Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(6): 11311–11319.
DOI: 10.3934/math.2022631
Received: 26 September 2021
Revised: 22 March 2022
Accepted: 25 March 2022
Published: 11 April 2022

Research article

Limit behaviour of constant distance boundaries of Jordan curves

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Abstract: For a Jordan curve Γ in the complex plane, its constant distance boundary Γ_λ is an inflated version of Γ. A flatness condition, $(1/2, r_0)$-chordal property, guarantees that Γ_λ is a Jordan curve when λ is not too large. We prove that Γ_λ converges to Γ, as λ approaching to 0, in the sense of Hausdorff distance if Γ has the $(1/2, r_0)$-chordal property.

Keywords: constant distance boundary; Hausdorff distance; Jordan curve
Mathematics Subject Classification: 53A04, 57N40

1. Introduction

Let $\Gamma \subset \mathbb{C}$ be a closed Jordan curve and let Ω be the bounded component of $\mathbb{C} \setminus \Gamma$. For any $\lambda > 0$, the set

$$\Gamma_\lambda := \{z \in \Omega : \mathrm{dist}(z, \Gamma) = \lambda\}.$$

is called the constant distance boundary of Γ. Meanwhile, let

$$\Omega_\lambda := \{z \in \Omega : \mathrm{dist}(z, \Gamma) > \lambda\}.$$

Here $\mathrm{dist}(z, \Gamma) := \inf\{|z - \zeta| : \zeta \in \Gamma\}$. In [2], Brown showed that for all but countable number of λ, every component of Γ_λ is a single point, or a simple arc, or a simple closed curve. It was also proved that Γ_λ is a 1-manifold for almost all λ in [3]. Blokh, Misiurewiczch and Oversteegen generalised Brown's result in [1], they provided that for all but countably many $\lambda > 0$ each component of Γ_λ is either a point or a simple closed curve. If Γ is smooth or having positive reach, Γ_λ is called the offset of Γ in [6]. For λ within a positive reach, most nice properties are fulfilled by the Γ_λ. For instance, Γ_λ shares the class of differentiability of the Γ, since there is no ambiguity about the nearest point on Γ in such region. And points of Γ_λ project onto Γ along the normal to Γ through such point.

In Figure 1, we display three examples to show the relationship between $\partial\Omega_\lambda$ and Γ_λ. In the left two graphs, Γ is the outside polygon. The interior "curve" of (A) is Γ_λ, which is not a real curve. The interior curve of (B) is $\partial\Omega_\lambda$. In graph (C), the outer curve is Γ and the interior two curves make up $\partial\Omega_\lambda$. So, in general Ω_λ is not necessarily to be connected and its boundary $\partial\Omega_\lambda$ may not be equal with Γ_λ. However, it is not hard to see that $\partial\Omega_\lambda \subset \Gamma_\lambda$. We would like to ask that what is the sufficient condition for Γ_λ to be a Jordan curve and what is the sufficient condition for $\partial\Omega_\lambda = \Gamma_\lambda$? These questions are studied in [7]. If Γ_λ is a Jordan curve when λ is small enough, we find that, with a flatness condition, Γ_λ is approaching to Γ in the sense of Hausdorff distance as λ goes to 0. This means that Γ_λ is similar to Γ when λ is small enough. Thus we may expect Γ_λ inherits the geometric properties of Γ. This approximation property of Γ_λ could be applied in the theory of integration. In another paper we are preparing for, we show that $\int_{\Gamma_\lambda} f \to \int_\Gamma f$ with some geometric restriction on Γ.

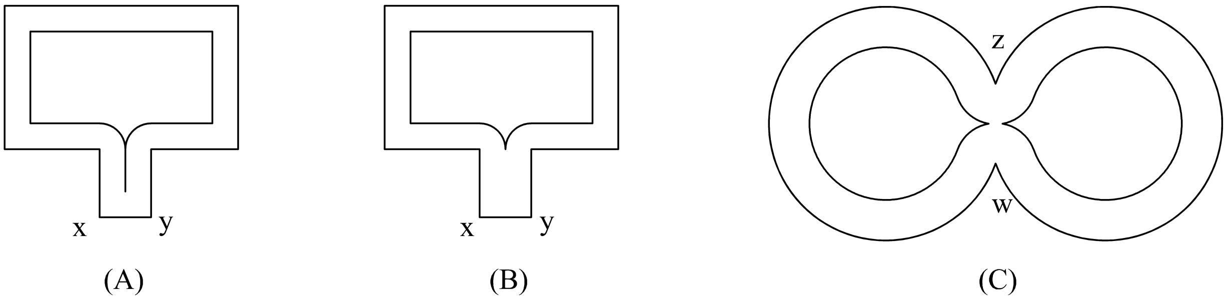

Figure 1. In (A) and (B) $\lambda = \frac{xy}{2}$. In (C) $\lambda > \frac{zw}{2}$.

