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Research article

$*$-Ricci tensor on (κ, μ)-contact manifolds

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Abstract: We introduce the notion of semi-symmetric $*$-Ricci tensor and illustrate that a non-Sasakian (κ, μ)-contact manifold is $*$-Ricci semi-symmetric or has parallel $*$-Ricci operator if and only if it is $*$-Ricci flat. Then we find that among the non-Sasakian (κ, μ)-contact manifolds with the same Boeckx invariant I_M, only one is $*$-Ricci flat, so we can think of it as the representative of such class. We also give two methods to construct $*$-Ricci flat (κ, μ)-contact manifolds.

Keywords: $*$-Ricci flat; $*$-Ricci semi-symmetric; parallel $*$-Ricci operator; (κ, μ)-contact manifolds
Mathematics Subject Classification: 53C21, 53D10, 53D15

1. Introduction

In differential geometry, curvature tensor R has a very important influence on the properties of manifolds. The authors of this paper have investigated some properties of Lorentzian generalized Sasakian space-forms which are related to the curvature tensor R in [1]. We know many manifolds have special curvature properties. For instance, if $(M^{2n+1}, \phi, \xi, \eta, g)$ is a Sasakian manifold, then it satisfies

$$R(X, U)\xi = \eta(U)X - \eta(X)U,$$

where X, U are vector fields on M. So we can classify manifolds according to their special curvature properties. As the meaningful generalization of the curvature condition of Sasakian manifold, D. E. Blair, T. Koufogiorgos and B. J. Papantoniou introduced (κ, μ)-contact manifold in [2]. If a contact manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ satisfies

$$R(X, U)\xi = \kappa(\eta(U)X - \eta(X)U) + \mu(\eta(U)hX - \eta(X)hU), \tag{1.1}$$

where $(\kappa, \mu) \in \mathbb{R}^2$, h is half of the Lie derivative of structure tensor ϕ along Reeb vector field ξ. This curvature condition of a non-Sasakian contact manifold is attractive to mathematicians not only because it determines the curvature tensor R completely but also remains unchange under D_a-homothetic deformation while κ and μ vary. Moreover (κ,μ)-contact manifolds are examples of H-contact manifolds (see [3]) or other remarkable contact manifolds. In Eq 1.1, when κ and μ are non constant smooth functions, it is a generalized (κ,μ)-contact manifold and this condition can only occur in dimension three (see [4]).

Many mathematicians have studied (κ,μ)-contact manifolds from many aspects. The most excellent work was done by E. Boeckx who showed that non-Sasakian (κ,μ)-contact manifolds are locally homogeneous and locally strongly ϕ-symmetric in [5]. Moreover E. Boeckx introduced an invariant I_M which called Boeckx invariant and he used it to give a full classification of non-Sasakian (κ,μ)-contact manifolds (see [6]). When $I_M \leq -1$, the classification of (κ,μ)-contact manifolds was given by E. Loiudice and A. Lotta in [7]. X. Chen investigated (κ,μ)-contact manifolds admit weakly Einstein metrics and gave the classification of them. D. S. Patra and A. Ghosh prove that if a non-Sasakian (κ,μ)-contact manifold is a non-trivial complete Einstein-type manifold, then it is flat for $n = 1$ or locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ in [8].

We know the Ricci tensor is an important tensor on the manifold induced by curvature tensor. In contact geometry, there is another important tensor, that is the $*$-Ricci tensor Ric^*. In 1959, S. Tachibana gave the notion of $*$-Ricci tensor in complex geometric (see [9, 10]). In 2002, T. Hamada extended this notion to almost contact manifolds in [11]. Like Ricci tensor, $*$-Ricci tensor is also a trace but may not be symmetric. We know that the asymmetry tensor makes no much geometric or physical meaning. So when the $*$-Ricci tensor is symmetric, we can study it directly; when it is not symmetric, we first want to know how can it be symmetric or we study the symmetric part of it.

