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Research article

Alternating reflection method on conics leading to inverse trigonometric and hyperbolic functions

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Abstract: An unusual alternating reflection method on conics is presented to evaluate inverse trigonometric and hyperbolic functions.

Keywords: reflection; conics; digits; inverse trigonometric function; inverse hyperbolic function
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1. Introduction

It is known that circular reflections on the unit circle leads to the computation of the digits of the trigonometric constant π [1, 2]. A similar analysis shows that hyperbolic reflections on the unit hyperbola leads to the computation of the digits of the logarithmic constant $\ln(2)$ [3]. Looking at those two situations, we obtain a unified way to present an unusual alternating reflection method on conics to evaluate inverse trigonometric and hyperbolic functions. A lot of research has been done on the efficient evaluation of elementary functions, see for example [4–6]. The method we obtain is much more a curiosity than an efficient practical method, but since it uses only an elementary geometric operation (reflection), its presentation can be interesting.

In Section 2, we introduce parametrization of the conics and express reflections and rotations in terms of the parameter. The numerical method is presented in Section 3. The way to compute the digits of the solution of the problem is explained in Section 4. In Section 5 we present some examples to illustrate the unusual methods we obtained to evaluate inverse trigonometric and inverse hyperbolic functions.

2. Reflection and rotation on conics

Let $G = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}$, where $\delta = \pm 1$, so $G^2 = I$, and

$$\mathcal{S} = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; (x, y) G \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + \delta y^2 = 1,\ (1 - \delta) x \geq 0 \right\}.$$

So $\mathcal{S}$ is the unit circle for $\delta = +1$, and $\mathcal{S}$ is the right branch ($x \geq 0$) of the unit hyperbola for $\delta = -1$.

Suppose we have two regular functions $c(\cdot)$ and $s(\cdot)$ defined in such a way that to every point $(x, y) \in \mathcal{S}$ there exist θ such that $(x, y) = (c(\theta), s(\theta))$, so

$$c^2(\theta) + \delta s^2(\theta) = 1.$$

It is clear here that for $\delta = +1$ we can take

$$(c(\theta), s(\theta)) = (\cos(\theta), \sin(\theta)),$$

and for $\delta = -1$ we can take

$$(c(\theta), s(\theta)) = (\cosh(\theta), \sinh(\theta)).$$

For a given point $(c(\cdot), s(\cdot)) \in \mathcal{S}$ we observe that

$$(c(\theta), s(\theta)) G \begin{pmatrix} -\delta s(\theta) \\ c(\theta) \end{pmatrix} = 0,$$

we say that $(-\delta s(\theta), c(\theta))$ is G-orthogonal to $(c(\cdot), s(\cdot))$. From the regularity of $c(\cdot)$ and $s(\cdot)$ we get that

$$0 = c(\theta) c'(\theta) + \delta s(\theta) s'(\theta) = (c(\theta), s(\theta)) G \begin{pmatrix} c'(\theta) \\ s'(\theta) \end{pmatrix},$$

and since $(c'(\theta), s'(\theta))$ and $(-\delta s(\theta), c(\theta))$ are both G-orthogonal to $(c(\theta), s(\theta))$, we can suppose that

$$(c'(\theta), s'(\theta)) = (-\delta s(\theta), c(\theta))$$

then $(-\delta s(\theta), c(\theta))$ is the direction of the tangent to $\mathcal{S}$ at $(c(\theta), s(\theta))$.

Let us also suppose that

$$\left\{ \begin{array}{rcl} c(\alpha + \beta) & = & c(\alpha) c(\beta) - \delta s(\alpha) s(\beta), \\ s(\alpha + \beta) & = & s(\alpha) c(\beta) + c(\alpha) s(\beta), \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcr} c(-\theta) & = & c(\theta), \\ s(-\theta) & = & -s(\theta). \end{array} \right.$$

In particular $(c(0), s(0)) = (1, 0)$.

