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Research article

On degenerate generalized Fubini polynomials

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Abstract: The n-th Fubini number counts the number of ordered partitions of a set with n elements and is the number of possible ways to write the Fubini formula for a summation of integration of order n. Further, Fubini polynomials are natural extensions of the Fubini numbers. Recently, the generalized Fubini polynomials were introduced by Sebaoui-Laissaoui-Guettai-Rahmani, as one of the variants of the Fubini polynomials. The aim of this paper is to study the degenerate generalized Fubini polynomials, which are a degenerate version of those polynomials, and to find an application of them to probability theory in connection with geometric random variable.

Keywords: egenerate generalized Fubini polynomials; degenerate Eulerian polynomials; degenerate Frobenius-Euler polynomials; geometric random variable
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1. Introduction

Carlitz initiated an exploration of degenerate Bernoulli and Euler polynomials, which are degenerate versions of the ordinary Bernoulli and Euler polynomials (see [2]). Along the same line as Carlitz's pioneering work, intensive studies have been done for degenerate versions of quite a few special polynomials and numbers. There are various ways of studying special numbers and polynomials, to mention a few, generating functions, p-adic analysis, umbral calculus, operator theory, combinatorial methods, differential equations, special functions, probability theory and analytic number theory.

The aim of this paper is to study the degenerate generalized Fubini polynomials $F_{n,\lambda}(x, y)$, which are a degenerate version of the recently introduced generalized Fubini polynomials, and to find an application to probability theory. In more detail, for those polynomials we derive an explicit expression, an identity involving those polynomials, an integral representation, connections with the two variable degenerate Fubini polynomials, the degenerate Eulerian polynomials and with the degenerate Frobenius-Euler polynomials, a recurrence relation and a convolution type identity. Further, we show that the degenerate generalized Fubini polynomial is given by the expectation of some random variable made out of the geometric random variable with success probability $p = \frac{1}{x+1}$. Here we recall that Fubini number $F_n = \sum_{k=0}^{n} k!S_2(n,k)$ enumerates the ordered partitions of the set $[n] = \{1, 2, \dots, n\}$, where $S_2(n,k)$ denotes the Stirling numbers of the second kind. It is the number of possible ways to write the Fubini formula for a summation of integration of order n. A natural extension of F_n is the Fubini polynomial $F_n(x) = \sum_{k=0}^{n} k!S_2(n,k)x^k$. Two variable variants of Fubini polynomials include the two variable Fubini polynomials and the generalized Fubini polynomials. For some of the recent works on Fubini polynomials, the reader may refer to [4].

The outline of this paper is as follows. In Section 1, we recall degenerate exponentials, degenerate logarithms, degenerate Stirling numbers of the first kind and degenerate Stirling numbers of the second kind which are respectively degenerate versions of exponentials, logarithms, Stirling numbers of the first kind and Stirling numbers of the second kind. Also, we remind the reader of Fubini polynomials and two variable Fubini polynomials, and their degenerate versions, namely degenerate Fubini polynomials and degenerate two variable Fubini polynomials. Finally, we recall the generalized Fubini polynomials which are recently introduced. The degenerate generalized Fubini polynomials are a degenerate version of that and the central thing in this paper. In Section 2, we state our main results. In Theorem 1, we express the degenerate generalized Fubini polynomials in terms of the degenerate Stirling numbers of the second kind. In Theorem 2, we obtain an identity involving the degenerate generalized Fubini polynomial and some differential operator. In Theorem 3, we have an integral representation of the degenerate generalized Fubini polynomial. In Theorem 4, the degenerate generalized Fubini polynomial is expressed in terms of the two variable degenerate Fubini polynomials. In Theorem 5, we deduce an identity involving the degenerate generalized Fubini polynomials and the two variable degenerate Fubini polynomials. A recurrence relation is derived for the degenerate generalized Fubini polynomials in Theorem 6. A convolution identity is obtained for the degenerate generalized Fubini polynomials in Theorem 7. The degenerate generalized Fubini polynomials are expressed in terms of the degenerate Eulerian polynomials in Theorem 9 and of the degenerate Frobenius-Euler polynomials in Theorem 10. In Section 3, we present an application to probability theory. Indeed, we show that the expectation of $(X-1)_{n,\lambda}$ is equal to $F_{n,\lambda}(x)$, if X is the geometric random variable with success probability $p = \frac{1}{x+1}$. In the rest of this section, we state what are needed throughout this paper.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponential functions are defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \qquad e_\lambda^1(t) = e_\lambda(t) = (1+\lambda t)^{\frac{1}{\lambda}}, \quad \text{(see [10])}. \tag{1.1}$$

From (1.1), we note that

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} \frac{(x)_{n,\lambda}}{n!} t^n, \quad \text{(see [7–12])}, \tag{1.2}$$

where

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1). \tag{1.3}$$

In [8], the degenerate Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k)(x)_{k,\lambda}, \quad (n \geq 0), \tag{1.4}$$

where $(x)_0 = 1,\ (x)_n = x(x-1)\cdots(x-n+1),\ (n \geq 1)$.

