Mathematics

Research article

Computation of numerical solutions to variable order fractional differential equations by using non-orthogonal basis

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Abstract: In this work, we present some numerical results about variable order fractional differential equations (VOFDEs). For the said numerical analysis, we use Bernstein polynomials (BPs) with non-orthogonal basis. The method we use does not need discretization and neither collocation. Hence omitting the said two operations sufficient memory and time can be saved. We establish operational matrices for variable order integration and differentiation which convert the consider problem to some algebraic type matrix equations. The obtained matrix equations are then solved by Matlab 13 to get the required numerical solution for the considered problem. Pertinent examples are provided along with graphical illustration and error analysis to validate the results. Further some theoretical results for time complexity are also discussed.

1. Introduction

Calculus of arbitrary order derivatives and integrations has been found very applicable in mathematical modeling of various real world problems. Such differential and integral operators have greater degree

of freedom. Therefore its dynamical behavior is global instead of local. Also various features related to represent memory and hereditary process can be comprehensively explained through the said area [1–3]. Therefore in last few decades Fractional order differential and integral equations (FODIEs) has got proper attention from researchers. Interesting applications regarding FODIEs have been investigated in variety of disciplines including engineering, technology and applied sciences [4–6]. Numerous features related to memory and hereditary characteristics corresponding to different materials and processes can be found [7]. Due to these applications FODIEs have appealed researchers for further analysis and progress.

Since arbitrary order differential equations are infact definite integrals which include the classical derivative as a special case. Therefore, arbitrary order derivative has not a unique definition. It is interesting that researchers have introduced various definitions for the said differential operators. Geometrically fractional order derivative provides a complete spectrum of a function which include the classical order curve as a special case. Among all the definitions of fractional derivative, the definitions of Caputo and Riemann-Liouville has gained more popularity among the researchers. On the other hand various aspects including qualitative analysis, stability theory and numerical treatment have been studied for FODIEs. For such investigations the authors have used fixed point approach, tools of nonlinear analysis to handle the mentioned aspects for various problems of FODIEs [8]. One of the most important area in the theory of FODIEs is known as numerical solutions of the mentioned area. In variety of situations it is difficult to obtain analytical solution of numerous FODIEs due to the complex behavior of fractional order. In such conditions approximating the solutions to that problem will be more suitable to be determined. For this purposes, large numbers of analytical and numerical methods have been established. In this regards, eigen function procedure [9], perturbation tools [10], iteration techniques [11], transform methods [12], decomposition schemes [13] have been established for analytical or semi-analytical results of FODIEs. Also for numerical solutions, difference methods [14], numerical method for multi-terms FODEs [15], Tau method [16], collocation techniques [17], wavelet analysis [18] have been introduced in literature. Also spectral methods based on operational matrices [19] have been constructed in large numbers. The aforementioned methods have been very well applied for classical differential and integral equations. Here we remark that the said tools have also applied in the area of fractional calculus very well. Among the mentioned methods, spectral techniques are the most powerful and significant for the numerical solutions to various problems. The said methods are based on some operational matrices of integration and differentiations. Based on these matrices, the proposed problem is converted to some algebraic equation known as Sylvester equation of the form

$$MX + XN = L, \tag{1.1}$$

where L, M, N are constant co-efficient matrices and X is an unknown matrix that has to be determined. Further a Sylvester equation is a linear matrix equation having a unique solution if the spectra of L and M has an empty intersection. If X has order $m \times n$, then the order of matrix L will be $m \times n$ and that of M will be $n \times n$, where the order of X and N will be $m \times n$. Sylvester linear matrix equation plays very important role in control and stability theory of many applied problems.

Keeping in mind the importance of numerical solutions to fractional order problems, various powerful techniques have been established in last two decades. For instance authors [20] have developed numerical scheme for two dimensional stochastic Volterra-Fredholm integral equations. In same line the authors [21] have developed a hybrid numerical scheme for partial-integro type problems. Further some more advanced numerical methods have been recently established for ordinary and partial fractional order

problems. For the mentioned problems the authors have used various procedure including BPs and Bernoulli and some other complex tools. For the mentioned numerical methods we refer [22–28].

