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Research article

Computation of eccentric topological indices of zero-divisor graphs based on their edges

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Abstract: The topological index of a graph gives its topological property that remains invariant up to graph automorphism. The topological indices which are based on the eccentricity of a chemical graph are molecular descriptors that remain constant in the whole molecular structure and therefore have a significant position in chemical graph theory. In recent years, various topological indices are intensively studied for a variety of graph structures. In this article, we will consider graph structures associated with zero-divisors of commutative rings, called zero-divisor graphs. We will compute the topological indices for a class of zero-divisor graphs of finite commutative rings that are based on their edge eccentricity. More precisely, we will compute the first and third index of Zagreb eccentricity, the eccentricity index of geometric arithmetic, the atomic bonding connectivity eccentricity index, and the eccentric harmonic index of the fourth type related to graphs constructed using zero-divisors of finite commutative rings $\mathbb{Z}_{p^n}$.

Keywords: topological indices; zero-divisor graph; commutative ring
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1. Introduction

The motivation of this study arises from the molecular descriptors and their conversion into number structures. It enables Mathematics to play its part in these studies. This work is useful in Chemistry, pharmacy, and environmental protection. The topological indices which are molecular descriptors that remain invariants in the molecular structure created a bridge between chemical structures through the characterization of chemical diagrams with graph theory and combinatorics. It also attracts algebraists and other researchers to correlate algebraic structures to enhance the parameter of mentioned applications.

The molecular graph can be considered as the structural formula for a chemical compound that can be seen as a graph structure. A molecular graph can be seen as a colored graph where atoms are considered to be vertices of the compound and edges are considered as atomic bonds. We consider a topological index for a graph of a molecular structure as a real digit that is assigned in a way that it represents the topological structure of the graph and remains constant up to the graph automorphism. The indices associated with a molecular graph structure infers several applications in chemistry. It also infers the study of nanotube structures that can be seen in [10, 12, 22, 24].

The characteristic of a topological index is the association of various types of topological indices to a graph. Topological indices are mainly associated with the degrees, distances and eccentricities of a graph. In particular, index related to Randi connectivity, harmonic indices, connectivity of atomic bond, Zagreb indices and geometric arithmetic indices are based on the degrees [1, 2, 5, 28].

Some of the examples of distance-based topological indices are the Hosaya index, Wiener index, Estrada index [36, 37]. The geometric-arithmetic eccentric index [20], Zagreb eccentric index [19, 35], connectivity of atomic bond eccentric index [15] and index related to eccentric harmonic [16, 17] are eccentric based indices.

From the application point of view, the ABC index gives an association between branched and linear alkanes stability. Also, it is used to compute the strain energy for Cycloalkanes [18, 33]. The GA index is used as a tool to correlate certain Physico-chemical characteristics which is proven to be more effective for predictive power in comparison with the index of Randic connectivity [11, 32, 34]. First, as well as second Zagreb indices are used for the computation of total energy of p-electron in a molecule [31]. The topological indices which are based on degrees are generally used to analyze the chemical properties of various molecular diagrams.

Motivated by the above works the authors study eccentric-based topological indices for a class of graphs that are being used to analyze the molecule's structure of a compound for the assessment of pharmacological, Physicochemical, and toxicological characteristics. Further details can be seen in [14, 25]. The QSAR, that is, quantitative structure and activity relationship is used for such analysis [21].

For the structural study, ring structures are associated with graphs in several ways, amongst those we shall consider zero-divisor graphs. Initially, these graphs over commutative rings were constructed by I. Beck [13] in 1988 and discussed the coloring problem on these graphs.

Classical and logical algebraic structures linked with Graph theory have been studied intensively in previous years for associated invariants and applications in various fields. For example, it is interesting to explore ring $\mathcal{R}$ so that the graph $\Omega(\mathcal{R})$ becomes isomorphic with the given graph Γ. By taking a commutative ring having 14 elements and assuming them as vertices, Redmond [29] constructed all possible zero-divisor graphs of this structure. Further, an algorithm is provided to compute rings (up to isomorphism) that produce the graph on zero-divisors of a ring having a fixed given number of vertices. Some of the recent studies on algebraic combinatorics with application in other sciences are available in [23, 26, 27]. Further, applications and the relation between the algebraic structures and chemical graphs can be seen in [3, 14]. The recent works due to Asir et al. [8], Selvakumar et al. [30] and Asir et al. [9] provided formulas for calculating the Wiener index of zero-divisor graphs of $\mathbb{Z}_n$. A detailed work on graphs associated with rings structure is given in [6].

Consider a class of commutative rings $\mathbb{Z}_{p^n}$ where p is a prime number and consider $\Omega(\mathbb{Z}_{p^n})$, the corresponding zero-divisor graph. In this text, we provide a method of calculation of eccentric topological indices of zero-divisor graphs $\Omega(\mathbb{Z}_{p^n})$ for a fixed positive integer n and any prime number p.