After the $*$-Ricci tensor of contact manifold being proposed, it attracts great interest of mathematicians. The notion of $*$-Ricci solitons was intrduced by G. Kaimakamis and K. Panagiotidou and it was first defined on real hypersurfaces in complex space form (see [12]). Since then many mathematicians have studied $*$-Ricci solitons in many aspects, and obtain important and meaningful results (see [13, 14]). Moreover some of the latest connected studies can be seen in [15–17]. For example, in [18], we know that if there is a $*$-Ricci soliton on three-dimensional Kenmotsu manifold, then the manifold is of constant curvature -1. In [19], it has been proved that if a three-dimensional Sasakian manifold admits a $*$-Ricci soliton on it, then it is a manifold with constant curvature. A. Ghosh and D. S. Patra give a complete classification of $*$-Ricci soliton of non-Sasakian (κ,μ)-contact manifolds (see [20]). We give the classification of trans-Sasakian three-manifolds with Reeb invariant $*$-Ricci operator in [21].

We investigate the $*$-Ricci tensor and $*$-Ricci operator on (κ,μ)-contact manifolds in this paper. Analogue to the notion of semi-symmetric Ricci tensor, we propose the definition of semi-symmetric $*$-Ricci tensor which is:

$$0 = (R(X,U)\mathrm{Ric}^*)(V,W) = -\mathrm{Ric}^*(R(X,U)V,W) - \mathrm{Ric}^*(V,R(X,U)W), \tag{1.2}$$

where X, U, V and W are arbitrary vector fields of the manifold. If the $*$-Ricci tensor is semi-symmetric, we also say that the manifold is $*$-Ricci semi-symmetric. We give the necessary and sufficient condition that the non-Sasakian (κ,μ)-contact manifold is $*$-Ricci semi-symmetric and has parallel $*$-Ricci operator. We find that if some (κ,μ)-contact manifolds sharing the same Boeckx invariant I_M, $*$-Ricci flat manifold can be considered as their representative, because the $*$-Ricci flat manifold is unique among them. Using D_a-homothetic deformation (see [22]) which has been investigated by

Y. Wang and H. Wu that they study invariant vector fields under it (see [23]), we also show how to construct $*$-Ricci flat (κ,μ)-contact manifolds.

2. Preliminaries

Firstly we recall some basic notations about contact metric manifolds. If a Riemannian manifold (M^{2n+1}, g) admits a triple (ϕ,ξ,η) which satisfies (see [24]):

$$\phi\xi = 0, \quad \eta\circ\phi = 0,$$
$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1,$$
$$\eta(X) = g(\xi, X),$$
$$g(\phi X, \phi U) = g(X, U) - \eta(X)\eta(U),$$

where we call ϕ structure tensor field, ξ Reeb vector field, η is a 1-form dual to ξ, for arbitrary vector fields X and U on M^{2n+1}, then it is an almost contact manifold and the triple (ϕ,ξ,η) called almost contact metric structure. Moreover if the metric g and the 2-form $d\eta$ satisfy:

$$d\eta(X, U) = g(X, \phi U),$$

then the manifold M is a contact manifold and denoted by $(M^{2n+1}, \phi, \xi, \eta, g)$.

The $*$-Ricci tensor Ric^* of an almost contact manifold is half of the trace (see [11]):

$$\mathrm{Ric}^*(X, U) = \frac{1}{2}\,\mathrm{trace}\{V \to R(X, \phi U)\phi V\}, \tag{2.1}$$

it is equivalent to

$$\mathrm{Ric}^*(X, U) = \frac{1}{2}\,\mathrm{trace}\{\phi\circ R(X, \phi U)\}.$$