Given two points $x, y \in \Gamma$, denotes $\Gamma(x, y)$ by the subarc of Γ containing and connecting x and y which has a smaller diameter, or, to be either subarc when both have the same diameter. Let $\ell_{x,y}$ be the infinite line through x and y, let

$$\zeta_\Gamma(x, y) = \frac{1}{|x - y|} \sup\{\mathrm{dist}(z, \ell_{x,y}) : z \in \Gamma(x, y)\}.$$

Following definition can be introduced.

Definition 1.1. *[7] A Jordan curve Γ is said to have the (ζ, r_0)-chordal property for a certain $\zeta > 0$ and $r_0 > 0$, if*

$$\sup\{\zeta_\Gamma(x, y) : x, y \in \Gamma \text{ and } |x - y| \le r_0\} \le \zeta.$$

Also, denote

$$\zeta_\Gamma = \lim_{r \to 0} \sup\{\zeta_\Gamma(x, y) : x, y \in \Gamma \text{ and } |x - y| \le r\}.$$

These quantities are used to measure the local deviation of the subarcs from their chords. It is not hard to see that Γ is smooth if and only if $\zeta_\Gamma = 0$. Therefore all smooth curves have the (ζ, r_0)-chordal property. Moreover, if a piecewise smooth Jordan curve only has corner points then it has the (ζ, r_0)-chordal property. However if a piecewise smooth Jordan curve has a cusp point then it does not have the (ζ, r_0)-chordal property.

Theorem 1.1. *[7] Let Γ be a Jordan curve in $\mathbb{R}^2$. If Γ has the $(1/2, r_0)$-chordal property for some $r_0 > 0$, then Γ has the level Jordan curve property i.e., there exists $\lambda_0 > 0$ such that Γ_λ is a Jordan curve for each $\lambda < \lambda_0$.*

This theorem provides us a method to verify whether Γ_λ is a Jordan curve. As we have seen in Figure 1, even through Γ is a simple Jordan curve, Γ_λ varies greatly. Based on this theorem, the authors of [7] also studied the quasi-circle property of Γ_λ. However, we are interested in the limit behaviour of Γ_λ as λ approaching to 0. The (ζ, r_0)-chordal property of Jordan curves is an essential condition in the proof of the main theorem, we show that Γ_λ converges to Γ if Γ has the $(1/2, r_0)$-chordal property.

The other parts of the paper will be organized as follows: In Section 2, we investigate some basic properties of constant distance boundary of Jordan curves. We prove that if Γ and Γ_λ are Jordan curves then there exist at least three points of Γ which have distance λ from Γ_λ. Also, we find out the relation between $\Gamma_{\lambda+\mu}$ and $(\Gamma_\mu)_\lambda$. This relation will be used in the proof of our main theorem. Section 3 is devoted to prove our main result, Theorem 3.1. The definition and some basic properties of Hausdorff distance, $d_H(\cdot, \cdot)$, are introduced firstly. We show that under the $(1/2, r_0)$-chordal property of Γ, the upper and lower bounds of $d_H(\Gamma, \Gamma_\lambda)$ are obtained. Consequently, the main theorem can be obtained.

2. Constant distance boundary

In this section, we investigate several fundamental properties according to the (ζ, r_0)-chordal property of Jordan curve Γ. In this paper, we always assume that $\lambda > 0$ and that Γ_λ is non-empty. First we introduce a notation which will be used frequently through the paper. For each $x \in \Gamma$, define

$$\Gamma_\lambda^x := \{y \in \Gamma_\lambda : |x - y| = \lambda\}.$$

And for any $y \in \Gamma_\lambda$, define

$$\Gamma^y := \{x \in \Gamma : |x - y| = \lambda\}.$$

In [5], the so called λ-parallel set of Γ is introduced. The definition is the following,

$$\Omega_\lambda^p := \{z \in \Omega : \mathrm{dist}(z, \Gamma) < \lambda\}.$$

Recall that we already have the set

$$\Omega_\lambda = \{z \in \Omega : \mathrm{dist}(z, \Gamma) > \lambda\}.$$

We have seen in Figure 1 that $\partial\Omega_\lambda$ is a proper subset of Γ_λ and Theorem 1.1 states that if Γ has the $(1/2, r_0)$-chordal property for some $r_0 > 0$ then $\partial\Omega_\lambda = \Gamma_\lambda$ whenever λ is small enough. However, the identical of Γ_λ and $\partial\Omega_\lambda^p$ can be obtained directly without the $(1/2, r_0)$-chordal property.

