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Research article

Variants of Julia and Mandelbrot sets as fractals via Jungck-Ishikawa fixed point iteration system with s-convexity

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Abstract: In this paper, we generate some non-classical variants of Julia and Mandelbrot sets, utilizing the Jungck-Ishikawa fixed point iteration system equipped with s-convexity. We establish a novel escape criterion for complex polynomials of a higher degree of the form $z^n + az^2 - bz + c$, where a, b and c are complex numbers and furnish some graphical illustrations of the generated complex fractals. In the sequel, we discuss the errors committed by the majority of researchers in developing the escape criterion utilizing distinct fixed point iterations equipped with s-convexity. We conclude the paper by examining variation in images and the impact of parameters on the deviation of dynamics, color and appearance of fractals. It is fascinating to notice that some of our fractals represent the traditional Kachhi Thread Works found in the Kutch district of Gujarat (India) which is useful in the Textile Industry.

Keywords: escape criterion; fixed point; Jungck-Ishikawa orbit; s-convexity
Mathematics Subject Classification: 28A10, 31E05, 37F10, 37F46, 47H10

1. Introduction

The Mandelbrot and Julia sets are celebrated graphics of a very complex chaotic system created by a straightforward mathematical practice. Julia sets were explored in the early twentieth century by French mathematician Gaston Maurice Julia [10] and a French mathematician and astronomer Pierre Joseph Louis Fatou [9]. However, Mandelbrot sets were proposed by a Polish-born French-American mathematician and polymath Benoit B. Mandelbrot [15] much later in the late 1970s. Mandelbrot visualized the Julia sets for $z^2 + c$, while working at IBM. Later on, numerous mathematicians explored distinct characteristics of Mandelbrot and Julia sets and generalized these in different directions. For instance, some generalizations of quadratic functions are anti-polynomials [5], transcendental functions [7], elliptic functions [14], rational functions [27], and so on. The most natural and first generalization is the function of the form $z^n + c$ which was utilized by Mandelbrot [15]. In 2004, Rani and Kumar [30, 31] initiated the use of fixed point iteration process in the visualization of some interesting fractals, by utilizing the Mann iteration process. Since then various iteration processes have been exploited like Ishikawa iteration [17], Noor iteration [23], S-iteration [11], Jungck-Ishikawa iteration [25], Jungck-Noor iteration [12], Jungck-CR iteration [34], Picard-Mann iteration with s-convexity [32], Jungck-SP iteration with s-convexity [19], implicit iterative scheme [40], and so on. Recently, Antal et al. [1] obtained interesting Julia sets of complex sine functions using celebrated fixed point iterations.

The Julia and Mandelbrot sets are fascinating from both a mathematical as well as application viewpoint having several applications in real life. In biology, fractals have been utilized to understand the nerve fibbers pattern [28], the growth culture of micro-organisms (like Chlamydomonas, bacteria, and amoeba) [13], and so on. In physics (in Fluid Mechanics), fractals have been utilized to understand turbulent flows [33]. In telecommunication, fractals have been utilized to manufacture antennae [4]. In computer networking, fractals have been utilized in architectural models, radar systems, and so on [24]. Some more applications are in computer graphics, dictionaries with domain blocks, image encryption algorithms, and the arts (see [16,26,36]). Recently, Usurelu et al. [35] initiated two Newton-like methods to provide necessary postulates for the convergence of the induced iterative procedures and obtained visual investigations of some modified Thakur iteration processes, via polynomiographic practices which are not the outcome of an artistic methodology but these graphics encode specific mathematical data.

There are numerous conclusions related to characteristics of fractional order dynamic in physical phenomena. For instance, heat transfer [2], a novel fractional order hydro-turbine-generator unit [39], and so on. The fractional order system also finds applications in cellulose degradation [8], digital cryptography [21], and so on. A modified DLA (diffusion-limited aggregation) model relying on a fractional diffusion mechanism, provides a new dimension to modeling fractal growth [38], and so on. A fractal simulation of flocculation processes utilizing the DLA model with different particle numbers, launch radii, motion step lengths, and finite motion steps are also significant. These four model parameters have a huge impact on the morphology and structures of the model parameters/the DLA aggregates [20]. Flocculation is a significant procedure in the area of water and wastewater treatment that may help in migrating, removing and transforming particulate matter in water. Recently, Wang et al. [37] addressed the significant problem of the parameter estimation of the Julia set because without knowing the precise system parameters, the existing theoretical results cannot be effectively applied to applications areas.

In this work, we concentrate just on the visualization of Julia and Mandelbrot sets via Jungck-Ishikawa fixed point iteration system with s-convexity. We establish a novel escape criterion for complex polynomials of a higher-order of the form $z^n + az^2 - bz + c$, where a, b and c are complex numbers. This polynomial is a slight modification of nth degree polynomial studied by Jolaoso et al. [18] for the development of biomorphs. The motivation behind this is the fact that on altering the iteration process the dynamics and behavior of the generated fractals are also altered, which are significant from the graphical as well as applications point of view. Our theorems and corollaries are generalizations, enrichment, and sharpened versions of some existing conclusions. Towards the end, we visualize aesthetic non-classical fractals and give concluding remarks to demonstrate the significance of our conclusion.

2. Preliminaries

Definition 2.1. *(Julia set [3, 10]) The set of all points in $\mathbb{C}$ for which the orbits do not converge to a point at infinity is a filled Julia set of the polynomial $p : \mathbb{C} \to \mathbb{C}$ of degree ≥ 2 and we denote it by F_p, that is,*

$$F_p = \{z \in \mathbb{C} : \{|p(z_k)|\}_{k=0}^{\infty} \text{ is bounded}\}.$$

The Julia set of p is the boundary of F_p, that is, $J_p = \partial F_p$.