As the inversion formula of (1.4), the degenerate Stirling numbers of the second kind are defined as

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k)(x)_k, \quad (n \geq 0), \quad \text{(see [8])}. \tag{1.5}$$

Let $\log_\lambda t$ be the compositional inverse of $e_\lambda(t)$. Then we have

$$\log_\lambda(1+t) = \frac{1}{\lambda}((1+t)^\lambda - 1), \quad \text{(see [8])}. \tag{1.6}$$

For (1.4) and (1.5), we note that

$$\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}, \tag{1.7}$$

and

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad \text{(see [8])}. \tag{1.8}$$

Note that $\lim_{\lambda\to 0} S_{1,\lambda}(n,k) = S_1(n,k)$ and $\lim_{\lambda\to 0} S_{2,\lambda}(n,k) = S_2(n,k),\ (n,k \geq 0)$, where $S_1(n,k)$ and $S_2(n,k)$ are the Stirling numbers of the first kind and the Stirling numbers of the second kind, respectively. Here we recall that

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k, \quad x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \quad (n \geq 0), \quad \text{(see [1,3,5–16,18,19])}.$$

It is well known that the Fubini polynomials are defined by

$$\frac{1}{1-x(e^t-1)} = \sum_{n=0}^{\infty} F_n(x)\frac{t^n}{n!}, \quad \text{(see [6,13,14,16])}. \tag{1.9}$$

For (1.9), we note that

$$F_n(x) = \sum_{k=0}^{n} k!S_2(n,k)x^k, \quad (n \geq 0), \quad \text{(see [5,6,13,16])}. \tag{1.10}$$

The two variable Fubini polynomials are given by

$$\frac{1}{1-x(e^t-1)}e^{yt} = \sum_{n=0}^{\infty} F_n(x|y)\frac{t^n}{n!}, \quad \text{(see [6,12])}. \tag{1.11}$$

By (1.9) and (1.10), we get

$$F_n(x|y)=\sum_{k=0}^{n}\binom{n}{k}F_k(x)y^{n-k},\quad (n\geq 0),\quad \text{(see [6])}.$$

Recently, the degenerate Fubini polynomials are defined as

$$\frac{1}{1-x(e_\lambda(t)-1)}=\sum_{n=0}^{\infty}F_{n,\lambda}(x)\frac{t^n}{n!},\quad \text{(see [12])}. \tag{1.12}$$

Thus, by (1.8) and (1.12), we get

$$F_{n,\lambda}(x)=\sum_{k=0}^{n}k!S_{2,\lambda}(n,k)x^k,\quad (n\geq 0),\quad \text{(see [12])}. \tag{1.13}$$

In view of (1.11), the two variable degenerate Fubini polynomials are defined as

$$\frac{1}{1-x(e_\lambda(t)-1)}e_\lambda^{y}(t)=\sum_{n=0}^{\infty}F_{n,\lambda}(x|y)\frac{t^n}{n!},\quad \text{(see [9])}. \tag{1.14}$$

By (1.14), we easily get

$$F_{n,\lambda}(x|y)=\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}(x)(y)_{n-k,\lambda},\quad (n\geq 0),\quad \text{(see [9])}.$$

Recently, a sequence of polynomials $F_n(x,y)$, called generalized Fubini polynomials, was introduced in [15], which are given by

$$\sum_{n=0}^{\infty}F_n(x,y)\frac{t^n}{n!}=\frac{1}{1-\frac{x}{y}(e^{ty}-1)},\quad \text{(see [6,18])}. \tag{1.15}$$

In this paper, we consider the two variable degenerate Fubini polynomials and investigate some properties of those polynomials associated with combinatorial numbers and polynomials. From our investigation, we derive some new formulae and identities related to degenerate Fubini and Eulerian polynomials.