Proposition 2.1. $\Gamma_\lambda = \partial\Omega_\lambda^p$.

Proof. According to the continuity of the distance function, the relation of $\partial\Omega_\lambda^p \subset \Gamma_\lambda$ is obvious.

Let $z \in \Gamma_\lambda$. Then there exists $x \in \Gamma^z$. Consider an arbitrary point y on the segment (x, z). We know that $\mathrm{dist}(y, \Gamma) \le |x - y| < \lambda$. Thus $y \in \Omega_\lambda^p$. Since the point z is the limit of points along the segment $[x, z]$, we know that $z \in \overline{\Omega_\lambda^p}$. Therefore we have $\Gamma_\lambda \subset \partial\Omega_\lambda^p$.

In the above proof, $[x, z]$ stands for the line segment connecting points x and z, while (x, z) is $[x, z] \setminus \{x, z\}$.

Proposition 2.2. *Let $x, y \in \Gamma_\lambda$ be different points and let $x' \in \Gamma^x$ and $y' \in \Gamma^y$. If the two segments $[x, x']$ and $[y, y']$ intersect at p, i.e., $[x, x'] \cap [y, y'] = \{p\}$ then $x' = y' = p$.*

Proof. If $x' \neq y'$ then $\{p\} = (x, x') \cap (y, y')$. We have

$$|x - p| + |p - x'| = \lambda \text{ and } |y - p| + |p - y'| = \lambda.$$

Since

$$|x - p| + |p - y'| \geq |x - y'| \geq \lambda.$$

It follows that

$$|x - p| \geq |y - p|.$$

Then

$$|x' - y| \leq |x' - p| + |p - y| \leq \lambda.$$

Because of $|x' - y| \geq \lambda$, we know that $|x' - y| = |x' - p| + |p - y| = \lambda$. This means that the points y, p and x' are collinear, i.e., $p \in (y, x')$. However $p \in (y, y')$, this is impossible unless $x' = y'$. Therefore, we must have $x' = y' = p$.

This proposition tells us that two such segments $[x, x']$ and $[y, y']$ could only intersect at the end points.

Proposition 2.3. *Let $x \in \Gamma$ and $y \in \Gamma^x_\lambda$. If $z \in (x, y)$ such that $|x - z| = \mu$ for some $0 < \mu < \lambda$ then $z \in \Gamma^x_\mu$.*

Proof. Since $y \in \Gamma^x_\lambda$ and $|x - z| = \mu$, we have $|y - z| = \lambda - \mu$ and $\text{dist}(z, \Gamma) \leq |x - z| = \mu$. Suppose that $\text{dist}(z, \Gamma) < \mu$, there exists $t \in \Gamma$ such that $|z - t| = \text{dist}(z, \Gamma) < \mu$. Then $|y - t| \leq |y - z| + |z - t| < \lambda - \mu + \mu = \lambda$. It follows that $\text{dist}(y, \Gamma) < \lambda$. This contradicts to the fact that $y \in \Gamma^x_\lambda \subset \Gamma_\lambda$. Thus $\text{dist}(z, \Gamma) = \mu$ and then $z \in \Gamma^x_\mu$.

In the proofs of the above three propositions, the set Γ is not necessarily to be a Jordan curve. So these properties are correct for any compact subset of $\mathbb{C}$. In the following context, we assume that Γ is a Jordan curve. The Lemma 4.2 of [7] states that if Γ_λ is a Jordan curve and if there exist distinct $x, y \in \Gamma^z_\lambda$ for some $z \in \Gamma$, then the subarc $\Gamma_\lambda(x, y)$ is a circular arc of the circle centred at z and with radius λ, which denoted by $\gamma(z, \lambda)$.