In particular, we found first and third Zagreb eccentric indices, geometrico arithmetic eccentric index, atomic bonding connectivity eccentric index and eccentric harmonic indices of fourth types for zero-divisor graphs associated with the rings $\mathbb{Z}_{p^n}$.

2. Related definitions and some notations

In this whole text Γ will denote a general connected graph with set of vertices $V(\Gamma)$ and edges set $E(\Gamma)$. For any $v \in V(\Gamma)$, the degree $d(v)$ denotes the number of edges connected to the vertex v. The numbers $\Delta(\Gamma)$ and $\delta(\Gamma)$ are maximum and minimum degree of a vertex in a graph Γ. The number of edges between the shortest path of any two given vertices v_1 and v_2 will be denoted by $d(v_1, v_2)$. For a vertex $v \in V(\Gamma)$, we define the eccentricity of v as:

$$\varrho_v = \max\{d(v, x) : x \in V(\Gamma)\}. \tag{2.1}$$

The $T(\Gamma)$ is a topological invariant of the eccentricity for the vertices of graph Γ.

$$T(\Gamma) = \sum_{vu \in E(\Gamma)} \phi(\varrho_v, \varrho_u), \tag{2.2}$$

where $\phi(\varrho_u, \varrho_v) = \phi(\varrho_v, \varrho_u)$ gives a real valued function between two eccentricities ϱ_u and ϱ_v.

- If $\phi(\varrho_u, \varrho_v) = (\varrho_u + \varrho_v)^\beta$ for $\beta \in \mathcal{R} \setminus 0$, then we say that $\phi(\Gamma)$ is the first Zagreb eccentric index if $\beta = 1$ [19, 35].
- If $\phi(\varrho_u, \varrho_v) = (\varrho_u \times \varrho_v)^\alpha$ for $\alpha \in \mathcal{R} \setminus 0$, then we say that $\phi(\Gamma)$ is the third Zagreb eccentric index if $\alpha = 1$ [19, 35].
- If $\phi(\varrho_u, \varrho_v) = \frac{2\sqrt{\varrho_u \times \varrho_v}}{\varrho_u + \varrho_v}$, it gives the geometric arithmetic eccentric index, $A_4(\Gamma)$ [20].
- If $\phi(\varrho_u, \varrho_v) = \sqrt{\frac{\varrho_u + \varrho_v - 2}{\varrho_u \times \varrho_v}}$, we obtain the atom-bond connectivity eccentric index $ABC_5(\Gamma)$ [15].
- If $\phi(\varrho_u, \varrho_v) = \frac{2}{\varrho_u + \varrho_v}$, then we obtained eccentric harmonic index, $H_4(\Gamma)$, called fourth eccentric harmonic index [16, 17].

We consider a ring $\mathcal{R}$ which is commutative with unity. A *zero-divisor* is a non-zero element $z \in \mathcal{R}$ if there exists another element $x \in \mathcal{R},\ x \neq 0$ and we have $z.x = 0$. Similarly, a *unit* is an element $a \in \mathcal{R},\ a \neq 0$ if there exists another element $b \in \mathcal{R},\ b \neq 0$ and we have $a.b = 1$. The set $Z(\mathcal{R})$ will denotes the collection of all zero-divisors elements in ring $\mathcal{R}$. If $\mathcal{R}$ is finite and commutative, then it is easy to associate $\mathcal{R}$ with $\Omega_{\mathcal{R}}$, the *zero-divisor graph* of $\mathcal{R}$ in a way that $V(\Omega_{\mathcal{R}}) = Z(\mathcal{R})$ becomes the set of vertices of $\Omega_{\mathcal{R}}$ and the set $E(\Omega_{\mathcal{R}})$ will represents the edge set of a zero-divisor graph $\Omega_{\mathcal{R}}$. Clearly, any $(x_1, x_2) \in E(\Omega_{\mathcal{R}})$ if $x_1, x_2 \in V(\Omega_{\mathcal{R}})$ and $x_1.x_2 = 0$. it is proved by Anderson and Livingston [7] that the graph $\Omega_{\mathcal{R}}$ remains a connected graph irrespective of any commutative ring $\mathcal{R}$ [4, 14].

