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Research article

Novel Noor iterations technique for solving nonlinear equations

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Abstract: The aim of this paper is to propose a novel Noor iteration technique, called the CT-iteration for approximating a fixed point of continuous functions on closed interval. Then, a necessary and sufficient condition for the convergence of the CT-iteration of continuous functions on closed interval is established. We also compare the rate of convergence between the proposed iteration and some other iteration processes in the literature. Specifically, our main result shows that CT-iteration converges faster than CP-iteration to the fixed point. We finally give numerical examples to compare the result with Mann, Ishikawa, Noor, SP and CP iterations. Our findings improve corresponding results in the contemporary literature.

Keywords: convergence theorem; convergence rate; continuous function; fixed point; closed interval
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1. Introduction

Fixed point theory takes a large amount of literature, since it provides useful tools to solve many problems that have applications in different fields like engineering, economics, chemistry and game theory etc. Iterative methods are popular tools to approximate fixed points of nonlinear mappings. In computational mathematics, it is of vital interest to know which of the given iterative procedures converge faster to a desired solution, commonly known as the rate of convergence. Thus, when studying an iterative procedure, we should consider two criteria which are the faster and the simplify. In this direction, some of notable studies were conducted by Mann, Ishikawa, Noor, Phuengrattana and Suantai, Cholamjiak and Pholasa (see [1–5]). In addition, the fixed point mappings were studied as much as studies on the iterative methods. Different varieties of these mappings are available in the literature. The well known of them, are contraction mappings, nonexpansive mappings and Lipschitzian mappings, and these are the continuous ones. Therefore, in this study, we handle the general mapping which is a class of continuous mapping.

Let C be a closed interval on the real line and let $f : C \to C$ be a continuous function. A point

$p \in C$ is called a *fixed point* of f if $f(p) = p$.

Now, we will consider some of these schemes related to this work. Mann [1] introduced Mann iteration, which generates a sequence $\{u_n\}$ as follows :

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n f(u_n) \tag{1.1}$$

for all $n \geq 1$, where $\alpha_n \in [0, 1]$. Such an iteration process is known as *Mann iteration.* In 1991, Borwein and Borwein [6] proved the convergence theorem for a continuous function on the closed and bounded interval in the real line by using iteration (1.1).

Another classical iteration process was introduced by Ishikawa [2] which is formulated as follows:

$$\begin{aligned} t_n &= (1 - \beta_n)s_n + \beta_n f(s_n), \\ s_{n+1} &= (1 - \alpha_n)s_n + \alpha_n f(t_n) \end{aligned} \tag{1.2}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1]. Such iterative method is called *Ishikawa iteration.* In 2006, Qing and Qihou [7] proved the convergence theorem of the sequence generated by iteration (1.2) for a continuous function on the closed interval in the real line (see also [8]).

In 2000, Noor [3] defined the following iterative scheme by $l_1 \in C$ and

$$\begin{aligned} m_n &= (1 - \mu_n)l_n + \mu_n f(l_n), \\ v_n &= (1 - \beta_n)l_n + \beta_n f(m_n), \\ l_{n+1} &= (1 - \alpha_n)l_n + \alpha_n f(v_n) \end{aligned} \tag{1.3}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\mu_n\}$ are sequences in [0,1], which is called *Noor iteration* [3] for continuous functions on an arbitrary interval in the real line. Clearly, the Mann and Ishikawa iteration processes are special cases of the Noor iteration process. Because of its simplicity, the method (1.3) has been widely utilized to solve the fixed point problem, and as a result, it has been enhanced by many works, as seen in [9–12].

In 1976, Rhoades [13] proved the convergence of the Mann and Ishikawa iterations for the class of continuous and nondecreasing functions on unit closed interval. After that in 1991, Borwein and Borwein [6] obtained the convergence result to Mann iteration for continuous functions on a bounded closed interval. Qing and Qihou [7] extended results in [6] to an arbitrary interval and to Ishikawa iteration and presented a necessary and sufficient condition for the convergence of Ishikawa iteration of continuous functions on an arbitrary interval (see also [8]). There are many articles have been published on the iterative methods using for approximation of fixed points of nonlinear mappings, see for instance [1–3, 6–8, 13]. However, there are only a few articles concerning comparison of those iterative methods in order to establish which one converges faster. As far as we know, there are two ways for comparison of the rate of convergence. The first one was introduced by Berinde [14]. He used this idea to compare the rate of convergence of Picard and Mann iterations for a class of Zamfirescu operators in arbitrary Banach spaces. Popescu [15] also used this concept to compare the rate of convergence of Picard and Mann iterations for a class of quasi-contractive operators. It was shown in [16] that the Mann and Ishikawa iterations are equivalent for the class of Zamfirescu operators. In 2006, Babu and Prasad [17] showed that the Mann iteration converges faster than the Ishikawa iteration for this class of operators. Two years later, Qing and Rhoades [18] provided an example to show that the claim of Babu and Prasad [17] is false. However, this concept is not suitable or cannot be applied to a class of continuous self-mappings defined on a closed interval. In order to compare the rate of convergence of continuous self-mappings defined on a closed interval, Rhoades [13] introduced the other concept which is slightly different from that of Berinde to compare iterative methods which one converges faster as follows.

Definition 1. *([13]) Let C be a closed interval on the real line and let $f : C \to C$ be a continuous mapping. Suppose that $\{x_n\}$ and $\{w_n\}$ are two iterations which converge to the fixed point p of f. Then $\{x_n\}$ is said to converge faster than $\{w_n\}$ if*

$$|x_n - p| \leq |w_n - p|$$

for all $n \geq 1$.

Phuengrattana and Suantai [4] introduced and studied the SP-iteration as follows: $h_1 \in C$ and

$$\begin{aligned} e_n &= (1-\mu_n)h_n + \mu_n f(h_n), \\ d_n &= (1-\beta_n)e_n + \beta_n f(e_n), \\ h_{n+1} &= (1-\alpha_n)d_n + \alpha_n f(d_n) \end{aligned} \tag{1.4}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\mu_n\}$ are sequences in [0,1]. They showed that (1.4) converges to a fixed point of f. Moreover, the rate of convergence is better than those of Mann (1.1), Ishikawa (1.2) and Noor (1.3) in the sense of Rhoades [13].

Clearly Mann iteration is special cases of SP-iteration. Some interesting results concerning fixed point theory of continuous functions can be found in [19].

Recently, by combining the SP-iteration and Noor iteration, Cholamjiak and Pholasa [5] proposed the CP-iteration as follows: $w_1 \in C$ and

$$\begin{aligned} r_n &= (1-\mu_n)w_n + \mu_n f(w_n), \\ q_n &= (1-\tau_n-\beta_n)w_n + \tau_n r_n + \beta_n f(r_n), \\ w_{n+1} &= (1-\gamma_n-\alpha_n)r_n + \gamma_n q_n + \alpha_n f(q_n) \end{aligned} \tag{1.5}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\tau_n\}$ and $\{\gamma_n\}$ are sequences in [0, 1]. They proved some convergence theorems of such iterations for continuous functions on an arbitrary interval. Also, they compared the rate of convergence of Mann, Ishikawa, Noor and CP iterations by numerical examples and concluded that CP-iteration converges faster than all of them.

Inspired and motivated by these facts, we introduce and study a new accelerated iteration process for solving a fixed point problem for continuous function on an arbitrary interval in the real line. The scheme is defined as follows.