In present literature, the operational matrices have been developed by using different orthogonal polynomials like Legendre, Laguerre, Jacobi, etc. All the operational matrices have been obtained from the aforesaid polynomials by using descritization techniques to solve FODIEs subject to some initial or boundary conditions [29]. However, boundary value problems are rarely investigated which constitute a very applicable branch of applied analysis. Also some new operational matrices for BPs have been developed recently [30]. Keeping in mind that the said polynomials are non orthogonal, some researchers have used discretization together with collocation method to construct some operational matrices for arbitrary order differentiation and integration. But the mentioned approach is limited to only initial value problems. Recently some simple problems of fractional order differential equations (FODEs) have been investigated by using the said nonorthogonal polynomials. Since in most of the literature discretization and collocation techniques have jointly used which exploit extra memory and consume much time. For some frequent results in this regards see [31, 32]. Therefore to omit discretization and collocation to save memory and time from wastage, we will directly construct the operational matrices for solving FODIEs numerically with some initial/boundary conditions by using BPs. Authors [33] have started the area of variable order integration and differentiation during 1993. Recently the mentioned area of variable order FODEs has attracted much attention. The reason is that such problems have more degree of freedom in choosing the most suitable order for the accurate description of a real world problems. Currently many valuable articles have been published in this regards (see [34–38]). Recently various articles related to fractional order dynamics of epidemiological disease, neural network and PD controller theory have been published (see [39–43]). In respect of numerical analysis BPs give more accurate results as compared to other polynomials. Because Bernstein polynomials have non orthogonal basis. For variable order the said polynomials are very rarely used. Further BPs have been developed by a Russian mathematician Sergie Natanovich Bernstein. According to the Weierstrass Approximation Theorem every continuous real valued function can be approximated uniformly with the help of polynomial function over R. In this regard, BPs play a very important role in function approximation. Though the BPs are non-orthogonal, but good approximation for real valued continues functions. BPs play significant roles in distribution functions theory. In recent time BPs estimators of density functions have got great popularity from researchers of statistics. Some authors have investigated various problems of estimating a multivariate distribution function by using BPs in multiple dimensions (see [44]).

Since it has been proved that spectral methods are stable and convergent. So far we know these type of methods have been used for traditional fractional order derivatives very well. But in case of variable order where the differential operators are more flexible and posses greater degree of freedom. The spectral methods have not so properly applied in past many years. Also the mentioned tools have been shamefully applied in many disciplines to perform simulations like heat conduction, quantum mechanics, fluid dynamics, weather prediction and so on (see [45]). Further the proposed method has some advantages like more accurate than finite difference method with the same number of degrees of freedom, can be used spatial filters of very high order easily, and obtain power spectra directly. Also the present method does not required any prior discretization of data like need by finite methods, Galerkin and wavelet methods. Since these methods are scaling techniques depend on scale level. So for larger scale level it need more time for compilation but produce more significant results (see [46]). Motivated by the aforesaid work, here we consider the following two cases of VOFDEs including initial and boundary value problems of

the form

$$\begin{cases} {}^{c}_{0}D_{t}^{\vartheta(t)}v(t) = g(t, v(t)), \ 0 < \vartheta(t) \leq 1, \\ v(0) = v_0, \end{cases} \tag{1.2}$$

and

$$\begin{cases} {}^{c}_{0}D_{t}^{\vartheta(t)}v(t) = g(t, v(t)), \ 1 < \vartheta(t) \leq 2, \\ v(0) = v_0, \ v(1) = v_1, \end{cases} \tag{1.3}$$

where in both cases $g : [0, 1] \times R \to R$ is linear continuous function. Here initially, we will construct operational matrices of fractional integration and differentiation by using BPs. Afterwards, with the help of developed matrices, we will transfer the proposed problems to algebraic equations of Sylvester type. The proposed problem (3.1) and (1.3) include some problems as special case like:

- For instance if $g(t, v(t)) = v(t)$ with $0 < \vartheta(t) \leq 1$ then the concerned problem (3.1) reduces to the famous growth type model. If $g(t, v(t)) = -v(t)$ with $0 < \vartheta(t) \leq 1$, then problem (3.1) becomes a decay model which has various applications in radioactivity process of various elements.
- Similarly if we consider $g(t, v(t)) = \pm v(t)$, with $1 < \vartheta(t) \leq 2$, the concerned problem (1.3) reduces to two point boundary value problems been studied in [47].

We use the computational software Matlab 13 to find the unknown matrices for the required numerical solution. Also the results are demonstrated graphically. A comparison between integer order and numerical solutions at various fractional order is also given. Further some theoretical analysis in the time complexity is provided.

2. Elementary results

Here we recall some basic results from [48].

Definition 2.1. If $s : [0, \infty) \to R$ is a function whose integral converges over R^+, then its fractional integral of order $\vartheta(t) > 0$ is given by

$$I^{\vartheta}s(t) = \frac{1}{\Gamma(\vartheta(t))} \int_{0}^{t} (t - v)^{\vartheta(v)-1} s(v)dv, \tag{2.1}$$

Definition 2.2. The Caputo fractional derivative of a function $s : R^+ \to R$ is defined as under

$$D^{\vartheta(t)}s(t) = \begin{cases} \dfrac{1}{\Gamma(n - \vartheta(t))} \displaystyle\int_{0}^{t} (t - v)^{n-\vartheta(v)-1} s^{(n)}(v)dv, \ n - 1 < \vartheta(t) < n, \\ \dfrac{d^n s}{dt^n}.n = \vartheta, \end{cases} \tag{2.2}$$

Lemma 2.3. *For $\vartheta(t) > 0$ fractional integral of a fractional derivative is defined as*

$$I^{\vartheta}(t)[{}^{c}_{0}D_{t}^{\vartheta}(t)v(t)] = v(t) - d_0 - d_1 t - d_3 t^2 + \ldots + d_m - 1t^{m-1}, \tag{2.3}$$

where $d_k \in R, k = 0, 1, 2, \ldots m - 1$.

