alpha^{+}}\mathfrak{p}(\vartheta)+{}^{\rho}I^{1-\xi}_{\alpha^{+}}\mathfrak{p}(\alpha)|_{\vartheta=\bar{\alpha}}\right\} \\ + & \mathfrak{p}(\vartheta)-\frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}\,{}^{\rho}I^{1-\xi}_{\alpha^{+}}\mathfrak{p}(\alpha) \\ = & \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}\,{}^{\rho}I^{1-\xi}_{\alpha^{+}}\mathfrak{p}(\alpha)+\mathfrak{p}(\vartheta)-\frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}\,{}^{\rho}I^{1-\xi}_{\alpha^{+}}\mathfrak{p}(\alpha) \\ = & \mathfrak{p}(\vartheta), \end{aligned}$$

with $\vartheta \in (\alpha, \bar{\alpha}]$, that is $\mathfrak{p}(\cdot)$ satisfies (3.1). This completes the proof of the Lemma.

Lemma 3.2. *Let* $\xi = \zeta_1+\zeta_2-\zeta_1\zeta_2$, *where* $0 < \zeta_1 < 1$ *and* $0 \le \zeta_2 \le 1$. *Let* $\Psi : (\alpha,\mu] \to \mathbb{R}$ *is a continuous function. A function* $\mathfrak{p} \in PC^{\xi}_{1-\xi,\rho}(\Theta)$ *is a solution of the fractional integral equation*

$$\mathfrak{p}(\vartheta) = \begin{cases} \dfrac{\tilde{\alpha}}{\Gamma(\xi)}\left(\dfrac{\vartheta^\rho-\alpha^\rho}{\rho}\right)^{\xi-1} + \left({}^\rho I^{\zeta_1}_{\alpha^+}\Psi\right)(\vartheta), & \vartheta \in (\alpha,\vartheta_1] \\ \dfrac{1}{\Gamma(\xi)}\left(\dfrac{\vartheta^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}\left(\tilde{\alpha}+\displaystyle\sum_{i=1}^{\nu}\chi_i\right) + \left({}^\rho I^{\zeta_1}_{\alpha^+}\Psi\right)(\vartheta), & \vartheta \in (\vartheta_\nu,\vartheta_{\nu+1}], \end{cases} \tag{3.6}$$

where $\nu = 1,\dots,\varsigma$, *if and only if* $\mathfrak{p}$ *is a solution of the problem:*

$$\left({}^\rho D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{p}\right)(\vartheta) = \Psi(\vartheta), \text{ for each } , \ \vartheta \in (\vartheta_\nu,\vartheta_{\nu+1}],\ \nu = 0,\dots,\varsigma, \tag{3.7}$$

$$\Delta {}^\rho I^{1-\xi}_{\alpha^+}\mathfrak{p}|_{\vartheta=\vartheta_\nu} = \chi_\nu \in \mathbb{R},\ \nu = 1,\dots,\varsigma, \tag{3.8}$$

$${}^\rho I^{1-\xi}_{\alpha^+}\mathfrak{p}(\alpha) = \tilde{\alpha} \in \mathbb{R}. \tag{3.9}$$

Proof. Assume that $\mathfrak{p} \in PC^{\xi}_{1-\xi,\rho}(\Theta,\mathbb{R})$ satisfies the problems (3.7)–(3.9).
If $\vartheta \in (\alpha,\vartheta_1]$ then

$$\begin{cases} \left({}^\rho D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{p}\right)(\vartheta) = \Psi(\vartheta) \\ {}^\rho I^{1-\xi}_{\alpha^+}\mathfrak{p}(\alpha) = \tilde{\alpha}. \end{cases} \tag{3.10}$$

Then the problem (3.10) is equivalent to the following fractional integral [22].

$$\mathfrak{p}(\vartheta) = \frac{\tilde{\alpha}}{\Gamma(\xi)}\left(\frac{\vartheta^\rho-\alpha^\rho}{\rho}\right)^{\xi-1} + \left({}^\rho I^{\zeta_1}_{\alpha^+}\Psi\right)(\vartheta), \quad \vartheta \in (\alpha,\vartheta_1]. \tag{3.11}$$

Now, if $\vartheta \in (\vartheta_1,\vartheta_2]$ then

$$\left({}^\rho D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{p}\right)(\vartheta) = \Psi(\vartheta);\ \ \vartheta \in (\vartheta_1,\vartheta_2] \ \text{ with } \ {}^\rho I^{1-\xi}_{\alpha^+}\mathfrak{p}(\vartheta_1^+) - {}^\rho I^{1-\xi}_{\alpha^+}\mathfrak{p}(\vartheta_1^-) = \chi_1.$$

By Lemma 3.1, we have

$$\begin{aligned} \mathfrak{p}(\vartheta) &= \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}\left\{\left({}^\rho I^{1-\xi}_{\alpha^+}\right)\mathfrak{p}(\vartheta_1^+) - \left({}^\rho I^{1-\xi+\zeta_1}_{\alpha^+}\Psi\right)(\vartheta)|_{\vartheta=\vartheta_1}\right\} + \left({}^\rho I^{\zeta_1}_{\alpha^+}\Psi\right)(\vartheta) \\ &= \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}\left\{\left({}^\rho I^{1-\xi}_{\alpha^+}\right)\mathfrak{p}(\vartheta_1^-) + \chi_1 - \left({}^\rho I^{1-\xi+\zeta_1}_{\alpha^+}\Psi\right)(\vartheta)|_{\vartheta=\vartheta_1}\right\} \\ &+ \left({}^\rho I^{\zeta_1}_{\alpha^+}\Psi\right)(\vartheta);\ \ \vartheta \in (\vartheta_1,\vartheta_2]. \end{aligned} \tag{3.12}$$

Now, from (3.11), we have

$${}^\rho I^{1-\xi}_{\alpha^+}\mathfrak{p}(\vartheta) = \tilde{\alpha} + {}^\rho I^{1-\xi+\zeta_1}_{\alpha^+}\Psi(\vartheta),\ \ \vartheta \in (\alpha,\vartheta_1].$$

This gives

$${}^\rho I^{1-\xi}_{\alpha^+}\mathfrak{p}(\vartheta_1^-) - {}^\rho I^{1-\xi+\zeta_1}_{\alpha^+}\Psi(\vartheta)|_{\vartheta=\vartheta_1} = \tilde{\alpha}. \tag{3.13}$$

Using (3.13) in (3.12), we obtain

$$\mathfrak{p}(\vartheta) = \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi-1}(\tilde{\alpha} + \chi_1) + {}^\rho I_{\alpha^+}^{\zeta_1}\Psi(\vartheta), \ \ \vartheta \in (\vartheta_1, \vartheta_2]. \tag{3.14}$$

Next, if $\vartheta \in (\vartheta_2, \vartheta_3]$ then

$${}^\rho D_{\alpha^+}^{\zeta_1,\zeta_2}\mathfrak{p}(\vartheta) = \Psi(\vartheta); \ \ \vartheta \in (\vartheta_2, \vartheta_3] \ \ \text{with} \ {}^\rho I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta_2^+) - {}^\rho I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta_2^-) = \chi_2.$$

Again, by Lemma 3.1, we have

$$\begin{aligned}
\mathfrak{p}(\vartheta) &= \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi-1}\left\{{}^\rho I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta_2^+) - {}^\rho I_{\alpha^+}^{1-\xi+\zeta_1}\Psi(\vartheta)|_{\vartheta=\vartheta_2}\right\} + {}^\rho I_{\alpha^+}^{\zeta_1}\Psi(\vartheta) \\
&= \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi-1}\left\{{}^\rho I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta_2^-) + \chi_2 - {}^\rho I_{\alpha^+}^{1-\xi+\zeta_1}\Psi(\vartheta)|_{\vartheta=\vartheta_2}\right\} \\
&+ {}^\rho I_{\alpha^+}^{\zeta_1}\Psi(\vartheta); \ \vartheta \in (\vartheta_2, \vartheta_3].
\end{aligned} \tag{3.15}$$

From (3.14), we have

$${}^\rho I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta) = (\tilde{\alpha} + \chi_1) + {}^\rho I_{\alpha^+}^{1-\xi+\zeta_1}\Psi(\vartheta); \ \ \vartheta \in (\vartheta_1, \vartheta_2],$$

which gives

$${}^\rho I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta_2^-) - {}^\rho I_{\alpha^+}^{1-\xi+\zeta_1}\Psi(\vartheta)|_{\vartheta=\vartheta_2} = \tilde{\alpha} + \chi_1. \tag{3.16}$$

Using (3.16) in (3.15), we get

$$\mathfrak{p}(\vartheta) = \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi-1}(\tilde{\alpha} + \chi_1 + \chi_2) + {}^\rho I_{\alpha^+}^{\zeta_1}\Psi(\vartheta), \ \ \vartheta \in (\vartheta_2, \vartheta_3].$$