Lemma 2.1. *If Γ and Γ_λ are Jordan curves then there exist at least three points of Γ which all have distance λ from Γ_λ.*

Proof. Suppose that there is no point on Γ has distance λ from Γ_λ. It means that for any $p \in \Gamma$ the distance $\text{dist}(p, \Gamma_\lambda) \neq \lambda$. It is clear that $\text{dist}(p, \Gamma_\lambda) < \lambda$ is incorrect. Thus $\text{dist}(p, \Gamma_\lambda) > \lambda$ for all $p \in \Gamma$. It follows that for a fixed point $q \in \Gamma_\lambda$ we know that $|p - q| > \lambda$ for all $p \in \Gamma$. Therefore $\text{dist}(q, \Gamma) > \lambda$. This contradicts the fact that $q \in \Gamma_\lambda$.

Suppose that there is only one point $x \in \Gamma$ which has distance λ from Γ_λ. Therefore $\operatorname{dist}(x, \Gamma_\lambda) = \lambda$ and $\operatorname{dist}(p, \Gamma_\lambda) > \lambda$ for any $p \in \Gamma$ when $p \neq x$. Thus for arbitrary $q \in \Gamma_\lambda$, we have $|q - p| > \lambda$ when $p \neq x$. It implies that $|q - x| = \lambda$. Then $\Gamma_\lambda \subset \Gamma_\lambda^x$. So Γ_λ is a circular arc of the circle with center at x and with radius λ, i.e., $\Gamma_\lambda \subset \gamma(x, \lambda)$. Because Γ_λ is a Jordan curve, we must have $\Gamma_\lambda = \gamma(x, \lambda)$. Therefore Γ is the union of $\{x\}$ and a certain subset of circle $\gamma(x, 2\lambda)$. In other words, Γ is separated by Γ_λ into two parts. This contradicts the fact that Γ is a Jordan curve.

Suppose that there are only two points $x, y \in \Gamma$ which have distance λ from Γ_λ. It means that $\operatorname{dist}(x, \Gamma_\lambda) = \lambda = \operatorname{dist}(y, \Gamma_\lambda)$ and $\operatorname{dist}(p, \Gamma_\lambda) > \lambda$ for any $p \in \Gamma$ when $p \neq x, y$. Similar to the one point case, we know that $\Gamma_\lambda \subset \Gamma_\lambda^x \cup \Gamma_\lambda^y$. Since Γ_λ is a Jordan curve, there are three situations we should consider.

(i) If $|x - y| = 2\lambda$ then Γ_λ lies in the two tangential circles $\gamma(x, \lambda)$ and $\gamma(y, \lambda)$. Because Γ_λ is a Jordan curve, it could only contained in one circle. Then x and y are separated by this circle which contradicts that fact that Γ is a Jordan curve.

(ii) If $|x - y| < 2\lambda$ then Γ_λ is the curve looks like number eight which enclose x and y at the inside area. While $\Gamma \setminus \{x, y\}$ is in the outside area otherwise $\Gamma = \{x, y\}$. The both situations contradict the fact that Γ is a Jordan curve.

(iii) If $|x - y| > 2\lambda$ then Γ_λ lies in one of the disjoint two circles $\gamma(x, \lambda)$ and $\gamma(y, \lambda)$. Therefore x and y are not connected which contradicts that fact that Γ is a Jordan curve.

By the above analysis we finished the proof.

The constant distance boundary Γ_λ of Γ will be a Jordan curve under specific conditions (see Theorem 1.1). Thus we can consider the constant distance boundary of Γ_λ, denoted by $(\Gamma_\lambda)_\mu$ if which is non-empty. Naturally we will investigate the relationship between $(\Gamma_\lambda)_\mu$ and $\Gamma_{\lambda+\mu}$.

Lemma 2.2. *Let $\lambda_0 > 0$. Suppose that Γ_λ is a Jordan curve for each $\lambda < \lambda_0$. Then for $0 < \mu < \lambda < \lambda_0$ we have*

$$\Gamma_\lambda \subset (\Gamma_\mu)_{\lambda-\mu}.$$

Proof. Since $0 < \mu < \lambda < \lambda_0$, it follows from Proposition 2.1 that $\Omega_\mu^p \subset \Omega_\lambda^p$. For any $y \in \Gamma_\lambda$, there is $x \in \Gamma$ such that $|x - y| = \lambda$. This means that $y \in \Gamma_\lambda^x$. Let z be a point of segment $[x, y]$ such that $|x - z| = \mu$. By Proposition 2.3, we conclude that $z \in \Gamma_\mu^x$, i.e., $z \in \Gamma_\mu$.