Definition 2.2. *(Mandelbrot set [6,15]) The Mandelbrot set M is the set of parameters $c \in \mathbb{C}$ for which the filled Julia set F_{p_c} of the polynomial $p_c(z) = z^2 + c$ is connected, that is,*

$$M = \{c \in \mathbb{C} : F_{p_c} \text{ is connected}\}.$$

Equivalently

$$M = \{c \in \mathbb{C} : \{|p_c(z_k)|\} \nrightarrow \infty \text{ as } k \to \infty\}.$$

Nazeer et al. [22] utilized Jungck-Mann and Jungck-Ishikawa iterations with s-convexity to generate Julia and Mandelbrot sets. If $S, T : \mathbb{C} \to \mathbb{C}$ is two complex-valued maps so that the degree of T is greater or equal to two and S is injective. Then sequence $\{z_k\}$ of iterates for any initial point $z_0 \in \mathbb{C}$, $u, v, s \in (0, 1]$, $k = 1, 2, \ldots$, is known as:

(i) Jungck-Mann orbit (JMOs) with s-convexity if

$$Sz_k : Sz_k = (1-u)^s Sz_{k-1} + u^s Tz_{k-1}.$$

Clearly, it is a function of four arguments (T, z_0, u, s) and is a one-step feedback procedure.

(ii) Jungck-Ishikawa orbit (JIOs) with s-convexity if

$$\begin{aligned} Sz_k : Sz_k &= (1-u)^s Sz_{k-1} + u^s Ty_{k-1}, \\ Sy_{k-1} &= (1-v)^s Sz_{k-1} + v^s Tz_{k-1}. \end{aligned}$$

Clearly, it is a function of five arguments (T, z_0, u, v, s) and is a two-step feedback procedure.

Remark 2.1. *The Jungck-Ishikawa orbit reduces to: Picard orbit [42] when $S(z) = z$, $u = 1$, $v = 0$ and $s = 1$; Mann orbit [41] when $S(z) = z$, $v = 0$ and $s = 1$; Ishikawa orbit [17] when $S(z) = z$ and $s = 1$; Jungck-Mann orbit with s-convexity [22] when $v = 0$.*

3. Escape time algorithm for the complex polynomials in Jungck-Ishikawa orbit

Motivated by numerous applications of complex polynomials in science and engineering, we establish the escape time algorithm via Jungck-Ishikawa fixed point iteration system extended with s-convex combination for complex polynomials of the type $p(z) = z^n + az^2 - bz + c$, where $n \geq 3$ and a, b and c are complex numbers. We break the polynomial $p(z)$ into two maps S and T so that $p(z) = Tz - Sz$, where Sz is injective. Consequently, we establish a novel threshold escape radii and utilize these to visualize some nonclassical variants of classical fractals in the following result.

Theorem 3.1. *Let $p(z) = z^n + az^2 - bz + c$ be a higher-order complex polynomial. Assume that $z_0 \in \mathbb{C}$, $|z_0| \geq |c| > \left(\frac{2(1+|b|)}{su|(c^{n-2}+a)|}\right)$ and $|z_0| \geq |c| > \left(\frac{2(1+|b|)}{sv|(c^{n-2}+a)|}\right)$, where $0 < u, v, s \leq 1$, and c is a complex parameter. Define*

$$Sz_k = (1-u)^s Sz_{k-1} + u^s Ty_{k-1},$$

where

$$Sy_{k-1} = (1-v)^s Sz_{k-1} + v^s Tz_{k-1},$$

where $Sz = bz$ is injective, $Tz = z^n + az^2 + c$ is a higher-order polynomial, and $k = 1, 2, 3, 4, 5, \ldots$. Then the orbit of z_0 tends to ∞ as k tends to ∞.

Proof. For $k = 1$,

$$Sy_{k-1} = (1-v)^s Sz_{k-1} + v^s Tz_{k-1}$$

implies

$$\begin{aligned} |Sy_0| &= |(1-v)^s Sz_0 + v^s Tz_0| \\ &= |(1-v)^s bz_0 + v^s(z_0^n + az_0^2 + c)|. \end{aligned}$$

Since $v, s \in (0, 1]$, so $v^s \geq sv$, we get

$$\begin{aligned} |Sy_0| &\geq |(1-v)^s bz_0 + sv(z_0^n + az_0^2 + c)| \\ &\geq |sv(z_0^n + az_0^2) + (1-v)^s bz_0| - sv|c| \\ &\geq |sv(z_0^n + az_0^2) + (1-v)^s bz_0| - sv|z_0|, \ |z_0| \geq |c| \\ &\geq |sv(z_0^n + az_0^2)| - |(1-v)^s bz_0| - sv|z_0|. \end{aligned}$$

Utilizing binomial expansion of $(1-v)^s$ up to linear terms of v, we attain

$$\begin{aligned} |Sy_0| &\geq |sv(z_0^n + az_0^2)| - |(1-sv)bz_0| - sv|z_0| \\ &= sv|(z_0^n + az_0^2)| - |bz_0| + |svbz_0| - sv|z_0| \\ &\geq sv|(z_0^n + az_0^2)| - |bz_0| - sv|z_0|, \quad |b| \geq |0|, \end{aligned}$$

which gives us

$$\begin{aligned}
|by_0| &\geq sv|(z_0^n + az_0^2)| - |z_0| - |bz_0|, \quad sv < 1 \\
&= sv|z_0^2||(z_0^{n-2} + a)| - |z_0|(1 + |b|) \\
&\geq |z_0|\Big(sv|z_0||(c^{n-2} + a)| - (1 + |b|)\Big) \\
&= |z_0|(1 + |b|)\Big(\frac{sv|z_0||(c^{n-2} + a)|}{(1 + |b|)} - 1\Big).
\end{aligned}$$

Thus,

$$|y_0| \geq \frac{|by_0|}{(1 + |b|)} \geq |z_0|\Big(\frac{sv|z_0||(c^{n-2} + a)|}{(1 + |b|)} - 1\Big).$$

Now

$$|z_0| \geq |c| > \frac{2(1 + |b|)}{sv|(c^{n-2} + a)|} \text{ implies that } \frac{sv|z_0||(c^{n-2} + a)|}{(1 + |b|)} - 1 > 1.$$

Hence $|y_0| > |z_0|$.