3. Main results

In this text, the under consideration rings are of the form $\mathcal{R} = \mathbb{Z}_m$ for a fix positive integer m. An element $x \in \mathbb{Z}_m \setminus \{0\}$ is a zero-divisor if and only if $gcd(x, m) > 1$ and an element $a \in \mathbb{Z}_m \setminus \{0\}$ is a unit if and only if $gcd(a, m) = 1$. Therefore any non-zero element in the ring $\mathbb{Z}_m$ is either a zero-divisor or

a unit.
Let p be any prime number and n be a a fix positive integer. We consider the finite commutative rings of the form $\mathcal{R} = \mathbb{Z}_{p^n}$. According to above construction, an element $a \in \mathbb{Z}_{p^n},\ a \neq 0$ is in fact a zero-divisor if and only if p divides a. We partition the set of zero-divisors $Z(\mathbb{Z}_{p^n})$ into the sets, $\Lambda_i = \{u.p^i :\ u \text{ is a unit in } \mathbb{Z}_{p^n}\} \subseteq \mathbb{Z}_{p^n}$ that in fact contains elements which are a multiple of p^i but not of p^{i+1}. Clearly, the set $Z(\mathbb{Z}_{p^n}) = \bigsqcup_{i=1}^{n-1} \Lambda_i$ and $|\Lambda_i| = p^{n-i} - p^{n-i-1}$ for each $i = 1, 2, \ldots, n-1$ and therefore $|Z(\mathbb{Z}_{p^n})| = \sum_{i=1}^{n-1} |\Lambda_i| = p^{n-1} - 1$.
We denote associated zero-divisor graph to the ring $\mathcal{R} = \mathbb{Z}_{p^n}$ by $\Omega_{\mathcal{R}} = \Omega_{\mathbb{Z}_{p^n}}$ with set of vertices $V(\Omega_{\mathcal{R}}) = Z(\mathbb{Z}_{p^n})$. Since we consider a zero-divisor to be a non-zero element, so $0 \notin V(\Omega_{\mathcal{R}})$.
Let us denote $d_{\Lambda_i}(x)$. The degree of a vertex $x \in \Lambda_i$ is obtained in the following result.

Proposition 3.1. *For a zero-divisor graph $\Omega_{\mathcal{R}}$ associated to the ring $\mathcal{R} = \mathbb{Z}_{p^n}$, the degree of vertex x is,*

$$d_{\Lambda_i}(x) = \begin{cases} p^i - 1, & \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1 \\ p^i - 2, & \text{for } \lceil \frac{n}{2} \rceil \leq i \leq n-1 \end{cases}$$

Proof. For any vertex $x \in \Lambda_i$, we have $x.y = 0$ if and only if $y \in \Lambda_j$ for $j \geq n-i$. For $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$ we get $d_{\Lambda_i}(x) = |\bigsqcup_{j=n-i}^{n-1} \Lambda_j| = \sum_{j=n-i}^{n-1} |\Lambda_j| = p^i - 1$. For $\lceil \frac{n}{2} \rceil \leq i \leq n-1$ we get $d_{\Lambda_i}(x) = |\bigsqcup_{j=n-i}^{n-1} \Lambda_j - \{x\}| = \sum_{j=n-i}^{n-1} |\Lambda_j| - 1 = p^i - 1 - 1 = p^i - 2$. □

By using the hand shaking lemma and after simplification, we obtained the size of the $\Omega_{\mathcal{R}}$ in the following theorem.

Proposition 3.2. *For a prime number p and a fix positive integer $n \geq 2$, the size of $\Omega_{\mathcal{R}}$ is,*

$$\frac{1}{2}\{\sum_{x \in V(\Omega_{\mathcal{R}})} d(x)\} = \frac{1}{2}\{p^{n-1}(np - n - p) - p^{n - \lceil \frac{n}{2} \rceil} + 2\},\ \textit{except } n = p = 2.$$

Theorem 3.3. *Consider a prime number p and a fixed integer $n \geq 2$. Then the vertex eccentricity of the associated graph $\Omega_{\mathcal{R}}$ with the ring $\mathcal{R} = \mathbb{Z}_{p^n}$ is either 1 or 2.*

Proof. Let $d(\Lambda_i, \Lambda_j)$ be the distance between sets Λ_i and Λ_j. For any vertex $\varrho_u^i \in \Lambda_i$ and $\varrho_u^j \in \Lambda_j$, then $d(\varrho_u^i, \varrho_v^j) = 1$ for $i = j$ and $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$. Also, $d(\varrho_u^i, \varrho_v^j) = 2$ for $i = j$ and $\lceil \frac{n}{2} \rceil \leq i \leq n-1$. Now, $d(\varrho_u^i, \varrho_v^j) = 2$ for $i \neq j$ & $1 \leq i, j \leq \lceil \frac{n}{2} \rceil - 1$ and $d(\varrho_u^i, \varrho_v^j) = 1$ for $i \neq j$ & $\lceil \frac{n}{2} \rceil \leq i, j \leq n-1$. Also, $d(\varrho_u^1, \varrho_v^j) = 2$ if $2 \leq j \leq n-2$ and $d(\varrho_u^1, \varrho_v^j) = 1$ if $j = n-1$. Therefore, the vertex eccentricity of the graph $\Omega_{\mathcal{R}}$ is 1 or at most 2. It also gives that the diameter, $diam(\Omega_{\mathcal{R}}) = 2$. □