BPs are formed by the linear combination of Bernstein basis [27]. The polynomial is defined as

$$P_{j,k}(t) = \binom{k}{j} t^j (1-t)^{k-j}, \ j = 0, 1, 2, \cdots, k,$$

where k is the order of the polynomial. The compact form of BPs [29] is obtained by using binomial expansion as

$$P_{j,k}(t) = \sum_{i=0}^{k} \sum_{l=0}^{k-j} (-1)^k \binom{k}{j} \binom{k-j}{l} t^{l+j}, \ j = 0, 1, 2, \cdots, k,$$

where

$$P_{i,l,k} = (-1)^k \binom{k}{j} \binom{k-j}{l}. \tag{2.4}$$

The set of BPs of degree m is expressed as

$$P_{i,m}(t) = \{P_{0,m}(t), P_{1,m}(t), P_{2,m}(t), P_{3,m}(t), \cdots P_{m,m}(t)\} \tag{2.5}$$

(2.5) is known as the Bernstein basis. Further some needful results are given bellow. Since, BPs are non-orthogonal, In this regard the inner product of two Bernstein basis is given by

$$
\begin{aligned}
\int_0^1 P_{i,m}(t) P_{j,m}(t) dt &= \int_0^1 \binom{k}{i} t^i (1-t)^{k-i} \binom{k}{j} t^j (1-t)^{k-j} dt, \\
&= \binom{k}{i} \binom{k}{j} \int_0^1 t^{i+j} (1-t)^{2k-i-j} dt, \\
&= \binom{k}{i} \binom{k}{j} \beta(i+j+1, 2k-i-j+1), \\
&= \binom{k}{i} \binom{k}{j} \frac{\Gamma(i+j+1)\Gamma(2k-i-j+1)}{\Gamma(i+j+1+2k-i-j+1)}, \\
&= \binom{k}{i} \binom{k}{j} \frac{\Gamma(i+j+1)\Gamma(2k-i-j+1)}{\Gamma(2k+2)}, \\
&= \frac{\binom{k}{i} \binom{k}{j}}{(2k+1)\frac{\Gamma(2k+1)}{\Gamma(i+j+1)\Gamma(2k-i-j+1)}}, \\
&= \frac{\binom{k}{i} \binom{k}{j}}{(2k+1) \binom{2k}{i+j}}.
\end{aligned}
$$

Hence

$$\int_0^1 P_{i,m}(t) P_{j,m}(t) dt = G_{i,j}. \tag{2.6}$$

Which upon representation in matrix form can be written as This results into

$$
\int_0^1 P_{i,m}(t)P_{j,m}(t)dt =
\begin{bmatrix}
G_{0,0} & G_{0,1} & \cdots & G_{0,r} & \cdots & G_{0,m} \\
G_{1,0} & G_{1,1} & \cdots & G_{1,r} & \cdots & G_{1,m} \\
G_{2,0} & G_{2,1} & \cdots & G_{2,r} & \cdots & G_{2,m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
G_{r,0} & G_{r,1} & \cdots & G_{r,r} & \cdots & G_{r,m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
G_{m,0} & G_{m,1} & \cdots & G_{m,r} & \cdots & G_{m,m}
\end{bmatrix}.
\tag{2.7}
$$

For the given two functions $\phi,\ \psi \in L^2[0,1]$, we can approximate any function $\phi(t)$ in term of Bernstein basis as

$$
\phi(t) = \sum_{i=0}^m s_i P_{i,m}(t).
\tag{2.8}
$$

It implies

$$
\begin{aligned}
R_{1xm} &= \int_0^1 \phi(t)P_{j,m}(t)dt \\
&= \int_0^1 \sum_{i=0}^m s_i P_{i,m}(t)P_{j,m}(t)d, \\
&= \sum_{i=0}^m s_i \int_0^1 P_{i,m}(t)P_{j,m}(t)d,
\end{aligned}
\tag{2.9}
$$

where $j = 0, 1, 2, \cdots$. Hence we get

$$
R_{1xm} =
\begin{bmatrix} s_0 & s_1 & \cdots & s_m \end{bmatrix}
\begin{bmatrix}
G_{0,0} & G_{0,1} & \cdots & G_{0,r} & \cdots & G_{0,m} \\
G_{1,0} & G_{1,1} & \cdots & G_{1,r} & \cdots & G_{1,m} \\
G_{2,0} & G_{2,1} & \cdots & G_{2,r} & \cdots & G_{2,m} \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
G_{3,0} & G_{3,1} & \cdots & G_{3,r} & \cdots & G_{3,m} \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
G_{m,0} & G_{m,1} & \cdots & G_{m,r} & \cdots & G_{m,m}
\end{bmatrix},
$$

where $S_{1xm} = \begin{bmatrix} s_0 & s_1 & \cdots & s_m \end{bmatrix}$ is the coefficient matrix and as (2.7) $G_{mxm} = G_{i,j}$ where

$$
S_{1xm} = R_{1xm}G_{i,j}^{-1}.
\tag{2.10}
$$

we provide the following two lemmas for our algorithm.