Let C be a closed interval on the real line and $f : C \to C$ given mapping. Then for an arbitrary $x_1 \in C$, the following iteration scheme is studied:

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\tau_n-\beta_n)x_n + \tau_n f(x_n) + \beta_n f(z_n), \\ x_{n+1} &= (1-\gamma_n-\alpha_n)z_n + \gamma_n f(z_n) + \alpha_n f(y_n),\ n \geq 1, \end{aligned} \tag{1.6}$$

where, $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ are appropriate real sequences in $[0, 1]$. The iterative scheme (1.6) is called the CT-iteration for continuous functions.

The first purpose of this article is to give a necessary and sufficient condition for the strong convergence of the CT-iteration of continuous functions on an arbitrary interval. The second purpose is to improve the rate of convergence compared to previous work. Specifically, our main result shows that CT-iteration converges faster than CP-iteration to the fixed point. Numerical examples are also presented to compare the result with Mann, Ishikawa, Noor, SP and CP iterations. Consequently, we have that CT-iteration converges faster than the other schemes in the same category.

2. Convergence theorem

In this section, we provide the convergence theorem of CT-iteration (1.6) for continuous functions on an arbitrary closed interval. Now, we will give some crucial lemmas for proofs of our main results.

Lemma 1. *Let C be a closed interval on the real line (can be unbounded) and let $f : C \to C$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \beta_n < \infty, \sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$. From an arbitrary initial guess $x_1 \in C$, define the sequence $\{x_n\}$ using (1.6). If $x_n \to a$, then a is a fixed point of f.*

Proof. Let $x_n \to a$, and suppose $a \neq f(a)$. Then $\{x_n\}$ is bounded. So, $\{f(x_n)\}$ is bounded by the continuity of f. So are $\{y_n\}, \{z_n\}, \{f(y_n)\}$ and $\{f(z_n)\}$. Moreover, $z_n \to a$ since $x_n \to a$ and $\mu_n \to 0$. We also have $y_n \to a$ since $x_n \to a, \beta_n \to 0$ and $\tau_n \to 0$. From (1.6), we get

$$\begin{aligned} x_{n+1} &= (1 - \gamma_n - \alpha_n)z_n + \gamma_n f(z_n) + \alpha_n f(y_n) \\ &= z_n + \gamma_n(f(z_n) - z_n) + \alpha_n(f(y_n) - z_n). \end{aligned} \tag{2.1}$$

Let $p_k = f(z_k) - z_k, q_k = f(y_k) - z_k$. Then, we have

$$\begin{aligned} \lim_{k\to\infty} p_k &= \lim_{k\to\infty}(f(z_k) - z_k) = f(a) - a \neq 0, \\ \lim_{k\to\infty} q_k &= \lim_{k\to\infty}(f(y_k) - z_k) = f(a) - a \neq 0. \end{aligned}$$

From (2.1) we get

$$\begin{aligned} x_n &= z_1 + \sum_{k=1}^{n} \gamma_k(f(z_k) - z_k) + \sum_{k=1}^{n} \alpha_k(f(y_k) - z_k) \\ &= z_1 + \sum_{k=1}^{n} \gamma_k p_k + \sum_{k=1}^{n} \alpha_k q_k. \end{aligned}$$

It is worth noting here that $\sum_{k=1}^{\infty} \gamma_k p_k < \infty$ since $\lim_{k\to\infty} p_k \neq 0$ and $\sum_{k=1}^{\infty} \gamma_k < \infty$. This shows that $\{x_n\}$ is a divergent sequence since $\lim_{k\to\infty} q_k \neq 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$. This contradicts to the convergence of $\{x_n\}$. Hence $f(a) = a$ and a is fixed point of f. □

Lemma 2. *Let C be a closed interval on the real line (can be unbounded) and let $f : C \to C$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \beta_n < \infty, \sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$. From an arbitrary initial guess $x_1 \in C$, define the sequence $\{x_n\}$ using (1.6). If $\{x_n\}$ is bounded, then $\{x_n\}$ is convergent.*

Proof. Suppose $\{x_n\}$ is not convergent. Let $a = \liminf_n x_n$ and $b = \limsup_n x_n$. Then $a < b$. We first show that if $a < m < b$, then $f(m) = m$. Suppose $f(m) \neq m$. Without loss of generality, we suppose $f(m) - m > 0$. Since f is continuous, there exists δ with $0 < \delta < b - a$ such that for $|x - m| \leq \delta$, $f(x) - x > 0$. By continuity of f and $\{x_n\}$ is bounded we have that $\{f(x_n)\}$ is bounded, so $\{z_n\}, \{y_n\}$, $\{f(z_n)\}$ and $\{f(y_n)\}$ are bounded sequences. Using

$$\begin{aligned}
x_{n+1} - x_n &= (1 - \gamma_n - \alpha_n)(z_n - x_n) + \gamma_n(f(z_n) - x_n) + \alpha_n(f(y_n) - x_n), \\
y_n - x_n &= \tau_n(f(x_n) - x_n) + \beta_n(f(z_n) - x_n), \\
z_n - x_n &= \mu_n(f(x_n) - x_n),
\end{aligned}$$

we can easily show that $|z_n - x_n| \to 0$, $|y_n - x_n| \to 0$ and $|x_{n+1} - x_n| \to 0$. Thus, there exists a positive integer N such that

$$|x_{n+1} - x_n| < \frac{\delta}{2},\ |y_n - x_n| < \frac{\delta}{2},\ |z_n - x_n| < \frac{\delta}{2}, \forall n > N. \tag{2.2}$$

Since $b = \limsup_n x_n > m$, there exists $k_1 > N$ such that $x_{n_{k_1}} > m$. Let $n_{k_1} = k$, then $x_k > m$. For x_k, there exist two cases as follows:

(i) $x_k > m + \frac{\delta}{2}$, then $x_{k+1} > x_k - \frac{\delta}{2} \geq m$ using (2.2). So, we have $x_{k+1} > m$.

(ii) $m < x_k < m + \frac{\delta}{2}$, then $m - \frac{\delta}{2} < y_k < m + \delta$ and $m - \frac{\delta}{2} < z_k < m + \delta$ by (2.2). So, we obtain $|x_k - m| < \frac{\delta}{2} < \delta, |y_k - m| < \delta, |z_k - m| < \delta$. Hence

$$f(x_k) - x_k > 0,\ f(y_k) - y_k > 0,\ f(z_k) - z_k > 0. \tag{2.3}$$

In addition,

$$\begin{aligned}
y_k - z_k &= (1 - \tau_k - \beta_k)(x_k - z_k) + \tau_k(f(x_k) - z_k) \\
&\quad + \beta_k(f(z_k) - z_k) \\
&= (1 - \tau_k - \beta_k)(x_k - z_k) + \tau_k(f(x_k) - x_k) \\
&\quad + \tau_k(x_k - z_k) + \beta_k(f(z_k) - z_k) \\
&= (1 - \beta_k)(x_k - z_k) + \tau_k(f(x_k) - x_k) \\
&\quad + \beta_k(f(z_k) - z_k) \\
&= (1 - \beta_k)\mu_k(x_k - f(x_k)) + \tau_k(f(x_k) - x_k) \\
&\quad + \beta_k(f(z_k) - z_k).
\end{aligned} \tag{2.4}$$