For a contact structure (ϕ,ξ,η), the induced tensor field h is defined by $h = \frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}$ is Lie differentiation. h vanishes if and only if ξ is a Killing vector field, that is the manifold M^{2n+1} is K-contact manifold. For metric g, h is self-adjoint and enjoys many properties such as:

$$h\xi = 0, \quad h\phi = -\phi h,$$

$$\mathrm{trace}(h) = \mathrm{trace}(h\phi) = 0,$$

and there holds

$$\nabla_X\xi = -\phi X - \phi h X.$$

For the pair $(\kappa,\mu)\in\mathbb{R}^2$, we can define a distribution on a contact metric manifold that

$$\begin{aligned} N(\kappa,\mu) : p \to N_p(\kappa,\mu) &= \{V \in T_pM \mid \\ R(X, U)V &= (\kappa + \mu h)(g(U, V)X - g(X, V)U)\}, \end{aligned}$$

which is called (κ,μ)-nullity distribution, where the mapping $N(\kappa,\mu)$ assigns to each point p of M a subspace $N_p(\kappa,\mu)$ of T_pM. If the Reeb vector field ξ of a contact metric manifold M^{2n+1} belongs to the

(κ, μ)-nullity distribution, then it is a (κ, μ)-contact manifold. That is the curvature tensor R of M^{2n+1} satisfies Eq 1.1.

For a (κ, μ)-contact manifold M^{2n+1}, $\kappa \leq 1$ and when $k = 1$ the manifold is Sasakian. In our paper, we consider the non-Sasakian situation, that is $\kappa < 1$. The tensor fields h and ϕ satisfy the relations:

$$h^2 = (k-1)\phi^2,$$

and

$$(\nabla_X \phi)U = g(X + hX, U)\xi - \eta(U)(X + hX).$$

The (κ, μ)-nullity distribution completely determines the curvature tensor of a contact manifold such that the $(0,4)$ curvature tensor is (see [5]):

$$\begin{aligned}
& g(R(X,U)V,W) \\
= \; & (1 - \frac{\mu}{2})(g(U,V)g(X,W) - g(X,V)g(U,W)) \\
& + g(U,V)g(hX,W) - g(X,V)g(hU,W) \\
& - g(U,W)g(hX,V) + g(X,W)g(hU,V) \\
& + \frac{1-\mu/2}{1-\kappa}(g(hU,V)g(hX,W) - g(hX,V)g(hU,W)) \\
& - \frac{\mu}{2}(g(\phi U,V)g(\phi X,W) - g(\phi X,V)g(\phi U,W)) \\
& + \frac{\kappa-\mu/2}{1-\kappa}(g(\phi hU,V)g(\phi hX,W) - g(\phi hX,V)g(\phi hU,W)) \\
& + \mu g(\phi X,U)g(\phi V,W) \\
& + \eta(X)\eta(W)((\kappa - 1 + \frac{\mu}{2})g(U,V) + (\mu-1)g(hU,V)) \\
& - \eta(X)\eta(V)((\kappa - 1 + \frac{\mu}{2})g(U,W) + (\mu-1)g(hU,W)) \\
& + \eta(U)\eta(V)((\kappa - 1 + \frac{\mu}{2})g(X,W) + (\mu-1)g(hX,W)) \\
& - \eta(U)\eta(W)((\kappa - 1 + \frac{\mu}{2})g(X,V) + (\mu-1)g(hX,V)).
\end{aligned} \tag{2.2}$$

Lemma 2.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a non-Sasakian (κ, μ)-contact manifold. Then its* $*$-Ricci *tensor* Ric^* *and* $*$-Ricci *operator Q^* are*

$$\mathrm{Ric}^*(X,U) = -g(\kappa + \mu n)g(\phi X, \phi U), \tag{2.3}$$

and

$$Q^* X = (\kappa + \mu n)\phi^2 X, \tag{2.4}$$

where $X, U \in \Gamma(TM)$.

Proof. Let $\{e_1, \ldots, e_{2n+1}\}$ be the local orthonormal basis of M. From the definition of $*$-Ricci tensor (Eq 2.1), we have:

$$\mathrm{Ric}^*(X,U) = \frac{1}{2}\sum_{i=1}^{2n+1} g(R(X,\phi U)\phi e_i, e_i)$$

$$
\begin{aligned}
&= \frac{1}{2}((2-\mu)g(\phi X, \phi U) + 2g(\phi U, \phi hX) + 2g(\phi X, h\phi U) \\
&\quad + \frac{2-\mu}{1-\kappa} g(h\phi U, \phi hX) - \mu g(\phi^2 X, \phi^2 U) \\
&\quad + \frac{2\kappa-\mu}{1-\kappa} g(\phi h\phi U, \phi^2 hX) - 2n\mu g(\phi X, \phi U)) \\
&= -g(\kappa + \mu n) g(\phi X, \phi U).
\end{aligned}
$$