Continuing the above process, we obtain

$$\mathfrak{p}(\vartheta) = \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi-1}\left(\tilde{\alpha} + \sum_{i=1}^{\nu}\chi_i\right) + \left({}^\rho I_{\alpha^+}^{\zeta_1}\Psi\right)(\vartheta), \quad \vartheta \in (\vartheta_\nu, \vartheta_{\nu+1}], \nu = 1\cdots\varsigma.$$

Conversely, let $\mathfrak{p} \in PC_{1-\xi,\rho}^{\xi}(\Theta, \mathbb{R})$ satisfies the fractional integral equation (3.11). Then, for $\vartheta \in (\alpha, \vartheta_1]$, we have

$$\mathfrak{p}(\vartheta) = \frac{\tilde{\alpha}}{\Gamma(\xi)}\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi-1} + {}^\rho I_{\alpha^+}^{\zeta_1}\Psi(\vartheta).$$

Applying ${}^\rho D_{\alpha^+}^{\zeta_1,\zeta_2}$ on both sides, we get

$${}^\rho D_{\alpha^+}^{\zeta_1,\zeta_2}\mathfrak{p}(\vartheta) = \frac{\tilde{\alpha}}{\Gamma(\xi)}\,{}^\rho D_{\alpha^+}^{\zeta_1,\zeta_2}\left(\frac{\varrho^\rho - \alpha^\rho}{\rho}\right)^{\xi-1}(\vartheta) + {}^\rho D_{\alpha^+}^{\zeta_1,\zeta_2}\,{}^\rho I_{\alpha^+}^{\zeta_1}\Psi(\vartheta).$$

From Lemma 2.4 and by the definition of the space $C_{1-\lambda,\rho}^1(\Theta)$, we can get

$${}^\rho I_{\alpha^+}^{1-\zeta_2(1-\zeta_1)}\Psi \in C_{1-\lambda,\rho}^1(\Theta). \tag{3.17}$$

Using Lemma 2.12 and Lemma 2.5, we obtain

$$ {}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{p}(\vartheta) = \Psi(\vartheta), \vartheta \in (\alpha, \vartheta_1]. $$

Now, for $\vartheta \in (\vartheta_\nu; \vartheta_{\nu+1}]$, we have

$$ \mathfrak{p}(\vartheta) = \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi-1}\left(\tilde{\alpha} + \sum_{i=1}^{\nu}\chi_i\right) + {}^{\rho}I^{\zeta_1}_{\alpha^+}\Psi(\vartheta). $$

Applying the operator ${}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}$ on both sides, we get

$$ {}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{p}(\vartheta) = \frac{1}{\Gamma(\xi)}\left(\tilde{\alpha} + \sum_{i=1}^{\nu}\chi_i\right){}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\left(\frac{\varrho^\rho - \alpha^\rho}{\rho}\right)^{\xi-1}(\vartheta) + {}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\,{}^{\rho}I^{\zeta_1}_{\alpha^+}\Psi(\vartheta). $$

From (3.17) and using Lemma 2.12 and Lemma 2.5, we obtain

$$ {}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{p}(\vartheta) = \Psi(\vartheta). $$

Thus, $\mathfrak{p}$ satisfies (3.7). Next, we demonstrate that $\mathfrak{p}$ also verify (3.8) and (3.9). Applying the operator ${}^{\rho}I^{1-\xi}_{\alpha^+}$ on both sides of Eq (3.11), we get

$$ {}^{\rho}I^{1-\xi}_{\alpha^+}\mathfrak{p}(\vartheta) = \frac{\tilde{\alpha}}{\Gamma(\xi)}\,{}^{\rho}I^{1-\xi}_{\alpha^+}\left(\frac{\varrho^\rho - \alpha^\rho}{\rho}\right)^{\xi-1}(\vartheta) + {}^{\rho}I^{1-\xi+\zeta_1}_{\alpha^+}\Psi(\vartheta). $$

By Lemma 2.5 we get

$$ {}^{\rho}I^{1-\xi}_{\alpha^+}\mathfrak{p}(\vartheta) = \tilde{\alpha} + {}^{\rho}I^{1-\xi+\zeta_1}_{\alpha^+}\Psi(\vartheta), $$

which implies that

$$ \begin{aligned} {}^{\rho}I^{1-\xi}_{\alpha^+}\mathfrak{p}(\vartheta) - {}^{\rho}I^{1-\xi+\zeta_1}_{\alpha^+}\Psi(\vartheta) &= \tilde{\alpha}, \\ {}^{\rho}I^{1-\xi}_{\alpha^+}\left(\mathfrak{p}(\vartheta) - {}^{\rho}I^{\zeta_1}_{\alpha^+}\Psi(\vartheta)\right) &= \tilde{\alpha}. \end{aligned} $$

Since $\mathfrak{p}$ satisfies (3.7) we have

$$ {}^{\rho}I^{1-\xi}_{\alpha^+}\left(\mathfrak{p}(\vartheta) - {}^{\rho}I^{\zeta_1}_{\alpha^+}\,{}^{\rho}D^{\zeta_1,\zeta_2}_{\alpha^+}\mathfrak{p}(\vartheta)\right) = \tilde{\alpha}. $$

Using Lemma 2.11 we have

$$ {}^{\rho}I^{1-\xi}_{\alpha^+}\left(\mathfrak{p}(\vartheta) - {}^{\rho}I^{\xi}_{\alpha^+}\,{}^{\rho}D^{\xi}_{\alpha^+}\mathfrak{p}(\vartheta)\right) = \tilde{\alpha}. $$

From the definition of $C^{\xi}_{1-\xi,\rho}(\Theta)$, Lemma 2.4 and using Definition 2.2, we have

$$ {}^{\rho}I^{1-\xi}_{\alpha^+}\mathfrak{p} \in C(\Theta) \quad \text{and} \quad {}^{\rho}D^{\xi}_{\alpha^+}\mathfrak{p} = \delta_\rho\,{}^{\rho}I^{1-\xi}_{\alpha^+}\mathfrak{p} \in C_{1-\xi,\rho}(\Theta). $$

Thus

$$ {}^{\rho}I^{1-\xi}_{\alpha^+}\mathfrak{p} \in C^{1}_{1-\xi,\rho}(\Theta). $$

By Lemma 2.7, with $\zeta_1 = \xi$ we can write

$$ {}^{\rho}I^{1-\xi}_{\alpha^+}\left(\mathfrak{p}(\vartheta) - \mathfrak{p}(\vartheta) + \frac{{}^{\rho}I^{1-\xi}_{\alpha^+}\mathfrak{p}(\alpha)}{\Gamma(\xi)}\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi-1}\right) = \tilde{\alpha}, $$

$$\frac{{}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}(\alpha)}{\Gamma(\xi)}\, {}^{\rho}I_{\alpha^+}^{1-\xi}\left(\frac{\varrho^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}(\vartheta)=\tilde{\alpha}.$$

By Lemma 2.5 we have

$${}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}(\alpha)=\tilde{\alpha},$$

which is the condition (3.9).

Further, from Eq (3.6), for $\vartheta\in(\vartheta_{\nu},\vartheta_{\nu+1}]$, we have

$$\begin{aligned}{}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta) &= \frac{1}{\Gamma(\xi)}\left(\tilde{\alpha}+\sum_{i=1}^{\nu}\chi_i\right){}^{\rho}I_{\alpha^+}^{1-\xi}\left(\frac{\varrho^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}(\vartheta)+{}^{\rho}I_{\alpha^+}^{1-\xi+\zeta_1}\Psi(\vartheta)\\ &= \tilde{\alpha}+\sum_{i=1}^{\nu}\chi_i+{}^{\rho}I_{\alpha^+}^{1-\xi+\zeta_1}\Psi(\vartheta),\end{aligned}\tag{3.18}$$

and for $\vartheta\in(\vartheta_{\nu-1},\vartheta_{\nu}]$, we have

$$\begin{aligned}{}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta) &= \frac{1}{\Gamma(\xi)}\left(\tilde{\alpha}+\sum_{i=1}^{\nu-1}\chi_i\right){}^{\rho}I_{\alpha^+}^{1-\xi}\left(\frac{\varrho^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}(\vartheta)+{}^{\rho}I_{\alpha^+}^{1-\xi+\zeta_1}\Psi(\vartheta)\\ &= \tilde{\alpha}+\sum_{i=1}^{\nu-1}\chi_i+{}^{\rho}I_{\alpha^+}^{1-\xi+\zeta_1}\Psi(\vartheta).\end{aligned}\tag{3.19}$$

Therefore, from (3.18) and (3.19), we obtain

$${}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta_{\nu}^{+})-{}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}(\vartheta_{\nu}^{-})=\sum_{i=1}^{\nu}\chi_i-\sum_{i=1}^{\nu-1}\chi_i=\chi_{\nu},$$

which condition (3.8). We have proved that $\mathfrak{p}$ satisfies the problems (3.7)–(3.9).

As a consequence of Lemma 3.2, we have the following lemma.