From (2.1), (2.3) and (2.4), we have

$$\begin{aligned}
x_{k+1} &= z_k + \gamma_k(f(z_k) - z_k) + \alpha_k(f(y_k) - z_k) \\
&= x_k + \mu_k(f(x_k) - x_k) + \gamma_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k) + \alpha_k(y_k - z_k)
\end{aligned}$$

$$
\begin{aligned}
&= x_k + \mu_k(f(x_k) - x_k) + \gamma_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k) \\
&\quad +\alpha_k(-(1-\beta_k)\mu_k(f(x_k) - x_k) + \tau_k(f(x_k) - x_k) + \beta_k(f(z_k) - z_k)) \\
&= x_k + \mu_k(f(x_k) - x_k) + \gamma_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k) \\
&\quad -\alpha_k(1-\beta_k)\mu_k(f(x_k) - x_k) + \alpha_k\tau_k(f(x_k) - x_k) + \alpha_k\beta_k(f(z_k) - z_k) \\
&= x_k + \mu_k(1-\alpha_k(1-\beta_k))(f(x_k) - x_k) + \gamma_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k) \\
&\quad +\alpha_k\tau_k(f(x_k) - x_k) + \alpha_k\beta_k(f(z_k) - z_k) \\
&= x_k + \mu_k(1-\alpha_k + \alpha_k\beta_k)(f(x_k) - x_k) + \gamma_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k) \\
&\quad +\alpha_k\tau_k(f(x_k) - x_k) + \alpha_k\beta_k(f(z_k) - z_k) \\
&> x_k.
\end{aligned}
$$

Thus $x_{k+1} > x_k > m$. This together with (i) and (ii), imply $x_{k+1} > m$. Similarly, we get that $x_{k+2} > m$, $x_{k+3} > m, \ldots$. Thus we have $x_n > m$ for all $n > k = n_{k_1}$. So $a = \lim_{k\to\infty} x_{n_k} \geq m$, which is a contradiction with $a < m$. Thus $f(m) = m$.

We next consider the following two cases.

(i) There exists x_M such that $a < x_M < b$. Then $f(x_M) = x_M$. It follows that

$$z_M = (1-\mu_M)x_M + \mu_M f(x_M) = x_M$$

and

$$
\begin{aligned}
y_M &= (1-\tau_M-\beta_M)z_M + \tau_M f(x_M) + \beta_M f(z_M) \\
&= (1-\tau_M-\beta_M)x_M + \tau_M f(x_M) + \beta_M f(x_M) \\
&= x_M.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
x_{M+1} &= (1-\gamma_M-\alpha_M)z_M + \gamma_M f(z_M) + \alpha_M f(y_M) \\
&= (1-\tau_M-\gamma_M)x_M + \gamma_M f(x_M) + \alpha_M f(x_M) \\
&= x_M.
\end{aligned}
$$

Similarly, we obtain $x_M = x_{M+1} = x_{M+2} = \ldots$. It clear that $x_n \to x_M$. Since there exists $x_{n_k} \to a, x_M = a$. This shows that $x_n \to a$, which is a contradiction.

(ii) For all $n, x_n \leq a$ or $x_n \geq b$. Since $b - a > 0$ and $\lim\limits_{n\to\infty} |x_{n+1} - x_n| = 0$, there exists $\tilde{N}$ such that $|x_{n+1} - x_n| < \frac{(b-a)}{2}$ for $n > \tilde{N}$. So, it is seen that $x_n \leq a$ for $n > \tilde{N}$, or it is always that $x_n \geq b$ for $n > \tilde{N}$. If $x_n \leq a$ for $n > \tilde{N}$, then $b = \lim\limits_{j\to\infty} x_{n_j} \leq a$, which is a contradiction with $a < b$. If $x_n \geq b$ for $n > \tilde{N}$, then $a = \lim\limits_{k\to\infty} x_{n_k} \geq b$, which is a contradiction with $a < b$. Thus we conclude that $x_n \to a$. The proof is completed. □

We are now ready to prove the main theorem.

Theorem 1. *Let C be a closed interval on the real line (can be unbounded) and let $f : C \to C$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1]$ such that*

$\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \beta_n < \infty, \sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \tau_n < \infty$. *From an arbitrary initial guess* $x_1 \in C$, *define the sequence* $\{x_n\}$ *using (1.6). Then* $\{x_n\}$ *is bounded if and only if it converges to a fixed point of* f .

Proof. Sufficiency is obvious. It suffices to show that if $\{x_n\}$ is bounded, then $\{x_n\}$ converges to a fixed point. Let $\{x_n\}$ be a bounded sequence. Using Lemma 2, we have $\{x_n\}$ is a convergent sequence. Hence, by Lemma 1, it converges to a fixed point of f. □

When $C = [a, b]$ in Theorem 1, we obtain the following result.

Corollary 1. *Let* $f : [a, b] \to [a, b]$ *be a continuous function. Let* $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ *and* $\{\tau_n\}$ *be sequences in* $[0, 1]$. *Let* $\{x_n\}$ *be a sequence generated iteratively by* $x_1 \in [a, b]$ *and*

$$\begin{aligned} z_n &= (1 - \mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1 - \tau_n - \beta_n)x_n + \tau_n f(x_n) + \beta_n f(z_n), \\ x_{n+1} &= (1 - \gamma_n - \alpha_n)z_n + \gamma_n f(z_n) + \alpha_n f(y_n),\ n \geq 1, \end{aligned}$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \beta_n < \infty, \sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \tau_n < \infty$.
Then $\{x_n\}$ *converges to a fixed point of* f.

3. Rate of convergence

In this section, we compare the convergence rate of (1.6) with the CP-iteration proposed in [5]. We show that the CT-iteration (1.6) converges faster than the CP-iteration (1.5) for the class of continuous nondecreasing functions on an arbitrary interval in the sense of Rhoades [13].

We next prove some crucial lemmas which will be used in the sequel.

Lemma 3. *Let* C *be a closed interval on the real line and let* $f : C \to C$ *be a continuous and nondecreasing function. Let* $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ *and* $\{\tau_n\}$ *be sequences in* $[0, 1)$. *Let* $\{w_n\}$ *and* $\{x_n\}$ *be sequences defined by (1.5) and (1.6), respectively. Then the following hold:*
(i) If $f(w_1) < w_1$, *then* $f(w_n) < w_n$ *for all* $n \geq 1$ *and* $\{w_n\}$ *is nonincreasing.*
(ii) If $f(w_1) > w_1$, *then* $f(w_n) > w_n$ *for all* $n \geq 1$ *and* $\{w_n\}$ *is nondecreasing.*
(iii) If $f(x_1) < x_1$, *then* $f(x_n) < x_n$ *for all* $n \geq 1$ *and* $\{x_n\}$ *is nonincreasing.*
(iv) If $f(x_1) > x_1$, *then* $f(x_n) > x_n$ *for all* $n \geq 1$ *and* $\{x_n\}$ *is nondecreasing.*

Proof. (i) Let $f(w_1) < w_1$. Then $f(w_1) < r_1 \leq w_1$. Since f is nondecreasing, we have $f(r_1) \leq f(w_1) < r_1 \leq w_1$. This implies $f(r_1) < q_1 \leq w_1$. Thus $f(q_1) \leq f(w_1) < r_1 \leq w_1$. For q_1, we consider the following two cases.

Case 1: $f(r_1) < q_1 \leq r_1$. Then $f(q_1) \leq f(r_1) < q_1 \leq r_1 \leq w_1$. This implies $f(q_1) < w_2 \leq w_1$. Thus $f(w_2) \leq f(w_1) < r_1 \leq w_1$. It follows that if $f(q_1) < w_2 \leq q_1$, then $f(w_2) \leq f(q_1) < w_2$, if $q_1 < w_2 \leq r_1$, then $f(w_2) \leq f(r_1) < q_1 < w_2$ and if $r_1 < w_2 \leq w_1$, then $f(w_2) \leq f(w_1) < r_1 < w_2$. Thus we have $f(w_2) < w_2$.