Lemma 3.4. *For any prime number p and the graph $\Omega_{\mathcal{R}}$ associated with the ring $\mathcal{R} = \mathbb{Z}_{p^n}$ we have the followings. For the even integers n,*

$$T(\Omega_{\mathcal{R}}) = \frac{p^n}{2}\left(\phi(1,1) + \phi(1,2)(n-2)\right) - \frac{n}{2}p^{n-1}\phi(1,2) + p^{\frac{n}{2}}\left(\phi(1,2) - \frac{3}{2}\phi(1,1)\right) + \phi(1,1).$$

For the odd integers n,

$$T(\Omega_{\mathcal{R}}) = \frac{p^{n-1}}{2}\left(\phi(1,1) - \phi(1,2)(pn-p-n-1)\right) + p^{\frac{n-1}{2}}\left(\phi(1,2) - \frac{3}{2}\phi(1,1)\right) + \phi(1,1).$$

Proof. The graph $\Omega_{\mathcal{R}}$ contains $p^{n-1}-1$ vertices and $\frac{1}{2}\{p^{n-1}(np-n-P) - p^{n-\lceil\frac{n}{2}\rceil} + 2$ edges except for $n = p = 2$. Partitioning the edges of $\Omega_{\mathcal{R}}$ into different sets we obtained the vertex eccentricity.

$$\mathcal{E}_{r,s} = \{uv \in E(\Omega_{\mathcal{R}}): \text{ where } \varrho_u = r, \varrho_v = s\}.$$

It is clear that the set $\mathcal{E}_{r,s}$ keeps those edges that are incident to vertices having eccentricities r and s. By using Proposition 3.1 and Theorem 3.3, we get $|\mathcal{E}_{1,1}| = \frac{p^{n-1}-3p^{\frac{n}{2}}+2}{2}$ for n even & $|\mathcal{E}_{1,1}| = \frac{p^{n-1}-3p^{\frac{n-1}{2}}+2}{2}$ for n odd, $|\mathcal{E}_{1,2}| = \frac{p^{n-1}(pn-n-2p)}{2} + p^{\frac{n}{2}}$ for n even & $|\mathcal{E}_{1,2}| = \frac{p^{n-1}(np-p-n-1)}{2} + p^{\frac{n-1}{2}}$ for n odd and $E(\Omega_{\mathcal{R}}) = \mathcal{E}_{1,1} \cup \mathcal{E}_{1,2}$. Then for even integers n,

$$\begin{aligned}
T(\Omega_{\mathcal{R}}) &= \sum_{uv\in E(\Omega_{\mathcal{R}})} \phi(\varrho_u, \varrho_v) \\
&= \sum_{uv\in\mathcal{E}_{1,1}} \phi(1,1) + \sum_{uv\in\mathcal{E}_{1,2}} \phi(1,2) \\
&= \frac{p^{n-1} - 3p^{\frac{n}{2}} + 2}{2}\phi(1,1) + \left(\frac{p^{n-1}(pn-n-2p)}{2} + p^{\frac{n}{2}}\right)\phi(2,2) \\
&= \frac{p^n}{2}\left(\phi(1,1) + \phi(1,2)(n-2)\right) - \frac{n}{2}p^{n-1}\phi(1,2) + p^{\frac{n}{2}}\left(\phi(1,2) - \frac{3}{2}\phi(1,1)\right) + \phi(1,1).
\end{aligned}$$

For odd integers n,

$$\begin{aligned}
T(\Omega_{\mathcal{R}}) &= \sum_{uv\in E(\Omega_{\mathcal{R}})} \phi(\varrho_u, \varrho_v) \\
&= \sum_{uv\in\mathcal{E}_{1,1}} \phi(1,1) + \sum_{uv\in\mathcal{E}_{1,2}} \phi(1,2) \\
&= \frac{p^{n-1} - 3p^{\frac{n-1}{2}} + 2}{2}\phi(1,1) + \left(\frac{p^{n-1}(np-p-n-1)}{2} + p^{\frac{n-1}{2}}\right)\phi(2,2) \\
&= \frac{p^{n-1}}{2}\left(\phi(1,1) + \phi(1,2)(pn-p-n-1)\right) + p^{\frac{n-1}{2}}\left(\phi(1,2) - \frac{3}{2}\phi(1,1)\right) + \phi(1,1).
\end{aligned}$$

□

The given theorem computes the edge-based values for the eccentric based topological indices of graphs $\Omega_{\mathcal{R}}$.