Now we have $\operatorname{dist}(y, \Gamma_\mu) \leq |y - z| = \lambda - \mu$. If the equality holds then $y \in (\Gamma_\mu)_{\lambda-\mu}$. If $\operatorname{dist}(y, \Gamma_\mu) < |y - z|$ then there exists $z' \in \Gamma_\mu$ such that $\operatorname{dist}(y, \Gamma_\mu) = |y - z'| < |y - z| = \lambda - \mu$. Because of $z' \in \Gamma_\mu$, there must exists $x' \in \Gamma$ such that $|z' - x'| = \mu$. Therefore $\operatorname{dist}(y, \Gamma) \leq |y - x'| \leq |y - z'| + |z' - x'| < \lambda - \mu + \mu = \lambda$ which contradicts to the fact of $y \in \Gamma_\lambda$. Therefore we must have $\operatorname{dist}(y, \Gamma_\mu) = |y - z| = \lambda - \mu$ which means $y \in (\Gamma_\mu)_{\lambda-\mu}$. It follows that $\Gamma_\lambda \subset (\Gamma_\mu)_{\lambda-\mu}$.

In Lemma 2.2 even though we assume that the sets Γ_λ and Γ_μ are Jordan curves, but Γ_μ does not necessarily satisfy the $(1/2, r_0)$-chordal property, thus the set $(\Gamma_\mu)_{\lambda-\mu}$ probably is not a Jordan curve (see Theorem 1.1).

Corollary 2.1. *Let Γ be a Jordan curve and has level Jordan curve property for some $\lambda_0 > 0$. If Γ_μ has $(1/2, r_0)$-chordal property for a $\mu < \lambda_0$, then $\Gamma_\lambda = (\Gamma_\mu)_{\lambda-\mu}$ when $0 < \mu < \lambda < \lambda_0$ and $\lambda - \mu < \delta$ for some $\delta > 0$.*

Proof. By the assumption of level Jordan curve property of Γ, we know that Γ_λ and Γ_μ are Jordan curves if $0 < \mu < \lambda < \lambda_0$. Because of Lemma 2.2 we have

$$\Gamma_\lambda \subset (\Gamma_\mu)_{\lambda-\mu}.$$

By Theorem 1.1 and by the assumption of Γ_μ has $(1/2, r_0)$-chordal property, we know that the curve Γ_μ has level Jordan curve property for some $\delta > 0$. Thus its constant distance boundary $(\Gamma_\mu)_{\lambda-\mu}$ is a Jordan curve when $\lambda-\mu < \delta$. Then both Γ_λ and $(\Gamma_\mu)_{\lambda-\mu}$ are Jordan curves, it implies that $\Gamma_\lambda = (\Gamma_\mu)_{\lambda-\mu}$.

Remark. In Corollary 2.1, the $(1/2, r_0)$-chordal property of Γ_μ is crucial, because it is a necessary condition for the set $(\Gamma_\mu)_{\lambda-\mu}$ to be a Jordan curve, i.e., the curve Γ_μ has level Jordan curve property. So far, we only know that if Γ_μ has $(1/2, r_0)$-chordal property then $(\Gamma_\mu)_{\lambda-\mu}$ is a Jordan curve when $\lambda-\mu < \delta$ for some $\delta > 0$.

3. Limit behaviour of Γ_λ

In this section, we study the limit behaviour of Γ_λ as λ tends to 0. All the limits are considered in the sense of Hausdorff distance. For the convenience of readers, we briefly introduce the concept and some elementary properties of Hausdorff distance, which can be found in [4].

Definition 3.1. *Let X and Y be two non-empty subsets of $\mathbb{C}$. The Hausdorff distance of X and Y, denoted by $d_H(X, Y)$, is defined by*

$$d_H(X, Y) := \max\{\sup_{x\in X} \inf_{y\in Y} |x-y|, \ \sup_{y\in Y} \inf_{x\in X} |x-y|\}.$$

Denote by

$$d(X, Y) := \sup_{x\in X} \operatorname{dist}(x, Y) \ \text{ and } \ d(Y, X) := \sup_{y\in Y} \operatorname{dist}(y, X)$$

the distance from X to Y and Y to X respectively. We could rewrite

$$d_H(X, Y) = \max\{d(X, Y), \ d(Y, X)\}. \tag{3.1}$$

Note that $d(X, Y) \neq d(Y, X)$ usually happens.