Case 2: $r_1 < q_1 \leq w_1$. Then $f(q_1) \leq f(w_1) < r_1 \leq w_1$. This implies $f(q_1) < w_2 \leq w_1$. Thus $f(w_2) \leq f(w_1) < r_1 < q_1 \leq w_1$. It follows that if $f(q_1) < w_2 \leq q_1$, then $f(w_2) \leq f(q_1) < w_2$ and if $q_1 < w_2 \leq w_1$, then $f(w_2) \leq f(w_1) < q_1 < w_2$. Hence, we have $f(w_2) < w_2$.

In conclusion by Case 1 and Case 2, we have $f(w_2) < w_2$. By continuing in this way, we can show that $f(w_n) < w_n$ for all $n \geq 1$. This implies $r_n \leq w_n$ for all $n \geq 1$. Since f is nondecreasing, we have $f(r_n) \leq f(w_n) < w_n$ for all $n \geq 1$. Thus $q_n \leq w_n$ for all $n \geq 1$, then $f(q_n) \leq f(w_n) < w_n$ for all $n \geq 1$. Hence, we have $w_{n+1} \leq w_n$ for all $n \geq 1$, that is $\{w_n\}$ is nonincreasing.

(ii) By using the same argument as in (i), we obtain the desired result.

(iii) Let $f(x_1) < x_1$. Then $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we have $f(z_1) \leq f(x_1) < z_1 \leq x_1$. This implies $f(z_1) < y_1 \leq x_1$. Thus $f(y_1) \leq f(x_1) < z_1 \leq x_1$. For y_1, we consider the following two cases.

Case 1: $f(z_1) < y_1 \leq z_1$. Then $f(y_1) \leq f(z_1) < z_1 < x_1$. It follows that if $f(y_1) < x_2 \leq y_1$, then $f(x_2) \leq f(y_1) < x_2$, if $y_1 < x_2 \leq z_1$, then $f(x_2) \leq f(z_1) < y_1 < x_2$ and if $z_1 < x_2 \leq x_1$, then $f(x_2) \leq f(x_1) < z_1 < x_2$. Thus we have $f(x_2) < x_2$.

Case 2: $z_1 < y_1 \leq x_1$. Then $f(y_1) \leq f(x_1) < z_1 \leq x_1$. This implies $f(y_1) < x_2 \leq x_1$. Thus $f(x_2) \leq f(x_1) < z_1 < y_1 \leq x_1$. It follows that if $f(y_1) < x_2 \leq y_1$, then $f(x_2) \leq f(y_1) < x_2$ and if $y_1 < x_2 \leq x_1$, then $f(x_2) \leq f(x_1) < y_1 < x_2$. Hence, we have $f(x_2) < x_2$.

In conclusion by Case 1 and Case 2, we have $f(x_2) < x_2$. By continuing in this way, we can show that $f(x_n) < x_n$ for all $n \geq 1$. This implies $z_n \leq x_n$ for all $n \geq 1$. Since f is nondecreasing, we have $f(z_n) \leq f(x_n) < x_n$ for all $n \geq 1$. Thus $y_n \leq x_n$ for all $n \geq 1$, then $f(y_n) \leq f(x_n) < x_n$ for all $n \geq 1$. Hence, we have $x_{n+1} \leq x_n$ for all $n \geq 1$, that is $\{x_n\}$ is nonincreasing.

(iv) Following the proof line as in (iii), we obtain the desired result. □

Lemma 4. *Let C be a closed interval on the real line and let $f : C \to C$ be a continuous and nondecreasing function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1)$. For $w_1 = x_1 \in C$, let $\{w_n\}$ and $\{x_n\}$ be sequences defined by the CP-iteration (1.5) and CT-iteration (1.6), respectively. Then the following are satisfied:*

(i) If $f(w_1) < w_1$, then $x_n \leq w_n$ for all $n \geq 1$.

(ii) If $f(w_1) > w_1$, then $x_n \geq w_n$ for all $n \geq 1$.

Proof. (i) Let $f(w_1) < w_1$. Then $f(x_1) < x_1$ since $w_1 = x_1$. From (1.6), we get $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we obtain $f(z_1) \leq f(x_1) < z_1 \leq x_1$. Hence $f(z_1) < y_1 \leq z_1$. Using the CP-iteration (1.5) and CT-iteration (1.6), we obtain the following estimation:

$$z_1 - r_1 = (1 - \mu_1)(x_1 - w_1) + \mu_1(f(x_1) - f(w_1)) = 0.$$

So, $z_1 = r_1$, and so

$$y_1 - q_1 = (1 - \tau_1 - \beta_1)(x_1 - w_1) + \tau_1(f(x_1) - r_1) + \beta_1(f(z_1) - f(r_1)) \leq 0.$$

Hence, we have $y_1 \leq q_1$. Since f is nondecreasing, we have $f(y_1) \leq f(q_1)$. We next obtain

$$x_2 - w_2 = (1 - \gamma_1 - \alpha_1)(z_1 - r_1) + \gamma_1(f(z_1) - q_1) + \alpha_1(f(y_1) - f(q_1)) \leq 0,$$

so, $x_2 \leq w_2$. Assume that $x_k \leq w_k$. Thus $f(x_k) \leq f(w_k)$. From Lemma 3 (i) and Lemma 3 (iii), we get $f(w_k) < w_k$ and $f(x_k) < x_k$. It follows that $f(x_k) < z_k \leq x_k$ and $f(z_k) \leq f(x_k) < z_k$. Thus

$$z_k - r_k = (1 - \mu_k)(x_k - w_k) + \mu_k(f(x_k) - f(w_k)) \leq 0.$$

So, $z_k \leq r_k$. Since $f(z_k) \leq f(r_k)$, we have

$$y_k - q_k = (1 - \tau_k - \beta_k)(x_k - w_k) + \tau_k(f(x_k) - r_k) + \beta_k(f(z_k) - f(r_k)) \leq 0,$$

so, $y_k \leq q_k$, which yields $f(y_k) \leq f(q_k)$. In addition, $f(z_k) \leq f(x_k) < z_k \leq x_k$, using (1.6), we have

$f(z_k) - y_k = (1 - \tau_k - \beta_k)(f(z_k) - x_k) + \tau_k(f(z_k) - f(x_k)) + \beta_k(f(z_k) - f(z_k)) \leq 0.$
So, $f(z_k) - q_k = (f(z_k) - y_k) + (y_k - q_k) \leq 0.$
This shows that

$x_{k+1} - w_{k+1} = (1 - \gamma_k - \alpha_k)(z_k - r_k) + \gamma_k(f(z_k) - q_k) + \alpha_k(f(y_k) - f(q_k)) \leq 0,$

which gives, $x_{k+1} \leq w_{k+1}$. By induction, we conclude that $x_n \leq w_n$ for all $n \geq 1$.

(ii) From Lemma 3 (ii), Lemma 3 (iv) and the same argument as in (i), we can show that $x_n \geq w_n$ for all $n \geq 1$. □

For convenience, we write algorithm (1.6) by $CT(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$.