Lemma 3.3. *Let $\xi=\zeta_1+\zeta_2-\zeta_1\zeta_2$ where $0<\zeta_1<1$ and $0\le\zeta_2\le1$, let $\psi:(\alpha,\mu]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function where $\psi(\cdot,\mathfrak{p}(\cdot),\mathfrak{x}(\cdot))\in PC_{1-\xi,\rho}(\Theta)$ for any $\mathfrak{p},\mathfrak{x}\in PC_{1-\xi,\rho}(\Theta)$. If $\mathfrak{p}\in PC^{\xi}_{1-\xi,\rho}(\Theta)$, then $\mathfrak{p}$ verifies* (1.1)–(1.3) *if and only if $\mathfrak{p}$ is the fixed point of the operator $\mathcal{S}:PC_{1-\xi,\rho}(\Theta)\to PC_{1-\xi,\rho}(\Theta)$ given by*

$$\mathcal{S}\mathfrak{p}(\vartheta)=\frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}\left(\tilde{\alpha}+\sum_{\alpha<\vartheta_{\nu}<\vartheta}\chi_{\nu}\right)+\left({}^{\rho}I_{\alpha^+}^{\zeta_1}\varkappa\right)(\vartheta),\tag{3.20}$$

where $\varkappa:(0,\mu]\to\mathbb{R}$ be a function verifying the functional equation

$$\varkappa(\vartheta)=\psi(\vartheta,\mathfrak{p}(\vartheta),\varkappa(\vartheta)).$$

It is obvious that $\varkappa\in PC_{1-\xi,\rho}(\Theta)$. Also, by Lemma 2.4, $\mathcal{S}\mathfrak{p}\in PC_{1-\xi,\rho}(\Theta)$.

Assume that the function $\psi:(\alpha,\mu]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and verifies the following:

(*H1*) $\psi(\cdot,\mathfrak{x}(\cdot),\mathfrak{y}(\cdot))\in PC^{\zeta_2(1-\zeta_1)}_{1-\xi,\rho}$ for any $\mathfrak{x},\mathfrak{y}\in PC_{1-\xi,\rho}(\Theta)$.

(H2) There exist constants $\theta_1 > 0$ and $0 < \theta_2 < 1$ such that

$$|\psi(\vartheta, \mathfrak{x}, \mathfrak{y}) - \psi(\vartheta, \bar{\mathfrak{x}}, \bar{\mathfrak{y}})| \leq \theta_1|\mathfrak{x} - \bar{\mathfrak{x}}| + \theta_2|\mathfrak{y} - \bar{\mathfrak{y}}|$$

for any $\mathfrak{x}, \mathfrak{y}, \bar{\mathfrak{x}}, \bar{\mathfrak{y}} \in PC_{1-\xi,\rho}(\Theta)$ and $\vartheta \in (\alpha, \mu]$.

We can now declare and demonstrate our existence result for problems (1.1)–(1.3) based on Banach's fixed point [15].

Theorem 3.4. *If* $(H1)$ *and* $(H2)$ *are met, and*

$$B := \frac{\theta_1\Gamma(\xi)}{\Gamma(\zeta_1 + \xi)(1 - \theta_2)}\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{\zeta_1} < 1, \tag{3.21}$$

then the problems (1.1)–(1.3) has unique solution in $PC^{\xi}_{1-\xi,\rho}(\Theta) \subset PC^{\zeta_1,\zeta_2}_{1-\xi,\rho}(\Theta)$.

Proof. The proof will be presented in two segments.

Step 1: We demonstrate that $\mathcal{S}$ defined in (3.20) has a unique fixed point $\mathfrak{p}^*$ in $PC_{1-\xi,\rho}(\Theta)$. Let $\mathfrak{p}, \mathfrak{x} \in PC_{1-\xi,\rho}(\Theta)$ and $\vartheta \in (\alpha, \mu]$, then, we have

$$\begin{aligned} |\mathcal{S}\mathfrak{p}(\vartheta) - \mathcal{S}\mathfrak{x}(\vartheta)| &= \left|{}^\rho I^{\zeta_1}_{\alpha^+}\varkappa(\vartheta) -{}^\rho I^{\zeta_1}_{\alpha^+}\gamma(\vartheta)\right| \\ &\leq \frac{1}{\Gamma(\zeta_1)}\int_\alpha^\vartheta \left(\frac{\vartheta^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}\,|\psi(\varrho, \mathfrak{p}(\varrho), \varkappa(\varrho)) - \psi(\varrho, \mathfrak{x}(\varrho), \gamma(\varrho))|\,d\varrho, \end{aligned}$$

where $\varkappa, \gamma \in PC_{1-\xi,\rho}(\Theta)$ such that

$$\begin{aligned} \varkappa(\vartheta) &= \psi(\vartheta, \mathfrak{p}(\vartheta), \varkappa(\vartheta)), \\ \gamma(\vartheta) &= \psi(\vartheta, \mathfrak{x}(\vartheta), \gamma(\vartheta)). \end{aligned}$$

By (H2), we have

$$\begin{aligned} |\varkappa(\vartheta) - \gamma(\vartheta)| &= |\psi(\vartheta, \mathfrak{p}(\vartheta), \varkappa(\vartheta)) - \psi(\vartheta, \mathfrak{x}(\vartheta), \gamma(\vartheta))| \\ &\leq \theta_1|\mathfrak{p}(\vartheta) - \mathfrak{x}(\vartheta)| + \theta_2|\varkappa(\vartheta) - \gamma(\vartheta)|. \end{aligned}$$

Then,

$$|\varkappa(\vartheta) - \gamma(\vartheta)| \leq \frac{\theta_1}{1 - \theta_2}|\mathfrak{p}(\vartheta) - \mathfrak{x}(\vartheta)|.$$

Therefore, for each $\vartheta \in (\alpha, \mu]$

$$\begin{aligned} |\mathcal{S}\mathfrak{p}(\vartheta) - \mathcal{S}\mathfrak{x}(\vartheta)| &\leq \frac{\theta_1}{(1 - \theta_2)\Gamma(\zeta_1)}\int_\alpha^\vartheta \left(\frac{\vartheta^\rho - \varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}|\mathfrak{p}(\varrho) - \mathfrak{x}(\varrho)|d\varrho \\ &\leq \frac{\theta_1}{(1 - \theta_2)}\left(I^{\zeta_1}_{\alpha^+}\left(\frac{\varrho^\rho - \alpha^\rho}{\rho}\right)^{\xi-1}\right)(\vartheta)\|\mathfrak{p} - \mathfrak{x}\|_{PC_{1-\xi,\rho}}. \end{aligned}$$

By Lemma 2.5, we have

$$|\mathcal{S}\mathfrak{p}(\vartheta) - \mathcal{S}\mathfrak{x}(\vartheta)| \leq \frac{\theta_1\Gamma(\xi)}{\Gamma(\zeta_1 + \xi)(1 - \theta_2)}\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\zeta_1+\xi-1}\|\mathfrak{p} - \mathfrak{x}\|_{PC_{1-\xi,\rho}},$$

hence

$$\begin{aligned}\left|\left(\frac{\vartheta^\rho-\alpha^\rho}{\rho}\right)^{1-\xi}(\mathcal{S}\mathfrak{p}(\vartheta)-\mathcal{S}\mathfrak{x}(\vartheta))\right| &\leq \frac{\theta_1\Gamma(\xi)}{\Gamma(\zeta_1+\xi)(1-\theta_2)}\left(\frac{\vartheta^\rho-\alpha^\rho}{\rho}\right)^{\zeta_1}\|\mathfrak{p}-\mathfrak{x}\|_{PC_{1-\xi,\rho}}\\ &\leq \frac{\theta_1\Gamma(\xi)}{\Gamma(\zeta_1+\xi)(1-\theta_2)}\left(\frac{\mu^\rho-\alpha^\rho}{\rho}\right)^{\zeta_1}\|\mathfrak{p}-\mathfrak{x}\|_{PC_{1-\xi,\rho}},\end{aligned}$$

which implies that

$$\|\mathcal{S}\mathfrak{p}-\mathcal{S}\mathfrak{x}\|_{PC_{1-\xi,\rho}} \leq \frac{\theta_1\Gamma(\xi)}{\Gamma(\zeta_1+\xi)(1-\theta_2)}\left(\frac{\mu^\rho-\alpha^\rho}{\rho}\right)^{\zeta_1}\|\mathfrak{p}-\mathfrak{x}\|_{PC_{1-\xi,\rho}}.$$

By (3.21), the operator $\mathcal{S}$ is a contraction. Hence, by Banach's contraction principle, $\mathcal{S}$ has a unique fixed point $\mathfrak{p}^* \in PC_{1-\xi,\rho}(\Theta)$.