Since $\mathrm{Ric}^*(X, U) = g(Q^* X, U)$, we have

$$Q^* X = (\kappa + \mu n)\phi^2 X.$$

□

From the expression of $*$-Ricci tensor, we can see that

Lemma 2.2. *A non-Sasakian (κ, μ)-contact manifold M^{2n+1} is $*$-Ricci flat, that is*

$$\mathrm{Ric}^*(X, U) = 0,$$

if and only if $\kappa + \mu n = 0$.

3. Main results

Theorem 3.1. *A non-Sasakian (κ, μ)-contact manifold M^{2n+1} is $*$-Ricci semi-symmetric if and only if it is $*$-Ricci flat.*

Proof. Putting Eqs 2.3 and 2.2 in Eq 1.2, we have

$$
\begin{aligned}
& (R(X, U)\mathrm{Ric}^*)(V, W) \\
&= -\mathrm{Ric}^*(R(X, U)V, W) - \mathrm{Ric}^*(V, R(X, U)W) \\
&= (\kappa + \mu n)(g(\phi R(X, U)V, \phi W) + g(\phi R(X, U)W, \phi V)) \\
&= (\kappa + \mu n)(g(R(X, U)\phi^2 V, W) - g(\phi R(X, U)V, \phi^2 W)) \\
&= (\kappa + \mu n)((1 - \frac{\mu}{2})(\eta(U)\eta(V)g(X, W) - \eta(X)\eta(W)g(U, V) \\
&\qquad - \eta(X)\eta(V)g(U, W) + \eta(U)\eta(W)g(X, V)) \\
&\qquad + \eta(U)\eta(V)g(hX, W) - \eta(X)\eta(V)g(hU, W) \\
&\qquad + \eta(U)\eta(W)g(hX, V) - \eta(X)\eta(W)g(hU, V) \\
&\quad -(\kappa - 1 + \frac{\mu}{2})\eta(X)\eta(W)g(U, V) - (\mu - 1)\eta(X)\eta(W)g(hU, V) \\
&\quad +(\kappa - 1 + \frac{\mu}{2})\eta(U)\eta(W)g(X, V) + (\mu - 1)\eta(U)\eta(W)g(hX, V) \\
&\quad -(\kappa - 1 + \frac{\mu}{2})\eta(X)\eta(V)g(U, W) - (\mu - 1)\eta(X)\eta(V)g(hU, W) \\
&\quad +(\kappa - 1 + \frac{\mu}{2})\eta(U)\eta(V)g(X, W) + (\mu - 1)\eta(U)\eta(V)g(hX, W)) \\
&= (\kappa + \mu n)(\kappa(\eta(U)\eta(V)g(X, W) - \eta(X)\eta(W)g(U, V)
\end{aligned}
$$

$$
\begin{aligned}
&- \eta(X)\eta(V)g(U,W) + \eta(U)\eta(W)g(X,V)) \\
&+ \mu(\eta(U)\eta(V)g(X,W) - \eta(X)\eta(W)g(U,V) \\
&- \eta(X)\eta(V)g(U,W) + \eta(U)\eta(W)g(X,V))).
\end{aligned}
$$

If $(R(X,U)\mathrm{Ric}^*)(V,W) = 0$, putting $X = V = \xi$, we have

$$
\begin{aligned}
0 &= (R(\xi,U)\mathrm{Ric}^*)(\xi,W) \\
&= (\kappa + \mu n)(\kappa(\eta(U)\eta(W) - g(U,W)) - \mu g(hU,W)) \\
&= (\kappa + \mu n)(-\kappa g(\phi U,\phi W) - \mu g(hU,W)) \\
&= (\kappa + \mu n)g(\kappa\phi^2 U - \mu hU, W).
\end{aligned}
$$