Proposition 1. *Let C be a closed interval on the real line and let $f : C \to C$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 > \sup\{p \in C : p = f(p)\}$. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1)$. If $f(x_1) > x_1$, then $\{x_n\}$ defined by $CP(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ and $CT(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ do not converge to a fixed point of f.*

Proof. From Lemma 3 ((ii), (iv)), we know that $\{x_n\}$ is nondecreasing. Since the initial point $x_1 > \sup\{p \in C : p = f(p)\}$, it follows that $\{x_n\}$ does not converge to a fixed point of f. □

Proposition 2. *Let C be a closed interval on the real line and let $f : C \to C$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 < \inf\{p \in C : p = f(p)\}$. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1)$. If $f(x_1) < x_1$, then $\{x_n\}$ defined by $CP(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ and $CT(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ do not converge to a fixed point of f.*

Proof. From Lemma 3 ((i), (iii)), we know that $\{x_n\}$ is nonincreasing. Since the initial point $x_1 < \inf\{p \in C : p = f(p)\}$, it follows that $\{x_n\}$ does not converge to a fixed point of f. □

Next, we compare the rate of convergence of CT-iteration with CP-iteration.

Theorem 2. *Let C be a closed interval on the real line and let $f : C \to C$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1)$. For $w_1 = x_1 \in C$, let $\{w_n\}$ and $\{x_n\}$ be sequences defined by the CP-iteration (1.5) and the CT-iteration (1.6), respectively. If the CP-iteration $\{w_n\}$ converges to $p \in F(f)$, then the CT-iteration $\{x_n\}$ converges to p. Moreover, the CT-iteration (1.6) converges faster than the CP-iteration (1.5).*

Proof. Assume that the CP-iteration $\{w_n\}$ converges to $p \in F(f)$. Put $L = \inf\{p \in C : p = f(p)\}$ and $U = \sup\{p \in C : p = f(p)\}$. For $w_1 = x_1$, we devide our proof into the following three cases:

Case 1: $w_1 = x_1 > U$, Case 2: $w_1 = x_1 < L$, Case 3: $L \leq w_1 = x_1 \leq U$.

Case 1: $w_1 = x_1 > U$. By Proposition 1, we get $f(w_1) < w_1$ and $f(x_1) < x_1$. So, by Lemma 4 (i), we have $x_n \leq w_n$ for all $n \geq 1$. By induction, we can show that $U \leq x_n$ for all $n \geq 1$. Then, we have $0 \leq x_n - p \leq w_n - p$, which yields $|x_n - p| \leq |w_n - p|$ for all $n \geq 1$. This shows that $x_n \to p$. By Definition 1, we conclude that the CT-iteration $\{x_n\}$ converges faster than the CP-iteration $\{w_n\}$.

Case 2: $w_1 = x_1 < L$. By Proposition 2, we get $f(w_1) > w_1$ and $f(x_1) > x_1$. This implies, by Lemma 4 (ii), that $x_n \geq w_n$ for all $n \geq 1$. So, by induction, we can show that $x_n \leq L$ for all $n \geq 1$. Then, we have $|x_n - p| \leq |w_n - p|$ for all $n \geq 1$. It follows that $x_n \to p$ and the CT-iteration $\{x_n\}$ converges faster than the CP-iteration $\{w_n\}$.

Case 3: $L \leq w_1 = x_1 \leq U$. Suppose that $f(w_1) \neq w_1$. If $f(w_1) < w_1$, we have, by Lemma 3 (i), that $\{w_n\}$ is nonincreasing with limit p. Lemma 4 (i) gives $p \leq x_n \leq w_n$ for all $n \geq 1$. It follows that $|x_n - p| \leq |w_n - p|$ for all $n \geq 1$. Therefore $x_n \to p$ and the result follows. If $f(w_1) > w_1$, by Lemma 3 (ii) and Lemma 4 (ii), then we can also show that the result holds. □

4. Numerical examples

In this section, some numerical examples are given to demonstrate the convergence of the algorithm defined in this paper. For convenience, we call the iteration (1.6) the CT-iteration.

Example 1. $f : [-1, 4] \to [-1, 4]$ *defined by* $f(x) = \frac{x^3+x-3}{19}$. *The fixed point of the function is* $p = -0.166925066$. *Initial point is* $x_1 = 4$ *and control conditions are* $\alpha_n = \frac{1}{(n+1)^{0.5}}$, $\beta_n = \frac{1}{(n+1)^{1.7}}$, $\mu_n = \frac{1}{(n+1)^{2.3}}$, $\gamma_n = \frac{1}{(n+1)^{1.5}}$ *and* $\tau_n = \frac{1}{(n+1)^{1.1}}$. *The stopping criteria is* $|x_n - p| < 10^{-8}$.

Example 2. $f : [1, \infty] \to [1, \infty]$ *defined by* $f(x) = x^{0.3} - (\sqrt{log(x+9)-1})^3$. *The fixed point of the function is* $p = 1$. *Initial point is* $x_1 = 9$ *and control conditions are* $\alpha_n = \frac{1}{(n+1)^{0.5}}$, $\beta_n = \frac{1}{(n+1)^{2.0}}$, $\mu_n = \frac{1}{(n+1)^{3.6}}$, $\gamma_n = \frac{1}{(n+1)^{2.5}}$ *and* $\tau_n = \frac{1}{(n+1)^{1.1}}$. *The stopping criteria is* $|x_n - p| < 10^{-6}$.

Tables 2 and 4 confirm that the proposed method performs favorably with rapid convergence and Tables 1, 3, Figures 1, 2, 3 and 4 show the behavior of six comparative methods consisting of Mann iteration, Ishikawa iteration, Noor iteration, CP-iteration, SP-iteration and CT-iteration in converging to the fixed point of the numerical experiments. The results of the both examples indicates that the CT-iteration converges faster than the other methods. Even though the initial points are differently selected as shown on Figures 2 and 4, the convergence of CT-iteration still be better than other methods. The effect of initial point being close to or far from p is not observed from these examples. The control sequnces $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ of the examples are chosen to satisfy on conditions with Corollary 1. The option of sequences is flexible for user application. However, the optimal chioce of them is an open problem to investigate.

Table 1. Mann, Ishikawa, Noor, CP, SP and CT iterations for $f(x) = \frac{x^3+x-3}{19}$.

n	Mann	Ishikawa	Noor	CP	SP	CT-iteration	
	u_n	s_n	l_n	w_n	h_n	x_n	$\|x_n - p\|$
1	4	4	4	4	4	4	4.1669251
5	1.3932393	0.7533402	0.6286365	0.451696663	0.2932282	0.0012006	0.1681257
10	0.0461983	-0.0497313	-0.0662004	-0.090005411	-0.1191936	-0.1529447	0.0139804
15	-0.1207432	-0.1415242	-0.1450865	-0.150271594	-0.1573078	-0.1643412	0.0025839
20	-0.1538461	-0.1597323	-0.1607405	-0.162211234	-0.1643099	-0.1662565	0.0006686
25	-0.1625577	-0.1645237	-0.1648603	-0.165351661	-0.1660745	-0.1667147	0.0002104
30	-0.1652901	-0.1660262	-0.1661522	-0.166336214	0.1666125	-0.1668496	0.0000755
35	-0.1662585	-0.1665587	-0.1666100	-0.166685055	-0.1667994	-0.1668953	0.0000298
No. of iterations	133	126	124	e119	113	97	

Table 2. The sequences generated by CT-iteration for given x_1= -0.3, -0.2, -0.1, 0, 0.1, 1, 2 and 3 in Example 1.