Step 2: We show that such a fixed point $\mathfrak{p}^* \in PC_{1-\xi,\rho}(\Theta)$ is actually in $PC^{\xi}_{1-\xi,\rho}(\Theta)$.
Since $\mathfrak{p}^*$ is the unique fixed point of operator $\mathcal{S}$ in $PC_{1-\xi,\rho}(\Theta)$, then, for each $\vartheta \in (\alpha,\mu]$, we have

$$\mathfrak{p}^*(\vartheta) = \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}\left(\tilde{\alpha}+\sum_{\alpha<\vartheta_\nu<\vartheta}\chi_\nu\right)+\left({}^\rho I^{\zeta_1}_{\alpha^+}\psi(\varrho,\mathfrak{p}^*(\varrho),\varkappa(\varrho))\right)(\vartheta).$$

Applying ${}^\rho D^{\xi}_{\alpha^+}$ to both sides and by Lemma 2.5, and Lemma 2.11, we have

$$\begin{aligned}{}^\rho D^{\xi}_{\alpha^+}\mathfrak{p}^*(\vartheta) &= \left({}^\rho D^{\xi}_{\alpha^+}\,{}^\rho I^{\zeta_1}_{\alpha^+}\psi(\varrho,\mathfrak{p}^*(\varrho),\varkappa(\varrho))\right)(\vartheta)\\ &= \left({}^\rho D^{\zeta_2(1-\zeta_1)}_{\alpha^+}\psi(\varrho,\mathfrak{p}^*(\varrho),\varkappa(\varrho))\right)(\vartheta).\end{aligned}$$

Since $\xi \geq \zeta_1$, by (H1), the right hand side is in $PC_{1-\xi,\rho}(\Theta)$ and thus ${}^\rho D^{\xi}_{\alpha^+}\mathfrak{p}^* \in PC_{1-\xi,\rho}(\Theta)$ which implies that $\mathfrak{p}^* \in PC^{\xi}_{1-\xi,\rho}(\Theta)$. As a consequence of Steps 1 and 2 together with Lemma 3.3, we can conclude that the problems (1.1)–(1.3) has a unique solution in $PC^{\xi}_{1-\xi,\rho}(\Theta)$.

Our second result is based on Krasnoselskii fixed point theorem [15].

Theorem 3.5. *Assume* $(H1)$ *and,*

(H3) There exist constants $0<\theta_1<\frac{(1-\theta_2)\Gamma(\zeta_1+\xi)}{2\Gamma(\xi)}\left(\frac{\mu^\rho-\alpha^\rho}{\rho}\right)^{-\zeta_1}$ *and* $0<\theta_2<1$ *such that*

$$|\psi(\vartheta,\mathfrak{x},\mathfrak{y})-\psi(\vartheta,\bar{\mathfrak{x}},\bar{\mathfrak{y}})| \leq \theta_1|\mathfrak{x}-\bar{\mathfrak{x}}|+\theta_2|\mathfrak{y}-\bar{\mathfrak{y}}|$$

for any $\mathfrak{x},\mathfrak{y},\bar{\mathfrak{x}},\bar{\mathfrak{y}} \in PC_{1-\xi,\rho}(\Theta)$ *and* $\vartheta \in (\alpha,\mu]$.

Then the problems (1.1)–(1.3) *has at least one solution.*

Proof. Consider the set

$$B_{\varepsilon^*} = \{\mathfrak{p} \in PC_{1-\xi,\rho}(\Theta) : \|\mathfrak{p}\|_{PC_{1-\xi,\rho}} \leq \varepsilon^*\},$$

where

$$\varepsilon^* \geq \frac{2}{\Gamma(\xi)}\left(|\tilde{\alpha}|+\sum_{\nu=1}^{\varsigma}|\chi_\nu|\right)+\frac{2\Gamma(\xi)\psi^*}{(1-\theta_2)\Gamma(\zeta_1+\xi)}\left(\frac{\mu^\rho-\alpha^\rho}{\rho}\right)^{1-\xi+\zeta_1},$$

where $\psi^* = \sup_{\vartheta\in\Theta} |\psi(\vartheta, 0, 0)|$.

We define the operators $\mathcal{S}_1$ and $\mathcal{S}_2$ on B_{ε^*} by

$$\mathcal{S}_1\mathfrak{p}(\vartheta) = \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi-1}\left(\tilde{\alpha} + \sum_{\alpha<\vartheta_\nu<\vartheta} \chi_\nu\right), \quad \vartheta \in \Theta, \tag{3.22}$$

$$\mathcal{S}_2\mathfrak{p}(\vartheta) = \frac{1}{\Gamma(\zeta_1)}\int_\alpha^\vartheta \left(\frac{\vartheta^\rho - \varrho^\rho}{\rho}\right)^{\zeta_1-1} \varrho^{\rho-1}\varkappa(\varrho)d\varrho, \quad \vartheta \in \Theta. \tag{3.23}$$

Then (3.20) can be written as

$$\mathcal{S}\mathfrak{p}(\vartheta) = \mathcal{S}_1\mathfrak{p}(\vartheta) + \mathcal{S}_2\mathfrak{p}(\vartheta), \quad \mathfrak{p} \in PC_{1-\xi,\rho}(\Theta).$$

Step 1: We demonstrate that $\mathcal{S}_1\mathfrak{p} + \mathcal{S}_2\mathfrak{x} \in B_{\varepsilon^*}$ for any $\mathfrak{p}, \mathfrak{x} \in B_{\varepsilon^*}$. For operator $\mathcal{S}_1$, multiplying both sides of (3.22) by $\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{1-\xi}$, we have

$$\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{1-\xi}\mathcal{S}_1\mathfrak{p}(\vartheta) = \frac{1}{\Gamma(\xi)}\left(\tilde{\alpha} + \sum_{\alpha<\vartheta_\nu<\vartheta} \chi_\nu\right),$$

then

$$\left|\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{1-\xi}\mathcal{S}_1\mathfrak{p}(\vartheta)\right| \leq \frac{1}{\Gamma(\xi)}\left(|\tilde{\alpha}| + \sum_{\nu=1}^{\varsigma} |\chi_\nu|\right).$$

This gives

$$\|\mathcal{S}_1\mathfrak{p}\|_{PC_{1-\xi,\rho}} \leq \frac{1}{\Gamma(\xi)}\left(|\tilde{\alpha}| + \sum_{\nu=1}^{\varsigma} |\chi_\nu|\right). \tag{3.24}$$

By (H3), we have for each $\vartheta \in (\alpha, \mu]$,

$$\begin{aligned}
|\varkappa(\vartheta)| &= |\psi(\vartheta, \mathfrak{x}(\vartheta), \varkappa(\vartheta)) - \psi(\vartheta, 0, 0) + \psi(\vartheta, 0, 0)| \\
&\leq |\psi(\vartheta, \mathfrak{x}(\vartheta), \varkappa(\vartheta)) - \psi(\vartheta, 0, 0)| + |\psi(\vartheta, 0, 0)| \\
&\leq \theta_1|\mathfrak{x}(\vartheta)| + \theta_2|\varkappa(\vartheta)| + \psi^*.
\end{aligned}$$

Multiplying both sides of the above inequality by $\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{1-\xi}$, we get

$$\begin{aligned}
\left|\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{1-\xi}\varkappa(\vartheta)\right| &\leq \left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{1-\xi}\psi^* + \theta_1\left|\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{1-\xi}\mathfrak{x}(\vartheta)\right| \\
&+ \theta_2\left|\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{1-\xi}\varkappa(\vartheta)\right| \\
&\leq \left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{1-\xi}\psi^* + \theta_1\varepsilon^* + \theta_2\left|\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{1-\xi}\varkappa(\vartheta)\right|.
\end{aligned}$$

Then, for each $\vartheta \in (\alpha, \mu]$, we have

$$\left|\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{1-\xi} \varkappa(\vartheta)\right| \leq \frac{\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{1-\xi} \psi^* + \theta_1 \varepsilon^*}{1 - \theta_2} := \theta_3. \tag{3.25}$$

Thus, (3.23) and Lemma 2.5, implies

$$|\mathcal{S}_2(\mathfrak{x})(\vartheta)| \leq \left[\frac{\Gamma(\xi)\psi^*}{(1-\theta_2)\Gamma(\zeta_1+\xi)}\left(\frac{\mu^\rho-\alpha^\rho}{\rho}\right)^{1-\xi} + \frac{\theta_1\Gamma(\xi)\varepsilon^*}{(1-\theta_2)\Gamma(\zeta_1+\xi)}\right]\left(\frac{\vartheta^\rho-\alpha^\rho}{\rho}\right)^{\zeta_1+\xi-1}.$$

Therefore

$$\begin{aligned}
\left|\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{1-\xi} \mathcal{S}_2\mathfrak{x}(\vartheta)\right| &\leq \left[\frac{\Gamma(\xi)\psi^*}{(1-\theta_2)\Gamma(\zeta_1+\xi)}\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{1-\xi}\right. \\
&+ \left.\frac{\theta_1\Gamma(\xi)\varepsilon^*}{(1-\theta_2)\Gamma(\zeta_1+\xi)}\right]\left(\frac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\zeta_1} \\
&\leq \frac{\Gamma(\xi)\psi^*}{(1-\theta_2)\Gamma(\zeta_1+\xi)}\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{1-\xi+\zeta_1} \\
&+ \frac{\theta_1\Gamma(\xi)\varepsilon^*}{(1-\theta_2)\Gamma(\zeta_1+\xi)}\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{\zeta_1}.
\end{aligned}$$