For non-empty subsets X and Y of $\mathbb{C}$, we know that

$$d(X, Y) = 0 \Leftrightarrow \forall x \in X, \operatorname{dist}(x, Y) = 0 \Leftrightarrow \forall x \in X, x \in \overline{Y} \Leftrightarrow X \subset \overline{Y}.$$

Here $\overline{Y}$ is the closure of Y in $\mathbb{C}$. We summarize these equivalence relations in the following proposition.

Proposition 3.1. *Let X and Y be two non-empty subsets of $\mathbb{C}$. Then $d(X, Y) = 0$ if and only if $X \subset \overline{Y}$. Furthermore, $d_H(X, Y) = 0$ if and only if $\overline{X} = \overline{Y}$.*

The triangle inequality is true not only for d_H but also for d. That is for any compact subsets A, B and C of $\mathbb{C}$ we have

$$d(A, B) \leq d(A, C) + d(C, B). \tag{3.2}$$

We left the proof of (3.2) for interested readers as an exercise.

Denote by Π the set of compact subsets of $\mathbb{C}$. Federer shows in [4] that (Π, d_H) is a complete metric space. According to our consideration, Γ is a Jordan curve, so it is compact. By the definition of constant distance boundary, Γ_λ is compact as well. Thus we have $\Gamma, \Gamma_\lambda \in \Pi$. Observe that $\Gamma_0 = \Gamma$, so we want to know whether the limit of Γ_λ, in (Π, d_H), is Γ or not as λ approaching to zero. The first proposition we obtained is the following.

Proposition 3.2. *If there exists $L \in \Pi$ such that $\lim_{\lambda\to 0} \Gamma_\lambda = L$ then $L \subset \Gamma$.*

Proof. If $\lim_{\lambda\to 0} \Gamma_\lambda = L$ then $\lim_{\lambda\to 0} d_H(\Gamma_\lambda, L) = 0$. It follows that $\lim_{\lambda\to 0} d(L, \Gamma_\lambda) = 0$. We know that $d(\Gamma_\lambda, \Gamma) = \lambda$, since

$$d(\Gamma_\lambda, \Gamma) = \sup_{x\in\Gamma_\lambda} \operatorname{dist}(x, \Gamma) = \lambda.$$

It follows from (3.2) that $d(L, \Gamma) \le d(L, \Gamma_\lambda) + d(\Gamma_\lambda, \Gamma)$. Letting λ tends to 0 implies that $d(L, \Gamma) = 0$, thus $L \subset \overline{\Gamma}$. By the compactness of Γ, we have that $L \subset \overline{\Gamma} = \Gamma$.

Proposition 3.2 states that if the limit of Γ_λ exists in Π then it must be a subset of Γ. But we still cannot confirm whether this limit is a proper subset of Γ or equal to Γ. While if Γ has the $(1/2, r_0)$-chordal property, we obtain the following result.

Lemma 3.1. *Suppose that Γ has $(1/2, r_0)$-chordal property and that $\lambda \le r_0/2$. Then $\lambda \le d_H(\Gamma_\lambda, \Gamma) \le (2\sqrt{5} + 1)\lambda$.*

Proof. Because that Γ has the $(1/2, r_0)$-chordal property, we may assume that Γ_λ in consideration is a Jordan curve. Recall that $d(\Gamma_\lambda, \Gamma) = \lambda$. By (3.1), we already have

$$d_H(\Gamma_\lambda, \Gamma) = \max\{d(\Gamma_\lambda, \Gamma), d(\Gamma, \Gamma_\lambda)\} \ge \lambda.$$

Because of $d(\Gamma, \Gamma_\lambda) \ge \lambda$ we assume that $d(\Gamma, \Gamma_\lambda) > \lambda$, otherwise $d_H(\Gamma_\lambda, \Gamma) = \lambda$.

Now suppose that there exists a $w \in \Gamma$ such that $\operatorname{dist}(w, \Gamma_\lambda) > \lambda$. By Lemma 2.1 there are at least three points of Γ which have distance λ from Γ_λ. So we can choose a subarc $\Gamma(x, y)$ of Γ such that $w \in \Gamma(x, y)$ and $d(p, \Gamma_\lambda) \ge \lambda$ for all $p \in \Gamma(x, y)$, especially, the equality holds only when $p \in \{x, y\}$. The reason is that if there is a $z \in \Gamma(x, y) \setminus \{x, y\}$ such that $d(z, \Gamma_\lambda) = \lambda$ then one of the two subarcs $\Gamma(x, z)$ or $\Gamma(z, y)$ contains w. Thus only need to replace $\Gamma(x, y)$ by this subarc. By the compactness of Γ_λ, there exist $x' \in \Gamma_\lambda^x$ and $y' \in \Gamma_\lambda^y$.