n	$\|x_n - p\|$							
	Initial points were close to p					Initial points were far from p		
	x_1= -0.3	x_1= -0.2	x_1= -0.1	x_1= 0	x_1= 0.1	x_1= 1	x_1= 2	x_1= 3
1	0.1330749	0.0330749	0.0669251	0.1669251	0.2669251	1.1669251	2.1669251	3.1669251
5	0.0001234	3.21×10^{-5}	6.72×10^{-5}	0.0001716	0.0002780	0.0008025	0.0023414	0.0209243
10	1.03×10^{-5}	2.71×10^{-6}	5.58×10^{-6}	1.43×10^{-5}	2.31×10^{-5}	6.70×10^{-5}	0.0001956	0.0017466
15	1.93×10^{-6}	5.29×10^{-7}	1.01×10^{-6}	2.61×10^{-6}	4.26×10^{-6}	1.23×10^{-5}	3.61×10^{-5}	0.0003229
20	5.26×10^{-7}	1.61×10^{-7}	2.35×10^{-7}	6.52×10^{-7}	1.07×10^{-6}	3.17×10^{-6}	9.38×10^{-6}	8.35×10^{-5}
25	1.88×10^{-7}	7.39×10^{-8}	5.10×10^{-8}	1.82×10^{-7}	3.16×10^{-7}	9.75×10^{-7}	2.97×10^{-6}	2.63×10^{-5}
30	8.91×10^{-8}	4.80×10^{-8}	3.21×10^{-9}	4.38×10^{-8}	9.18×10^{-8}	3.28×10^{-7}	1.08×10^{-6}	9.45×10^{-6}
35	5.54×10^{-8}	1.43×10^{-8}	2.15×10^{-8}	3.00×10^{-9}	1.59×10^{-8}	1.09×10^{-7}	4.49×10^{-7}	3.75×10^{-6}

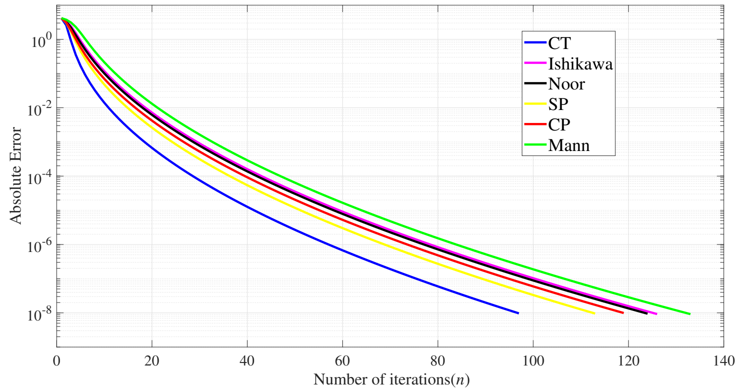

Figure 1. Error values obtained from CT, Ishikawa, Noor, SP, CP and Mann iterations for $f(x) = \frac{x^3+x-3}{19}$.

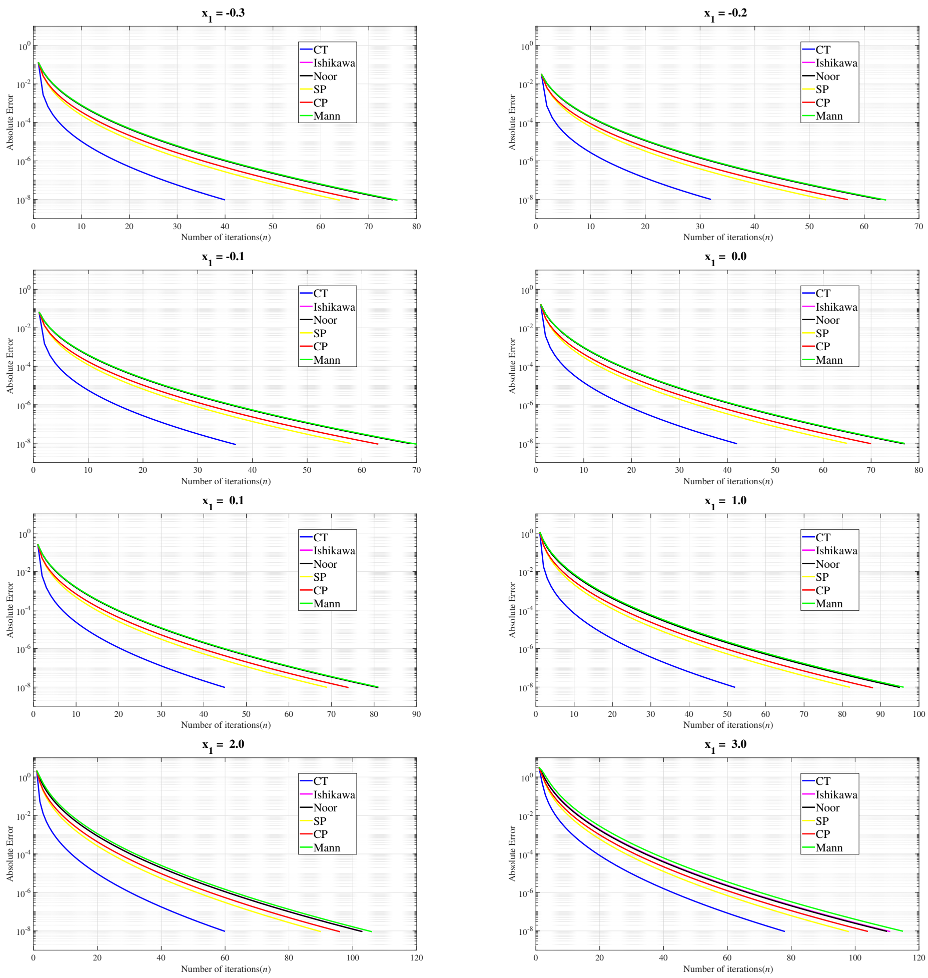

Figure 2. Convergence behaviors for given x_1= -0.3, -0.2, -0.1,0, 0.1, 1, 2 and 3 in Example 1.

Table 3. Mann, Ishikawa, Noor, CP, SP and CT iterations for $f(x) = x^{0.3} - (\sqrt{log(x+9) - 1})^3$ and $x_1 = 9$.

n	Mann	Ishikawa	Noor	CP	SP	CT-iteration	
	u_n	s_n	l_n	w_n	h_n	x_n	$\|x_n - p\|$
1	9	9	9	9	9	9	8
5	1.5901350	1.5674767	1.5674116	1.4934240	1.3822427	1.2214054	0.2214054
10	1.1277881	1.1222912	1.1222778	1.1071548	1.0795988	1.0458060	0.0458060
15	1.0423403	1.0404612	1.0404568	1.0355209	1.0259652	1.0148665	0.0148665
20	1.0170313	1.0162644	1.0162626	1.0142875	1.0103511	1.0059037	0.0059037
25	1.0077061	1.0073563	1.0073555	1.0064638	1.0046562	1.0026478	0.0026478
30	1.0037823	1.0036097	1.0036093	1.0031722	1.002276	1.0012913	0.0012913
35	1.0019726	1.0018822	1.0018820	1.0016542	1.0011834	1.0006701	0.0006701
40	1.0010788	1.0010292	1.0010291	1.0009046	1.0006457	1.0003650	0.0003650
45	1.0006131	1.0005849	1.0005849	1.0005141	1.0003663	1.0002068	0.0002068
No. of iterations	124	123	123	121	116	108	

Table 4. The sequences generated by CT-iteration for given x_1= 1.5, 2, 2.5, 3, 3.5, 15, 40 and 70 in Example 2.