Lemma 2.4. *[30] Let $P_{i,m}(t)$ be function vector defined in* (2.4) *the then the fractional order integration over the function is given as*

$$
I^{\varphi(t)}P_{i,k}(t) = V_{k\times k}^{\varphi(t)}P_{i,k}(t),
\tag{2.11}
$$

where $V_{k \times k}^{\varphi(t)}$ is the Operational Matrix for integration, and

$$V_{k \times k}^{\varphi(t)} = \Upsilon_{k \times k} G_{k \times k}^{-1}.$$

Moreover

$$\Upsilon_{k \times k}^{\varphi(t)} = \begin{bmatrix} \Lambda_{0,0} & \Lambda_{0,1} & \cdots & \Lambda_{0,r} & \cdots & \Lambda_{0,k} \\ \Lambda_{1,0} & \Lambda_{1,1} & \cdots & \Lambda_{1,r} & \cdots & \Lambda_{1,k} \\ \Lambda_{2,0} & \Lambda_{2,1} & \cdots & \Lambda_{2,r} & \cdots & \Lambda_{2,k} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \Lambda_{r,0} & \Lambda_{r,1} & \cdots & \Lambda_{r,r} & \cdots & \Lambda_{r,k} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \Lambda_{k,0} & \Lambda_{k,1} & \cdots & \Lambda_{k,r} & \cdots & \Lambda_{k,k} \end{bmatrix}, \tag{2.12}$$

where by (2.4)

$$\Lambda_{i,j} = \sum_{l=0}^{k-i} \sum_{c=0}^{k-j} P_{i,l,k} P_{j,c,k} \frac{\Gamma(k + i + 1)}{(i + j + k + l + \varphi(t) + 1)\Gamma(k + i + \varphi(t) + 1)} \tag{2.13}$$

Lemma 2.5. *[30] Let $P_{i,k}(t)$ be a vector function then the fractional order differentiation of the function is given by*

$$D^{\varphi(t)} P_{i,k}(t) = W_{k \times k}^{\varphi(t)} P_{i,k}(t), \tag{2.14}$$

where

$$W_{k \times k}^{\varphi(t)} = \P_{k \times k}^{\varphi(t)} G_{k \times k}^{-1}, \tag{2.15}$$

where

$$\P_{k \times k}^{\varphi(t)} = \begin{bmatrix} \gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,r} & \cdots & \gamma_{0,k} \\ \gamma_{1,0} & \gamma_{1,1} & \cdots & \gamma_{1,r} & \cdots & \gamma_{1,k} \\ \gamma_{2,0} & \gamma_{2,1} & \cdots & \gamma_{2,r} & \cdots & \gamma_{2,k} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \gamma_{r,0} & \gamma_{r,1} & \cdots & \gamma_{r,r} & \cdots & \gamma_{r,k} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \gamma_{k,0} & \gamma_{k,1} & \cdots & \gamma_{k,r} & \cdots & \gamma_{k,k} \end{bmatrix}, \tag{2.16}$$

where

$$\gamma{i,j} = \begin{cases} \displaystyle\sum_{w=\varphi(t)}^{k-i} \sum_{z=0}^{k-j} P_{i,w,k} P_{j,z,k} \frac{\Gamma(w + i - \varphi(t))}{\Gamma(w + i - \varphi(t) + 1)\Gamma(w + z + i + j - \varphi(t) + 1)}, & i < [\varphi(t)] \\ \displaystyle\sum_{w=0}^{k-i} \sum_{z=0}^{k-j} P_{i,w,k} P_{j,z,k} \frac{\Gamma(w + i - \varphi(t))}{\Gamma(w + i - \varphi(t) + 1)\Gamma(w + z + i + j - \varphi(t) + 1)}, & i >= [\varphi(t)]. \end{cases}$$

3. Numerical scheme

Here we divide this section to two subsection. In one subsection we establish numerical scheme for initial values problems. In second subsection, we derive the scheme for boundary value problems under variable order.

3.1. Numerical algorithm for initial value problem

Initial value problems have their own importance in the field of mathematics. We discuss different cases of initial value problems to investigated for numerical analysis.

