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Research article

Inequalities for different type of functions via Caputo fractional derivative

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Abstract: In this paper, we obtain some new inequalities for different type of functions that are connected with the Caputo fractional derivative. We extend and generalize some important inequalities to this interesting calculus including Hermite-Hadamard inequality.

Keywords: Hermite-Hadamard inequality; Caputo fractional derivative; convex functions; different types of functions
Mathematics Subject Classification: 26A33, 26D15

1. Introduction

Inequalities develop tools for analyzing the problems in pure and applied mathematics. Thus, inequalities have become the centre of interest for many branches of science. A large number of extensions, various generalizations have been investigated in the theory of mathematical inequalities. We refer the reader to the references [1–7]. Fractional calculus is the generalization of derivatives and integrals of arbitrary non-integer order. In the last few years, some new definitions of fractional operators have been introduced to provide the best method for fractional calculus. Such type of studies promotes future researches to obtain new ideas to unify the fractional derivatives and fractional inequalities. For a review of this topic we direct the reader to the monographes [8, 9].

In [10], Samraiz et al. developed a new class of trapezium-type inequalities up to twice differentiable h-convex mappings for fractional integrals of Riemann-type. Ali et al. established a new version of generalized fractional Hadamard and Fejér-Hadamard type integral inequalities in [11]. Rahman et al. investigated some novel inequalities for a class of differentiable functions related to Chebyshev's functionals in [12]. In [13], Mohammed et al. considered some Hermite-Hadamard-Fejér fractional integral inequalities and related results for the weighted fractional operators.

Some authors extended fractional inequalities using time scale calculus. For instance, Mohammed et al. introduced new time scales on $\mathbb{Z}$ and investigated the discrete inequality of Hermite-Hadamard type for discrete convex functions in [14]. A variety of various types of some classical inequalities have been established utilizing the fractional derivative and integral operators and their extensions are found in [15–18].

In [19], authors established several inequalities connected with the Riemann-Liouville fractional integrals. Our aim in this paper is to extend and establish some new inequalities using Caputo fractional derivative for different classes of convex functions and mappings. Now we give the definitions of different classes of functions and Caputo fractional derivative to obtain our main results.

Definition 1.1. *A function $f:[a,b]\to\mathbb{R}$ is said to be convex if the inequality*

$$f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y)$$

holds for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. If the inequality holds in the reverse direction, the function f is called concave.

Definition 1.2. *A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is said to belong to the class of $Q(I)$, if it is nonnegative and for all $x,y\in I$ and $\lambda\in(0,1)$ satisfies the following inequality:*

$$f(\lambda x+(1-\lambda)y)\le\frac{f(x)}{\lambda}+\frac{f(y)}{(1-\lambda)}. \tag{1.1}$$

Definition 1.3. *A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is P function or that f belongs to the class of $P(I)$, if it is nonnegative and for all $x,y\in I$ and $\lambda\in[0,1]$ satisfies the following inequality:*

$$f(\lambda x+(1-\lambda)y)\le f(x)+f(y). \tag{1.2}$$

The power mean $M_r(x,y;\lambda)$ of order r of positive numbers x,y is defined by

$$M_r(x,y;\lambda)=\begin{cases}(\lambda x^r+(1-\lambda)y^r)^{\frac{1}{r}}, & r\neq 0,\\ x^{\lambda}y^{1-\lambda}, & r=0.\end{cases}$$

In [1], Pearce et al. generalized this inequality to r-convex positive function f which is defined on an interval $[a,b]$, for all $x,y\in[a,b]$ and $\lambda\in[0,1]$,

$$f(\lambda x+(1-\lambda)y)\le M_r(f(x),f(y);\lambda).$$

Definition 1.4. *Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ be a positive function. We say that $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is h-convex or that f belongs to the class of $SX(h,I)$. If f is nonnegative and for all $x,y\in I$ and $\lambda\in(0,1)$, we have*

$$f(\lambda x+(1-\lambda)y)\le h(\lambda)f(x)+h(1-\lambda)f(y). \tag{1.3}$$

If the above inequality is reversed, then f is said to be h-concave, i.e. $f\in SV(h,I)$.