n	$\|x_n - p\|$							
	Initial points were close to p					Initial points were far from p		
	x_1= 1.5	x_1= 2	x_1= 2.5	x_1= 3	x_1= 3.5	x_1= 15	x_1= 40	x_1= 70
1	0.5	1	1.5	2	2.5	14	39	69
5	0.0227557	0.0421025	0.0594148	0.0753317	0.0902155	0.3361598	0.7234206	1.1203709
10	0.0048476	0.008938	0.0125785	0.0159083	0.0190081	0.0685687	0.1417523	0.2124014
15	0.0015833	0.0029173	0.0041027	0.0051859	0.0061934	0.0221860	0.0454512	0.0675607
20	0.0006301	0.0011607	0.0016320	0.0020625	0.0024627	0.0088014	0.0179776	0.0266527
25	0.0002829	0.0005210	0.0007325	0.0009256	0.0011051	0.0039458	0.0080498	0.0119216
30	0.0001380	0.0002542	0.0003573	0.0004515	0.0005391	0.0016798	0.0039226	0.0058065
35	7.16×10^{-5}	0.0001319	0.0001854	0.0002343	0.0002798	0.0009983	0.0020349	0.0030114

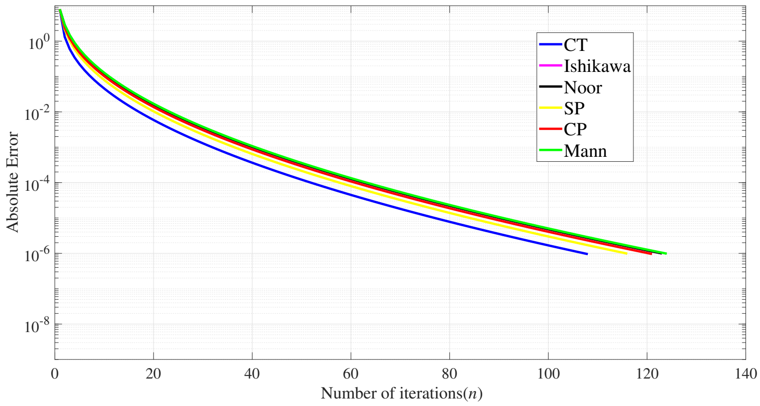

Figure 3. Mann, Ishikawa, Noor, SP, CP and CT iterations for given x_1= 9 of $f(x) = x^{0.3} - (\sqrt{log(x + 9) - 1})^3$.

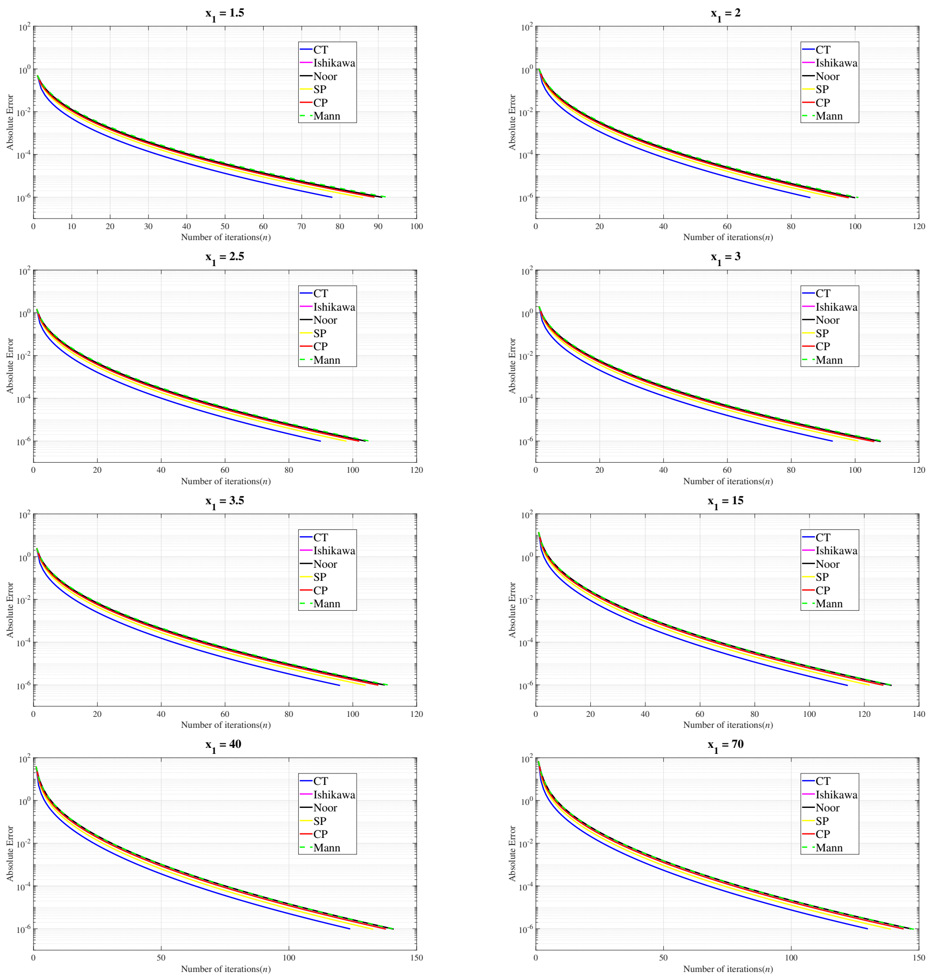

Figure 4. Convergence behaviors for given x_1= 1.5, 2, 2.5, 3, 3.5, 15, 40 and 70 in Example 2.

Next, we will consider on the rate of convergence between the CT-iteration and the algorithm defined in this paper. The Definition 1 will be used to indicate the rate of convergence in the numerical aspects and results are scoped only on the Example 1 and Example 2.

We also give a graphic to compare the rates of convergence of the iterations mentioned in Example 1 visually, as Figure 5.

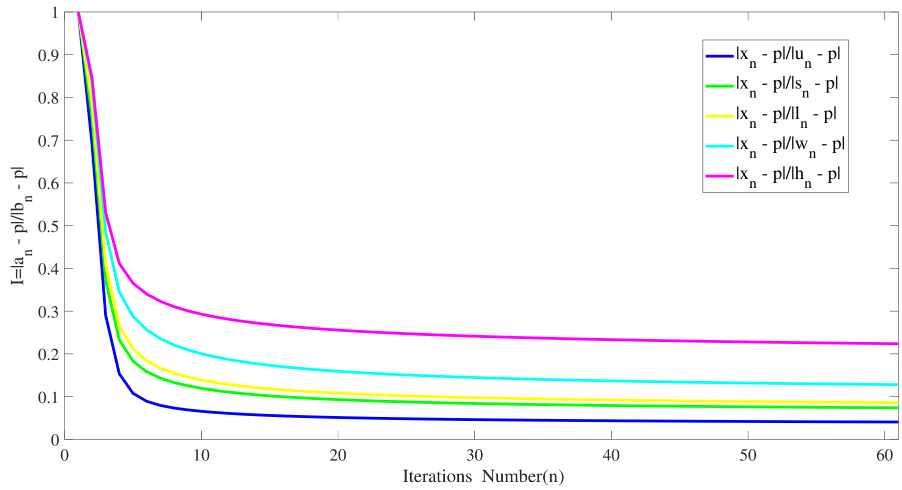

Figure 5. Convergence comparison of sequence generated by Mann iteration (u_n), Ishikawa iteration (s_n), Noor iteration (l_n), CP-iteration (w_n) and SP-iteration (h_n) with CT-iteration (x_n) for Example 1.

We also give a graphic to compare the rates of convergence of the iterations mentioned in Example 2 visually, as Figure 6.