Theorem 3.5. *For a prime number $p \neq 3$, the zero-divisor graph $\Omega_{\mathcal{R}}$ for $\mathcal{R} = \mathbb{Z}_{p^n}$. The first Zagreb eccentric index of the graph $\Omega_{\mathcal{R}}$ is;*

$$M_1^*(\Omega_{\mathcal{R}}) = \begin{cases} \frac{p^n}{2}(3n-4) - \frac{3n}{2}p^{n-1} + 2, & \text{for } n \text{ even} \\ \frac{p^n}{2}(3n-3) - \frac{p^{n-1}}{2}(3n-1) + 2, & \text{for } n \text{ odd} \end{cases}$$

the third Zagreb eccentric index is;

$$M_3^*(\Omega_{\mathcal{R}}) = \begin{cases} \frac{p^n}{2}(2n-1) - np^{n-1} + \frac{1}{2}p^{\frac{n}{2}} + 1, & \text{for } n \textit{ even} \\ p^n(n-1) - \frac{p^{n-1}}{2}(2n+1) + \frac{1}{2}p^{\frac{n-1}{2}} + 1, & \text{for } n \textit{ odd} \end{cases}$$

the geometric-arithmetic eccentric index is;

$$GA_4(\Omega_{\mathcal{R}}) = \begin{cases} \frac{p^n}{2}\left(\frac{2n\sqrt{2}-4\sqrt{2}+3}{3}\right) - \frac{n\sqrt{2}}{3}p^{n-1} + p^{\frac{n}{2}}\left(\frac{4\sqrt{2}-9}{6}\right) + 1, & \text{for } n \textit{ even} \\ \\ \frac{p^n}{2}\left(\frac{2\sqrt{2}}{3}(n-1)\right) + \frac{p^{n-1}}{2}\left(\frac{3-2\sqrt{2}-2n\sqrt{2}}{3}\right) + p^{\frac{n-1}{2}}\left(\frac{4\sqrt{2}-9}{6}\right) + 1, & \text{for } n \textit{ odd} \end{cases}$$

the atom-bond connectivity eccentric index is;

$$ABC_5(\Omega_{\mathcal{R}}) = \begin{cases} \frac{p^n}{2}\left(\frac{n-2}{\sqrt{2}}\right) - \frac{n}{2\sqrt{2}}p^{n-1} + \frac{1}{\sqrt{2}}p^{\frac{n}{2}}, & \text{for } n \textit{ even} \\ \frac{\sqrt{2}p^n}{4}(n-1) - \frac{\sqrt{2}p^{n-1}}{4}(n+1) + \frac{\sqrt{2}}{2}p^{\frac{n-1}{2}}, & \text{for } n \textit{ odd} \end{cases}$$

the eccentric harmonic index of type four is as below:

$$H_4(\Omega_{\mathcal{R}}) = \begin{cases} \frac{p^n}{6}(2n-1) - \frac{n}{3}p^{n-1} - \frac{5}{6}p^{\frac{n}{2}} + 1, & \text{for } n \textit{ even} \\ \frac{p^n}{3}(n-1) + \frac{p^{n-1}}{6}(1-2n) - \frac{5}{6}p^{\frac{n-1}{2}} + 1, & \text{for } n \textit{ odd} \end{cases}$$

Proof. For the first Zagreb eccentric indices $M_1^*(\Omega_{\mathcal{R}})$ of graph $\Omega_{\mathcal{R}}$ we have $\phi(\varrho_u, \varrho_v) = \varrho_u + \varrho_v$. Therefore, $\phi(1,1) = 2$ and $\phi(1,2) = 3$. Thus by Lemma 3.4,
For even integers n,

$$\begin{aligned} M_1^*(\Omega_{\mathcal{R}}) &= \frac{p^n}{2}(2 + 3(n-2)) - \frac{n}{2}p^{n-1}(3) + p^{\frac{n}{2}}\left(3 - \frac{3}{2}(2)\right) + 2 \\ &= \frac{p^n}{2}(3n-4) - \frac{3n}{2}p^{n-1} + 2. \end{aligned}$$

For odd integers n,

$$\begin{aligned} M_1^*(\Omega_{\mathcal{R}}) &= \frac{1}{2}(p^{n-1})(2 + 3(pn - p - n - 1)) + p^{\frac{n-1}{2}}\left(3 - \frac{3}{2}(2)\right) + 2 \\ &= \frac{p^n}{2}(3n-3) - \frac{p^{n-1}}{2}(3n-1) + 2. \end{aligned}$$