Now we suppose that $\kappa + \mu n \neq 0$, then there must be $\kappa\phi^2 U - \mu hU = 0$, from which we will have $\kappa\phi^3 U = \mu\phi hU = \mu h\phi U = -\mu\phi hU$. So $\mu\phi hU = 0$ and $-\kappa\phi U = \kappa\phi^3 U = \mu\phi hU = 0$. Thus $\kappa = 0$ and $\mu hU = 0$. So $\mu = 0$ since $h \neq 0$. We get $\kappa + \mu n = 0$, which is contradict to our assumption $\kappa + \mu n \neq 0$. So if M^{2n+1} is $*$-Ricci semi-symmetric, then $\kappa + \mu n = 0$, it is $*$-Ricci flat.

Conversely, if M^{2n+1} is $*$-Ricci flat, then

$$
\begin{aligned}
&(R(X,U)\mathrm{Ric}^*)(V,W) \\
= \; &-\mathrm{Ric}^*(R(X,U)V,W) - \mathrm{Ric}^*(V,R(X,U)W) = 0,
\end{aligned}
$$

it is $*$-Ricci semi-symmetric. Thus we have completed the proof of the theorem. □

Theorem 3.2. *The* $*$-Ricci *operator of a non-Sasakian* (κ,μ)*-contact manifold is parallel if and only if the manifold is* $*$-Ricci *flat.*

Proof. Since

$$
\begin{aligned}
\nabla_X(Q^*U) &= (\nabla_X Q^*)U + Q^*(\nabla_X U) \\
&= (\kappa + \mu n)((\nabla_X\phi)(\phi U) + \phi((\nabla_X\phi)U) + \phi^2(\nabla_X U)),
\end{aligned}
$$

thus we have

$$
\begin{aligned}
(\nabla_X Q^*)U &= (\kappa + \mu n)((\nabla_X\phi)(\phi U) + \phi((\nabla_X\phi)U)) \\
&= (\kappa + \mu n)(g(X + hX, \phi U)\xi - \eta(U)(\phi X + \phi hX)).
\end{aligned}
$$

If $\nabla_X Q^* = 0$, putting $U = \xi$, we have

$$
(\nabla_X Q^*)\xi = (\kappa + \mu n)(-\phi X - \phi hX) = 0,
$$

Suppose $\kappa + \mu n \neq 0$, there must be $\phi X + \phi hX = 0$. Then we have

$$
\phi^2 X = -\phi^2 hX = hX,
$$

and

$$
\phi^2 X = -\phi h\phi X = h\phi^2 X = -hX.
$$

From above two equations we have $h = 0$, M^{2n+1} is Sasakian. This is a contradiction since we have assumed M^{2n+1} is non-Sasakian manifold. So we have $\kappa + \mu n = 0$, then M^{2n+1} is $*$-Ricci flat.

Conversely if M^{2n+1} is $*$-Ricci flat, then $Q^* = 0$. Obviously Q^* is parallel. The proof is completed. □

From Theorems 3.1 and 3.2, we have the following theorem:

Theorem 3.3. *For non-Sasakian* (κ, μ)*-contact manifold* M*, the following are equivalent:*
(1) M *is* $*$-Ricci *flat.*
(2) M *is* $*$-Ricci *semi-symmetric.*
(3) M *has parallel* $*$-Ricci *operator.*

The classification of three dimensional non-Sasakian complete (κ, μ)-contact manifold M^3 can be found in [2]. According to the values of $c_2 = 1 - \lambda - \frac{\mu}{2}$ and $c_3 = 1 + \lambda - \frac{\mu}{2}$, where $\lambda = \sqrt{1-\kappa}$ is the eigenvalue of h such that $hX = \lambda X$ and $\eta(X) = 0, g(X, X) = 1$, M^3 can be putted in five classes. That is if $c_2 > 0$ and $c_3 > 0$, then M^3 is locally isometric to $SU(2)$ or $SO(3)$; if $c_2 < 0, c_3 > 0$ or $c_2 < 0, c_3 < 0$, then M^3 is locally isometric to $\mathrm{SL}(2, \mathbb{R})$ or $\mathrm{O}(1, 2)$; if $c_2 = 0$ and $\mu < 2$, then M^3 is locally isometric to E(2), where E(2) is the group of rigid motions of the Euclidean 2-space, when $\kappa = \mu = 0$ then it is flat; if $c_3 = 0$ and $\mu > 2$, then M^3 is locally isometric to $\mathrm{E}(1, 1)$, where $\mathrm{E}(1, 1)$ is the group of rigid motions of the Minkowski 2-space. All of these Lie groups are equipped with left invariant metric.