Thus

$$\|\mathcal{S}_2\mathfrak{x}\|_{PC_{1-\xi,\rho}} \leq \frac{\Gamma(\xi)\psi^*}{(1-\theta_2)\Gamma(\zeta_1+\xi)}\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{1-\xi+\zeta_1} + \frac{\theta_1\Gamma(\xi)\varepsilon^*}{(1-\theta_2)\Gamma(\zeta_1+\xi)}\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{\zeta_1}. \tag{3.26}$$

Linking (3.24) and (3.26) for every $\mathfrak{p}, \mathfrak{x} \in B_{\varepsilon^*}$ we obtain

$$\begin{aligned}
\|\mathcal{S}_1\mathfrak{p} + \mathcal{S}_2\mathfrak{x}\|_{PC_{1-\xi,\rho}} &\leq \|\mathcal{S}_1\mathfrak{p}\|_{PC_{1-\xi,\rho}} + \|\mathcal{S}_2\mathfrak{x}\|_{PC_{1-\xi,\rho}} \leq \frac{\Gamma(\xi)\psi^*}{(1-\theta_2)\Gamma(\zeta_1+\xi)}\left(\frac{\mu^\rho-\alpha^\rho}{\rho}\right)^{1-\xi+\zeta_1} \\
&+ \frac{\theta_1\Gamma(\xi)\varepsilon^*}{(1-\theta_2)\Gamma(\zeta_1+\xi)}\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{\zeta_1} + \frac{1}{\Gamma(\xi)}\left(|\tilde{\alpha}| + \sum_{\nu=1}^{\varsigma}|\chi_\nu|\right).
\end{aligned}$$

Since

$$\varepsilon^* \geq \frac{2}{\Gamma(\xi)}\left(|\tilde{\alpha}| + \sum_{\nu=1}^{\varsigma}|\chi_\nu|\right) + \frac{2\Gamma(\xi)\psi^*}{(1-\theta_2)\Gamma(\zeta_1+\xi)}\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{1-\xi+\zeta_1},$$

and

$$\theta_1 < \frac{(1-\theta_2)\Gamma(\zeta_1+\xi)}{2\Gamma(\xi)}\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{-\zeta_1},$$

we have

$$\|\mathcal{S}_1\mathfrak{p} + \mathcal{S}_2\mathfrak{x}\|_{PC_{1-\xi,\rho}} \leq \varepsilon^*.$$

which infers that $\mathcal{S}_1\mathfrak{p} + \mathcal{S}_2\mathfrak{x} \in B_{\varepsilon^*}$.

Step 2: Clearly $\mathcal{S}_1$ is a contraction.

Step 3: $\mathcal{S}_2$ is compact and continuous.

The continuity of $\mathcal{S}_2$ follows from the continuity of ψ. Next we prove that $\mathcal{S}_2$ is uniformly bounded on B_{ε^*}. Let any $\mathfrak{x} \in B_{\varepsilon^*}$. Then by (3.26) we have

$$\|\mathcal{S}_2\mathfrak{x}\|_{PC_{1-\xi,\rho}} \leq \frac{\Gamma(\xi)\psi^*}{(1-\theta_2)\Gamma(\zeta_1+\xi)}\left(\frac{\mu^\rho-\alpha^\rho}{\rho}\right)^{1-\xi+\zeta_1} + \frac{\theta_1\Gamma(\xi)\varepsilon^*}{(1-\theta_2)\Gamma(\zeta_1+\xi)}\left(\frac{\mu^\rho-\alpha^\rho}{\rho}\right)^{\zeta_1}.$$

This means that $\mathcal{S}_2$ is uniformly bounded on B_{ε^*}. Next, we show that $\mathcal{S}_2 B_{\varepsilon^*}$ is equicontinuous. Let any $\mathfrak{x} \in B_{\varepsilon^*}$ and $0 < \alpha < \tau_1 < \tau_2 \leq \mu$. Then

$$\begin{aligned}
|\mathcal{S}_2\mathfrak{x}(\tau_1) - \mathcal{S}_2\mathfrak{x}(\tau_2)| &= \frac{1}{\Gamma(\zeta_1)}\left|\int_\alpha^{\tau_1}\left(\frac{\tau_1^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}\varkappa(\varrho)d\varrho\right. \\
&- \left.\int_\alpha^{\tau_2}\left(\frac{\tau_2^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}\varkappa(\varrho)d\varrho\right| \\
&\leq \frac{1}{\Gamma(\zeta_1)}\left|\int_\alpha^{\tau_1}\left(\frac{\tau_1^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}\left(\frac{\varrho^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}\left(\frac{\varrho^\rho-\alpha^\rho}{\rho}\right)^{1-\xi}\varkappa(\varrho)d\varrho\right. \\
&- \left.\int_\alpha^{\tau_2}\left(\frac{\tau_2^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}\left(\frac{\varrho^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}\left(\frac{\varrho^\rho-\alpha^\rho}{\rho}\right)^{1-\xi}\varkappa(\varrho)d\varrho\right|,
\end{aligned}$$

by using (3.25) we have

$$\begin{aligned}
|\mathcal{S}_2\mathfrak{x}(\tau_1) - \mathcal{S}_2\mathfrak{x}(\tau_2)| &\leq \frac{\theta_3}{\Gamma(\zeta_1)}\left|\int_\alpha^{\tau_1}\left(\frac{\tau_1^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}\left(\frac{\varrho^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}d\varrho\right. \\
&- \left.\int_\alpha^{\tau_2}\left(\frac{\tau_2^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}\left(\frac{\varrho^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}d\varrho\right| \\
&\leq \frac{\theta_3\Gamma(\xi)}{\Gamma(\zeta_1+\xi)}\left|\left(\frac{\tau_1^\rho-\alpha^\rho}{\rho}\right)^{\zeta_1+\xi-1} - \left(\frac{\tau_2^\rho-\alpha^\rho}{\rho}\right)^{\zeta_1+\xi-1}\right|.
\end{aligned}$$

Note that

$$|\mathcal{S}_2\mathfrak{x}(\tau_1) - \mathcal{S}_2\mathfrak{x}(\tau_2)| \to 0 \quad \text{as} \quad \tau_2 \to \tau_1.$$

This shows that $\mathcal{S}_2$ is equicontinuous on Θ. Therefore $\mathcal{S}_2$ is relatively compact on B_{ε^*}. By $PC_{1-\xi}$; type Arzela-Ascoli theorem $\mathcal{S}_2$ is compact on B_{ε^*}.

By Krasnoselskii's fixed point theorem, $\mathcal{S}$ has at least a fixed point $\mathfrak{p}^* \in C_{1-\xi,\rho}(\Theta)$ and by the same way of the proof of Theorem 3.4, $\mathfrak{p}^* \in C^\xi_{1-\xi,\rho}(\Theta)$. Using Lemma 3.3, we conclude that the problems (1.1)–(1.3) has at least one solution in the space $C^\xi_{1-\xi,\rho}(\Theta)$.

4. Ulam-Hyers-Rassias stability

In what follows, we give the following result on Ulam-Hyers-Rassias stability.

Theorem 4.1. *Assume that (H1), (H2), (3.21) hold and,*

(H4) There exists a nondecreasing function $\omega \in PC_{\xi-1}(\Theta)$ and there exists $\lambda_\omega > 0$ such that for any $\vartheta \in (\alpha, \mu]$:

$${}^\rho I^{\zeta_1}_{\alpha^+}(\omega(\varrho))(\vartheta) \leq \lambda_\omega \omega(\vartheta).$$

Then, the Eq (1.1) *is Ulam-Hyers-Rassias stable with respect to (ω, ϖ).*

Proof. Let $\mathfrak{w} \in PC_{1-\xi,\rho}(\Theta)$ be a solution of the inequality (2.2). Denote by $\mathfrak{p}$ the unique solution of the problem:

$$\left({}^{\rho}D_{\alpha^+}^{\zeta_1,\zeta_2}\mathfrak{p}\right)(\vartheta) = \psi\left(\vartheta, \mathfrak{p}(\vartheta), \left({}^{\rho}D_{\alpha^+}^{\zeta_1,\zeta_2}\mathfrak{p}\right)(\vartheta)\right), \text{ for each , } \vartheta \in (\vartheta_\nu, \vartheta_{\nu+1}], \ \nu = 0, \ldots, \varsigma,$$

$$\Delta^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}|_{\vartheta=\vartheta_\nu} = \chi_\nu \in \mathbb{R}, \ \nu = 1, \ldots, \varsigma,$$
$$\left({}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{w}\right)(\alpha) = \left({}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}\right)(\alpha) = \tilde{\alpha} \in \mathbb{R}.$$

Using Lemma 3.3, we obtain

$$\mathfrak{p}(\vartheta) = \begin{cases} \dfrac{\tilde{\alpha}}{\Gamma(\xi)}\left(\dfrac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi-1} + \left({}^{\rho}I_{\alpha^+}^{\zeta_1}\varkappa(\varrho)\right)(\vartheta), & \vartheta \in (\alpha, \vartheta_1] \\ \dfrac{1}{\Gamma(\xi)}\left(\dfrac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi-1}\left(\tilde{\alpha} + \displaystyle\sum_{i=1}^{\nu}\chi_i\right) + \left({}^{\rho}I_{\alpha^+}^{\zeta_1}\varkappa(\varrho)\right)(\vartheta), & \vartheta \in (\vartheta_\nu, \vartheta_{\nu+1}], \end{cases}$$

where $\varkappa : (0, \mu] \to \mathbb{R}$ be a function satisfying the functional equation

$$\varkappa(\vartheta) = \psi(\vartheta, \mathfrak{p}(\vartheta), \varkappa(\vartheta)).$$