We introduce the following definitions and notations

$$\operatorname{rot}(\alpha) = \begin{pmatrix} c(\alpha) & -\delta s(\alpha) \\ s(\alpha) & c(\alpha) \end{pmatrix},$$

and

$$\mathrm{ref}(\alpha) = \begin{pmatrix} c(2\alpha) & \delta s(2\alpha) \\ s(2\alpha) & -c(2\alpha) \end{pmatrix}.$$

As expected we have

$$\mathrm{rot}(\alpha)\begin{pmatrix} c(\theta) \\ s(\theta) \end{pmatrix} = \begin{pmatrix} c(\theta+\alpha) \\ s(\theta+\alpha) \end{pmatrix},$$

and

$$\mathrm{ref}(\alpha)\begin{pmatrix} c(\alpha+\theta) \\ s(\alpha+\theta) \end{pmatrix} = \begin{pmatrix} c(\alpha-\theta) \\ s(\alpha-\theta) \end{pmatrix}.$$

We can verify that $\mathrm{rot}(\alpha)\mathrm{rot}(\beta) = \mathrm{rot}(\alpha+\beta)$, and

$$\begin{cases} \mathrm{ref}(\alpha) & = & \mathrm{rot}(2\alpha)\mathrm{ref}(0), \\ \mathrm{ref}(0)\mathrm{ref}(\alpha) & = & \mathrm{rot}(-2\alpha), \\ \mathrm{ref}(\alpha)\mathrm{rot}(\beta) & = & \mathrm{rot}(2\alpha+\beta)\mathrm{ref}(0). \end{cases}$$

3. The method

Let $(c(\alpha), s(\alpha)) \in \mathcal{S}$ be a fixed point in the first quadrant, so $c(\alpha) > 0$ and $s(\alpha) > 0$. Let us start with an admissible x value such that there exists a point $P = (x, y) \in \mathcal{S}$ with $y \geq 0$. Then we look for θ such that $c(\theta) = x$ and $y = s(\theta) = \sqrt{\delta(1-x^2)} \geq 0$, for an unknown θ. We apply to $P = P_0$ a sequence of reflections. We start with $\mathrm{ref}(\alpha)$ followed by $\mathrm{ref}(0)$, and we repeat the process. Starting with $P_0 = P \in \mathcal{S}$, after k reflections we get the point $P_k = (c(\theta_k), s(\theta_k)) \in \mathcal{S}$. It follows that after $2n$ reflections $P_{2n} = (c(\theta - 2n\alpha), s(\theta - 2n\alpha))$, so $\theta_{2n} = \theta - 2n\alpha$, and after $2n+1$ reflections $P_{2n+1} = (c(2(n+1)\alpha - \theta), s(2(n+1)\alpha - \theta))$, so $\theta_{2n+1} = 2(n+1)\alpha - \theta$.

Starting with $\theta = \theta_0 > \alpha$, we would like to stop when $\theta_k \in [0, \alpha]$ for the first time. The trajectory of the points P_k $(k = 0, 1, 2, \ldots)$ is illustrated for $\delta = +1$ on Figure 1 and for $\delta = -1$ on Figure 2, where P_f is the final point.