Now

$$Sz_k = (1 - u)^s Sz_{k-1} + u^s Ty_{k-1}$$

implies

$$\begin{aligned}
|Sz_1| &= |(1 - u)^s Sz_0 + u^s Ty_0| \\
&= |(1 - u)^s bz_0 + u^s(y_0^n + ay_0^2 + c)|.
\end{aligned}$$

Since $u, s \in (0, 1]$, so $u^s \geq su$, we get

$$\begin{aligned}
|Sz_1| &\geq |(1 - u)^s bz_0 + su(z_0^n + az_0^2 + c)| \\
&\geq |su(y_0^n + ay_0^2) + (1 - u)^s bz_0| - su|c| \\
&\geq |su(y_0^n + ay_0^2) + (1 - u)^s bz_0| - su|z_0|, \ |z_0| \geq |c| \\
&\geq |su(y_0^n + ay_0^2)| - |(1 - u)^s bz_0| - su|z_0|.
\end{aligned}$$

Utilizing binomial expansion of $(1 - u)^s$ up to linear terms of u, we attain

$$\begin{aligned}
|Sz_1| &\geq |su(y_0^n + ay_0^2)| - |(1 - su)bz_0| - su|z_0| \\
&= su|(y_0^n + ay_0^2)| - |bz_0| + |subz_0| - su|z_0| \\
&\geq su|(y_0^n + ay_0^2)| - |bz_0| - su|z_0|, \quad |b| \geq |0|,
\end{aligned}$$

which gives us

$$\begin{aligned}
|bz_1| &\geq su|(y_0^n + ay_0^2)| - |z_0| - |bz_0|, \quad su < 1 \\
&= su|y_0^2||(y_0^{n-2} + a)| - |z_0|(1 + |b|) \\
&> su|z_0^2||(z_0^{n-2} + a)| - |z_0|(1 + |b|), \ |y_0| > |z_0| \\
&\geq |z_0|\Big(su|z_0||(c^{n-2} + a)| - (1 + |b|)\Big) \\
&= |z_0|(1 + |b|)\Big(\frac{su|z_0||(c^{n-2} + a)|}{(1 + |b|)} - 1\Big).
\end{aligned}$$

Thus,

$$|z_1| \geq \frac{|bz_1|}{(1+|b|)} \geq |z_0|\Big(\frac{su|z_0||(c^{n-2}+a)|}{(1+|b|)} - 1\Big).$$

Since $|z_0| \geq |c| > \frac{2(1+|b|)}{su|(c^{n-2}+a)|}$, there exists $\lambda > 0$ so that $\frac{su|z_0||(c^{n-2}+a)|}{(1+|b|)} - 1 > 1 + \lambda$.

As a result,

$$|z_1| \geq (1+\lambda)|z_0|.$$

Following the same pattern repeatedly, we obtain $|z_k| > (1+\lambda)^k|z_0|$. Consequently, $|z_k| \to \infty$ as $k \to \infty$, that is, the orbit of z_0 tends to infinity. □

Corollary 3.1. *If we consider* $|z_m| > max\Big\{|c|, \frac{2(1+|b|)}{su|(c^{n-2}+a)|}, \frac{2(1+|b|)}{sv|(c^{n-2}+a)|}\Big\}$, $m \geq 0$, *then* $|z_{m+k}| > (1+\lambda)^k|z_k|$ *and the Jungck-Ishikawa orbit with s-convexity of sequence* $\{z_k\}$ *of iterates for any initial point* $z_0 \in \mathbb{C}$ *tends to* ∞ *as k tends to* ∞.

Remark 3.1. *(i) Jolaoso et al. [18] considered* $a = \{0, 1\}$, *and* b, c *to be complex numbers. However, we have considered* a, b, c *to be complex numbers by taking* $-b$ *instead of* b. *The selection of these parameters is new and has not been studied till now in this perspective.*

(ii) Similarly, we can easily derive the escape criterion for Jungck-Mann fixed point iteration with s-convexity for a higher-order complex polynomial of the type $p(z) = z^n + az^2 - bz + c$, *where* $n \geq 3$ *and* $a, b, c \in \mathbb{C}$ *by taking* $v = 0$ *and making use of simple algebraic calculations. The term* az^2, *where a is any complex number makes our conclusions very fascinating. On the other hand, Nazeer et al. [22] had investigated the escape criterion via Jungck-Mann and Jungck-Ishikawa iterations with s-convexity for* $p(z) = z^n - az + c$, *without actually exploring the fractals. Further, in the proof of Theorem 4.9 (respectively Theorem 3.9) [22], the authors claimed that they follow the mathematical induction but this is not true. They used Theorems 4.1 and 4.5 (respectively Theorems 3.1 and 3.5) to show that for the first steps the theorem is true, then they made an inductive assumption. But in the whole proof for* $n+1$, *they did not use the inductive assumption, so actually, they did not use the principle of mathematical induction. Besides this, the proofs of other theorems also contain mathematical mistakes since authors have utilized binomial series up to linear terms which is not true for the assumed parameters. One may easily check that neither* $(1-v)^s \geq 1 - sv$ *nor* $(1-(1-v))^s \geq 1 - s(1-v)$ *is true in* $(0, 1]$. *If we take* $a = 0$, *then we get the function* $p(z) = z^n - bz + c$ *which is utilized by Nazeer et al. [22]. However, when we take* $a = 0$ *in Theorem 3.1, we do not get the desired conclusion [22]. This discrepancy is due to the involved mathematical technique. In the sequel, we have suggested the technique to obtain the escape criterion for more general higher-order polynomials. Similar mathematical errors may also be seen in papers involving complex polynomials equipped with convexity parameter s, for instance Kumari et al. [19], Kwun et al. [29] and many others existing in the literature.*

(iii) Corollary 3.1 provides algorithms for exploring Julia and Mandelbrot sets of T_c. *If* $|z_0| \leq |c|$, *we obtain the orbit JIOs of z. If* $|z_k|$ *lies in the exterior of the circle of radius* $\max\Big\{|c|, \frac{2(1+|b|)}{su|(c^{n-2}+a)|}, \frac{2(1+|b|)}{sv|(c^{n-2}+a)|}\Big\}$ *for some n, then the orbit JIOs escapes meaning thereby that* z_0 *is not inside the Julia or Mandelbrot sets. However, if* $|z_k|$ *does not exceed this bound, then utilizing the definitions of the Julia and Mandelbrot sets, we utilize these algorithms to generate fractals in the next section.*

4. Generation of Julia and Mandelbrot sets

We use MATLAB 7.5.0 (R2007b) via colourmap (Figure 1), Algorithm 1 (for Julia set), and Algorithm 2 (for Mandelbrot set) for developing fractals for higher-order polynomials. During the simulation process, we have obtained and analyzed many fractals. But in this paper, we discuss the behavior of only a limited number of fractals for the different parameter values associated with it. The parameters u, v and the convexity s perform a very significant role in giving the vibrant colors and exploring the characteristics of the associated Julia sets and Mandelbrot sets. The structure of the fractals is very much dependent on the iterative method we choose. The number of petals in a standard fractal is $(n-1)$, for the n^{th} order polynomial. As n increases the fractal takes a circular shape. Throughout the paper, we are using a maximum number of iterations $K = 30$.