Definition 1.5. *$f:I\to\mathbb{R}$ is a convex function on the interval I of real numbers and $u,v\in I$ with $u<v$, the inequality*

$$f\left(\frac{u+v}{2}\right)\le\frac{1}{v-u}\int_u^v f(x)\,dx\le\frac{f(u)+f(v)}{2}$$

is known as the Hermite-Hadamard inequality.

Definition 1.6. *Suppose that* $\alpha > 0,\ t > a,\ \alpha, a, t \in \mathbb{R}$. *The fractional operator*

$$^{C}D^{\alpha} f(t) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int\limits_{a}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, & n-1 < \alpha < n \in \mathbb{N}, \\ \frac{d^{n}}{dt^{n}} f(t), & \alpha = n \in \mathbb{N}, \end{cases}$$

is called the Caputo fractional derivative or Caputo fractional differential operator of order α. *This operator is introduced by the Italian mathematician Caputo in 1967.*

Definition 1.7. *Let* $\alpha > 0$ *and* $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in C^{n}[a, b]$ *be a continuous n order differentiable function. Right-sided and left-sided Caputo fractional derivatives are defined as follows:*

$$\left(^{C}D_{a^{+}}^{\alpha} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int\limits_{a}^{x} \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, \quad x > a,$$

and

$$\left(^{C}D_{b^{-}}^{\alpha} f\right)(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int\limits_{x}^{b} \frac{f^{(n)}(\tau)}{(\tau-x)^{\alpha-n+1}} d\tau, \quad x < b.$$

Definition 1.8. *Let* $p \geq 1,\ 0 < \int\limits_{a}^{b} f^{p}(x)\, dx < \infty$ *and* $0 < \int\limits_{a}^{b} g^{p}(x)\, dx < \infty$. *Then Minkowski integral inequality is given as follow:*

$$\left(\int\limits_{a}^{b} (f(x) + g(x))^{p}\, dx \right)^{\frac{1}{p}} \leq \left(\int\limits_{a}^{b} f^{p}(x)\, dx \right)^{\frac{1}{p}} + \left(\int\limits_{a}^{b} g^{p}(x)\, dx \right)^{\frac{1}{p}}.$$

Lemma 1.1. *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable mapping on* (a, b) *with* $a < b$. *If* $f^{(n+1)} \in L[a, b]$, *then the following fractional equality holds:*

$$\frac{f^{(n)}(a) + f^{(n)}(b)}{2} - \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}} \left[^{C}D_{a^{+}}^{\alpha} f(b) + (-1)^{n}\ ^{C}D_{b^{-}}^{\alpha} f(a) \right]$$
$$= \frac{b-a}{2} \int\limits_{0}^{1} \left[(1-t)^{n-\alpha} - t^{n-\alpha} \right] f^{(n+1)}(ta + (1-t)b)\, dt.$$

2. Results

In this section, we establish some new inequalities for different type of functions using Caputo derivatives.

Theorem 2.1. *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable mapping on* (a, b). *If* $\left| f^{(n+1)} \right|$ *belongs to the class of* $Q(I)$, $a, b \in I$ *with* $0 \leq a < b$. *Then we have the following inequality:*

$$\left| \frac{f^{(n)}(a) + f^{(n)}(b)}{2} - \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}} \left[^{C}D_{a^{+}}^{\alpha} f(b) + (-1)^{n}\ ^{C}D_{b^{-}} f(a) \right] \right|$$
$$\leq (b-a) \left\{ \left[f^{(n+1)}(a) + f^{(n+1)}(b) \right] \left[\beta(0, n-\alpha+1) + \frac{1}{n-\alpha} \right] \right\}.$$