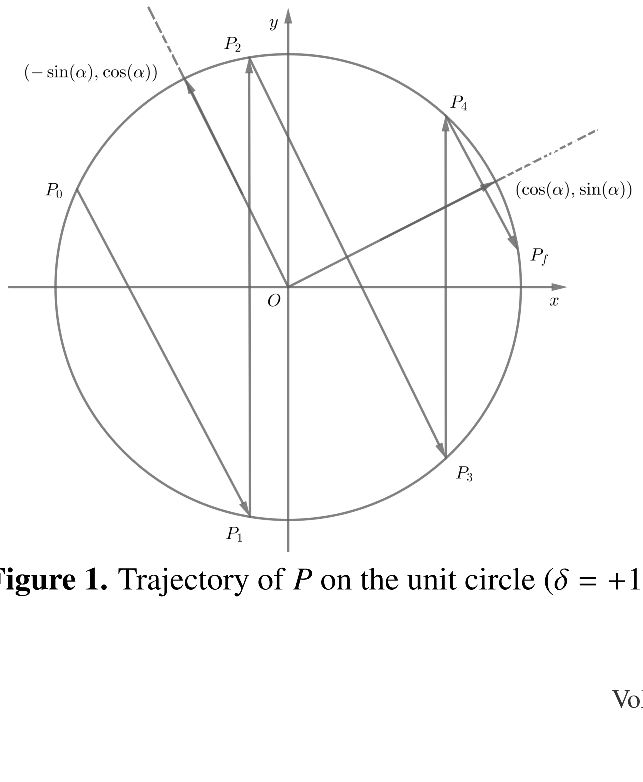

Figure 1. Trajectory of P on the unit circle ($\delta = +1$).

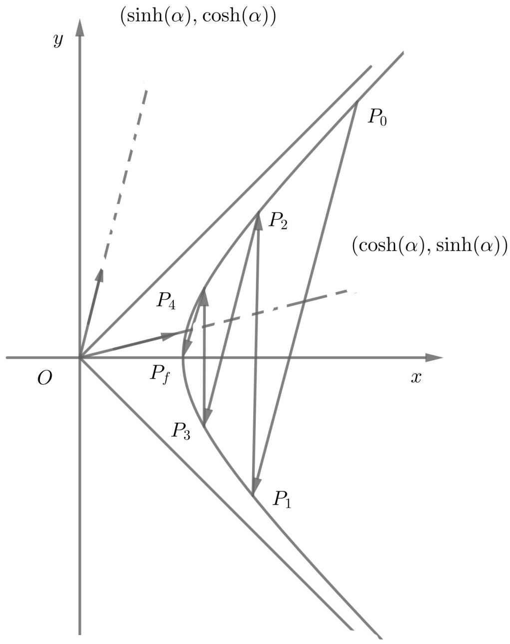

Figure 2. Trajectory of P on the unit hyperbola ($\delta = -1$).

If K is the total number of reflections ($2n$ or $2n+1$), it means that

$$0 < \theta - K\alpha \leq \alpha.$$

Indeed, for an even number of reflections $K = 2n$, the last reflection is with respect to the axis $y = 0$, so θ_{2n} cannot be 0 and we have $\theta_{2n} \in (0, \alpha]$, or $0 < \theta - 2n\alpha \leq \alpha$. For an odd number of reflections $K = 2n+1$, the last reflection is with respect to the line of direction α, so θ_{2n+1} cannot be α, and we have $\theta_{2n+1} \in [0, \alpha)$, so $0 \leq 2(n+1)\alpha - \theta < \alpha$ or $0 < \theta - (2n+1)\alpha \leq \alpha$.

So the total number K of reflections is

$$K = \begin{cases} \frac{\theta}{\alpha} - 1 & \text{if } \frac{\theta}{\alpha} \text{ is an integer,} \\ \left\lfloor \frac{\theta}{\alpha} \right\rfloor & \text{if } \frac{\theta}{\alpha} \text{ is not an integer.} \end{cases}$$

In the applications the coordinates of P are known but not θ. Moreover, α is not known and only some information about it is given that allow us to determine $c(\alpha)$ and $s(\alpha)$. The problem here is to find an approximation of θ from K and an approximation of α. So as a result we get a method to find $\theta = c^{-1}(x)$, i.e. an evaluation of the inverse function $c^{-1}(\cdot)$.

4. The digits of θ

The preceding result suggests a way to compute the digits of θ. Since the integer part of $\theta \cdot 10^N$, noted $\lfloor \theta \cdot 10^N \rfloor$, add the first N digits of the fractional part of θ, for an angle $\alpha \approx 10^{-N}$ we will be near the goal.