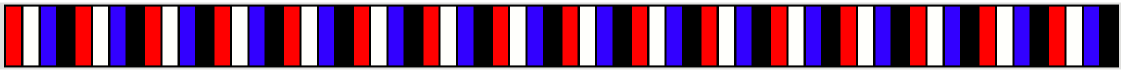


Figure 1. Colourmap used in the graphical examples.

Algorithm 1 Geometry of Julia-Set

Input: $T(z) = z^n + az^2 + c$, $Sz = bz$ where $a, b, c \in \mathbb{C}, b \neq 0$ and $n = 1, 2, 3, \ldots$; $A \subset \mathbb{C}-$ area; $K-$ a maximum number of iterations; $u, v, s \in (0, 1]-$parameter of the Jungck-Ishikawa iteration with s-convexity; $colourmap[0..C-1]-$color with C colors.
Output: Julia set for area A.

for $z_0 \in A$ **do**
  $R_1 = \left(\frac{2(1+|b|)}{su|(c^{n-2}+a)|}\right)$
  $R_2 = \left(\frac{2(1+|b|)}{sv|(c^{n-2}+a)|}\right)$
  $R = \max(|c|, R_1, R_2)$
  $n = 0$
  while $n \leq K$ **do**
    $y_n = \frac{(1-v)^s S z_n + v^s T(z_n)}{b}$
    $z_{n+1} = \frac{(1-u)^s S z_n + u^s T(y_n)}{b}$
    if $|z_{n+1}| > R$ **then**
      break
    end if
    $n = n + 1$
  end while
  $i = \lfloor (C-1)\frac{n}{K} \rfloor$
  color z_0 with $colourmap[i]$
end for

Algorithm 2 Geometry of Mandelbrot-Set

Input: $T(z) = z^n + az^2 + c$, $Sz = bz$ where $a, b, c \in \mathbb{C}, b \neq 0$ and $n = 1, 2, 3, \ldots$; $A \subset \mathbb{C}-$ area; $K-$ a maximum number of iterations; $u, v, s \in (0, 1]-$parameter of the Jungck-Ishikawa iteration with s-convexity; $colourmap[0..C-1]-$color with C colors.
Output: Mandelbrot set for area A.

```
for c ∈ A do
    R_1 = ( 2(1+|b|) / (su|(c^{n-2}+a)|) )
    R_2 = ( 2(1+|b|) / (sv|(c^{n-2}+a)|) )
    R = max(|c|, R_1, R_2)
    n = 0
    z_0 = 0
    while n ≤ K do
        y_n = ((1-v)^s S z_n + v^s T(z_n)) / b
        z_{n+1} = ((1-u)^s S z_n + u^s T(y_n)) / b
        if |z_{n+1}| > R then
            break
        end if
        n = n + 1
    end while
    i = ⌊(C-1) n/K⌋
    color z_0 with colourmap[i]
end for
```

4.1. Julia sets

For $n = 3$, the parameter values as given in Table 1, we get amazing fractal objects, which are visible in Figure 2(i,ii).

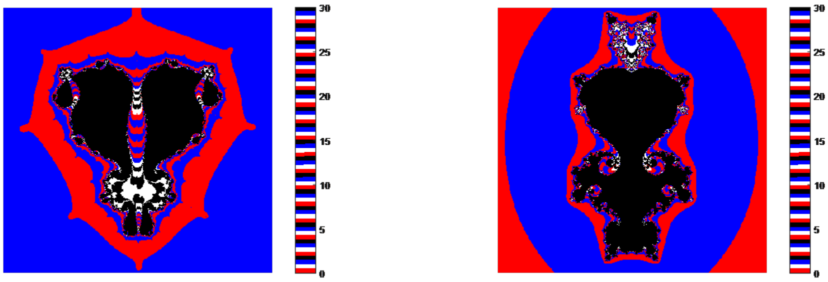
The parameter s, which denotes the convexity is responsible for the volume of the fractal. As s decreases from 0.95 to 0.55555, the resulting fractal shrinks as shown in Figure 3(i,ii). The parameters used in Figure 3(i,ii) are as in Table 2.

Table 1. Effect of parameters a and s on cubic Julia sets.

	a	b	c	u	v	n	s	A
(i)	$-1+90i$	100	$-60i$	0.045	0.045	3	0.45	$[-7,7]\times[-5.5,9.5]$
(ii)	$8i$	100	$-60i$	0.045	0.045	3	0.10	$[-17,17]\times[-19.5,15.5]$

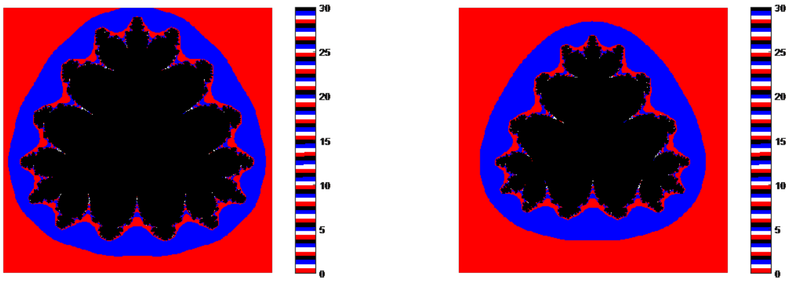
Table 2. Effect of parameter s on Quartic Julia sets.

	a	b	c	u	v	n	s	A
(i)	0	$-190i$	$-6.0+60i$	0.0005	0.045	4	0.95	$[-25,25]\times[-25,25]$
(ii)	0	$-190i$	$-6.0+60i$	0.0005	0.045	4	0.55555	$[-18,18]\times[-18,18]$



(i) (ii)

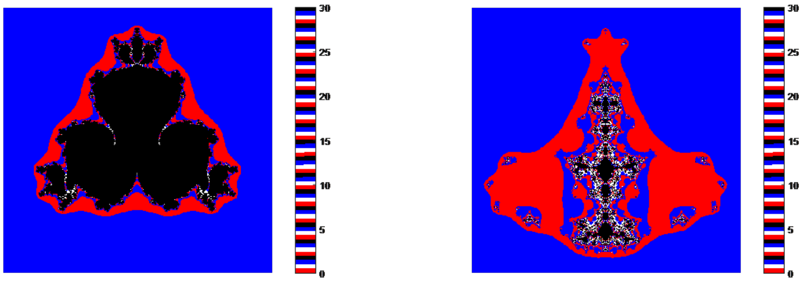
Figure 2. Cubic Julia sets.