Proof. By using the Lemma 1.1 and the Definition 1.2, we have

$$
\begin{aligned}
&\left|\tfrac{f^{(n)}(a)+f^{(n)}(b)}{2}-\tfrac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{a^{+}}f(b)+(-1)^{n}\ {}^{C}D_{b^{-}}f(a)\right]\right| \\
= &\quad \left|\tfrac{b-a}{2}\int_{0}^{1}\left[(1-t)^{n-\alpha}-t^{n-\alpha}\right]f^{(n+1)}(ta+(1-t)b)\,dt\right| \\
\leq &\quad \tfrac{b-a}{2}\int_{0}^{1}\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left[\left|\tfrac{f^{(n+1)}(a)}{t}\right|+\left|\tfrac{f^{(n+1)}(b)}{1-t}\right|\right]dt \\
\leq &\quad \tfrac{b-a}{2}\left\{\int_{0}^{1}\left|(1-t)^{n-\alpha}\ t^{-1}\ f^{(n+1)}(a)\right|dt+\int_{0}^{1}\left|t^{n-\alpha-1}\ f^{(n+1)}(a)\right|dt\right. \\
&\quad \left.+\int_{0}^{1}\left|(1-t)^{n-\alpha-1}\ f^{(n+1)}(b)\right|dt+\int_{0}^{1}\left|t^{n-\alpha}(1-t)^{-1}\ f^{(n+1)}(b)\right|dt\right\} \\
= &\quad (b-a)\left\{\left[f^{(n+1)}(a)+f^{(n+1)}(b)\right]\left[\int_{0}^{1}\left|(1-t)^{n-\alpha}\ t^{-1}\ \right|dt+\tfrac{1}{n-\alpha}\right]\right\} \\
= &\quad (b-a)\left\{\left[f^{(n+1)}(a)+f^{(n+1)}(b)\right]\left[\beta(0,n-\alpha+1)+\tfrac{1}{n-\alpha}\right]\right\}.
\end{aligned}
$$

This ends the proof.

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If the function $\left|f^{(n+1)}\right|$ belongs to the class of $P(I)$, then f the following inequality holds:*

$$
\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{a^{+}}f(b)+(-1)^{n}\ {}^{C}D_{b^{-}}f(a)\right]\right|\leq\frac{(b-a)}{n-\alpha}\left[f^{(n+1)}(a)+f^{(n+1)}(b)\right].
$$

Proof. As in the proof of the Theorem 2.1, using the definition of $P(I)$, we have

$$
\begin{aligned}
&\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{a^{+}}f(b)+(-1)^{n}\ {}^{C}D_{b^{-}}f(a)\right]\right| \\
\leq&\frac{b-a}{2}\int_{0}^{1}\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left|f^{(n+1)}(ta+(1-t)b)\right|dt \\
\leq&\frac{b-a}{2}\int_{0}^{1}\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left|f^{(n+1)}(a)+f^{(n+1)}(b)\right|dt \\
\leq&\frac{b-a}{2}\left\{f^{(n+1)}(a)\int_{0}^{1}\left|(1-t)^{n-\alpha}\right|dt+f^{(n+1)}(b)\int_{0}^{1}\left|(1-t)^{n-\alpha}\right|dt\right. \\
&\left.+f^{(n+1)}(a)\int_{0}^{1}\left|t^{n-\alpha}\right|dt+f^{(n+1)}(b)\int_{0}^{1}\left|t^{n-\alpha}\right|dt\right\} \\
=&\frac{b-a}{n-\alpha+1}\left[f^{(n+1)}(a)+f^{(n+1)}(b)\right].
\end{aligned}
$$

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b). If $\left|f^{(n+1)}\right|$ is a r-convex positive function, then we have the following inequality:*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D_{a^{+}}^{\alpha}f(b)+(-1)^{n}\ {}^{C}D_{b^{-}}f(a)\right]\right|$$
$$\leq\frac{b-a}{2}\left\{\left[f^{(n+1)}(a)\right]^{r}\left[\beta\left(\frac{r+1}{r},n-\alpha+1\right)+\frac{1}{n-\alpha+\frac{1}{r}+1}\right]^{r}\right.$$
$$\left.+\left[f^{(n+1)}(b)\right]^{r}\left[\frac{1}{n-\alpha+\frac{1}{r}+1}+\beta\left(n-\alpha+1,\frac{r+1}{r}\right)\right]^{r}\right\}^{1/r}.$$

Proof. Using the definition of r-convex function and the Minkowski integral inequality, we write