Applying the above classification theorem, if the non-Sasakian (κ, μ)-contact manifold M^3 is $*$-Ricci flat, we have classification theorem in the following:

Theorem 3.4. *Let* M^3 *be a* $*$-Ricci *flat non-Sasakian complete* (κ, μ)*-contact manifold. Then we can classify* M^3 *according to the value of* μ*:*
(1) $-1 < \mu < 0$*,* M^3 *is locally isometric to* $SU(2)$ *or* $SO(3)$*.*
(2) $\mu = 0$*,* M^3 *is flat.*
(3) $\mu > 0, \mu \neq 8$*,* M^3 *is locally isometric to* $SL(2, \mathbb{R})$ *or* $O(1, 2)$*.*
(4) $\mu = 8$*,* M^3 *is locally isometric to* $E(1, 1)$ *(the group of rigid motions of the Minkowski* 2*-space).*

Proof. From Lemma 2.2, if M^3 is $*$-Ricci flat, then $\kappa + \mu = 0$. Since $\kappa < 1$, then $\mu > -1$.
Case I. If $-1 < \mu < 0$, then $c_2 = 1 - \sqrt{1-\kappa} - \frac{\mu}{2} > 0$ and $c_3 = 1 + \sqrt{1-\kappa} - \frac{\mu}{2} > 0$, M^3 is locally isometric to $SU(2)$ or $SO(3)$.
Case II. If $\mu = 0$, then $\kappa = 0$, M^3 is flat.
Case III. If $0 < \mu < 8$, then $c_2 = 1 - \sqrt{1-\kappa} - \frac{\mu}{2} < 0$ and $c_3 = 1 + \sqrt{1-\kappa} - \frac{\mu}{2} > 0$; if $\mu > 8$, then $c_2 = 1 - \sqrt{1-\kappa} - \frac{\mu}{2} < 0$ and $c_3 = 1 + \sqrt{1-\kappa} - \frac{\mu}{2} < 0$, both of these cases M^3 is locally isometric to $\mathrm{SL}(2, \mathbb{R})$ or $\mathrm{O}(1, 2)$.
Case IV. If $\mu = 8$, then $c_3 = 1 + \sqrt{1-\kappa} - \frac{\mu}{2} = 0$, M^3 is locally isometric to $\mathrm{E}(1, 1)$. □

If the dimension of a $(0, 0)$-contact manifold is bigger then 3, then it is locally isometric to $E^{n+1} \times S^n(4)$ (see [24]), in additional of above theorem, we have

Theorem 3.5. *A* $*$-Ricci *flat non-Sasakian complete* (κ, μ)*-contact manifold* M^{2n+1} *with* $\kappa = 0$ *or* $\mu = 0$ *is flat for* $n = 1$ *or locally isometric to* $E^{n+1} \times S^n(4)$ *for* $n > 1$*.*

In the following we give two ways to construct $*$-Ricci flat non-Sasakian (κ, μ)-contact manifolds.

Example 1. *Let* $M^{n+1} (n > 1)$ *be a manifold of constant curvature* c*.* $T_1 M$ *is the unit tangent sphere bundle, then it is a* $(c(2-c), -2c)$*-contact manifold with the standard contact metric. If* M^{n+1} *is of constant curvature* $c = 2(1-n)$*, then* $T_1 M$ *is a* $*$-Ricci *flat* $(4n(1-n), -4(1-n))$*-contact manifold.*