(i) (ii)

Figure 3. Quartic Julia sets (showing the effect of a decrease in value of parameter s).

The parameters used in Figure 4(i,ii) are as in Table 3. The parameter b affects the volume of the entire fractal. As the absolute value of b decreases, the volume of the fractal decreases, as shown in Figure 4(i,ii).



(i) (ii)

Figure 4. Quartic Julia sets (showing the effect of a decrease in the absolute value of complex parameter b).

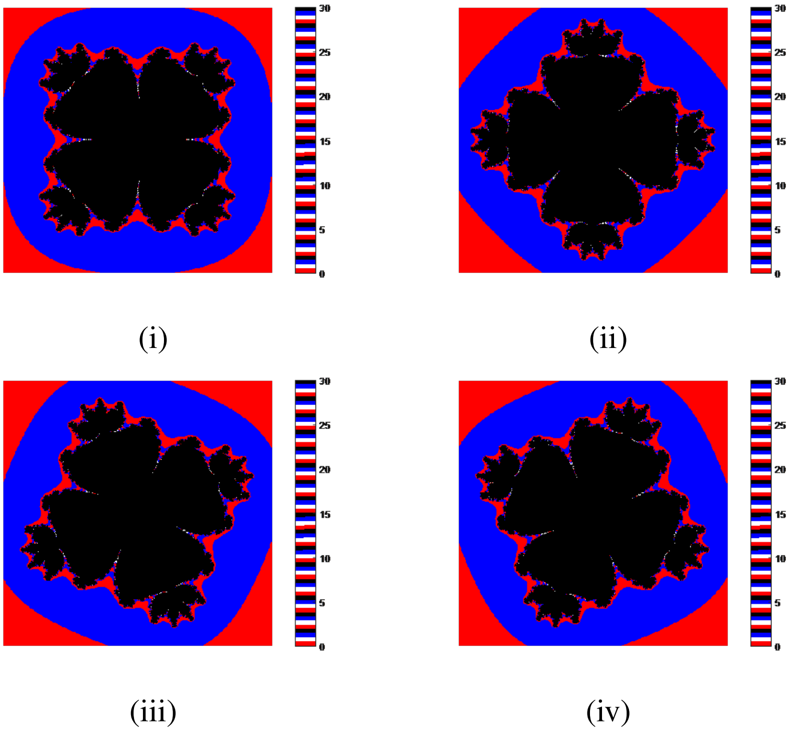
Table 3. Effect of parameter b on Quartic Julia sets.

	a	b	c	u	v	n	s	A
(i)	0	$-190i$	-17	0.000005	0.00045	4	0.0195	$[-9,9]\times[-9,9]$
(ii)	0	$-9.2i$	-17	0.000005	0.00045	4	0.0195	$[-3,3]\times[-3.5,2.5]$

The parameter b also gives rotational symmetry when it is purely real (imaginary) and changes the sign. For the same set of parameters and different values of b (change in sign in the real and complex values of b as in Table 4), the resultant fractals can be seen in Figure 5(i–iv).

Table 4. Effect of change in sign in parameter b (real and complex) on Quintic Julia sets.

	a	b	c	u	v	n	s	A
(i)	0	44	-19	0.000005	0.00045	5	0.33	$[-6.5,6.5]\times[-6.5,6.5]$
(ii)	0	-44	-19	0.000005	0.00045	5	0.33	$[-6.5,6.5]\times[-6.5,6.5]$
(iii)	0	$44i$	-19	0.000005	0.00045	5	0.33	$[-6.5,6.5]\times[-6.5,6.5]$
(iv)	0	$-44i$	-19	0.000005	0.00045	5	0.33	$[-6.5,6.5]\times[-6.5,6.5]$



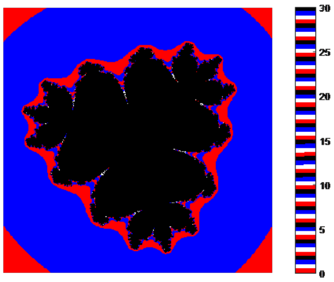
(i) (ii) (iii) (iv)

Figure 5. Quintic Julia sets

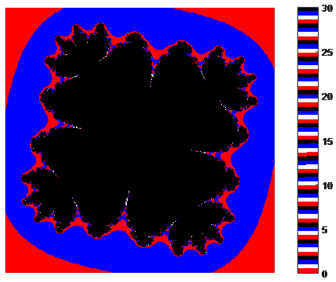
The parameters used in Figure 6(i–v) are as in Table 5.

Table 5. Effect of change in the degree (n) of polynomials on Julia sets.

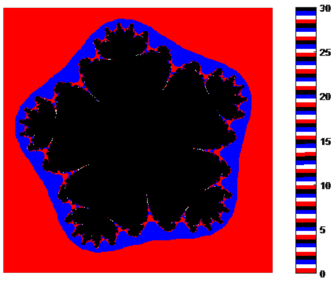
	a	b	c	u	v	n	s	A
(i)	0	$44+44i$	$-30i$	0.000005	0.000545	4	0.55	$[-25,25]\times[-25,25]$
(ii)	0	$44+44i$	$-30i$	0.000005	0.000545	5	0.55	$[-10,10]\times[-10,10]$
(iii)	0	$44+44i$	$-30i$	0.000005	0.000545	6	0.55	$[-7,7]\times[-7,7]$
(iv)	0	$44+44i$	$-30i$	0.000005	0.000545	9	0.55	$[-3.5,3.5]\times[-3.5,3.5]$
(v)	0	$44+44i$	$-30i$	0.000005	0.000545	15	0.55	$[-2,2]\times[-2,2]$