For the third Zagreb eccentric indices $M_3^*(\Omega_{\mathcal{R}})$ of $\Omega_{\mathcal{R}}$ we get $\phi(\varrho_u, \varrho_v) = \varrho_u \times \varrho_v$. Therefore, $\phi(1,1) = 1$ and $\phi(1,2) = 2$ So by Lemma 3.4,
For n even

$$M_3^*(\Omega_{\mathcal{R}}) = \frac{p^n}{2}(1 + 2(n-2)) - \frac{n}{2}p^{n-1}(2) + p^{\frac{n}{2}}\left(2 - \frac{3}{2}\right) + 1 = \frac{p^n}{2}(2n-1) - np^{n-1} + \frac{1}{2}p^{\frac{n}{2}} + 1.$$

For n odd

$$M_3^*(\Omega_{\mathcal{R}}) = \frac{p^{n-1}}{2}(1 + 2(pn - p - n - 1)) + p^{\frac{n-1}{2}}\left(2 - \frac{3}{2}\right) + 1$$

$$=p^n(n-1)-\frac{p^{n-1}}{2}(2n+1)+\frac{1}{2}p^{\frac{n-1}{2}}+1.$$

For the geometric arithmetic eccentric index $GA_4(\Omega_{\mathcal{R}})$ of the graph $\Omega_{\mathcal{R}}$, we obtained,

$$\phi(\varrho_u,\varrho_v)=\frac{2\sqrt{\varrho_u\times\varrho_v}}{\varrho_u+\varrho_v}.$$

Therefore, $\phi(1,1)=1$ and $\phi(1,2)=\frac{2\sqrt{2}}{3}$.
So, for even integers n,

$$\begin{aligned}GA_4(\Omega_{\mathcal{R}})=&\frac{p^n}{2}\left(1+\frac{2\sqrt{2}}{3}(n-2)\right)-\frac{n}{2}p^{n-1}(\frac{2\sqrt{2}}{3})+p^{\frac{n}{2}}\left(\frac{2\sqrt{2}}{3}-\frac{3}{2}\right)+1\\=&\frac{p^n}{2}\left(\frac{2n\sqrt{2}-4\sqrt{2}+3}{3}\right)-\frac{n\sqrt{2}}{3}p^{n-1}+p^{\frac{n}{2}}\left(\frac{4\sqrt{2}-9}{6}\right)+1.\end{aligned}$$

For odd integers n,

$$\begin{aligned}GA_4(\Omega_{\mathcal{R}})=&\frac{p^{n-1}}{2}\left(1+\frac{2\sqrt{2}}{3}(pn-p-n-1)\right)+p^{\frac{n-1}{2}}\left(\frac{2\sqrt{2}}{3}-\frac{3}{2}\right)+1\\=&\frac{p^n}{2}\left(\frac{2\sqrt{2}}{3}(n-1)\right)+\frac{p^{n-1}}{2}\left(\frac{3-2\sqrt{2}-2n\sqrt{2}}{3}\right)+p^{\frac{n-1}{2}}\left(\frac{4\sqrt{2}-9}{6}\right)+1.\end{aligned}$$

For the atom-bond connectivity eccentric index $ABC_5(\Omega_{\mathcal{R}})$ of $\Omega_{\mathcal{R}}$, we obtain $\phi(\varrho_u,\varrho_v)=\sqrt{\frac{\varrho_u+\varrho_v-2}{\varrho_u\times\varrho_v}}$. Thus $\phi(1,1)=0$ and $\phi(1,2)=\frac{1}{\sqrt{2}}$ Therefore,
For n even

$$\begin{aligned}ABC_5(\Omega_{\mathcal{R}})=&\frac{p^n}{2}\left(0+\frac{1}{\sqrt{2}}(n-2)\right)-\frac{n}{2}p^{n-1}(\frac{1}{\sqrt{2}})+p^{\frac{n}{2}}\left(\frac{1}{\sqrt{2}}-\frac{3}{2}(0)\right)+0\\=&\frac{p^n}{2}\left(\frac{n-2}{\sqrt{2}}\right)-\frac{n}{2\sqrt{2}}p^{n-1}+\frac{1}{\sqrt{2}}p^{\frac{n}{2}}.\end{aligned}$$

For n odd

$$\begin{aligned}ABC_5(\Omega_{\mathcal{R}})=&\frac{p^{n-1}}{2}\left(0+\frac{1}{\sqrt{2}}(pn-p-n-1)\right)+p^{\frac{n-1}{2}}\left(\frac{1}{\sqrt{2}}-\frac{3}{2}(0)\right)+0\\=&\frac{\sqrt{2}\,p^n}{4}(n-1)-\frac{\sqrt{2}\,p^{n-1}}{4}(n+1)+\frac{\sqrt{2}}{2}p^{\frac{n-1}{2}}.\end{aligned}$$