4.1. *Known exact value of α*

To get the digits of θ we could take $\alpha = 10^{-N}$, then

$$K = \begin{cases} \theta \cdot 10^N - 1 & \text{if} \quad \theta \cdot 10^N \quad \text{is an integer,} \\ \\ \lfloor \theta \cdot 10^N \rfloor & \text{if} \quad \theta \cdot 10^N \quad \text{is not an integer.} \end{cases}$$

Unfortunately it is not the interesting situation for the applications.

4.2. *Approximate value of α*

In the applications, α is not exactly given, but there is a function $\mathrm{T}(\cdot)$ for which the value $\mathrm{T}(\alpha) = \sigma$ can be fixed, so $\alpha = \mathrm{T}^{-1}(\sigma) \approx \sigma$. It is possible to take $\sigma = 10^{-N}$. We will consider the next two situations using the Taylor's expansion of $\mathrm{T}^{-1}(\cdot)$.

(a) Let us consider $\mathrm{T}(\cdot) = \sin(\cdot)$ for $\delta = +1$, and $\mathrm{T}(\cdot) = \tanh(\cdot)$ for $\delta = -1$. Taylor's expansions of their corresponding $\mathrm{T}^{-1}(\cdot)$ are

$$\arcsin(\sigma) = \sum_{\ell=0}^{+\infty} \frac{(2\ell)!}{4^\ell (\ell!)^2} \frac{\sigma^{2\ell+1}}{(2\ell+1)},$$

and

$$\operatorname{arctanh}(\sigma) = \sum_{\ell=0}^{+\infty} \frac{\sigma^{2\ell+1}}{(2\ell+1)}.$$

Thanks to the geometric series, we can consider the following lower and upper bounds for both series

$$\sigma < \mathrm{T}^{-1}(\sigma) < \sigma \left[1 + \frac{\sigma^2}{3} \left(\frac{1}{1-\sigma^2} \right) \right],$$

for $0 < \sigma < 1$. Moreover

$$1 + \frac{\sigma^2}{3} \left(\frac{1}{1-\sigma^2} \right) \leq \frac{1}{1 - \frac{\sigma^2}{2}},$$

for $0 < \sigma < 1/\sqrt{2}$. So we can write

$$\frac{1}{\sigma} - \frac{\sigma}{2} < \frac{1}{\mathrm{T}^{-1}(\sigma)} < \frac{1}{\sigma}$$

for $0 < \sigma < \rho$ where $1/2 < \rho = 1/\sqrt{2} < 1$. We set $\sigma = 10^{-N}$, $\alpha = \mathrm{T}^{-1}(10^{-N})$, and multiply by θ to get

$$\theta \cdot 10^N - \frac{\theta}{2} \cdot 10^{-N} < \frac{\theta}{\alpha} < \theta \cdot 10^N$$

As long as $\frac{\theta}{\alpha}$ is not an integer and that $\theta \cdot 10^N - \frac{\theta}{2} \cdot 10^{-N} > \left\lfloor \theta \cdot 10^N \right\rfloor$ we have $K = \left\lfloor \theta \cdot 10^N \right\rfloor$. In general in this case we get $K = \left\lfloor \theta \cdot 10^N \right\rfloor - 1$ or $K = \left\lfloor \theta \cdot 10^N \right\rfloor$.

(b) Let us consider $\mathrm{T}(\cdot) = \tan(\cdot)$ for $\delta = +1$, and $\mathrm{T}(\cdot) = \sinh(\cdot)$ for $\delta = -1$. Taylor's expansions of their corresponding $\mathrm{T}^{-1}(\cdot)$ are

$$\arctan(\sigma) = \sum_{\ell=0}^{+\infty} (-1)^\ell \frac{\sigma^{2\ell+1}}{(2\ell+1)},$$

and

$$\operatorname{arcsinh}(\sigma) = \sum_{\ell=0}^{+\infty} (-1)^\ell \frac{(2\ell)!}{4^\ell (\ell!)^2} \frac{\sigma^{2\ell+1}}{(2\ell+1)}.$$