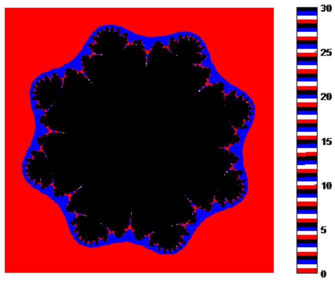
(i) Quartic



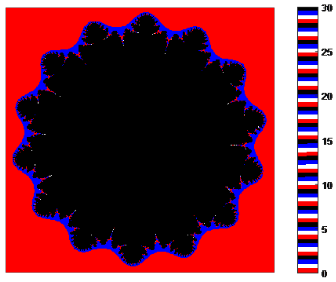
(ii) Quintic



(iii) Sextic



(iv) Nonic



(v) Higher-order

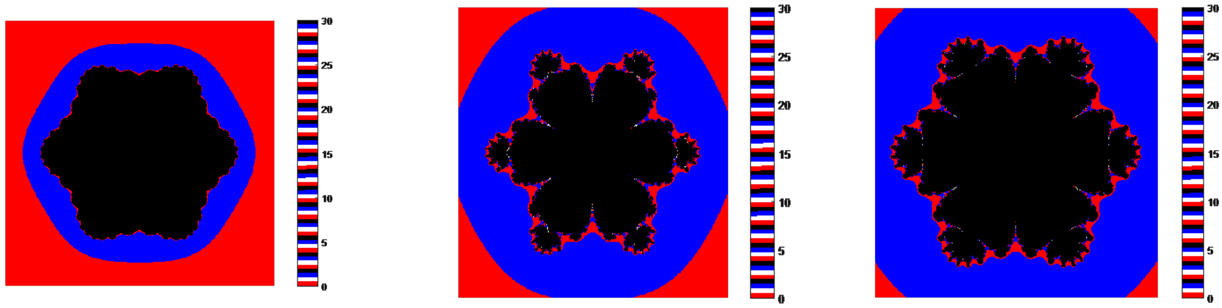
Figure 6. Julia sets.

During the generation of fractals, it is surprising to see that, for the same parameter values of a, b, c, u, v, s and different values of n artistic fractals are obtained which are similar to Kachhi Tread Work found in the Kutch district of Gujarat (India). Noticeably, the number of petals in each figure is $(n-1)$ (one less than the degree of polynomial) which represents the effect of Threadwork of these.

For a variety of parameter sets (see Table 6), we obtain aesthetic Julia fractals (see Figure 7(i–iii)). Figure 7 demonstrates the effect of a decrease in values of u and v on Septic Julia sets. To be specific, Figure 7(i) is for higher values of u and v (greater than 0.5) whereas Figure 7(ii,iii) is for lower values of u and v (nearly 0). Noticeably each figure has 6 leaves (one less than the degree of polynomial).

Table 6. Effect of parameters u and v on Septic Julia sets.

	a	b	c	u	v	n	s	
(i)	0	-19	$0.285 + 0.01i$	0.6	0.6	7	0.51	$[-2.5, 2.5] \times [-2.5, 2.5]$
(ii)	0	-19	$0.285 + 0.01i$	0.0001	0.0001	7	0.51	$[-5, 5] \times [-5, 5]$
(iii)	0	-19	$0.285 + 0.01i$	0.0000056	0.00054	7	0.51	$[-4, 4] \times [-4, 4]$



(i) (ii) (iii)

Figure 7. Septic Julia sets.

As u tends to 0 and $a = 0$, the stunning fractal can be seen in Figure 8. The parameter used in the Figure 8 are $a = 0,\ b = -44 - 44i,\ c = 88,\ v = 0.0045,\ u = 0.000000045,\ n = 5,\ s = 0.264973$, and $A = [-10, 10] \times [-10, 10]$.

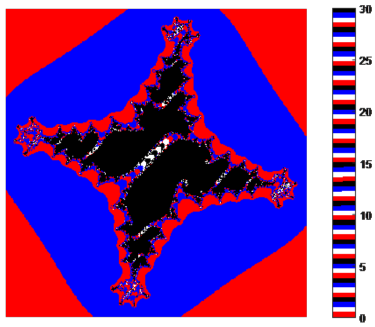


Figure 8. Quintic Julia set.

Remark 4.1. *It has been observed that not only does the polynomial of degree n give a beautiful fractal but sometimes the simple expressions having a non-integer value as a degree also gives beautiful fractal, as seen in Figure 9. It is a great task to select appropriate parameters to obtain the desired fractal pattern. The parameter used in Figure 9 are* $a = 0,\ b = 90.8i,\ c = 0,\ u = 0.067,\ v = 0.067,\ n = 3.14,\ s = 0.4$, *and* $A = [-17, 17] \times [-17, 17]$.

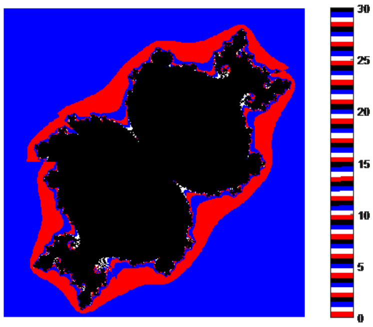


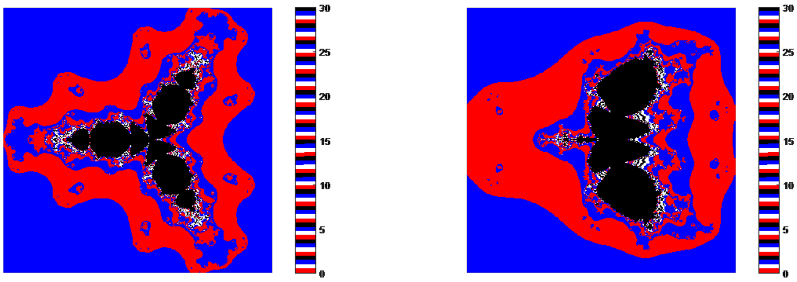
Figure 9. Julia set for $n = 3.14$.

4.2. Mandelbrot sets

To generate Mandelbrot sets, the parameters used in Figure 10(i,ii) are as in Table 7.