Case I:

$$\begin{cases} {}_0^c D_t^{\varphi(t)} v(t) = g(t, v), \ 0 < \varphi(t) \le 1, \\ v(0) = v_0, \end{cases} \tag{3.1}$$

Assume that

$$ {}_0^c D_t^{\varphi(t)} v(t) = A_M P_M^T(t). $$

Applying $\varphi(t)$ order integral on both sides, we get

$$ I^{\varphi(t)} {}_0^c D_t^{\varphi(t)} v(t) = I^{\varphi(t)} A_M P_M^T(t) $$

$$ v(t) - d_0 = I^{\varphi(t)} A_M P_M^T(t), $$
$$ v(t) = d_0 + I^{\varphi(t)} A_M P_M^T(t), $$

$v(0) = v_0$ implies

$$ v(t) = v_0 + I^{\varphi(t)} A_M P_M^T(t). $$

By using function approximation consider

$$
\begin{aligned}
v_0 &\approx D_M P_M^T, \\
A_M I^{\varphi(t)} P_M^T(t) &\approx A_M V_{k \times k}^{\varphi(t)} P_M^T(t).
\end{aligned}
$$

Hence

$$ v(t) = D_M P_M(t) + A_M V_{k \times k}^{\varphi(t)} P_M^T(t), \tag{3.2} $$

using $v(t)$ in the given (3.1) we can get required matrix equation.

Case II: If we take $g(t, v) = v(t)$, then one has

$$\begin{cases} {}_0^c D_t^{\varphi(t)} v(t) = v(t), \ 0 < \varphi(t) \le 1, \\ v(0) = v_0, \end{cases} \tag{3.3}$$

Considering

$$ {}_0^c D_t^{\varphi(t)} v(t) = A_M P_M^T(t) $$

and applying $\varphi(t)$ order integral, we get the following function $v(t)$ with the given initial condition

$$v(t) = v_0 + A_M V_{k \times k}^{\varphi(t)} P_M^T(t).$$

So one has

$$v(t) = D_M P_M^T(t) + A_M V_{k \times k}^{\varphi(t)} P_M^T(t), \tag{3.4}$$

$$A_M P_M^T(t) = D_M P_M^T(t) + A_M V_{k \times k}^{\varphi(t)} P_M^T(t). \tag{3.5}$$

Therefore (3.18) implies that

$$A_M P_M^T(t) - D_M P_M(t) - A_M V_{k \times k}^{\varphi(t)} P_M^T(t) = 0. \tag{3.6}$$

from (3.6), we can write

$$\begin{aligned} A_M - D_M - A_M V_{k \times k}^{\varphi(t)} &= 0, \\ A_M - A_M V_{k \times k}^{\varphi(t)} - D_M &= 0. \end{aligned} \tag{3.7}$$

The given Eq (3.7) is a Sylvester type equation that can be solved by Matlab.

Case III: If the given differential equation is non-homogeneous then the given linear function will be as

$$g(v) = Kv(t) + f(t),$$

then the given differential equation will be as

$$\begin{cases} {}_0^c D_t^{\varphi(t)} v(t) = Kv(t) + f(t), \ 0 < \varphi(t) \le 1, \\ v(0) = v_0. \end{cases} \tag{3.8}$$

In such case using the function $v(t)$ in Eq (3.4), we get

$${}_0^c D_t^{\varphi(t)} v(t) = K[D_M P_M^T(t) + A_M V_{k \times k}^{\varphi(t)} P_M^T(t)] + f(t), \tag{3.9}$$

approximating $f(t) \approx F_M P_M^T(t)$ in (3.9), we get

$${}_0^c D_t^{\varphi(t)} v(t) = K[D_M P_M^T(t) + A_M V_{k \times k}^{\varphi(t)} P_M^T(t)] + F_M P_M^T(t).$$

After rearrangement we have

$$\begin{aligned} A_M P_M^T(t) - K[D_M P_M^T(t) + A_M V_{k \times k}^{\varphi(t)} P_M^T(t)] - F_M P_M^T(t) &= 0, \\ A_M P_M^T(t) - K D_M P_M^T(t) - K A_M V_{k \times k}^{\varphi(t)} P_M^T(t) - F_M P_M^T(t) &= 0, \\ A_M P_M^T(t) - K A_M V_{k \times k}^{\varphi(t)} P_M^T(t) - K D_M P_M(t) F_M P_M^T(t) &= 0, \end{aligned}$$

which implies that

$$A_M - K A_M V_{k \times k}^{\varphi(t)} - (K D_M + F_M) = 0. \tag{3.10}$$

Hence (3.10) is the required Sylvester equation that can be solved with the help of Matlab.

Case IV: Consider another version

$$\begin{cases} {}^{c}_{0}D^{\varphi(t)}_{t}v(t) = g(v), \ 1 < \varphi(t) \le 2, \\ v(0) = v_0, v\prime(0) = v_1 \end{cases} \tag{3.11}$$

Let assume that

$$ {}^{c}_{0}D^{\varphi(t)}_{t}v(t) = A_M P^{T}_{M}(t). $$

Then taking $\varphi(t)$ order integral of both sides yields

$$ I^{\varphi(t)}{}^{c}_{0}D^{\varphi(t)}_{t}v(t) = I^{\varphi(t)}A_M P^{T}_{M}(t) $$

$$ v(t) - d_0 - d_1 t = I^{\varphi(t)}A_M P^{T}_{M}(t), $$
$$ v(t) = d_0 + d_1 t + I^{\varphi(t)}A_M P^{T}_{M}(t), $$