2. On degenerate generalized Fubini polynomials

As a degenerate version of (1.15), we consider the *degenerate generalized Fubini polynomials* which are given by

$$\frac{y}{y-x(e_\lambda(ty)-1)}=\sum_{n=0}^{\infty}F_{n,\lambda}(x,y)\frac{t^n}{n!}. \tag{2.1}$$

From (2.1), we note that

$$\sum_{n=0}^{\infty} F_{n,\lambda}(x,y)\frac{t^n}{n!} = \frac{y}{y-x(e_\lambda(ty)-1)} = \frac{1}{1-\frac{x}{y}(e_\lambda(ty)-1)} \tag{2.2}$$
$$= \sum_{k=0}^{\infty} \frac{x^k}{y^k} k!\frac{1}{k!}(e_\lambda(yt)-1)^k = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} k! S_{2,\lambda}(n,k)x^k y^{n-k}\Big)\frac{t^n}{n!}.$$

By comparing the coefficients on both sides of (2.2), we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$F_{n,\lambda}(x,y) = \sum_{k=0}^{n} S_{2,\lambda}(n,k)k!x^k y^{n-k}.$$

In particular, for $y = 1$, we have

$$F_{n,\lambda}(x,1) = \sum_{k=0}^{n} S_{2,\lambda}(n,k)k!x^k.$$

By (1.9) and (2.1), we get

$$\sum_{n=0}^{\infty} F_{n,\lambda}(x,y)\frac{t^n}{n!} = \frac{y}{y-x(e_\lambda(yt)-1)} = \frac{1}{1-\frac{x}{y}(e_\lambda(ty)-1)} = \sum_{n=0}^{\infty} F_{n,\lambda}\Big(\frac{x}{y}\Big)y^n\frac{t^n}{n!}. \tag{2.3}$$

Comparing the coefficients on both sides of (2.3), we have

$$F_{n,\lambda}(x,y) = y^n F_{n,\lambda}\Big(\frac{x}{y}\Big), \quad (n \geq 0).$$

We observe that

$$\Big(x\frac{d}{dx}\Big)_{n,\lambda}\frac{1}{1-x} = \sum_{k=0}^{\infty}\Big(x\frac{d}{dx}\Big)_{n,\lambda}x^k = \sum_{k=0}^{\infty}(k)_{n,\lambda}x^k = \frac{1}{1-x}F_{n,\lambda}\Big(\frac{x}{1-x}\Big), \quad (n \geq 0). \tag{2.4}$$

Here we note that the differential operator $(x\frac{d}{dx})_{n,\lambda}$ is defined by (1.3). From (2.1), we note that

$$\frac{y}{y-x}\sum_{n=0}^{\infty} F_{n,\lambda}\Big(\frac{yx}{y-x},y\Big)\frac{t^n}{n!}$$
$$= \frac{y}{y-x}\Big(\frac{1}{1-\frac{x}{y-x}(e_\lambda(yt)-1)}\Big) = \frac{y}{y-x-x(e_\lambda(yt)-1)} \tag{2.5}$$
$$= \frac{y}{y-xe_\lambda(yt)} = \frac{1}{1-\frac{x}{y}e_\lambda(yt)} = \sum_{k=0}^{\infty}\Big(\frac{x}{y}\Big)^k e_\lambda^k(yt)$$
$$= \sum_{k=0}^{\infty}\Big(\frac{x}{y}\Big)^k\sum_{n=0}^{\infty}(k)_{n,\lambda}\frac{y^n t^n}{n!} = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{\infty}\Big(\frac{x}{y}\Big)^k(k)_{n,\lambda}y^n\Big)\frac{t^n}{n!}.$$

Thus, by comparing the coefficients on both sides of (2.5), we get

$$\sum_{k=0}^{\infty}(k)_{n,\lambda}x^k y^{n-k} = \frac{y}{y-x}F_{n,\lambda}\Big(\frac{yx}{y-x},y\Big), \quad (n\geq 0). \tag{2.6}$$

On the other hand, we get

$$\Big(x\frac{d}{dx}\Big)_{n,\lambda}\frac{y}{y-x} = \Big(x\frac{d}{dx}\Big)_{n,\lambda}\frac{1}{1-\frac{x}{y}} = \Big(x\frac{d}{dx}\Big)_{n,\lambda}\sum_{k=0}^{\infty}y^{-k}x^k = \sum_{k=0}^{\infty}\Big(\frac{x}{y}\Big)^k(k)_{n,\lambda}, \quad (n\geq 0). \tag{2.7}$$

Therefore, by (2.6) and (2.7), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$\frac{y}{y-x}F_{n,\lambda}\Big(\frac{xy}{y-x},y\Big) = \sum_{k=0}^{\infty}(k)_{n,\lambda}x^k y^{n-k} = y^n\Big(x\frac{d}{dx}\Big)_{n,\lambda}\Big(\frac{y}{y-x}\Big).$$

When $y = 1$ in Theorem 2, we see that

$$\frac{1}{1-x}F_{n,\lambda}\Big(\frac{x}{1-x},1\Big) = \frac{1}{1-x}F_{n,\lambda}\Big(\frac{x}{1-x}\Big) = \sum_{k=0}^{\infty}(k)_{n,\lambda}x^k = \Big(x\frac{d}{dx}\Big)_{n,\lambda}\Big(\frac{1}{1-x}\Big), \quad (n\geq 0).$$