Since $\mathfrak{w}$ solution of the inequality (2.2) and by Remark 2.20, we have

$$\left({}^{\rho}D_{\alpha^+}^{\zeta_1,\zeta_2}\mathfrak{w}\right)(\vartheta) = \psi\left(\vartheta, \mathfrak{w}(\vartheta), \left({}^{\rho}D_{\alpha^+}^{\zeta_1,\zeta_2}\mathfrak{w}\right)(\vartheta)\right) + \eta(\vartheta), \text{ for each, } \vartheta \in (\vartheta_\nu, \vartheta_{\nu+1}], \tag{4.1}$$

$$\Delta^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{w}|_{\vartheta=\vartheta_\nu} = \chi_\nu + \eta_\nu, \ \nu = 1, \ldots, \varsigma. \tag{4.2}$$

Clearly, the solution of the problems (4.1) and (4.2) is given by

$$\mathfrak{w}(\vartheta) = \begin{cases} \dfrac{\tilde{\alpha}}{\Gamma(\xi)}\left(\dfrac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi-1} + \left({}^{\rho}I_{\alpha^+}^{\zeta_1}(\gamma(\varrho) + \eta(\varrho))\right)(\vartheta), & \vartheta \in (\alpha, \vartheta_1] \\ \dfrac{1}{\Gamma(\xi)}\left(\dfrac{\vartheta^\rho - \alpha^\rho}{\rho}\right)^{\xi-1}\left(\tilde{\alpha} + \displaystyle\sum_{i=1}^{\nu}(\chi_i + \eta_i)\right) & \\ + \left({}^{\rho}I_{\alpha^+}^{\zeta_1}(\gamma(\varrho) + \eta(\varrho))\right)(\vartheta), \quad \vartheta \in (\vartheta_\nu, \vartheta_{\nu+1}], & \end{cases}$$

where $\gamma : (0, \mu] \to \mathbb{R}$ be a function satisfying the functional equation

$$\gamma(\vartheta) = \psi(\vartheta, \mathfrak{w}(\vartheta), \gamma(\vartheta)).$$

If $\vartheta \in (\alpha, \vartheta_1]$, it follows that

$$\begin{aligned} |\mathfrak{w}(\vartheta) - \mathfrak{p}(\vartheta)| &\leq \frac{1}{\Gamma(\zeta_1)}\int_\alpha^\vartheta \left(\frac{\vartheta^\rho - \varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}|\eta(\varrho)|d\varrho \\ &+ \frac{1}{\Gamma(\zeta_1)}\int_\alpha^\vartheta \left(\frac{\vartheta^\rho - \varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}|\gamma(\varrho) - \varkappa(\varrho)|d\varrho \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\kappa}{\Gamma(\zeta_1)}\int_\alpha^\vartheta\left(\frac{\vartheta^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}\omega(\varrho)d\varrho\\
&+ \frac{1}{\Gamma(\zeta_1)}\int_\alpha^\vartheta\left(\frac{\vartheta^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}|\gamma(\varrho)-\varkappa(\varrho)|d\varrho\\
&\leq \kappa\lambda_\omega\omega(\vartheta)+\frac{1}{\Gamma(\zeta_1)}\int_\alpha^\vartheta\left(\frac{\vartheta^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}|\gamma(\varrho)-\varkappa(\varrho)|d\varrho,
\end{aligned}
$$

where $\varkappa,\gamma\in C_{1-\xi,\rho}(\Theta)$ such that

$$
\begin{aligned}
\varkappa(\vartheta)&=\psi(\vartheta,\mathfrak{p}(\vartheta),\varkappa(\vartheta)),\\
\gamma(\vartheta)&=\psi(\vartheta,\mathfrak{w}(\vartheta),\gamma(\vartheta)).
\end{aligned}
$$

By (H2), we have

$$
\begin{aligned}
|\gamma(\vartheta)-\varkappa(\vartheta)| &= |\psi(\vartheta,\mathfrak{w}(\vartheta),\gamma(\vartheta))-\psi(\vartheta,\mathfrak{p}(\vartheta),\varkappa(\vartheta))|\\
&\leq \theta_1|\mathfrak{w}(\vartheta)-\mathfrak{p}(\vartheta)|+\theta_2|\gamma(\vartheta)-\varkappa(\vartheta)|.
\end{aligned}
$$

Then,

$$
|\gamma(\vartheta)-\varkappa(\vartheta)|\leq\frac{\theta_1}{1-\theta_2}|\mathfrak{w}(\vartheta)-\mathfrak{p}(\vartheta)|.
$$

Therefore, for each $\vartheta\in(\alpha,\vartheta_1]$

$$
|\mathfrak{w}(\vartheta)-\mathfrak{p}(\vartheta)|\leq\kappa\lambda_\omega\omega(\vartheta)+\frac{\theta_1}{(1-\theta_2)\Gamma(\zeta_1)}\int_\alpha^\vartheta\left(\frac{\vartheta^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}|\mathfrak{w}(\varrho)-\mathfrak{p}(\varrho)|d\varrho.
$$

Applying Corollary 2.15, we get

$$
\begin{aligned}
|\mathfrak{w}(\vartheta)-\mathfrak{p}(\vartheta)| &\leq \kappa\lambda_\omega\omega(\vartheta)E_{\zeta_1}\left(\frac{\theta_1}{1-\theta_2}\left(\frac{\vartheta^\rho-\alpha^\rho}{\rho}\right)^{\zeta_1}\right)\\
&\leq \kappa\lambda_\omega\omega(\vartheta)E_{\zeta_1}\left(\frac{\theta_1}{1-\theta_2}\left(\frac{\mu^\rho-\alpha^\rho}{\rho}\right)^{\zeta_1}\right)\\
&\leq \kappa\lambda_\omega(\psi+\omega(\vartheta))E_{\zeta_1}\left(\frac{\theta_1}{1-\theta_2}\left(\frac{\mu^\rho-\alpha^\rho}{\rho}\right)^{\zeta_1}\right)=\kappa c_1(\psi+\omega(\vartheta)),
\end{aligned}
$$

where

$$
c_1 = \lambda_\omega E_{\zeta_1}\left(\frac{\theta_1}{1-\theta_2}\left(\frac{\mu^\rho-\alpha^\rho}{\rho}\right)^{\zeta_1}\right).
$$

If $\vartheta\in(\vartheta_\nu,\vartheta_{\nu+1}],\ \nu=1,\dots,\varsigma$, then we have

$$
\begin{aligned}
|\mathfrak{w}(\vartheta)-\mathfrak{p}(\vartheta)| &\leq \frac{1}{\Gamma(\xi)}\left(\tfrac{\vartheta^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}\sum_{\nu=1}^{\varsigma}|\eta_\nu|+\frac{1}{\Gamma(\zeta_1)}\int_\alpha^\vartheta\left(\tfrac{\vartheta^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}|\eta(\varrho)|d\varrho\\
&+ \frac{1}{\Gamma(\zeta_1)}\int_\alpha^\vartheta\left(\frac{\vartheta^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}|\gamma(\varrho)-\varkappa(\varrho)|d\varrho
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\varsigma\kappa\psi}{\Gamma(\xi)}\left(\frac{\vartheta_1^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}+\frac{\kappa}{\Gamma(\zeta_1)}\int_\alpha^\vartheta\left(\frac{\vartheta^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}\omega(\varrho)d\varrho\\
&+\frac{1}{\Gamma(\zeta_1)}\int_\alpha^\vartheta\left(\frac{\vartheta^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}|\gamma(\varrho)-\varkappa(\varrho)|d\varrho\\
&\leq \frac{\varsigma\kappa\psi}{\Gamma(\xi)}\left(\frac{\vartheta_1^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}+\kappa\lambda_\omega\omega(\vartheta)\\
&+\frac{1}{\Gamma(\zeta_1)}\int_\alpha^\vartheta\left(\frac{\vartheta^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}|\gamma(\varrho)-\varkappa(\varrho)|d\varrho,
\end{aligned}
$$

where $\varkappa,\gamma\in C_{1-\xi,\rho}(\Theta)$ such that

$$
\begin{aligned}
\varkappa(\vartheta)&=\psi(\vartheta,\mathfrak{p}(\vartheta),\varkappa(\vartheta)),\\
\gamma(\vartheta)&=\psi(\vartheta,\mathfrak{w}(\vartheta),\gamma(\vartheta)).
\end{aligned}
$$