(i) Consider the case when $x' = y' = q$. It is not hard to know that $|x - y| \le |x - q| + |y - q| = 2\lambda$ and thus $|x - y| \le r_0$. By the $(1/2, r_0)$-chordal property of Γ, we obtain that $\operatorname{dist}(p, \ell_{x,y}) \le 1/2|x - y| \le \lambda$ for every $p \in \Gamma(x, y)$. The straight line $\ell_{x,y}$ separates the complex plane into two parts, which denoted by $\mathbb{C}^R$ and $\mathbb{C}^L$.

Firstly, we assume that $\Gamma \cap \mathbb{C}^R$ and $\Gamma \cap \mathbb{C}^L$ are non-empty. Let $p_0 \in \mathbb{C}^R \cap \Gamma(x, y)$ such that

$$\operatorname{dist}(p_0, \ell_{x,y}) = \max\{\operatorname{dist}(p, \ell_{x,y}) : p \in \mathbb{C}^R \cap \Gamma(x, y)\}.$$

Similarly, let $p_1 \in \mathbb{C}^L \cap \Gamma(x, y)$ such that

$$\operatorname{dist}(p_1, \ell_{x,y}) = \max\{\operatorname{dist}(p, \ell_{x,y}) : p \in \mathbb{C}^L \cap \Gamma(x, y)\}.$$

Construct straight lines ℓ_{p_0} and ℓ_{p_1} pass through p_0 and p_1 respectively and parallel to $\ell_{x,y}$. Thus the arc $\Gamma(x, y)$ is bounded in the strip region between ℓ_{p_0} and ℓ_{p_1} which has width at most 2λ. It is needed to explain that $\Gamma(x, y)$ may only at one side of the line $\ell_{x,y}$. Thus p_0 or p_1 may does not exist. However, $\Gamma(x, y)$ can not be a straight line otherwise x' and y' must be different. Therefore, at least, one of p_0 or p_1 must exists. Then the mentioned strip region now is between ℓ_{p_1} and $\ell_{x,y}$ if p_0 does not exist, while the strip region is between ℓ_{p_0} and $\ell_{x,y}$ if p_1 does not exist. In these cases, the width of the strip region is at most λ.

Construct straight lines ℓ_x and ℓ_y pass through x and y respectively and perpendicular to $\ell_{x,y}$. Choose x' and x'' on $\ell_x \cap \Gamma(x, y)$ such that

$$|x' - x''| = \max\{|s - t| : s, t \in \ell_x \cap \Gamma(x, y)\}.$$

Because $\Gamma(x, y)$ is bounded in the strip region with width at most 2λ, we must have $|x' - x''| \leq 2\lambda \leq r_0$. Thus the $(1/2, r_0)$-chordal property implies that the arc $\Gamma(x', x'')$ is bounded in a strip region which has width at most 2λ. By the similar argument for ℓ_y, we obtain that $\Gamma(x, y)$ is bounded in a rectangular with width 2λ and length 4λ. We denote this rectangular by Δ. Thus $|w - x| \leq \operatorname{diam}\Delta = 2\sqrt{5}\lambda$. Here $\operatorname{diam}\Delta$ is the diameter of Δ. So $|w - q| \leq |w - x| + |x - q| \leq 2\sqrt{5}\lambda + \lambda = (2\sqrt{5} + 1)\lambda$. This implies that $\operatorname{dist}(w, \Gamma_\lambda) \leq (2\sqrt{5} + 1)\lambda$.

(ii) Consider the case when $x' \neq y'$. For every $q \in \Gamma_\lambda(x', y')$ there exists $p \in \Gamma^q$.

If $p \in \Gamma(x, y)$ the selection condition of $\Gamma(x, y)$ implies that $p \in \{x, y\}$. We can see that if $p = x$ then replace x' by q, also denoted by x', and if $p = y$ then replace y' by q, also denoted by y'. Choose another point $q' \in \Gamma_\lambda(x', y')$ and continuous the above process, finally we must have that $x' = y'$. Then repeat the proof of case (i), we also have $\operatorname{dist}(w, \Gamma_\lambda) \leq (2\sqrt{5} + 1)\lambda$.