We can consider the following lower and upper bounds for both series

$$\sigma - \frac{\sigma^3}{3} < \mathrm{T}^{-1}(\sigma) < \sigma,$$

for $0 < \sigma < 1$. Moreover

$$1 - \frac{\sigma^2}{3} > \frac{1}{1 + \frac{\sigma^2}{2}},$$

for $0 < \sigma < 1$. So we can write

$$\frac{1}{\sigma} < \frac{1}{\mathrm{T}^{-1}(\sigma)} < \frac{1}{\sigma} + \frac{\sigma}{2}$$

for $0 < \sigma < \rho$ where $1/2 < \rho < 1$. We set $\sigma = 10^{-N}$, $\alpha = \mathrm{T}^{-1}(10^{-N})$, and multiply by θ to get

$$\theta \cdot 10^N < \frac{\theta}{\alpha} < \theta \cdot 10^N + \frac{\theta}{2} \cdot 10^{-N}$$

As long as $\frac{\theta}{\alpha}$ is not an integer and that $\theta \cdot 10^N + \frac{\theta}{2} \cdot 10^{-N} < \left\lfloor \theta \cdot 10^N \right\rfloor + 1$ we have $K = \left\lfloor \theta \cdot 10^N \right\rfloor$. In general in this case we get $K = \left\lfloor \theta \cdot 10^N \right\rfloor$ or $K = \left\lfloor \theta \cdot 10^N \right\rfloor + 1$.

In both cases, the result $K = \left\lfloor \theta \cdot 10^N \right\rfloor$ depends of the expansion of θ. let

$$\begin{cases} \theta &= a_0 \;.\; a_1 a_2 \cdots a_N a_{N+1} \cdots a_{2N} a_{2N+1} \cdots \\ \frac{\theta}{2} &= \tilde{a}_0 \;.\; \tilde{a}_1 \tilde{a}_2 \cdots \tilde{a}_N \tilde{a}_{N+1} \cdots \tilde{a}_{2N} \tilde{a}_{2N+1} \cdots \end{cases}$$

and

$$\begin{cases} \theta \cdot 10^N &= a_0 a_1 a_2 \cdots a_N \;.\; a_{N+1} \cdots a_{2N-1} \quad a_{2N} a_{2N+1} \cdots \\ \frac{\theta}{2} \cdot 10^{-N} &= 0 \;.\; \underbrace{0 \cdots 0}_{(N-1) times} \quad \tilde{a}_0 \;\; \tilde{a}_1 \cdots . \end{cases}$$

A sufficient condition to get $K = \left\lfloor \theta \cdot 10^N \right\rfloor$ can be given for the two cases. The conditions are:
(a) there exists an index $n \in [N+1, 2N-1]$ such that $a_n > 0$,
(b) there exists an index $n \in [N+1, 2N-1]$ such that $a_n < 9$.

For example, up to now and with modern computational facilities and methods [7], for small values of N and up to very large values of N it has not been observed sequences such that:

(a') $a_n = 0$ for $n \in [N+1, 2N-1]$,

(b') $a_n = 9$ for $n \in [N+1, 2N-1]$,

in the expansion of π and $\ln(2)$. So we could claim that $K = \lfloor \theta \cdot 10^N \rfloor$ holds up to very large values of N for $\theta = \pi$ and $\theta = \ln(2)$.

5. Algorithm and examples

5.1. Algorithm

The algorithm is as follows :

Alternating reflection method

Step 0. Enter $P = (x, y) \in \mathcal{S}$ with $y \geq 0$, and $\sigma = T(\alpha)$.

Step 1. Determine $c(\alpha)$ and $s(\alpha)$ from the data σ.

Step 2. The target zone is $\{ P \in \mathcal{S} \mid P$ is between $(1, 0)$ and $(c(\alpha), s(\alpha)) \}$.

Step 3. Compute $c(2\alpha) = c^2(\alpha) - \delta s^2(\alpha)$ and $s(2\alpha) = 2s(\alpha)c(\alpha)$.