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D_{a^{+}}^{\alpha}f(b)+(-1)^{n}\ {}^{C}D_{b^{-}}f(a)\right]\right|$$
$$\leq\frac{b-a}{2}\int_{0}^{1}\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left|f^{(n+1)}(ta+(1-t)b)\right|dt$$
$$\leq\frac{b-a}{2}\int_{0}^{1}\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left|t\left[f^{(n+1)}(a)\right]^{r}+(1-t)\left(f^{(n+1)}(b)\right)^{r}\right|^{1/r}dt$$
$$\leq\frac{b-a}{2}\left\{\left[\int_{0}^{1}\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left|t^{1/r}f^{(n+1)}(a)\right|dt\right]^{r}\right.$$
$$\left.+\left[\int_{0}^{1}\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left|(1-t)^{1/r}f^{(n+1)}(b)\right|dt\right]^{r}\right\}^{1/r}$$
$$\leq\frac{b-a}{2}\left\{\left[f^{(n+1)}(a)\int_{0}^{1}\left|(1-t)^{n-\alpha}t^{1/r}\right|dt+f^{(n+1)}(a)\int_{0}^{1}\left|t^{n-\alpha+1/r}\right|dt\right]^{r}\right.$$
$$\left.+\left[f^{(n+1)}(b)\int_{0}^{1}\left|(1-t)^{n-\alpha+1/r}\right|dt+\ f^{(n+1)}(b)\int_{0}^{1}\left|t^{n-\alpha}(1-t)^{1/r}\right|dt\right]^{r}\right\}^{1/r}$$
$$\leq\frac{b-a}{2}\left\{\left[f^{(n+1)}(a)\right]^{r}\left[\beta\left(\frac{r+1}{r},n-\alpha+1\right)+\frac{1}{n-\alpha+\frac{1}{r}+1}\right]^{r}\right.$$
$$\left.+\left[f^{(n+1)}(b)\right]^{r}\left[\frac{1}{n-\alpha+\frac{1}{r}+1}+\beta\left(n-\alpha+1,\frac{r+1}{r}\right)\right]^{r}\right\}^{1/r}.$$

Theorem 2.4. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b). If the function $\left|f^{(n+1)}\right|$ belongs* *to the class of $SX(h,I)$, then we have the following inequality:*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D_{a^+}^{\alpha}f(b)+(-1)^n\ {}^{C}D_{b^-}f(a)\right]\right|$$
$$\leq(b-a)\ \left[f^{(n+1)}(a)+f^{(n+1)}(b)\right]\int_0^1(1-t)^{n-\alpha}h(t)\,dt.$$

Proof. From Lemma 1.1, we have

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D_{a^+}^{\alpha}f(b)+(-1)^n\ {}^{C}D_{b^-}f(a)\right]\right|$$
$$\leq\frac{b-a}{2}\int_0^1\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left|f^{(n+1)}(ta+(1-t)b)\right|dt$$
$$\leq\frac{b-a}{2}\left\{\int_0^1\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left|h(t)f^{(n+1)}(a)+h(1-t)f^{(n+1)}(b)\right|dt\right\}$$
$$\leq\frac{b-a}{2}\left\{\int_0^1\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left|h(t)f^{(n+1)}(a)\right|dt\right.$$
$$\left.+\int_0^1\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left|h(1-t)f^{(n+1)}(b)\right|dt\right\}$$
$$\leq\frac{b-a}{2}\left\{\int_0^1\left|(1-t)^{n-\alpha}h(t)f^{(n+1)}(a)\right|dt+\int_0^1\left|t^{n-\alpha}h(t)f^{(n+1)}(a)\right|dt\right.$$
$$\left.+\int_0^1\left|(1-t)^{n-\alpha}h(1-t)f^{(n+1)}(b)\right|dt+\int_0^1\left|t^{n-\alpha}h(1-t)f^{(n+1)}(b)\right|dt\right\}$$
$$=(b-a)\ \left[f^{(n+1)}(a)+f^{(n+1)}(b)\right]\int_0^1(1-t)^{n-\alpha}h(t)\,dt.$$

Theorem 2.5. *Let f be a function belongs to the class $Q(I)$, $a,b\in I$ with $0\leq a<b$ and $f\in L_1[a,b]$. Then we have*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[{}^{C}D_{\alpha^+}^{\alpha}f(b)\ +(-1)^n\ {}^{C}D_{b^-}^{\alpha}f(a)\right],\tag{2.1}$$

for $\alpha>0$.