For eccentric harmonic index of type four $H_4(\Omega_{\mathcal{R}})$ of the graph $\Omega_{\mathcal{R}}$, we obtained, $\phi(\varrho_u,\varrho_v)=\frac{2}{\varrho_u+\varrho_v}$, hence, $\phi(1,1)=1$ and $\phi(1,2)=\frac{2}{3}$.
Thus, for even integers n,

$$H_4(\Omega_{\mathcal{R}})=\frac{p^n}{2}\left(1+\frac{2}{3}(n-2)\right)-\frac{n}{2}p^{n-1}\frac{2}{3}+p^{\frac{n}{2}}\left(\frac{2}{3}-\frac{3}{2}\right)+1$$

$$=\frac{p^n}{6}(2n-1)-\frac{n}{3}p^{n-1}-\frac{5}{6}p^{\frac{n}{2}}+1.$$

For odd integers n,

$$\begin{aligned}H_4(\Omega_{\mathcal{R}}) &=\frac{p^{n-1}}{2}\left(1+\frac{2}{3}(pn-p-n-1)\right)+p^{\frac{n-1}{2}}\left(\frac{2}{3}-\frac{3}{2}\right)+1\\ &=\frac{p^n}{3}(n-1)+\frac{p^{n-1}}{6}(1-2n)-\frac{5}{6}p^{\frac{n-1}{2}}+1.\end{aligned}$$

□

4. Conclusions

We have computed Zagreb eccentric index for the first type, the Zagreb eccentric index for the third type, geometrico-arithmetic eccentric index, atomic bonding connectivity eccentric indices and the eccentric harmonic indices of the fourth type of the graphs that are related to the ring $\mathbb{Z}_{p^n}$. Our work can be used to study different physical and chemical structures such as carbohydrate, silicon structure, polymer, coating, paint constituent and for various computer network problems.

Conflict of interest

The authors declare no conflict of interest.

References

1. A. Ahmad, Computation of certain topological properties of honeycomb networks and Graphene, *Discret. Math. Algorit.*, **9** (2017), 1750064. https://doi.org/10.1142/S1793830917500641
2. A. Ahmad, On the degree based topological indices of benzene ring embedded in p-type-surface in 2D network, *Hacet. J. Math. Stat.*, **47** (2018), 9–18. https://doi.org/10.15672/HJMS.2017.443
3. A. Ahmad, A. Haider, Computing the radio labeling associated with zero-divisor graph of a commutative ring, *U. Politeh. Buch. Ser. A*, **81** (2019), 65–72.
4. S. Akbari, A. Mohammadian, On the zero-divisor graph of a commutative ring, *J. Algebra*, **274** (2004), 847–855. https://doi.org/10.1016/S0021-8693(03)00435-6
5. S. Akhter, W. Gao, M. Imran, M. R. Farahani, On topological indices of honeycomb networks and graphene networks, *Hacet. J. Math. Stat.*, **47** (2018), 19–35. https://doi.org/10.15672/HJMS.2017.464
6. D. F. Anderson, T. Asir, A. Badawi, T. T. Chelvam, *Graphs from rings,* Springer International Publishing, 2021. https://doi.org/10.1007/978-3-030-88410-9
7. D. F. Anderson, P. S. Livingston, The zero-divisor graph of commutative ring, *J. Algebra*, **217** (1999), 434–447. https://doi.org/10.1006/jabr.1998.7840
8. T. Asir, V. Rabikka, The Wiener index of the graph Zn, *Discrete Appl. Math.*, 2022.

9. T. Asir, V. Rabikka, H. D. Su, On Wiener Index of unit graph associated with a commutative ring, *Algebra Colloq.*, to appear.

10. M. Bača, J. Horvràthovà, M. Mokrišovà, A. Suhànyiovà, On topological indices of fullerenes, *Appl. Math. Comput.*, **251** (2015), 154–161.

11. A. Q. Baig, M. Imran, H. Ali, On topological indices of poly oxide, poly silicate, DOX, and DSL networks, *Can. J. Chem.*, **93** (2015), 730–739.

12. A. Q. Baig, M. Imran, H. Ali, S. U. Rehman, Computing topological polynomial of certain nanostructures, *J. Optoelectron. Adv. Mat.*, **17** (2015), 877–883.

13. I. Beck, Coloring of a commutative ring, *J. Algebra*, **116** (1988), 208–226. https://doi.org/10.1016/0021-8693(88)90202-5

14. K. Elahi, A. Ahmad, R. Hasni, Construction algorithm for zero-divisor graphs of finite commutative rings and their vertex-based eccentric topological indices, *Mathematics*, **301** (2018). https://doi.org/10.3390/math6120301

15. M. R. Farahani, Eccentricity version of atom bond connectivity index of benzenoid family ABC_5(Hk), *World Appl. Sci. J. Chem.*, **21** (2013), 1260–1265.