Example 2. *For contact metric structure, we recall the notion of D_a-homothetic deformation. If there is a contact metric structure $(\phi_0, \xi_0, \eta_0, g_0)$ on a manifold M^{2n+1}, then*

$$\phi' = \phi_0, \ \ \xi' = \frac{1}{a}\xi_0, \ \ \eta' = a\eta_0, \ \ g' = ag_0 + a(a-1)\eta_0 \otimes \eta_0,$$

where a is a positive constant, is also a contact metric structure on M^{2n+1} . From [6], we know that a D_a-homothetic deformation preserve the condition of Eq 1.1 but change (κ, μ) to $(\overline{\kappa}, \overline{\mu})$ where

$$\overline{\kappa} = \frac{\kappa + a^2 - 1}{a^2} \quad \text{and} \quad \overline{\mu} = \frac{\mu + 2a - 2}{a}.$$

*Now the question is, given a (κ, μ)-contact manifold, whether there is a positive number a so that the $(\overline{\kappa}, \overline{\mu})$-contact manifold obtained after the D_a-homothetic deformation is $*$-*Ricci *flat. The answer is positive since if $(\overline{\kappa}, \overline{\mu})$-contact manifold is $*$-*Ricci *flat then $\overline{\kappa} + \overline{\mu}n = 0$, putting above two equations in it we will have the quadratic equation*

$$(2n+1)a^2 + (\mu - 2)na + \kappa - 1 = 0,$$

the unique positive solution is

$$a = \frac{\sqrt{(\mu-1)^2n^2 + 4(2n+1)(1-\kappa)} - (\mu-2)n}{2(2n+1)}.$$

*In this way, we know that using the appropriate D_a-homothetic deformation, we can transform any (κ, μ)-contact manifold into $*$-*Ricci *flat manifold. Moreover this D_a-homothetic deformation is unique.*

In [6], for (κ, μ)-contact manifold, E.Boeckx introduced an invariant I_M:

$$I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1-\kappa}},$$

and $I_M \in \mathbb{R}$. We call it Boeckx invariant. He proved that the D_a-homothetic deformation preserved the invariant I_M. Then E.Boeckx gave a full classification of (κ, μ)-contact manifolds that two (κ, μ)-contact manifolds $(M_i^{2n+1}, \phi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$, are locally isometric as contact metric spaces if and only if $I_{M_1} = I_{M_2}$ up to a D_a-homothetic deformation.

Theorem 3.6. *For arbitrary $I \in \mathbb{R}$, there exists a unique $*$-*Ricci *flat (κ, μ)-contact manifold M^{2n+1} such that the Boeckx invariant $I_M = I$.*

Proof. We just need to prove the flowing binary equations have a unique solution:

$$\begin{cases} \kappa + \mu n = 0 \\ \dfrac{1 - \frac{\mu}{2}}{\sqrt{1-\kappa}} = I_M = I. \end{cases}$$

Putting $\sqrt{1-\kappa} = t > 0$, then we have $\kappa = 1 - t^2$ and $\mu = 2(1 - tI)$. The above binary equations come to

$$t^2 + 2nIt - 2n - 1 = 0,$$

the unique positive solution is $t = (\sqrt{4n^2I^2 + 4(2n+1)} - 2nI)/2$. So the binary equations have a unique solution. □

From above theorem, we can regard a $*$-Ricci flat $(\overline{\kappa}, \overline{\mu})$-contact manifold as the representation of some (κ, μ)-contact manifolds sharing the same Boeckx invariant I_M.

Actually from above theorem and Example 2, we have:

Corollary 3.7. *A* (κ, μ)*-contact manifold* M^{2n+1} *is locally isometric to a unique* $*$-Ricci *flat* $(\overline{\kappa}, \overline{\mu})$*-contact manifold.*

4. Conclusions

Firstly we propose the definition of $*$-Ricci semi-symmetry and prove that a non-Sasakian (κ, μ)-contact manifold is $*$-Ricci semi-symmetric if and only if it is $*$-Ricci flat. Then we give the classification of three dimension $*$-Ricci flat non-Sasakian (κ, μ)-contact manifolds. We find that the $*$-Ricci flat non-Sasakian (κ, μ)-contact manifolds are representatives of each Boeckx invariant I_M class.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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