Proof. Since $f\in Q(I)$, using Definition 1.2, we have

$$f\left(\frac{x+y}{2}\right)\leq 2\left(f(x)+f(y)\right),$$

for all $x, y \in I$ and $\lambda = \frac{1}{2}$. Substituting $x = ta + (1-t)b$, $y = (1-t)a + tb$, we obtain

$$f\left(\frac{a+b}{2}\right) \leq 2\left(f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right). \tag{2.2}$$

Then multiplying both sides of (2.2) by $t^{n-\alpha-1}$ and integrating the inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_0^1 t^{n-\alpha-1}dt \leq& 2\int_0^1 t^{n-\alpha-1}\left[f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right]dt,\\
\frac{1}{n-\alpha}f\left(\frac{a+b}{2}\right) \leq& 2\int_a^b\left(\frac{b-x}{b-a}\right)^{n-\alpha-1} f(x)\frac{dx}{b-a}+2\int_a^b\left(\frac{y-a}{b-a}\right)^{n-\alpha-1} f(y)\frac{dy}{b-a},\\
f\left(\frac{a+b}{2}\right) \leq& \frac{2(n-\alpha)}{(b-a)^{n-\alpha}}\int_a^b f(x)\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right]dx,\\
f\left(\frac{a+b}{2}\right) \leq& \frac{2\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{\alpha^{+}}f(b)+(-1)^{n}\ {}^{C}D^{\alpha}_{b^{-}}f(a)\right].
\end{aligned}$$

This completes the proof.

Theorem 2.6. *Let f be a function belongs to the class $P(I)$, $a, b \in I$ with $0 \leq a < b$ and $f \in L_1[a,b]$. Then we have*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{\alpha^{+}}f(b)+(-1)^{n}\ {}^{C}D^{\alpha}_{b^{-}}f(a)\right]\\
&\leq 2\left(f(a)+f(b)\right),
\end{aligned} \tag{2.3}$$

for $\alpha > 0$.

Proof. Using Definition 1.3 and substituting with $x = ta + (1-t)b$, $y = (1-t)a + tb$ and $\lambda = \frac{1}{2}$, we have

$$f\left(\frac{a+b}{2}\right) \leq f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right), \tag{2.4}$$

for all $t \in [0,1]$.

As in the proof of Theorem 2.5, multiplying both sides of inequality (2.4) by $t^{n-\alpha-1}$ and integrating the inequality with respect to t over $[0,1]$, we have

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_0^1 t^{n-\alpha-1}dt \leq& \int_0^1 t^{n-\alpha-1}\left[f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right]dt,\\
\frac{1}{n-\alpha}f\left(\frac{a+b}{2}\right) \leq& \frac{1}{(b-a)^{n-\alpha}}\int_a^b f(x)\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right]dx,
\end{aligned}$$

$$f\left(\frac{a+b}{2}\right) \leq \frac{n-\alpha}{(b-a)^{n-\alpha}} \int_a^b f(x)\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right] dx,$$

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{\alpha^+} f(b)+(-1)^n \; {}^{C}D^{\alpha}_{b^-} f(a)\right],$$

which completes the first part of the proof of inequality (2.3).

Since $f \in P(I)$, we have

$$f(ta+(1-t)b) \leq f(a)+f(b)$$

and

$$f((1-t)a+tb) \leq f(a)+f(b).$$

Using above inequalities, we obtain

$$f(ta+(1-t)b)+f((1-t)a+tb) \leq 2[f(a)+f(b)]. \tag{2.5}$$

If we multiply both sides of inequality (2.5) by $t^{n-\alpha-1}$ and integrate with respect to t from 0 to 1, we get

$$\begin{aligned}
\int_0^1 t^{n-\alpha-1}[f(ta+(1-t)b)+f((1-t)a+tb)]\,dt &\leq 2[f(a)+f(b)]\int_0^1 t^{n-\alpha-1} dt, \\
\frac{n-\alpha}{(b-a)^{n-\alpha}} \int_a^b f(x)\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right] dx &\leq 2[f(a)+f(b)], \\
\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{\alpha^+} f(b)+(-1)^n \; {}^{C}D^{\alpha}_{b^-} f(a)\right] &\leq 2[f(a)+f(b)],
\end{aligned}$$

and thus the second part of the proof of inequality (2.3) is proved.