It is well known that the degenerate Bell polynomials are defined by

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty}\phi_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [11])}. \tag{2.8}$$

For (1.7) and (2.8), we note that

$$\phi_{n,\lambda}(x) = \sum_{k=0}^{n}S_{2,\lambda}(n,k)x^k, \quad (n\geq 0), \quad \text{(see [11])}. \tag{2.9}$$

From Theorem 1 and (2.9), we note that

$$\begin{aligned} F_{n,\lambda}(x,y) &= \sum_{k=0}^{n}S_{2,\lambda}(n,k)k!x^k y^{n-k} = y^n\sum_{k=0}^{n}S_{2,\lambda}(n,k)\Big(\frac{x}{y}\Big)^k\Gamma(k+1) \\ &= y^n\sum_{k=0}^{n}S_{2,\lambda}(n,k)\Big(\frac{x}{y}\Big)^k\int_0^\infty e^{-t}t^k dt = y^n\int_0^\infty\sum_{k=0}^{n}S_{2,\lambda}(n,k)\Big(\frac{xt}{y}\Big)^k e^{-t}dt \\ &= y^n\int_0^\infty\phi_{n,\lambda}\Big(\frac{x}{y}t\Big)e^{-t}dt, \quad (n\geq 0). \end{aligned} \tag{2.10}$$

Thus, by (2.10), we obtain the following theorem.

Theorem 3. *For $n \geq 0$, we have*

$$F_{n,\lambda}(x,y) = y^n\int_0^\infty\phi_{n,\lambda}\Big(\frac{x}{y}t\Big)e^{-t}dt.$$

By Theorem 3, we get

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n,\lambda}(x,y)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} y^n \int_0^{\infty} \phi_{n,\lambda}\Big(\frac{x}{y}z\Big)e^{-z}dz\frac{t^n}{n!} \\ &= \int_0^{\infty} e^{-z}\sum_{n=0}^{\infty}\phi_{n,\lambda}\Big(\frac{x}{y}z\Big)\frac{(yt)^n}{n!}dz = \int_0^{\infty} e^{-z}e^{\frac{x}{y}z(e_\lambda(yt)-1)}dz \\ &= \int_0^{\infty} e^{-z(1-\frac{x}{y}(e_\lambda(yt)-1))}dz = \frac{1}{1-\frac{x}{y}(e_\lambda(ty)-1)}.\end{aligned} \tag{2.11}$$

From (1.14) and (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n,\lambda}(x,y)\frac{t^n}{n!} &= \frac{1}{1-\frac{x}{y}(e_\lambda(ty)-1)} = e_\lambda^{-y}(yt)\frac{1}{1-\frac{x}{y}(e_\lambda(yt)-1)}e_\lambda^{y}(yt) \\ &= e_\lambda^{-y}(yt)\sum_{n=0}^{\infty}F_{n,\lambda}\Big(\frac{x}{y}\Big|y\Big)\frac{y^nt^n}{n!} = \sum_{k=0}^{\infty}y^k(-y)_{k,\lambda}\frac{t^k}{k!}\sum_{m=0}^{\infty}F_{m,\lambda}\Big(\frac{x}{y}\Big|y\Big)\frac{y^mt^m}{m!} \\ &= \sum_{n=0}^{\infty}\Big(y^n\sum_{k=0}^{n}\binom{n}{k}(-1)^k\langle y\rangle_{k,\lambda}F_{n-k,\lambda}\Big(\frac{x}{y}\Big|y\Big)\Big)\frac{t^n}{n!},\end{aligned} \tag{2.12}$$

where $\langle x\rangle_{0,\lambda} = 1,\ \langle x\rangle_{n,\lambda} = x(x+\lambda)\cdots(x+(n-1)\lambda),\ (n \geq 1)$.

Therefore, by comparing the coefficients on both sides of (2.12), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$F_{n,\lambda}(x,y) = y^n\sum_{k=0}^{n}\binom{n}{k}(-1)^k\langle y\rangle_{k,\lambda}F_{n-k,\lambda}\Big(\frac{x}{y}\Big|y\Big).$$