Table 7. A pair of Quartic Mandelbrot set for random parameters.

	a	b	u	v	n	s	A
(i)	0.02	−3	0.02	0.02	4	0.394	$[-42, 34] \times [-38, 38]$
(ii)	−1.87897	−3	0.04045	0.04045	4	0.00612	$[-5, 5] \times [-5, 5]$



(i) (ii)

Figure 10. Quartic Mandelbrot set.

In the overall analysis, we say that the effect of even minor changes in one parameter causes a major effect on the appearance of the resultant fractal. The parameters used in Figure 11(i–iii) are as in Table 8. Figure 11(ii,iii) shows the variation in shape when only a is varied and all the other parameters are fixed.

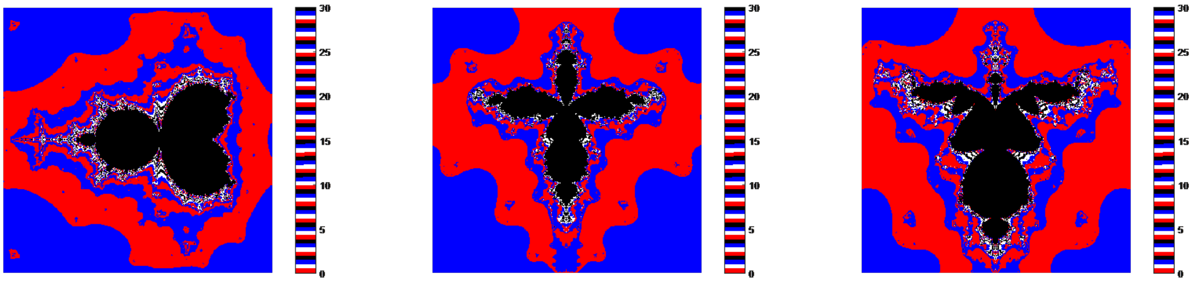
For values of the parameter as shown in Table 9 given below, we found very interesting fractal patterns in the Mandelbrot set of $n = 3, 4, 5, 6, 9$ and 50 (Figure 12(i–vi)).

Table 8. The effect of parameters a and b on Quintic Mandelbrot set with a higher value of u.

	a	b	u	v	n	s	A
(i)	3	1	0.77	0.77	5	0.394	$[-0.9, 0.3] \times [-0.6, 0.4]$
(ii)	$2i$	−3	0.0102	0.0102	5	0.394	$[-38, 38] \times [-30, 35]$
(iii)	$5i$	−3	0.0102	0.0102	5	0.394	$[-27, 27] \times [-21, 33]$

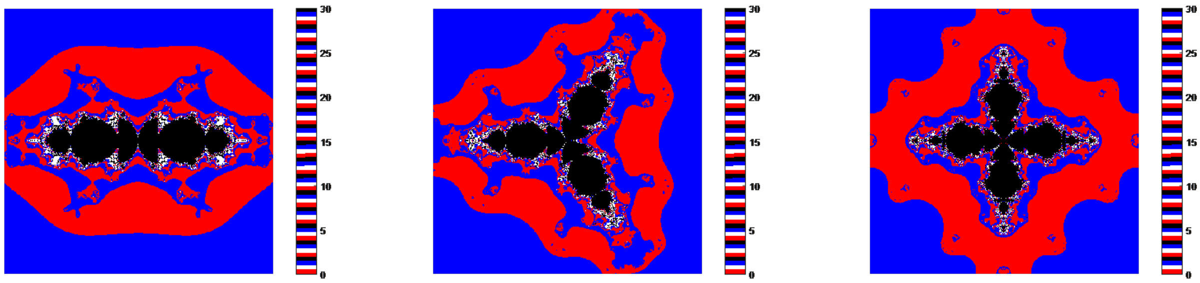
Table 9. Effect of change in the degree (n) of polynomials on Mandelbrot set.

	a	b	u	v	n	s	A
(i)	0	−6.9	0.094045	0.094045	3	0.00612	$[-25, 25] \times [-25, 25]$
(ii)	0	−6.9	0.094045	0.094045	4	0.00612	$[-15, 15] \times [-15, 15]$
(iii)	0	−6.9	0.094045	0.094045	5	0.00612	$[-11, 11] \times [-11, 11]$
(iv)	0	−6.9	0.094045	0.094045	6	0.00612	$[-7, 7] \times [-7, 7]$
(v)	0	−6.9	0.094045	0.094045	9	0.00612	$[-4, 4] \times [-4, 4]$
(vi)	0	−6.9	0.094045	0.094045	50	0.00612	$[-1.5, 1.5] \times [-1.5, 1.5]$

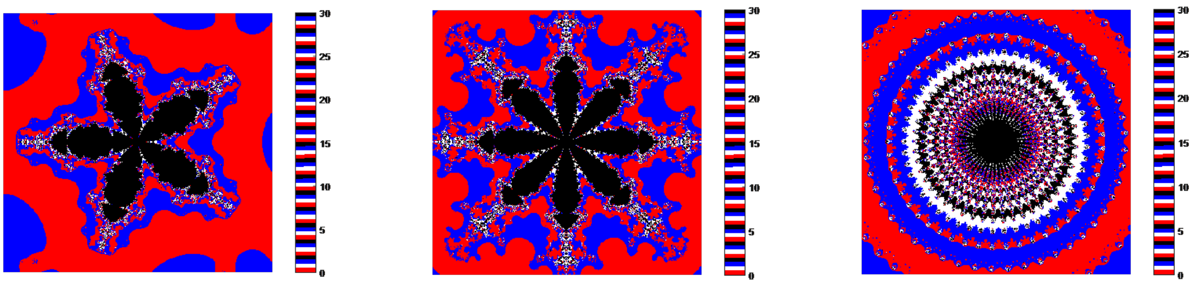


(i) (ii) (iii)

Figure 11. Quintic Mandelbrot set.