Theorem 2.7. *Let $f:[a,b]\to(0,\infty)$ be r-convex function on $[a,b]$ with $a<b$ and $0<r\leq 1$. Then the following inequality for fractional derivative holds:*

$$\begin{aligned}
&\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{\alpha^+} f(b)+(-1)^n \; {}^{C}D^{\alpha}_{b^-} f(a)\right] \\
\leq\; &\left[\left(\frac{1}{n-\alpha+\frac{1}{r}}\right)^r [f(a)]^r+\left(\beta\left(n-\alpha,\frac{r+1}{r}\right)\right)^r [f(b)]^r\right]^{\frac{1}{r}} \\
&+\left[\left(\beta\left(n-\alpha,\frac{r+1}{r}\right)\right)^r [f(a)]^r+\left(\frac{1}{n-\alpha+\frac{1}{r}}\right)^r [f(b)]^r\right]^{\frac{1}{r}},
\end{aligned}$$

where β is the beta function.

Proof. Since f is r-convex and $r>0$, we have

$$f(ta+(1-t)b) \leq \left(t[f(a)]^r+(1-t)[f(b)]^r\right)^{\frac{1}{r}}$$

and

$$f((1-t)a+tb) \leq ((1-t)[f(a)]^r + t[f(b)]^r)^{\frac{1}{r}},$$

for all $t \in [0,1]$. By using these inequalities, we have

$$\begin{aligned} & f(ta+(1-t)b) + f((1-t)a+tb) \\ \leq \; & (t[f(a)]^r + (1-t)[f(b)]^r)^{\frac{1}{r}} + ((1-t)[f(a)]^r + t[f(b)]^r)^{\frac{1}{r}}. \end{aligned}$$

If we multiply both sides of above inequality by $t^{n-\alpha-1}$ and integrate with respect to t from 0 to 1, we obtain

$$\begin{aligned} & \int_0^1 t^{n-\alpha-1}[f(ta+(1-t)b) + f((1-t)a+tb)]\,dt \\ \leq \; & \int_0^1 t^{n-\alpha-1}(t[f(a)]^r + (1-t)[f(b)]^r)^{\frac{1}{r}}\,dt + \int_0^1 t^{n-\alpha-1}((1-t)[f(a)]^r + t[f(b)]^r)^{\frac{1}{r}}\,dt. \end{aligned}$$

We know that

$$\int_0^1 t^{n-\alpha-1}[f(ta+(1-t)b) + f((1-t)a+tb)]\,dt = \frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}\left[{}^C D^{\alpha}_{\alpha^+} f(b) + (-1)^n \; {}^C D^{\alpha}_{b^-} f(a)\right].$$

By using the Minkowski inequality, we have

$$\begin{aligned} & \int_0^1 t^{n-\alpha-1}(t[f(a)]^r + (1-t)[f(b)]^r)^{\frac{1}{r}} \\ \leq & \left[\left(\int_0^1 t^{n-\alpha+\frac{1}{r}-1} f(a)\,dt\right)^r + \left(\int_0^1 t^{n-\alpha-1}(1-t)^{\frac{1}{r}} f(b)\,dt\right)^r\right]^{\frac{1}{r}} \\ = & \left[\left(\frac{1}{n-\alpha+\frac{1}{r}}\right)^r [f(a)]^r + \left(\beta\left(n-\alpha, \frac{r+1}{r}\right)\right)^r [f(b)]^r\right]^{\frac{1}{r}}, \end{aligned}$$

and similarly we obtain

$$\begin{aligned} & \int_0^1 t^{n-\alpha-1}((1-t)[f(a)]^r + t[f(b)]^r)^{\frac{1}{r}} \\ \leq & \left[\left(\int_0^1 t^{n-\alpha-1}(1-t)^{\frac{1}{r}} f(a)\,dt\right)^r + \left(\int_0^1 t^{n-\alpha+\frac{1}{r}-1} f(b)\,dt\right)^r\right]^{\frac{1}{r}} \\ = & \left[\left(\beta\left(n-\alpha, \frac{r+1}{r}\right)\right)^r [f(a)]^r + \left(\frac{1}{n-\alpha+\frac{1}{r}}\right)^r [f(b)]^r\right]^{\frac{1}{r}}. \end{aligned}$$