Suppose that $p \in \Gamma(y, x)$. Here $\Gamma(y, x) = \Gamma \setminus \Gamma(x, y)$. Because the segment $[p, q]$ does not intersect with $\Gamma_\lambda(y', x')$. Then $\Gamma_\lambda(y', x')$ is enclosed by the union of arcs $L := \Gamma(x, y) \cup [y, y'] \cup \Gamma_\lambda(x', y') \cup [x', x]$. It implies that for every $q' \in \Gamma_\lambda(y', x')$ there must exists $p' \in \Gamma^{q'}$ such that $p' \in \Gamma(x, y)$. If this is not the case then $p' \in \Gamma(y, x)$, and then $[p', q']$ intersects L which is impossible. Repeat the analysis of the case when $p \in \Gamma(x, y)$ for $p' \in \Gamma(x, y)$, it follows that $\operatorname{dist}(w, \Gamma_\lambda) \leq (2\sqrt{5} + 1)\lambda$.

In the above analysis, we have considered all the possible situations. As a conclusion, we obtain that $d(\Gamma, \Gamma_\lambda) \leq (2\sqrt{5} + 1)\lambda$. Therefore we have the inequalities $\lambda \leq d_H(\Gamma_\lambda, \Gamma) \leq (2\sqrt{5} + 1)\lambda$.

In Lemma 3.1, the condition which Γ has $(1/2, r_0)$-chordal property is crucial, because of the $(1/2, r_0)$-chordal property the curve Γ has level Jordan property and then the upper bound of $d_H(\Gamma_\lambda, \Gamma)$ can be decided. However this condition is rigorous, we should consider the questions for curves without this restriction in the future work.

Theorem 3.1. *If Jordan curve Γ has $(1/2, r_0)$-chordal property then $\lim_{\lambda\to 0} \Gamma_\lambda = \Gamma$ in (Π, d_H).*

Proof. By Lemma 3.1, we immediately obtain that $d_H(\Gamma_\lambda, \Gamma) \leq (2\sqrt{5} + 1)\lambda$ when $2\lambda \leq r_0$. It implies that $\lim_{\lambda\to 0} \Gamma_\lambda = \Gamma$.

This theorem provides us a sufficient condition for Γ such that its constant distance boundaries converging to itself. Now let λ take discrete values $\{\frac{1}{n}\}_{n=1}^{\infty}$, we have the following corollary.

Corollary 3.1. *Let Γ be a Jordan curve and has level Jordan curve property for some $\lambda_0 > 0$. If $\Gamma_{\frac{1}{n}}$ has $(1/2, r_0)$-chordal property when $1/n < \lambda_0$. Then the limit $\lim_{n\to\infty} \Gamma_{\frac{1}{n}}$ exists.*

Proof. The Corollary 2.1 implies that $\Gamma_{\frac{1}{n}} = (\Gamma_{\frac{1}{m}})_{\frac{1}{n}-\frac{1}{m}}$ when $0 < \frac{1}{m} < \frac{1}{n} < \lambda_0$ and $\frac{1}{n} - \frac{1}{m} < \delta$ for some $\delta > 0$. By Lemma 3.1, it follows that

$$d_H(\Gamma_{\frac{1}{m}}, (\Gamma_{\frac{1}{m}})_{\frac{1}{n}-\frac{1}{m}}) \leq (2\sqrt{5}+1)(\frac{1}{n} - \frac{1}{m}),$$

when $\frac{1}{n} - \frac{1}{m} < r_0/2$. Therefore when $\frac{1}{n} - \frac{1}{m} < \min\{\delta, r_0/2\}$, we obtain that

$$d_H(\Gamma_{\frac{1}{m}}, \Gamma_{\frac{1}{n}}) = d_H(\Gamma_{\frac{1}{m}}, (\Gamma_{\frac{1}{m}})_{\frac{1}{n}-\frac{1}{m}}) \leq (2\sqrt{5}+1)(\frac{1}{n} - \frac{1}{m}).$$

Therefore $\{\Gamma_{\frac{1}{n}}\}$ is a Cauchy sequence in (Π, d_H), and then the limit $\lim_{n\to\infty} \Gamma_{\frac{1}{n}}$ exists.

Acknowledgments

The first author is supported by NSFC (12101453).

Conflict of interest

The authors declared that they have no conflicts of interest to this work.

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