Step 4. Set

$$\mathrm{ref}(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathrm{ref}(\alpha) = \begin{pmatrix} c(2\alpha) & \delta s(2\alpha) \\ s(2\alpha) & -c(2\alpha) \end{pmatrix}$$

Step 5. With $P_0 = P$, compute P_{k+1} as long as P_k is not in the target zone

(i) for $k = 2n$, compute $P_{k+1} = P_{2n+1} = \mathrm{ref}(\alpha)P_{2n} = \mathrm{ref}(\alpha)P_k$,

(ii) for $k = 2n+1$, compute $P_{k+1} = P_{2n+2} = \mathrm{ref}(0)P_{2n+1} = \mathrm{ref}(0)P_k$.

5.2. First numerical examples

We have considered the two situations $\delta = \pm 1$ and we got the results given in Table 1 for π and $\ln(2)$.

Table 1. Number of reflections needed to get $\theta_K \in [0, \alpha]$.

		$\theta = \pi = 3.14159265358979...$	$\theta = \ln 2 = 0.69314718055995...$
N	$\sigma = 10^{-N}$	$K = \lfloor \pi \cdot 10^{-N} \rfloor$	$K = \lfloor \ln(2) \cdot 10^{-N} \rfloor$
1	10^{-1}	31	6
2	10^{-2}	314	69
3	10^{-3}	3141	693
4	10^{-4}	31415	6931
5	10^{-5}	314159	69314

5.2.1. Circular case: $\delta = +1$.

$\mathcal{S}$ is the unit circle, and $(c(\cdot), s(\cdot)) = (\cos(\cdot), \sin(\cdot))$. To get $\theta = \pi$, we start with $x = \cos(\pi) = -1$, so we consider $P = (-1, 0)$. K is computed using $\sigma = 10^{-N}$ for $N = 1, 2, 3, 4, 5$, for $T(\alpha) = \sin(\alpha) = \sigma$ and $T(\alpha) = \tan(\alpha) = \sigma$. As expected, both situations generate the same value of K.

5.2.2. Hyperbolic case: $\delta = -1$.

$\mathcal{S}$ is the right branch ($x \geq 0$) of the unit hyperbola, and $(c(\cdot), s(\cdot)) = (\cosh(\cdot), \sinh(\cdot))$. To get $\theta = \ln(2)$, we start with $x = \cosh(\ln(2)) = 5/4$, so we consider $P = (5/4, 3/4)$. K is computed using $\sigma = 10^{-N}$ for $N = 1, 2, 3, 4, 5$, for $T(\alpha) = \tanh(\alpha) = \sigma$ and $T(\alpha) = \sinh(\alpha) = \sigma$. As expected, both situations generate the same value of K.

5.3. Other numerical examples

We not only have an unusual method to find the digits of the trigonometric constant π and the logarithmic constant $\ln(2)$ but also to evaluate the inverse trigonometric functions (for $\delta = +1$) and inverse hyperbolic functions (for $\delta = -1$). Indeed the method can be extended to solve for θ the equation $c(\theta) = x$. If the given data for α allows us to say that $\alpha = 10^{-N} + O(10^{-2N})$, we get that

$$\left| \theta - \left(K + \frac{1}{2}\right) \cdot 10^{-N} \right| = \frac{1}{2} 10^{-N} + O(10^{-2N}).$$

The next examples illustrate that both values of K can be obtained. Numerical results are reported in Table 2 where examples were chosen to get the same figures.

Table 2. Values of K.