Thus, we have

$$\begin{aligned} & \frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[{}^{C}D_{\alpha^{+}}^{\alpha} f(b)+(-1)^{n}\ {}^{C}D_{b^{-}}^{\alpha} f(a)\right] \\ \leq\ & \left[\left(\frac{1}{n-\alpha+\frac{1}{r}}\right)^{r}[f(a)]^{r}+\left(\beta\left(n-\alpha, \frac{r+1}{r}\right)\right)^{r}[f(b)]^{r}\right]^{\frac{1}{r}} \\ & +\left[\left(\beta\left(n-\alpha, \frac{r+1}{r}\right)\right)^{r}[f(a)]^{r}+\left(\frac{1}{n-\alpha+\frac{1}{r}}\right)^{r}[f(b)]^{r}\right]^{\frac{1}{r}} . \end{aligned}$$

This completes the proof.

Theorem 2.8. *Let $f \in SX(h, I)$, $a, b \in I$ with $a < b$ and $L_1[a, b]$. Then we have inequality for h-convex functions via fractional derivative:*

$$\begin{aligned} \frac{1}{(n-\alpha) h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) & \leq \frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}\left[{}^{C}D_{\alpha^{+}}^{\alpha} f(b)+(-1)^{n}\ {}^{C}D_{b^{-}}^{\alpha} f(a)\right] \\ & \leq [f(a)+f(b)] \int_{0}^{1} t^{n-\alpha-1}[h(t)+h(1-t)]\, dt. \end{aligned} \tag{2.6}$$

Proof. According to inequality (1.3), if we replace $x = ta + (1-t)b$, $y = (1-t)a + tb$ and $n - \alpha = \frac{1}{2}$, we have

$$f\left(\frac{a+b}{2}\right) \leq h\left(\frac{1}{2}\right)[f(ta+(1-t)b)+f((1-t)a+tb)]. \tag{2.7}$$

If we multiply both sides of inequality (2.7) by $t^{n-\alpha-1}$ and integrate with respect to t from 0 to 1, we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \int_{0}^{1} t^{n-\alpha-1} dt & \leq h\left(\frac{1}{2}\right) \int_{0}^{1} t^{n-\alpha-1}[f(ta+(1-t)b)+f((1-t)a+tb)]\, dt, \\ \frac{1}{(n-\alpha) h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) & \leq \frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}\left[{}^{C}D_{\alpha^{+}}^{\alpha} f(b)+(-1)^{n}\ {}^{C}D_{b^{-}}^{\alpha} f(a)\right], \end{aligned}$$

and the first part of the inequality (2.6) is proved.

Since $f \in SX(h, I)$, we have

$$f(tx+(1-t)y) \leq h(t) f(x)+h(1-t) f(y)$$

and

$$f((1-t)x+ty) \leq h(1-t) f(x)+h(t) f(y).$$

Using these inequalities, we get

$$f(tx+(1-t)y)+f((1-t)x+ty) \leq [h(t)+h(1-t)][f(x)+f(y)]. \tag{2.8}$$

If we replace $x = a$ and $y = b$ in inequality (2.8), we find

$$f(ta + (1-t)b) + f((1-t)a + tb) \leq [h(t) + h(1-t)][f(a) + f(b)]. \tag{2.9}$$

Then multiplying both sides of the inequality (2.9) by $t^{n-\alpha-1}$ and integrating the inequality with respect to t over $[0, 1]$, we have

$$\int_0^1 t^{n-\alpha-1}[f(ta + (1-t)b) + f((1-t)a + tb)]\,dt \leq \int_0^1 t^{n-\alpha-1}[h(t) + h(1-t)][f(a) + f(b)]\,dt$$

and

$$\frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{\alpha^+}f(b) + (-1)^n\ {}^{C}D^{\alpha}_{b^-}f(a)\right] \leq [f(a) + f(b)]\int_0^1 t^{n-\alpha-1}[h(t) + h(1-t)]\,dt,$$

so that the second part of the inequality (2.6) is proved. Hence, the proof is complete.

3. Conclusions

Convex functions play an important role in the advancement of many inequalities. Most of the well-known inequalities are the consequences of convex functions. In this work, we have motivated by different type of convex functions which are integrable and we have obtained new results for Caputo fractional derivative. Fractional inequalities have gained considerable importance and popularity. Thus, this paper presents a new approach to new versions of different inequalities. We conclude that the results presented in this study would be a guide for further investigations concerning inequalities for various kinds of fractional calculus.