By (H2), we have

$$
\begin{aligned}
|\gamma(\vartheta)-\varkappa(\vartheta)| &= |\psi(\vartheta,\mathfrak{w}(\vartheta),\gamma(\vartheta))-\psi(\vartheta,\mathfrak{p}(\vartheta),\varkappa(\vartheta))|\\
&\leq \theta_1|\mathfrak{w}(\vartheta)-\mathfrak{p}(\vartheta)|+\theta_2|\gamma(\vartheta)-\varkappa(\vartheta)|.
\end{aligned}
$$

Then,

$$
|\gamma(\vartheta)-\varkappa(\vartheta)|\leq\frac{\theta_1}{1-\theta_2}|\mathfrak{w}(\vartheta)-\mathfrak{p}(\vartheta)|.
$$

Therefore, for each $\vartheta\in(\vartheta_\nu,\vartheta_{\nu+1}],\ \nu=1,\ldots,\varsigma$,

$$
\begin{aligned}
|\mathfrak{w}(\vartheta)-\mathfrak{p}(\vartheta)| &\leq \frac{\varsigma\kappa\varpi}{\Gamma(\xi)}\left(\frac{\vartheta_1^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}+\kappa\lambda_\omega\omega(\vartheta)\\
&+\frac{\theta_1}{(1-\theta_2)\Gamma(\zeta_1)}\int_\alpha^\vartheta\left(\frac{\vartheta^\rho-\varrho^\rho}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}|\mathfrak{w}(\varrho)-\mathfrak{p}(\varrho)|d\varrho.
\end{aligned}
$$

Applying Corollary 2.15, we get

$$
\begin{aligned}
|\mathfrak{w}(\vartheta)-\mathfrak{p}(\vartheta)| &\leq \left[\frac{\varsigma\kappa\psi}{\Gamma(\xi)}\left(\frac{\vartheta_1^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}+\kappa\lambda_\omega\omega(\vartheta)\right]E_{\zeta_1}\left(\frac{\theta_1}{1-\theta_2}\left(\frac{\vartheta^\rho-\alpha^\rho}{\rho}\right)^{\zeta_1}\right)\\
&\leq \left[\frac{\varsigma\kappa\psi}{\Gamma(\xi)}\left(\frac{\vartheta_1^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}+\kappa\lambda_\omega\omega(\vartheta)\right]E_{\zeta_1}\left(\frac{\theta_1}{1-\theta_2}\left(\frac{\mu^\rho-\alpha^\rho}{\rho}\right)^{\zeta_1}\right)\\
&\leq \kappa\left[\frac{\varsigma}{\Gamma(\xi)}\left(\frac{\vartheta_1^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}+\lambda_\omega\right](\omega(\vartheta)+\psi)E_{\zeta_1}\left(\frac{\theta_1}{1-\theta_2}\left(\frac{\mu^\rho-\alpha^\rho}{\rho}\right)^{\zeta_1}\right)\\
&= \kappa c_2(\omega(\vartheta)+\psi),
\end{aligned}
$$

where

$$
c_2 = \left[\frac{\varsigma}{\Gamma(\xi)}\left(\frac{\vartheta_1^\rho-\alpha^\rho}{\rho}\right)^{\xi-1}+\lambda_\omega\right]E_{\zeta_1}\left(\frac{\theta_1}{1-\theta_2}\left(\frac{\mu^\rho-\alpha^\rho}{\rho}\right)^{\zeta_1}\right).
$$

Thus, the Eq (1.1) is Ulam-Hyers-Rassias stable with respect to (ω,ϖ). The proof is complete.

The following theorem gives Ulam-Hyers stable result.

Theorem 4.2. *Assume that (H1), (H2) and* (3.21) *hold. Then, the Eq* (1.1) *is Ulam-Hyers stable.*

Proof. Let $\mathfrak{w} \in C_{1-\xi,\rho}(\Theta)$ be a solution of the inequality (2.1). Denote by $\mathfrak{p}$ the unique solution of the problem:

$$\left({}^{\rho}D_{\alpha^+}^{\zeta_1,\zeta_2}\mathfrak{p}\right)(\vartheta) = \psi\left(\vartheta, \mathfrak{p}(\vartheta), \left({}^{\rho}D_{\alpha^+}^{\zeta_1,\zeta_2}\mathfrak{p}\right)(\vartheta)\right), \text{ for each }, \ \vartheta \in (\vartheta_\nu, \vartheta_{\nu+1}], \ \nu = 0, \ldots, \varsigma,$$

$$\Delta^{\rho} I_{\alpha^+}^{1-\xi}\mathfrak{p}|_{\vartheta=\vartheta_\nu} = \chi_\nu \in \mathbb{R}, \ \nu = 1, \ldots, \varsigma,$$
$$\left({}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{w}\right)(\alpha) = \left({}^{\rho}I_{\alpha^+}^{1-\xi}\mathfrak{p}\right)(\alpha) = \tilde{\alpha} \in \mathbb{R}.$$

By the same way of the proof of Theorem 4.1, we can easily show that:

If $\vartheta \in (\alpha, \vartheta_1]$, it follows that

$$\begin{aligned}
|\mathfrak{w}(\vartheta) - \mathfrak{p}(\vartheta)| &\leq \frac{1}{\Gamma(\zeta_1)}\int_\alpha^\vartheta \left(\frac{\vartheta^\rho - \varrho^\rho}{\rho}\right)^{\zeta_1-1} \varrho^{\rho-1}|\eta(\varrho)|d\varrho \\
&+ \frac{1}{\Gamma(\zeta_1)}\int_\alpha^\vartheta \left(\frac{\vartheta^\rho - \varrho^\rho}{\rho}\right)^{\zeta_1-1} \varrho^{\rho-1}|\gamma(\varrho) - \varkappa(\varrho)|d\varrho \\
&\leq \frac{\kappa}{\Gamma(\zeta_1)}\int_\alpha^\vartheta \left(\frac{\vartheta^\rho - \varrho^\rho}{\rho}\right)^{\zeta_1-1} \varrho^{\rho-1}d\varrho \\
&+ \frac{\theta_1}{(1-\theta_2)\Gamma(\zeta_1)}\int_\alpha^\vartheta \left(\frac{\vartheta^\rho - \varrho^\rho}{\rho}\right)^{\zeta_1-1} \varrho^{\rho-1}|\mathfrak{w}(\varrho) - \mathfrak{p}(\varrho)|d\varrho \\
&\leq \frac{\kappa}{\Gamma(\zeta_1+1)}\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{\zeta_1} \\
&+ \frac{\theta_1}{(1-\theta_2)\Gamma(\zeta_1)}\int_\alpha^\vartheta \left(\frac{\vartheta^\rho - \varrho^\rho}{\rho}\right)^{\zeta_1-1} \varrho^{\rho-1}|\mathfrak{w}(\varrho) - \mathfrak{p}(\varrho)|d\varrho.
\end{aligned}$$

Applying Lemma 2.14, we get

$$\begin{aligned}
|\mathfrak{w}(\vartheta) - \mathfrak{p}(\vartheta)| &\leq \frac{\kappa}{\Gamma(\zeta_1+1)}\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{\zeta_1}\left[1 + \int_\alpha^\vartheta \sum_{\beta=1}^{\infty} \frac{\left(\frac{\theta_1}{1-\theta_2}\right)^\beta}{\Gamma(\beta\zeta_1)}\left(\frac{\vartheta^\rho - \varrho^\rho}{\rho}\right)^{\beta\zeta_1-1}\varrho^{\rho-1}d\varrho\right] \\
&\leq \frac{\kappa}{\Gamma(\zeta_1+1)}\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{\zeta_1}\left[1 + \sum_{\beta=1}^{\infty} \frac{\left(\frac{\theta_1}{1-\theta_2}\right)^\beta}{\Gamma(\beta\zeta_1+1)}\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{\beta\zeta_1}\right] \\
&= \frac{\kappa}{\Gamma(\zeta_1+1)}\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{\zeta_1}\left[1 + \sum_{\beta=1}^{\infty} \frac{1}{\Gamma(\beta\zeta_1+1)}\left[\left(\frac{\theta_1}{1-\theta_2}\right)\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{\zeta_1}\right]^\beta\right] \\
&= \frac{\kappa}{\Gamma(\zeta_1+1)}\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{\zeta_1}\left[1 + E_{\zeta_1}\left(\left(\frac{\theta_1}{1-\theta_2}\right)\left(\frac{\mu^\rho - \alpha^\rho}{\rho}\right)^{\zeta_1}\right)\right] = b_1\kappa.
\end{aligned}$$