$v(0) = v_0$ implies

$$ v(t) = v_0 + d_1 t + I^{\varphi(t)}A_M P^{T}_{M}(t). \tag{3.12}$$

Taking first order derivative of (3.13), we get

$$ v\prime(t) = d_1 + I^{\varphi(t)-1}A_M P^{T}_{M}(t), $$

applying initial condition we have,

$$ v\prime(0) = d_1 + I^{\varphi(t)-1}A_M P^{T}_{M}(0).v_1 = d_1, $$

By using values of v_0 and v_1 in (3.13), we get

$$ v(t) = v_0 + v_1 t + I^{\varphi(t)}A_M P^{T}_{M}(t). \tag{3.13}$$

consider

$$ v_0 + v_1 t \ \approx \ D_M P^{T}_{M}(t), $$
$$ A_M I^{\varphi(t)} P^{T}_{M}(t) \ \approx \ A_M V^{\varphi(t)}_{k \times k} P^{T}_{M}(t). $$

Hence

$$ v(t) = D_M P^{T}_{M}(t) + A_M V^{\varphi(t)}_{k \times k} P^{T}_{M}(t). \tag{3.14}$$

The above equation is used to solve the (3.11) with the help of Matlab.

Case V: Let the Eq (3.11) be considered as

$$\begin{cases} {}^{c}_{0}D^{\varphi(t)}_{t}v(t) = v(t), \ 1 < \varphi(t) \le 2, \\ v(0) = v_0, v\prime(0) = v_1. \end{cases} \tag{3.15}$$

Let us assume that

$$_0^c D_t^{\varphi(t)} v(t) = A_M P_M^T(t)$$

and applying $\varphi(t)$ order integral, we get the following function $v(t)$,

$$I^{\varphi(t)} {}_0^c D_t^{\varphi(t)} v(t) = I^{\varphi(t)} A_M P_M^T(t),$$
$$v(t) = d_0 + d_1 t + I^{\varphi(t)} A_M P_M^T(t). \tag{3.16}$$

Taking derivative of (3.16), we get

$$v\prime(t) = d_1 + I^{\varphi(t)} A_M P_M^T(t).$$

Applying initial conditions in (3.16) implies $v(0) = d_0 + d_1(0) + I^{\varphi(t)-1} A_M P_M^T(0)$. Hence we get $v_0 = d_0$. Also using boundary condition $v\prime(0) = d_1 + I^{\varphi(t)} A_M P_M^T(0)$, implies $v_1 = d_1$. Then the given Eq (3.16) become,

$$v(t) = v_0 + v_1 t + I^{\varphi(t)} A_M P_M^T(t). \tag{3.17}$$

On using Lemma 2.4 and Lemma 2.5, one has

$$A_M I^{\varphi(t)} P_M^T(t) \quad \approx \quad A_M V_{k \times k}^{\varphi(t)} P_M^T(t),$$
$$v_0 + v_1 t \quad \approx \quad D_M P_M^T(t),$$

it implies

$$v(t) = D_M P_M^T(t) + A_M V_{k \times k}^{\varphi(t)} P_M^T(t),$$

$$A_M P_M^T(t) = D_M P_M^T(t) + A_M V_{m \times m}^{\varphi(t)} P_M^T(t),$$
$$A_M P_M^T(t) - D_M P_M^T(t) - A_M V_{k \times k}^{\varphi(t)} P_M^T(t) = 0,$$

from above equation we can have

$$A_M - D_M - A_M V_{k \times k}^{\varphi(t)} = 0,$$
$$A_M - A_M V_{k \times k}^{\varphi(t)} - D_M = 0. \tag{3.18}$$

The given Eq (3.18) is the transformed Sylvester equation that can be solved by Matlab.
Case VI: If the given differential equation is non-homogeneous then the given linear function will be as,

$$g(v) = Kv(t) + f(t),$$

and the given differential equation will be as

$$\begin{cases} _0^c D_t^{\varphi(t)} v(t) = Kv(t) + f(t), \ 1 < \varphi(t) \leq 2, \\ v(0) = v_0, v\prime(0) = v_1 \end{cases} \tag{3.19}$$

in such case using the function $v(t)$ reference to Eq (3.17) we get

$$_0^c D_t^{\varphi(t)} v(t) = K[D_M P_M^T(t) + A_M V_{m \times m}^{\varphi(t)} P_M^T(t)] + f(t), \tag{3.20}$$

approximating $f(t) \approx F_M P_M^T(t)$ in (3.20), we get

$$_0^c D_t^{\varphi(t)} v(t) = K[D_M P_M^T(t) + A_M V_{m \times m}^{\varphi(t)} P_M^T(t)] + F_M P_M^T(t),$$

it implies

$$A_M P_M^T(t) - K[D_M P_M^T(t) + A_M V_{m \times m}^{\varphi(t)} P_M^T(t)] - F_M P_M^T(t) = 0,$$
$$A_M P_M^T(t) - K D_M P_M^T(t) - K A_M V_{m \times m}^{\varphi(t)} P_M^T(t) - F_M P_M^T(t) = 0,$$
$$A_M P_M^T(t) - K A_M V_{m \times m}^{\varphi(t)} P_M^T(t) - K D_M P_M^T(t) F_M P_M^T(t) = 0,$$

Thus we have

$$A_M - K A_M V_{m \times m}^{\varphi(t)} - (K D_M + F_M) = 0.$$

$$(3.21)$$

Hence (3.21) is the required Sylvester equation.