By (2.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n+1,\lambda}(x,y)\frac{t^n}{n!} &= \frac{d}{dt}\Big(\frac{1}{1-\frac{x}{y}(e_\lambda(yt)-1)}\Big) \\ =&\frac{xe_\lambda^{1-\lambda}(yt)}{(1-\frac{x}{y}(e_\lambda(yt)-1))^2} = \Big(\frac{x+y}{1-\frac{x}{y}(e_\lambda(yt)-1)}-y\Big)\frac{e_\lambda^{-\lambda}(yt)}{1-\frac{x}{y}(e_\lambda(yt)-1)} \\ =&(x+y)\sum_{k=0}^{\infty}F_{k,\lambda}(x,y)\frac{t^k}{k!}\sum_{l=0}^{\infty}F_{l,\lambda}\Big(\frac{x}{y}\Big|-\lambda\Big)y^l\frac{t^l}{l!} - y\sum_{n=0}^{\infty}F_{n,\lambda}\Big(\frac{x}{y}\Big|-\lambda\Big)y^n\frac{t^n}{n!}.\end{aligned} \tag{2.13}$$

Thus, by (2.13), we see that

$$F_{n+1,\lambda}(x,y) + y^{n+1}F_{n,\lambda}\Big(\frac{x}{y}\Big|-\lambda\Big) = (x+y)\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}(x,y)F_{n-k,\lambda}\Big(\frac{x}{y}\Big|-\lambda\Big)y^{n-k}. \tag{2.14}$$

Therefore, by (2.14), we obtain the following theorem.

Theorem 5. *For $n \geq 0$, we have*

$$F_{n+1,\lambda}(x,y) + y^{n+1}F_{n,\lambda}\Big(\frac{x}{y}\Big|-\lambda\Big) = (x+y)\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}(x,y)F_{n-k,\lambda}\Big(\frac{x}{y}\Big|-\lambda\Big)y^{n-k}.$$

It is easy to show that

$$\frac{d}{dt}\Big(\frac{1}{1-\frac{x}{y}(e_\lambda(yt)-1)}\Big)=\frac{1}{1-\frac{x}{y}(e_\lambda(ty)-1)}\frac{e_\lambda^{1-\lambda}(yt)}{\frac{1}{x}-\frac{1}{y}(e_\lambda(yt)-1)}. \tag{2.15}$$

From (2.15), we have

$$\Big(\frac{1}{x}-\frac{1}{y}(e_\lambda(ty)-1)\Big)\frac{d}{dt}\Big(\frac{1}{1-\frac{x}{y}(e_\lambda(ty)-1)}\Big)=\frac{e_\lambda^{1-\lambda}(yt)}{1-\frac{x}{y}(e_\lambda(yt)-1)} \tag{2.16}$$
$$=\sum_{k=0}^{\infty}F_{k,\lambda}(x,y)\frac{t^k}{k!}\sum_{m=0}^{\infty}(1-\lambda)_{m,\lambda}y^m\frac{t^m}{m!}=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\binom{n}{k}(1-\lambda)_{n-k,\lambda}y^{n-k}F_{k,\lambda}(x,y)\Big)\frac{t^n}{n!}.$$

On the other hand, by (2.1), we get

$$\begin{aligned}
&\Big(\frac{1}{x}-\frac{1}{y}(e_\lambda(ty)-1)\Big)\frac{d}{dt}\Big(\frac{1}{1-\frac{x}{y}(e_\lambda(ty)-1)}\Big)\\
=&\Big(\frac{1}{x}-\frac{1}{y}(e_\lambda(ty)-1)\Big)\sum_{k=0}^{\infty}F_{k+1,\lambda}(x,y)\frac{t^k}{k!}\\
=&\sum_{n=0}^{\infty}\Big(\frac{1}{x}+\frac{1}{y}\Big)F_{n+1,\lambda}(x,y)\frac{t^n}{n!}-\frac{1}{y}\sum_{l=0}^{\infty}(1)_{l,\lambda}\frac{y^lt^l}{l!}\sum_{k=0}^{\infty}F_{k+1,\lambda}(x,y)\frac{t^k}{k!}\\
=&\sum_{n=0}^{\infty}\Big(\frac{1}{x}+\frac{1}{y}\Big)F_{n+1,\lambda}(x,y)\frac{t^n}{n!}-\frac{1}{y}\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\binom{n}{k}(1)_{n-k,\lambda}y^{n-k}F_{k+1,\lambda}(x,y)\Big)\frac{t^n}{n!}.
\end{aligned} \tag{2.17}$$

Thus, by (2.16) and (2.17), we get

$$\Big(\frac{1}{x}+\frac{1}{y}\Big)F_{n+1,\lambda}(x,y)=\sum_{k=0}^{n}\binom{n}{k}y^{n-k}\Big((1-\lambda)_{n-k,\lambda}F_{k,\lambda}(x,y)+\frac{(1)_{n-k,\lambda}}{y}F_{k+1,\lambda}(x,y)\Big). \tag{2.18}$$

Therefore, by (2.18), we obtain the following theorem.