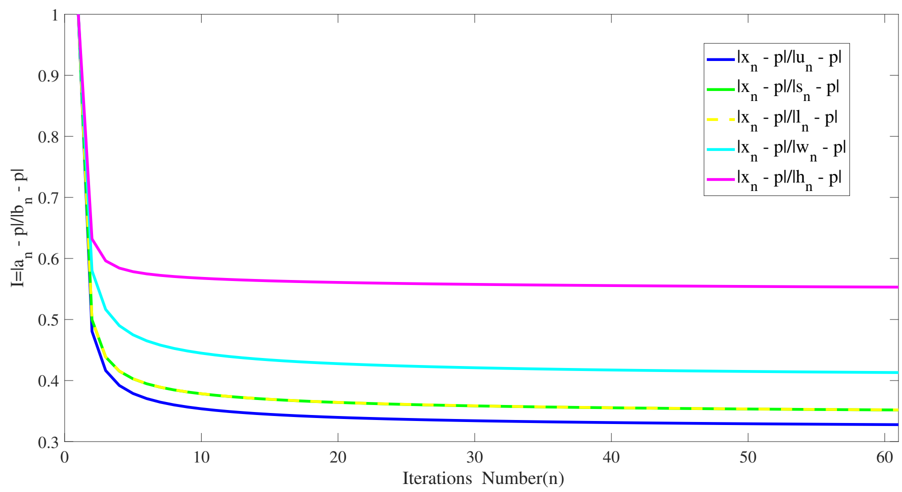

Figure 6. Convergence comparison of sequence generated by Mann iteration (u_n), Ishikawa iteration (s_n), Noor iteration (l_n), CP-iteration (w_n) and SP-iteration (h_n) with CT-iteration (x_n) for Example 2.

Tables 5 and 7 show the absolute errors of Mann, Ishikawa, Noor, CP, SP and CT itertations of the Example 1 and Example 2, respectively. Tables 6 and 8 show ratios between the absolute error of CT-iteration and those of other methods and graphs of Tables 6 and 8 are represented on Figures 5 and 6. Clearly, the graphs on both figures converge to constants less than 1. It indicates that the sequences of absolute error of CT-iteration are less than those sequences of other methods. By Definition 1, we can conclude that CT-iteration converges to the fixed point faster than other method. These results verify the proof on the section 3 which show that CT-iteration converge faster than Mann, Ishikawa, Noor, CP, and SP iterations.

Table 5. The rate of convergence of Mann, Ishikawa, Noor, CP, SP and CT iterations for $f(x) = \frac{x^3+x-3}{19}$ given in Example 1.

	Mann	Ishikawa	Noor	CP	SP	CT-iteration
n	$\lvert u_n - p\rvert$	$\lvert s_n - p\rvert$	$\lvert l_n - p\rvert$	$\lvert w_n - p\rvert$	$\lvert h_n - p\rvert$	$\lvert x_n - p\rvert$
1	4.1669251	4.1669251	4.1669251	4.1669251	4.1669251	4.1669251
...	...	...	...	...	...	...
22	8.2979337E-03	4.5630635E-03	3.9235100E-03	2.6579550E-03	1.6397483E-03	4.1302723E-04
23	6.6660003E-03	3.6655020E-03	3.1517630E-03	2.1335990E-03	1.3103941E-03	3.2789365E-04
24	5.3826746E-03	2.9597038E-03	2.5448946E-03	1.7216513E-03	1.0529971E-03	2.6185811E-04
...	...	...	...	...	...	...
58	2.1520207E-05	1.1825202E-05	1.0168014E-05	6.8241504E-06	3.9132032E-06	8.7880715E-07
59	1.8878297E-05	1.0373410E-05	8.9196769E-06	5.9858026E-06	3.4293688E-06	8.7880712E-07
60	1.6580113E-05	9.1105165E-06	7.8337653E-06	5.2566342E-06	3.0089633E-06	6.7356212E-07

Table 6. Convergence comparison of sequences generated by Mann iteration, Ishikawa iteration, Noor iteration, CP-iteration and SP-iteration with CT-iteration (see in Table 5) for numerical experiment of Example 1.

Rate of convergence between two sequences

n	$\frac{\lvert x_n - p\rvert}{\lvert u_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert s_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert l_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert w_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert h_n - p\rvert}$
1	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.1077615	0.1826927	0.2113296	0.2876608	0.3653690
10	0.0655970	0.1192918	0.1387968	0.2003645	0.2928943
20	0.0511132	0.0929416	0.1080929	0.1592907	0.2556291
30	0.0461208	0.0838958	0.0975696	0.1446444	0.2412792
40	0.0434849	0.0791196	0.0920146	0.1367792	0.2331997
50	0.0418128	0.0760882	0.0884893	0.1317378	0.2278379
60	0.0406258	0.0739361	0.0859869	0.1281362	0.2238863

Table 7. The rate of convergence of Mann, Ishikawa, Noor, CP, SP and CT iterations for $f(x) = x^{0.3} - (\sqrt{log(x+9)} - 1)^3$ given in Example 2.

n	Mann $\lvert u_n - p\rvert$	Ishikawa $\lvert s_n - p\rvert$	Noor $\lvert l_n - p\rvert$	CP $\lvert w_n - p\rvert$	SP $\lvert h_n - p\rvert$	CT-iteration $\lvert x_n - p\rvert$
1	8	8	8	8	8	8
...	...	...	...	...	...	...
35	1.9725555E-03	1.8822249E-03	1.8820245E-03	1.5884142E-03	1.1834264E-03	6.7014401E-04
36	1.7422239E-03	1.6623962E-03	1.6622192E-03	1.4029143E-03	1.0447074E-03	5.9139201E-04
37	1.5415695E-03	1.4708985E-03	1.4707419E-03	1.2413182E-03	9.2393975E-04	5.2285805E-04
...	...	...	...	...	...	...
86	1.4487895E-05	1.3817536E-05	1.3816066E-05	1.1662020E-05	8.5946173E-06	4.8221305E-06
87	1.3400554E-05	1.2780466E-05	1.2779106E-05	1.0786712E-05	7.9488484E-06	4.4593406E-06
88	1.2400552E-05	1.1826702E-05	1.1825443E-05	9.9817516E-06	7.3550156E-06	4.1257981E-06

Table 8. Convergence comparison of sequences generated by Mann iteration, Ishikawa iteration, Noor iteration, CP-iteration and SP-iteration with CT-iteration (see in Table 7) for numerical experiment of Example 2.

Rate of convergence between two sequences

n	$\frac{\lvert x_n - p\rvert}{\lvert u_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert s_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert l_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert w_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert h_n - p\rvert}$
1	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.3751776	0.3901578	0.3902026	0.4675022	0.5792274
10	0.3584529	0.3745653	0.3746062	0.4453414	0.5754611
20	0.3466405	0.3629859	0.3630247	0.4303450	0.5703508
40	0.3384200	0.3547035	0.3547413	0.4202962	0.5653881
60	0.3350960	0.3513080	0.3513454	0.4162480	0.5629381
80	0.3332460	0.3494062	0.3494434	0.4139888	0.5614158
100	0.3537939	0.3709719	0.3710114	0.4220645	0.5970553

5. Conclusions

In this article, the novel Noor iteration technique, called the CT-iteration is proposed for approximating a fixed point of continuous functions on closed interval. The convergence theorems are also established. The numerical examples comparing with Mann, Ishikawa, Noor, SP and CP iterations are demonstrated. From Examples 1 and 2, we observe that the sequence generated by the CT-iteration converges to a fixed point faster than Mann, Ishikawa, Noor, SP and CP iterations.

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Conflict of interest

The authors declare no conflict of interest.

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