	$\theta = 1.00000000$			$\theta = 0.99999999$		
		K			K	
N	$\lfloor \theta \cdot 10^N \rfloor$	(a)	(b)	$\lfloor \theta \cdot 10^N \rfloor$	(a)	(b)
1	10	9	10	9	9	10
2	100	99	100	99	99	100
3	1000	999	1000	999	999	1000
4	10000	9999	10000	9999	9999	10000
5	100000	99999	100000	99999	99999	99999

5.3.1. Circular case: $\delta = +1$.

We fix x and compute $y = \sqrt{1-x^2} \geq 0$ for the point $P = P_0$. We use for (a) $T(\alpha) = \sin(\alpha)$, and for (b) $T(\alpha) = \tan(\alpha)$. We consider two situations to illustrate that we can get the two values of K. Firstly, we consider $x = 0.540302305... = \cos(1)$ (and $y = \sqrt{1-x^2} \geq 0$), so $\theta = 1 = \arccos(x)$. We get for (a) $K = \lfloor 1 \cdot 10^N \rfloor - 1$ and for (b) $K = \lfloor 1 \cdot 10^N \rfloor$. Secondly we consider $x = 0.540302314... = \cos(0.99999999)$, so $\theta = 0.99999999 = \arccos(x)$. We get for (a) $K = \lfloor 0.99999999 \cdot 10^N \rfloor$ for $N = 1, 2, 3, 4$, and for (b) $K = \lfloor 0.99999999 \cdot 10^N \rfloor + 1$ for $N = 1, 2, 3, 4$, but not for $N \geq 5$.

5.3.2. Hyperbolic case: $\delta = -1$.

We fix x and compute $y = \sqrt{x^2-1} \geq 0$ for the point $P = P_0$. We use for (a) $T(\alpha) = \tanh(\alpha)$, and for (b) $T(\alpha) = \sinh(\alpha)$. We consider two situations to illustrate the different values of K. Firstly, we consider $x = 1.543080635... = \cosh(1)$, so $\theta = 1 = \operatorname{arccosh}(x)$. We get for (a) $K = \lfloor 1 \cdot 10^N \rfloor - 1$ and for (b) $K = \lfloor 1 \cdot 10^N \rfloor$. Secondly, we consider $x = 1.543080623... = \cosh(0.99999999)$, so $\theta = 0.99999999 = \operatorname{arccosh}(x)$. We get for (a) $K = \lfloor 0.99999999 \cdot 10^N \rfloor$ and for (b) $K = \lfloor 0.99999999 \cdot 10^N \rfloor + 1$ for $N = 1, 2, 3, 4$, but not for $N \geq 5$.

5.4. Remarks on the algorithm

The computational cost of this method is quite low at each iteration. It requires 4 multiplications and 2 additions to compute P_{2n+1} from P_{2n}, and only a sign change to get P_{2n+2} from P_{2n+1}. Also, at each iteration a test is required to eventually terminate the process.

We used MATLAB with single-precision computation for the numerical examples. For $\sigma = 10^{-N}$, we had $c(\alpha) = 1 + O(10^{-2N})$ and $s(\alpha) = O(10^{-N})$, and the matrix $\operatorname{ref}(\alpha)$ looked like

$$\begin{pmatrix} 1 + O(10^{-2N}) & \delta O(10^{-N}) \\ O(10^{-N}) & 1 + O(10^{-2N}) \end{pmatrix},$$

so it explains why we stopped at $N = 5$ in the computation. We could increase N with multi-precision computation.

6. Conclusions

In this paper we have presented an unusual method to find $\theta = c^{-1}(x)$. We have considered the following problem:

$$\begin{cases} \text{Suppose given an admissible value } x, \text{ and set } y = \sqrt{\delta(1-x^2)} \geq 0, \\ \text{then find } \theta \text{ such that } \theta = c^{-1}(x), \text{ so } (x, y) = (c(\theta), s(\theta)). \end{cases}$$

We can extend this problem to find also $\theta = s^{-1}(y)$ and $\theta = t^{-1}(z)$, where $t(\cdot) = s(\cdot)/c(\cdot)$. Indeed we have $\theta = s^{-1}(y) = \operatorname{sign}(y) c^{-1}\left(\sqrt{\delta(1-\delta y^2)}\right)$ for an appropriate y value, and $\theta = t^{-1}(z) = \operatorname{sign}(z) c^{-1}\left(1/\sqrt{1+\delta z^2}\right)$ for an appropriate value of z.

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Conflict of interests

The author declares no conflict of interests.

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