Theorem 6. *For $n\geq 0$, we have*

$$\Big(\frac{1}{x}+\frac{1}{y}\Big)F_{n+1,\lambda}(x,y)=\sum_{k=0}^{n}\binom{n}{y}y^{n-k}\Big((1-\lambda)_{n-k,\lambda}F_{k,\lambda}(x,y)+\frac{(1)_{n-k,\lambda}}{y}F_{k+1,\lambda}(x,y)\Big).$$

By elementary computation, we get

$$\begin{aligned}
&\frac{1}{1-\frac{a}{y}(e_\lambda(yt)-1)}\frac{1}{1-\frac{b}{y}(e_\lambda(yt)-1)}\\
=&\Big(\frac{b}{b-a}\Big)\frac{1}{1-\frac{b}{y}(e_\lambda(yt)-1)}-\Big(\frac{a}{b-a}\Big)\frac{1}{1-\frac{a}{y}(e_\lambda(yt)-1)}\\
=&\sum_{n=0}^{\infty}\Big(\frac{b}{b-a}F_{n,\lambda}(b,y)-\frac{a}{b-a}F_{n,\lambda}(a,y)\Big)\frac{t^n}{n!}.
\end{aligned} \tag{2.19}$$

From (2.1) and (2.19), we note that

$$\begin{aligned}&\frac{1}{1-\frac{a}{y}(e_\lambda(yt)-1)}\frac{1}{1-\frac{b}{y}(e_\lambda(yt)-1)}\\&=\sum_{k=0}^{\infty}F_{k,\lambda}(a,y)\frac{t^k}{k!}\sum_{l=0}^{\infty}F_{l,\lambda}(b,y)\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}(a,y)F_{n-k}(b,y)\Big)\frac{t^n}{n!}.\end{aligned}\tag{2.20}$$

Therefore, by comparing the coefficients on both sides of (2.20), we obtain the following theorem.

Theorem 7. *For $n \geq 0$ and $a, b \in \mathbb{R}$ with $a \neq b$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}(a,y)F_{n-k,\lambda}(b,y)=\frac{bF_{n,\lambda}(b,y)-aF_{n,\lambda}(a,y)}{b-a}.$$

It is well known that the Eulerian polynomials, $A_n(x)$, $(n \geq 0)$, are defined by

$$\sum_{n=0}^{\infty}A_n(x)\frac{t^n}{n!}=\frac{1-x}{e^{t(x-1)}-x},\quad (x\neq 1),\quad (\text{see [3, 14]}).$$

Here we remark that there are two different definitions for Eulerian polynomials. The other ones, denoted by $E_n(x)$, are defined by $E_n(x) = xA_n(x)$, for $n \geq 1$, and $E_0(x) = 1$ (see [17]). For $u \in \mathbb{C}$ with $u \neq 1$, the Frobenius-Euler polynomials are given by

$$\frac{1-u}{e^t-u}e^{xt}=\sum_{n=0}^{\infty}H_n(u|x)\frac{t^n}{n!},\quad (\text{see [3,14,19,20]}).$$

For any $\lambda \in \mathbb{R}$, the degenerate Eulerian polynomials are defined by

$$\frac{1-x}{e_\lambda(t(x-1))-x}=\sum_{n=0}^{\infty}A_{n,\lambda}(x)\frac{t^n}{n!},\quad (x\neq 1).\tag{2.21}$$

Note that $\lim_{\lambda\to 0}A_{n,\lambda}(x) = A_n(x)$, $(n \geq 0)$.

It is known that the degenerate Frobenius-Euler polynomials are defined by

$$\frac{1-u}{e_\lambda(t)-u}e_\lambda^x(t)=\sum_{n=0}^{\infty}H_{n,\lambda}(u|x)\frac{t^n}{n!}.\tag{2.22}$$

For $x = 0$, $H_{n,\lambda}(u) = H_{n,\lambda}(u|0)$ are called the degenerate Frobenius-Euler numbers.

From (2.21) and (2.22), we note that

$$\sum_{n=0}^{\infty}A_{n,\lambda}(x)\frac{t^n}{n!}=\frac{1-x}{e_\lambda(t(x-1))-x}=\sum_{n=0}^{\infty}H_{n,\lambda}(x)\frac{(x-1)^nt^n}{n!}.\tag{2.23}$$

By comparing the coefficients on both sides of (2.23), we obtain the following lemma.