16. M. R. Farahani, S. Ediz, M. Imran, On novel harmonic indices of certain nanotubes, *Int. J. Adv. Biotechnol. Res.*, **8** (2017), 277–282.

17. Y. Gao, S. Ediz, M. R. Farahani, M. Imran, On the second harmonic index of titania nanotubes, *Drug Des. Int. Prop. Int. J.*, **1** (2018). https://doi.org/10.32474/DDIPIJ.2018.01.000102

18. W. Gao, M. K. Siddiqui, M. Naeem, N. A. Rehman, Topological characterization of carbon graphite and crystal cubic carbon structures, *Molecules*, **22** (2017), 1496. https://doi.org/10.3390/molecules22091496

19. M. Ghorbani, M. A. Hosseinzadeh, A new version of Zagreb indices, *Filomat*, **26** (2012), 93–100. https://doi.org/10.2298/FIL1201093G

20. M. Ghorbani, A. Khaki, A note on the fourth version of geometric-arithmetic index, *Optoelectron. Adv. Mater. Rapid Commum.*, **4** (2010), 2212–2215.

21. S. Gupta, M. Singh, A. K. Madan, Application of graph theory: Relationship of eccentric connectivity index and wiener's index with Anti-inflammatory Activity, *J. Math. Anal. Appl.*, **266** (2002), 259–268. https://doi.org/10.1006/jmaa.2000.7243

22. I. Gutman, O. E. Polansky, *Mathematical concepts in organic chemistry,* Springer-Verlag, New York, 1986.

23. A. Haider, U. Ali, M. A. Ansari, Properties of Tiny braids and the associated commuting graph, *J. Algebr. Comb.*, **53** (2021), 147–155. https://doi.org/10.1007/s10801-019-00923-5

24. S. Hayat, M. Imran, Computation of topological indices of certain networks, *Appl. Math. Comput.*, **240** (2014), 213–228.

25. M. Imran, M. K. Siddiqui, A. A. E. Abunamous, D. Adi, S. H. Rafique, A. Q. Baig, Eccentricity based topological indices of an oxide network, *Mathematics*, **6** (2018). https://doi.org/10.3390/math6070126

26. A. N. A. Koam, Ali Ahmad, Azeem Haider, On eccentric topological indices based on edges of zero divisor graphs, *Symmetry*, **11** (2019), 907. https://doi.org/10.3390/sym11070907
27. A. N. A. Koam, Ali Ahmad, Azeem Haider, Radio number associated with zero divisor graph, *Mathematics*, **8** (2020), 2187. https://doi.org/10.3390/math8122187
28. M. F. Nadeem, S. Zafar, Z. Zahid, On certain topological indices of the line graph of subdivision graphs, *Appl. Math. Comput.*, **271** (2015), 790–794.
29. S. P. Redmond, On zero-divisor graphs of small finite commutative rings, *Discrete Math.*, **307** (2007), 1155–1166. https://doi.org/10.1016/j.disc.2006.07.025
30. K. Selvakumar, P. Gangaeswari, G. Arunkumar, The wiener index of the zero-divisor graph of a finite commutative ring with unity, *Discrete Appl. Math.*, 2022, In press.
31. Z. Shao, M. K. Siddiqui, M. H. Muhammad, Computing zagreb indices and zagreb polynomials for symmetrical nanotubes, *Symmetry*, **10** (2018), 244. https://doi.org/10.3390/sym10070244
32. Z. Shao, P. Wu, Y. Gao, I. Gutman, X. Zhang, On the maximum *ABC* index of graphs without pendent vertices, *Appl. Math. Comput.*, **315** (2017), 298–312. https://doi.org/10.1016/j.amc.2017.07.075
33. Z. Shao, P. Wu, X. Zhang, D. Dimitrov, J. Liu, On the maximum *ABC* index of graphs with prescribed size and without pendent vertices, *IEEE Access*, **6** (2018), 27604–27616. https://doi.org/10.1109/ACCESS.2018.2831910
34. M. K. Siddiqui, M. Imran, A. Ahmad, On Zagreb indices, Zagreb polynomials of some nanostar dendrimers, *Appl. Math. Comput.*, **280** (2016), 132–139.
35. D. Vukičević, A. Graovac, Note on the comparison of the first and second normalized Zagreb eccentricity indices, *Acta Chim. Slov.*, **57** (2010), 524–528.
36. S. Wang, M. R. Farahani, M. R. R. Kanna, M. K. Jamil, R. P. Kumar, The wiener index and the hosoya polynomial of the Jahangir graphs, *Appl. Comput. Math.*, **5** (2016), 138–141.
37. H. Wiener, Structural determination of paraffin boiling points, *J. Am. Chem. Soc.*, **69** (1947), 17–20.