(i) Cubic (ii) Quartic (iii) Quintic



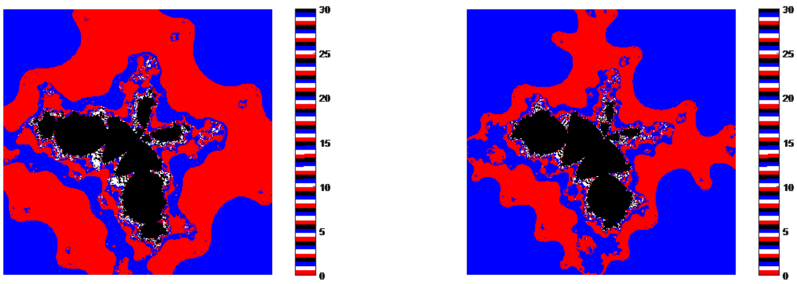
(iv) Sextic (v) Nonic (vi) Higher-order

Figure 12. Mandelbrot set.

Figure 13(i,ii) demonstrate the effect of a increase in values of parameters u and v (see Table 10) on Quintic Mandelbrot sets. Noticeably as the values of both the parameters u and v increase from 0.0006 to 0.9 the Quintic Mandelbrot set become fatter.

Table 10. Effect of parameters u and v on Quintic Mandelbrot sets.

	a	b	u	v	n	s	A
(i)	$2+2i$	-10	0.0006	0.0006	5	0.05	$[-25,25]\times[-25,25]$
(ii)	$2+2i$	-10	0.9	0.9	5	0.05	$[-20,20]\times[-20,20]$



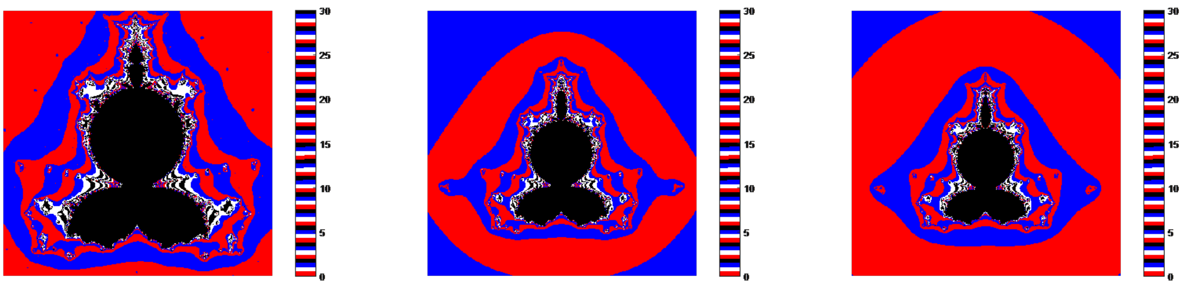
(i) (ii)

Figure 13. Quintic Mandelbrot set.

As the convexity parameter s decreases from 0.99 to 0.1 (see Table 11) the Quartic Mandelbrot set shrinks (see Figure 14). Noticeably each figure has 3 branches (one less than the degree of polynomial).

Table 11. Effect of parameter s on Quartic Mandelbrot sets.

	a	b	u	v	n	s	A
(i)	90	$1.7+1.7i$	0.05	0.05	4	0.99	$[-25,25]\times[-40,10]$
(ii)	90	$1.7+1.7i$	0.05	0.05	4	0.5	$[-2,2]\times[-3,1]$
(iii)	90	$1.7+1.7i$	0.05	0.05	4	0.1	$[-0.2,0.2]\times[-0.3,0.1]$



(i) (ii) (iii)

Figure 14. Quartic Mandelbrot set.

Remark 4.2. *(i) It is sometimes difficult, for anybody, to command perfectly on the results and discussion section, as the fixed point iteration (Jungck-Ishikawa) itself, involves three parameters* (u, v, s)*. Moreover, some additional parameters come from the polynomial* $p(z)$ *under consideration (like* a, b, c*). It is again a topic of research to comment on the effect of resulting fractals when we make changes in any of these parameters. Even if we keep all the parameters fixed and vary only any one of them (for instance,* s*), then also we may obtain a variety of fractals. This comment applies to other parameters also (see, Table 10 in which all the parameters are fixed and only* n *varies. For* $n = 3, 4, 5, 6, 9$ *and* 10 *we have obtained a variety of stunning variants of classical Mandelbrot set). Therefore, we have restricted our discussion to a limited type of combination of parameters. However, we have tried to cover the maximum possible combination of parameters involved in developing the algorithm (escape criterion) in Corollary 3.1. We have generated Mandelbrot and Julia sets of various polynomials and showed some selected fractals only. We notice that the role of each parameter is distinct. For instance, as* n *increases the fractal takes a circular shape (see, Figure 12(vi) looks like a stunning sunflower of a higher-order Mandelbrot set) and the area covered by the fractal decreases.*

(ii) Our theorem and corollary demonstrate the significance of the Jungck-Ishikawa iteration with s*-convexity in the generation of complex graphs of fractal sets. For* $s = 1$*, we obtain the improvements of analogous conclusions for the Jungck-Ishikawa iteration existing in the literature. Noticeably, the value of* s *has a great impact on the shape and the obtained sets are completely different from the original Julia set (see Figure 3).*

5. Conclusions

We have established the escape radii and escape criteria of Julia and Mandelbrot sets for the n^{th} degree complex polynomials of a special kind in Jungck-Ishikawa orbit equipped with s-convexity. Consequently, we have provided the technique to obtain an escape criterion for a more general higher-order polynomial that is not correct in related results (for instance [19, 22, 29], and so on). We have developed and analyzed the behavior of variants of the Julia set and the Mandelbrot set for different parameter values after obtaining fascinating non-classical variants of classical Mandelbrot and Julia fractals using MATLAB software. It is worth mentioning here that polynomials are often utilized to encode information about different objects and are useful in distinct branches of science and engineering. Also, we have observed that as we zoom in on the edges of the petals of the Mandelbrot set, we come across the Julia set meaning thereby each point of the Mandelbrot set includes massive image data of a Julia set. Also, the size of fractals explored using the Jungck-Ishikawa iteration depends on the parameter u and the convexity parameter s, whereas the shape and symmetry depend on the parameter a, b and c. As n increases the area occupied by the fractals decreases. In the sequel, we have provided some interesting features to compare with the existing one and demonstrate the significance of our outcomes such as some of our fractals resemble the traditional Kachhi Thread Works (see Figure 15) found in the Kutch district of Gujarat (India) which are useful in the textile industry.

Figure 15. Real Kachhi work images-beauty of Julia sets.

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Conflict of interest

The authors declare no conflicts of interest.

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