Lemma 8. *For $n \geq 0$, we have*

$$H_{n,\lambda}(x) = \frac{1}{(x-1)^n} A_{n,\lambda}(x).$$

From (2.1), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n,\lambda}(x,y)\frac{t^n}{n!} &= \frac{1}{1-\frac{x}{y}(e_\lambda(yt)-1)} = \frac{1}{\frac{x}{y}(\frac{y}{x}-(e_\lambda(yt)-1))} \\ &= \frac{1}{-\frac{x}{y}(e_\lambda(yt)-(1+\frac{y}{x}))}.\end{aligned} \tag{2.24}$$

Thus, by (2.24), we get

$$\sum_{n=0}^{\infty}\Big(-\frac{x}{y}\Big)F_{n,\lambda}(x,y)\frac{t^n}{n!} = \frac{1}{e_\lambda(yt)-(1+\frac{y}{x})}. \tag{2.25}$$

From (2.21), we note that

$$-\sum_{n=0}^{\infty}\frac{A_{n,\lambda}(x)}{(x-1)^{n+1}}\frac{t^n}{n!} = \frac{1}{e_\lambda(t)-x}. \tag{2.26}$$

By (2.25) and (2.26), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\Big(-\frac{x}{y}\Big)F_{n,\lambda}(x,y)\frac{t^n}{n!} &= \frac{1}{e_\lambda(yt)-(1+\frac{y}{x})} \\ = -\sum_{n=0}^{\infty}\frac{y^nA_{n,\lambda}(1+\frac{y}{x})}{(\frac{y}{x})^{n+1}}\frac{t^n}{n!} &= -\sum_{n=0}^{\infty}\frac{x^{n+1}}{y}A_{n,\lambda}\Big(1+\frac{y}{x}\Big)\frac{t^n}{n!}.\end{aligned} \tag{2.27}$$

By comparing the coefficients on both sides of (2.27), we get

$$-\frac{x}{y}F_{n,\lambda}(x,y) = -\frac{x^{n+1}}{y}A_{n,\lambda}\Big(1+\frac{y}{x}\Big), \quad (n \geq 0). \tag{2.28}$$

Therefore, by (2.29), we obtain the following theorem.

Theorem 9. *For $n \geq 0$, we have*

$$F_{n,\lambda}(x,y) = x^nA_{n,\lambda}\Big(1+\frac{y}{x}\Big).$$

Let $\frac{y}{x}+1 = t$ in Theorem 9. Then we have

$$A_{n,\lambda}(t) = \Big(\frac{t-1}{y}\Big)^n F_{n,\lambda}\Big(\frac{y}{t-1},y\Big) = \Big(\frac{1}{x}\Big)^n F_{n,\lambda}(x,x(t-1)), \quad (n \geq 0).$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n,\lambda}(x,y)\frac{t^n}{n!} &= \frac{1}{1-\frac{x}{y}(e_\lambda(yt)-1)} = \frac{1}{-\frac{x}{y}(e_\lambda(yt)-(1+\frac{y}{x}))} \\ &= \frac{-\frac{y}{x}}{e_\lambda(yt)-(1+\frac{y}{x})} = \frac{1-(1+\frac{y}{x})}{e_\lambda(yt)-(1+\frac{y}{x})} = e_\lambda^{-y}(yt)\frac{1-(1+\frac{y}{x})}{e_\lambda(yt)-(1+\frac{y}{x})}e_\lambda^{y}(yt)\end{aligned} \tag{2.29}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty}(-1)^k\langle y\rangle_{k,\lambda}y^k\frac{t^k}{k!}\sum_{m=0}^{\infty}H_{m,\lambda}\Big(1+\frac{y}{x}\Big|y\Big)y^m\frac{t^m}{m!}\\
&= \sum_{n=0}^{\infty}\Big(y^n\sum_{k=0}^{n}\binom{n}{k}(-1)^k\langle y\rangle_{k,\lambda}H_{n-k,\lambda}\Big(1+\frac{y}{x}\Big|y\Big)\Big)\frac{t^n}{n!}.
\end{aligned}
$$

Therefore, by comparing the coefficients on both sides of (2.29), we obtain the following theorem.

Theorem 10. *For $n \geq 0$, we have*

$$F_{n,\lambda}(x,y) = y^n\sum_{k=0}^{n}\binom{n}{k}(-1)^k\langle y\rangle_{k,\lambda}H_{n-k,\lambda}\Big(1+\frac{y}{x}\Big|y\Big).$$

3. Further remark

In probability theory, the geometric distribution gives the probability that the first occurrence of success requires k independent trials, each with success probability p. If the probability of success on each trial is p, the probability that the k-th trial is the first success is

$$P\{X=k\} = (1-p)^{k-1}p, \quad (k=1,2,3,\dots).$$

The geometric random variable X is the number of experiments that were performed before obtaining a success. This random variable can take values of $1,2,3,\dots$. The probability mass function of X is given by

$$p(k) = P\{X=k\} = (1-p)^{k-1}p, \quad (k=1,2,3,\dots), \quad \text{(see [15])},$$

where p is the probability of success on each experiment.