3.2. General algorithm for boundary value problems

Here we establish algorithm for boundary value problems.
Case I: We consider the following problem as:

$$\begin{cases} _0^c D_t^{\varphi(t)} v(t) = g(v), \ 1 < \varphi(t) \le 2, \\ v(0) = v_0, \ v(1) = v_1, \ \text{where } v_0, v_1 \in R, \end{cases} \tag{3.22}$$

we will approximate the given VOFDEs in Bernstein Basis as per previous practice. Let us assume that

$$D^{\varphi(t)} v(t) = C_M P_M^T(t), \tag{3.23}$$

applying $\varphi(t)$ order integral on both sides of (3.23), we get

$$\begin{aligned} _0 I_t^{\varphi(t)} D^{\varphi(t)} v(t) &= {}_0 I_t^{\varphi(t)} C_M P_M^T(t), \\ v(t) - d_0 - d_1 t &= C_{M0} I_t^{\varphi(t)} P_M^T(t), \\ v(t) &= d_0 + d_1 t + C_{M0} I_t^{\varphi(t)} P_M^T(t), \end{aligned} \tag{3.24}$$

applying initial conditions given in (3.22), we can find the values of d_0 and d_1 as

$$v(0) = d_0 + d_1(0) + C_{M0} I_0^{\varphi(t)} P_M^T(0).$$

Hence we get $v_0 = d_0$. Similarly the value of d_1 is found in the same manner by applying the boundary condition $v(1) = v_1$,

$$\begin{aligned} v(1) &= d_0 + d_1(1) + C_{M0} I_1^{\varphi(t)} P_M^T(1), \\ v_1 &= v_0 + d_1 + C_{M0} I_1^{\varphi(t)} P_M^T(1), \\ d_1 &= v_1 - v_0 - C_{M0} I_1^{\varphi(t)} P_M^T(1), \end{aligned} \tag{3.25}$$

putting the values of d_0 and d_1 in (3.24) we can find our function $v(t)$ as

$$v(t) = v_0 + (v_1 - v_0 - C_{M0} I_1^{\varphi(t)} P_M^T(1))t + C_{M0} I_t^{\varphi(t)} P_M^T(t). \tag{3.26}$$

Which implies

$$
\begin{aligned}
v(t) &= v_0 + \left(v_1 - v_0 - C_{M0} I_1^{\varphi(t)} P_M^T(1)\right) + C_{M0} I_t^{\varphi(t)} P_M^T(t), \\
v(t) &= v_0 + (v_1 - v_0)t - t(C_{M0} I_1^{\varphi(t)} P_M^T(1)) + C_{M0} I_t^{\varphi(t)} P_M^T(t).
\end{aligned}
\tag{3.27}
$$

We approximate $v_0 + (v_1 - v_0)t$ in terms of BPs as

$$v_0 + (v_1 - v_0)t \cong R_M P_M^T(t),$$

where R_M is the coefficient matrix. We proceed as

$$t(_0 I_1^{\varphi(t)} C_M P_M^T(t)) \approx D_M B_{m \times m}^{\varphi(t)} P_M^T(t), \tag{3.28}$$

$$_0 I_t^{\varphi(t)} P_M^T(t) \approx C_M V_{m \times m}^{\varphi(t)} P_M^T(t). \tag{3.29}$$

Hence the Eq (3.27) become

$$v(t) = R_M P_M^T(t) + D_M B_{m \times m}^{\varphi(t)} P_M^T(t) + C_M V_{m \times m}^{\varphi(t)} P_M^T(t). \tag{3.30}$$

Therefore, the given scalar problem in (3.22) is transformed to

$$C_M P_M^T(t) = g\left(R_M P_M^T(t) + D_M B_{m \times m}^{\varphi(t)} P_M^T(t) + C_M V_{m \times m}^{\varphi(t)} P_M^T(t),\right). \tag{3.31}$$

$$g(R_M P_M^t(t) + D_M B_{m times m}^{\varphi(t)} P_M^T(t) + C_M V_{m \times m}^{\varphi(t)} P_M^T(t)) - C_M P_M^T(t) = 0. \tag{3.32}$$

Hence the system in Eq (3.22) can be solved with the help of Matlab. Every model that is in the form of the given system can be solved with the help of algorithm given in (3.32) by algebraic Sylvester matrix equation.