If $\vartheta \in (\vartheta_\nu, \vartheta_{\nu+1}], \ \nu = 1, \ldots, \varsigma$, then we have

$$
\begin{aligned}
|\mathfrak{w}(\vartheta)-\mathfrak{p}(\vartheta)| &\leq \frac{1}{\Gamma(\xi)}\left(\frac{\vartheta^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}\sum_{\nu=1}^{\varsigma}|\eta_{\nu}|+\frac{1}{\Gamma(\zeta_1)}\int_{\alpha}^{\vartheta}\left(\frac{\vartheta^{\rho}-\varrho^{\rho}}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}|\eta(\varrho)|d\varrho\\
&+\frac{1}{\Gamma(\zeta_1)}\int_{\alpha}^{\vartheta}\left(\frac{\vartheta^{\rho}-\varrho^{\rho}}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}|\gamma(\varrho)-\varkappa(\varrho)|d\varrho\\
&\leq \frac{\varsigma\kappa}{\Gamma(\xi)}\left(\frac{\vartheta_1^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}+\frac{\kappa}{\Gamma(\zeta_1)}\int_{\alpha}^{\vartheta}\left(\frac{\vartheta^{\rho}-\varrho^{\rho}}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}d\varrho\\
&+\frac{1}{\Gamma(\zeta_1)}\int_{\alpha}^{\vartheta}\left(\frac{\vartheta^{\rho}-\varrho^{\rho}}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}|\gamma(\varrho)-\varkappa(\varrho)|d\varrho\\
&\leq \frac{\varsigma\kappa}{\Gamma(\xi)}\left(\frac{\vartheta_1^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}+\frac{\kappa}{\Gamma(\zeta_1+1)}\left(\frac{\mu^{\rho}-\alpha^{\rho}}{\rho}\right)^{\zeta_1}\\
&+\frac{\theta_1}{(1-\theta_2)\Gamma(\zeta_1)}\int_{\alpha}^{\vartheta}\left(\frac{\vartheta^{\rho}-\varrho^{\rho}}{\rho}\right)^{\zeta_1-1}\varrho^{\rho-1}|\mathfrak{w}(\varrho)-\mathfrak{p}(\varrho)|d\varrho.
\end{aligned}
$$

Applying Lemma 2.14, we get

$$
\begin{aligned}
|\mathfrak{w}(\vartheta)-\mathfrak{p}(\vartheta)| &\leq \kappa\left[\frac{\varsigma}{\Gamma(\xi)}\left(\frac{\vartheta_1^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}+\frac{1}{\Gamma(\zeta_1+1)}\left(\frac{\mu^{\rho}-\alpha^{\rho}}{\rho}\right)^{\zeta_1}\right]\\
&\left[1+\int_{\alpha}^{\vartheta}\sum_{\beta=1}^{\infty}\frac{\left(\frac{\theta_1}{1-\theta_2}\right)^{\beta}}{\Gamma(\beta\zeta_1)}\left(\frac{\vartheta^{\rho}-\varrho^{\rho}}{\rho}\right)^{\beta\zeta_1-1}\varrho^{\rho-1}d\varrho\right]\\
&\leq \kappa\left[\frac{\varsigma}{\Gamma(\xi)}\left(\frac{\vartheta_1^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}+\frac{1}{\Gamma(\zeta_1+1)}\left(\frac{\mu^{\rho}-\alpha^{\rho}}{\rho}\right)^{\zeta_1}\right]\\
&\left[1+\sum_{\beta=1}^{\infty}\frac{\left(\frac{\theta_1}{1-\theta_2}\right)^{\beta}}{\Gamma(\beta\zeta_1+1)}\left(\frac{\mu^{\rho}-\alpha^{\rho}}{\rho}\right)^{\beta\zeta_1}\right]\\
&= \kappa\left[\frac{\varsigma}{\Gamma(\xi)}\left(\frac{\vartheta_1^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}+\frac{1}{\Gamma(\zeta_1+1)}\left(\frac{\mu^{\rho}-\alpha^{\rho}}{\rho}\right)^{\zeta_1}\right]\\
&\left[1+\sum_{\beta=1}^{\infty}\frac{1}{\Gamma(\beta\zeta_1+1)}\left[\left(\frac{\theta_1}{1-\theta_2}\right)\left(\frac{\mu^{\rho}-\alpha^{\rho}}{\rho}\right)^{\zeta_1}\right]^{\beta}\right]\\
&= \kappa\left[\frac{\varsigma}{\Gamma(\xi)}\left(\frac{\vartheta_1^{\rho}-\alpha^{\rho}}{\rho}\right)^{\xi-1}+\frac{1}{\Gamma(\zeta_1+1)}\left(\frac{\mu^{\rho}-\alpha^{\rho}}{\rho}\right)^{\zeta_1}\right]\\
&\left[1+E_{\zeta_1}\left(\left(\frac{\theta_1}{1-\theta_2}\right)\left(\frac{\mu^{\rho}-\alpha^{\rho}}{\rho}\right)^{\zeta_1}\right)\right]=b_2\kappa,
\end{aligned}
$$

which completes the proof of the theorem. Moreover, if we set $\varpi_{\psi,\varsigma}(\kappa)=(b_1+b_2)\kappa$; $\varpi_{\psi,\varsigma}(0)=0$, then, the Eq (1.1) is generalized Ulam-Hyers stable.

5. An example

Consider the following initial value problem with impulse

$$ {}^{\frac{1}{2}}D_{1^+}^{\frac{1}{2},0}\mathfrak{p}(\vartheta) = \frac{2+|\mathfrak{p}(\vartheta)|+\left|{}^{\frac{1}{2}}D_{0^+}^{\frac{1}{2},0}\mathfrak{p}(\vartheta)\right|}{108e^{-\vartheta+3}\left(1+|\mathfrak{p}(\vartheta)|+\left|{}^{\frac{1}{2}}D_{0^+}^{\frac{1}{2},0}\mathfrak{p}(\vartheta)\right|\right)} + \frac{\ln(\sqrt{\vartheta}+1)}{3\sqrt{\sqrt{\vartheta}-1}}, \quad \vartheta \in J_0 \cup J_1, \tag{5.1} $$

$$ \Delta^{\rho} I_{\alpha^+}^{1-\xi}\mathfrak{p}\left(\frac{3}{2}\right) = \eta \in \mathbb{R}, \tag{5.2} $$

$$ {}^{\frac{1}{2}}I_{1^+}^{1-\xi}\mathfrak{p}(1) = 0, \tag{5.3} $$

where $J_0 = \left(1, \frac{3}{2}\right]$, $J_1 = \left(\frac{3}{2}, 2\right]$.

Set

$$ \psi(\vartheta, \mathfrak{x}, \mathfrak{y}) = \frac{2+|\mathfrak{x}|+|\mathfrak{y}|}{108e^{-\vartheta+3}(1+|\mathfrak{x}|+|\mathfrak{y}|)} + \frac{\ln(\sqrt{\vartheta}+1)}{3\sqrt{\vartheta}}, \ \vartheta \in (1,2], \ \mathfrak{x}, \mathfrak{y} \in \mathbb{R}. $$

We have

$$ PC_{1-\xi,\rho}^{\zeta_2(1-\zeta_1)}([1,2]) = PC_{\frac{1}{2},\frac{1}{2}}^{0}([1,2]) = PC_{\frac{1}{2},\frac{1}{2}}([1,2]), $$

with $\xi = \zeta_1 = \rho = \frac{1}{2}$ and $\zeta_2 = 0$. Clearly, the function $\psi \in PC_{\frac{1}{2},\frac{1}{2}}([1,2])$.

Hence condition (H1) is satisfied.

For each $\mathfrak{x}, \bar{\mathfrak{x}}, \mathfrak{y}, \bar{\mathfrak{y}} \in \mathbb{R}$ and $\vartheta \in (1,2]$:

$$ \begin{aligned} |\psi(\vartheta,\mathfrak{x},\mathfrak{y}) - \psi(\vartheta,\bar{\mathfrak{x}},\bar{\mathfrak{y}})| &\leq \frac{1}{108e^{-\vartheta+3}}(|\mathfrak{x}-\bar{\mathfrak{x}}| + |\mathfrak{y}-\bar{\mathfrak{y}}|) \\ &\leq \frac{1}{108e}\left(|\mathfrak{x}-\bar{\mathfrak{x}}| + |\mathfrak{y}-\bar{\mathfrak{y}}|\right). \end{aligned} $$

Hence condition $(H2)$ is satisfied with $\theta_1 = \theta_2 = \dfrac{1}{108e}$.

The condition

$$ \frac{\theta_1\Gamma(\xi)}{\Gamma(\zeta_1+\xi)(1-\theta_2)}\left(\frac{\mu^{\rho}-\alpha^{\rho}}{\rho}\right)^{\zeta_1} \approx 0.0055 < 1, $$

is satisfied with with $\alpha = 1$ and $\mu = 2$. It follows from Theorem 3.4 that the problems (5.1)–(5.3) has a unique solution in the space $PC_{\frac{1}{2},\frac{1}{2}}^{\frac{1}{2}}([1,2])$. Moreover, Theorem 4.2, implies that the Eq (1.1) is Ulam-Hyers stable.

6. Conclusions

We have investigated the existence, uniqueness and stability of solutions for a class of nonlinear impulsive Hilfer-Katugampola problems. Our reasoning is founded on the Banach contraction principle and Krasnoselskii's fixed point theorem. In addition, an example is provided to demonstrate the effectiveness of the main results. We plan to consider for for a futur study the same problem in infinite dimensional Banach space and make us of Darbo and Monch's fixed point theorems associated with the notion of measure of noncompactness.

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Conflict of interest

The authors declare that there is no conflict of interest.

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