Assume that X is the geometric random variable with success probability p. Let $f(x)$ be a real valued function. Then the expectation of $f(X)$ is defined by

$$E[f(X)] = \sum_{n=1}^{\infty}f(n)p(n), \quad \text{(see [15])}, \tag{3.1}$$

where $p(n) = P\{X=n\} = (1-p)^{n-1}p$ is the probability mass function of X.

For a fixed $\lambda \in \mathbb{R}$, let $f(x) = (x)_{n,\lambda} \in \mathbb{R}[x]$, $(n \geq 0)$. Then we have

$$E[(X)_{n,\lambda}] = \sum_{k=1}^{\infty}(k)_{n,\lambda}p(k) = p\sum_{k=1}^{\infty}(k)_{n,\lambda}(1-p)^{k-1} = p\sum_{k=0}^{\infty}(k+1)_{n,\lambda}(1-p)^k, \quad (n\geq 0). \tag{3.2}$$

Let X be the geometric random variable with success probability $p=\frac{1}{x+1}$. Then, by (3.2), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}E[(X-1)_{n,\lambda}]\frac{t^n}{n!} &= \frac{1}{x+1}\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{\infty}\Big(\frac{x}{x+1}\Big)^k(k)_{n,\lambda}\Big)\frac{t^n}{n!} \qquad (3.3)\\
&= \frac{1}{x+1}\sum_{k=0}^{\infty}\Big(\frac{x}{x+1}\Big)^k\sum_{n=0}^{\infty}(k)_{n,\lambda}\frac{t^n}{n!}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{x+1}\sum_{k=0}^{\infty}\Big(\frac{x}{x+1}\Big)^k e_\lambda^k(t) = \frac{1}{x+1}\Big(\frac{1}{1-\frac{x}{x+1}e_\lambda(t)}\Big)\\
&= \frac{1}{x+1-xe_\lambda(t)} = \frac{1}{1-x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} F_{n,\lambda}(x)\frac{t^n}{n!}.
\end{aligned}
$$

By comparing the coefficients on both sides of (3.3), we obtain the following theorem.

Theorem 11. *For $n \geq 0$, let X be the geometric random variable with success probability $p = \frac{1}{x+1}$. Then we have*

$$E[(X-1)_{n,\lambda}] = F_{n,\lambda}(x).$$

Remark. Assume that X is the geometric random variable with success probability $p = \frac{y}{x+y}$. Then we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} y^n E[(X-1)_{n,\lambda}]\frac{t^n}{n!} &= \frac{y}{x+y}\sum_{n=0}^{\infty}\Big(y^n\sum_{k=0}^{\infty}\Big(\frac{x}{x+y}\Big)^k (k)_{n,\lambda}\Big)\frac{t^n}{n!} \qquad (3.4)\\
&= \frac{y}{x+y}\sum_{k=0}^{\infty}\Big(\frac{x}{x+y}\Big)^k \sum_{n=0}^{\infty}(k)_{n,\lambda}y^n\frac{t^n}{n!}\\
&= \frac{y}{x+y}\sum_{k=0}^{\infty}\Big(\frac{x}{x+y}\Big)^k e_\lambda^k(yt) = \frac{y}{x+y}\frac{1}{1-\frac{x}{x+y}e_\lambda(yt)}\\
&= \frac{y}{x+y-xe_\lambda(yt)} = \frac{y}{y-x(e_\lambda(yt)-1)} = \sum_{n=0}^{\infty}F_{n,\lambda}(x,y)\frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficients on both sides of (3.4), we have

$$E[(X-1)_{n,\lambda}] = y^{-n}F_{n,\lambda}(x,y), \quad (n \geq 0).$$

4. Conclusions

In recent years, many mathematicians have drawn the attention to studies of degenerate versions of some special numbers and polynomials, which were initiated by Carlitz. These degenerate versions include the degenerate Stirling numbers of the first and second kinds, the degenerate Bernoulli numbers of the second kind and the degenerate Bell numbers and polynomials. In this paper, we studied the degenerate generalized Fubini polynomials which are a degenerate version of the generalized Fubini polynomials and found an application to probability theory. In more detail, for those polynomials we derived an explicit expression, an identity involving those polynomials, an integral representation, connections with the two variable degenerate Fubini polynomials, the degenerate Eulerian polynomials and with the degenerate Frobenius-Euler polynomials, a recurrence relation and a convolution type identity. Further, we showed that the degenerate generalized Fubini polynomial is given by the expectation of some random variable made out of the geometric random variable with success probability $p = \frac{1}{x+1}$.

It is one of our future projects to continue to explore various degenerate versions of many special polynomials and numbers and their applications to physics, science and engineering by using various tools aforementioned in Introduction.

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Conflict of interest

The authors declare no conflicts of interest.

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