Conflict of interest

The author declares no conflict of interest.

References

1. C. E. M. Pearce, J. Pečarić, V. Šimić, Stolarsky means and Hadamard's inequality, *J. Math. Anal. Appl.*, **220** (1998), 99–109. https://doi.org/10.1006/jmaa.1997.5822
2. M. Bombardelli, S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities, *Comput. Math. Appl.*, **58** (2009), 1869–1877. https://doi.org/10.1016/j.camwa.2009.07.073
3. M. Z. Sarikaya, E. Set, M. E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions, *Acta Math. Univ. Comenianae*, **79** (2010), 265–272.
4. N. P. G. Ngoc, N. V. Vinh, P. T. T. Hien, Integral inequalities of Hadamard type for r-convex functions, *Int. Math. Forum*, **4** (2009), 1723–1728.

5. P. M. Gill, C. E. M. Pearce, J. Pečarić, Hadamard's inequality for r-convex functions, *J. Math. Anal. Appl.*, **215** (1997), 461–470. https://doi.org/10.1006/jmaa.1997.5645
6. S. S. Dragomir, Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces, *Proyecciones*, **34** (2015), 323–341. https://doi.org/10.4067/S0716-09172015000400002
7. S. S. Dragomir, J. Pečarić, L. E. Persson, Some inequalities of Hadamard type, *Soochow J. Math.*, **21** (1995), 335–341.
8. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.
9. B. Ross, *Fractional calculus and its applications*, Berlin, Heidelberg: Springer-Verlag, 1975.
10. M. Samraiz, F. Nawaz, B. Abdalla, T. Abdeljawad, G. Rahman, Estimates of trapezium-type inequalities for h-convex functions with applications to quadrature formulae, *AIMS Math.*, **6** (2021), 7625–7648. https://doi.org/10.3934/math.2021443
11. R. S. Ali, A. Mukheimer, T. Abdeljawad, S. Mubeen, S. Ali, G. Rahman, et al., Some new harmonically convex function type generalized fractional integral inequalities, *Fractal Fract.*, **5** (2021), 1–12. https://doi.org/10.3390/fractalfract5020054
12. G. Rahman, A. Hussain, A. Ali, K. S. Nisar, R. N. Mohamed, More general weighted-type fractional integral inequalities via Chebyshev functionals, *Fractal Fract.*, **5** (2021), 1–14. https://doi.org/10.3390/fractalfract5040232
13. P. O. Mohammed, T. Abdeljawad, A. Kashuri, Fractional Hermite-Hadamard-Fejer inequalities for a convex function with respect to an increasing function involving a positive weighted symmetric function, *Symmetry*, **12** (2020), 1–17. https://doi.org/10.3390/sym12091503
14. P. O. Mohammed, T. Abdeljawad, M. A. Alqudah, F. Jarad, New discrete inequalities of Hermite-Hadamard type for convex functions, *Adv. Differ. Equ.*, **2021** (2021), 1–10. https://doi.org/10.1186/s13662-021-03290-3
15. F. X. Chen, Extensions of the Hermite-Hadamard inequality for convex functions via fractional integrals, *J. Math. Inequal.*, **10** (2016), 75–81. https://doi.org/10.7153/jmi-10-07
16. G. Farid, A. Javed, S. Naqvi, Hadamard and Fejér-Hadamard inequalities and related results via Caputo fractional derivatives, *Bull. Math. Anal. Appl.*, **9** (2017), 16–30.
17. T. Abdeljawad, M. A. Ali, P. O. Mohammed, A. Kashuri, On inequalities of Hermite-Hadamard-Mercer type involving Riemann-Liouville fractional integrals, *AIMS Math.*, **6** (2021), 712–725. https://doi.org/10.3934/math.2021043
18. T. Abdeljawad, P. O. Mohammed, A. Kashuri, New modified conformable fractional integral inequalities of Hermite-Hadamard type with applications, *J. Funct. Space.*, **2020** (2020), 1–14. https://doi.org/10.1155/2020/4352357
19. Ç. Yildiz, M. E. Ozdemir, H. K. Onelan, Fractional integral inequalities for different functions, *New Trends Math. Sci.*, **3** (2015), 110–117.

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