Case II: If in (3.22) the linear function $g : [0, 1] \to R$ is selected such that $g(v(t)) = v(t)$, then the given problem become a homogenous VOFDE and can be converted into following the matrix equation by using functions $v(t)$ in (3.30) as

$$
\begin{aligned}
C_M P_M^T(t) - R_M P_M^T(t) &- D_M B_{m \times m}^{\varphi(t)} P_M^T(t) - C_M V_{m \times m}^{\varphi(t)} P_M^T(t) = 0, \\
P_M^T(C_M - R_M &- D_M B_{m \times m}^{\varphi(t)} - C_M V_{m \times m}^{\varphi(t)}) = 0,
\end{aligned}
$$

since BPs are positive therefore

$$C_M - C_M V_{m \times m}^{\varphi(t)} - (D_M B_{m \times m}^{\varphi(t)} - R_M) = 0,$$

which is the required matrix equation that can be solved for unknown matrix C_M with Matlab.

Case III: If in (3.22) the linear function $g : [0, 1] \to R$ is selected in a way that it become a non-homogeneous differential equation, that is , let $g(v(t)) = Kv(t) + f(t)$, then the given problem become a non-homogenous VOFDE as

$$
\begin{cases}
{}_0^c D_t^{\varphi(t)} v(t) = Kv(t) + f(t), \ 1 < \varphi(t) \le 2, \\
v(0) = v_0, \ v(1) = v_1, \ \text{where } v_0, v_1 \in R,
\end{cases}
\tag{3.33}
$$

$$D^{\varphi(t)}v(t) = C_M P_M^T(t), \tag{3.34}$$

applying fractional integral of order $\varphi(t)$, we get

$$I^{\varphi(t)}D^{\varphi(t)}v(t) = I^{\varphi(t)}C_M P_M^T(t),$$
$$v(t) - d_0 - d_1 t = C_M I^{\varphi(t)} P_M^T(t),$$
$$v(t) = d_0 + d_1 t + C_M I^{\varphi(t)} P_M^T(t), \tag{3.35}$$

with the given initial conditions $v(0) = v_0$, we get $d_0 = v_0$. Similarly when $v(1) = v_1$, we have $d_1 = v_1 - v_0 - C_M I^{\varphi(t)} P_M^T(1)$ hence the given unknown function become

$$v(t) = v_0 + (v_1 - v_0 - C_M {}_0 I_1^{\varphi(t)} P_M^T(1))t + C_M {}_0 I_t^{\varphi(t)} P_M^T(t). \tag{3.36}$$

We take some approximations for $v_0 + (v_1 - v_0)t$ and $f(t)$ as under

$$v_0 + (v_1 - v_0)t \approx R_M P_M^T(t),$$
$$f(t) \approx F_M P_M^T(t), \tag{3.37}$$

where by using Lemma 2.4 and Lemma 2.5, we get required approximations as

$$t({}_0 I_1^{\varphi(t)} C_M P_M^T(t)) \approx D_M B_{m \times m}^{\varphi(t)} P_M^T(t),$$
$${}_0 I_t^{\varphi(t)} P_M^T(t) \approx C_M V_{m \times m}^{\varphi(t)} P_M^T(t). \tag{3.38}$$

Thus the Eq (3.36) becomes

$$v(t) = R_M P_M^T(t) - D_M B_{m \times m}^{\varphi(t)} P_M^T(t) + C_M V_{m \times m}^{\varphi(t)} P_M^T(t). $$

Also the given Eq (3.33) becomes

$$C_M P_M^T(t) = K[R_M P_M^T(t) - D_M B_{m \times m}^{\varphi(t)} P_M^T(t) + C_M V_{m \times m}^{\varphi(t)} P_M^T(t)] + F_M P_M^T(t),$$
$$C_M P_M^T(t) - K[R_M P_M^T(t) - D_M B_{m \times m}^{\varphi(t)} P_M^T(t) + C_M V_{m \times m}^{\varphi(t)} P_M^T(t)] - F_M P_M^T(t) = 0. \tag{3.39}$$

From the Eq (3.39), we have

$$C_M - K[R_M - D_M B_{m \times m}^{\varphi(t)} + C_M V_{m \times m}^{\varphi(t)}] - F_M = 0,$$
$$C_M + K C_M V_{m \times m}^{\varphi(t)} + (K D_M B_{m \times m}^{\varphi(t)} - F_M - K R_M) = 0. \tag{3.40}$$

The given equation in (3.40) is the required Sylvester equation, which can be solved for the unknown matrix C_M by using Matlab.

4. Numerical examples

Here we enrich this part by providing some examples. In this section we provide some examples for the demonstration of our proposed single valued problems of fractional order via using the suggested scheme.

Example 4.1. Consider the given problem with boundary conditions

$$\begin{cases} D^{\varphi(t)}v(t) = v(t), \ t \in [0,1], \ 1 < \varphi(t) \le 2, \\ v(0) = 0, \ v(1) = 1. \end{cases} \tag{4.1}$$

The exact solution at fixed fractional order $\varphi \in (1,2]$ of the problem is given by

$$v(t) = \frac{e}{e^2 - 1}\Big(E_\varphi(t^\varphi) - E_\varphi(-t^\varphi)\Big).$$

We approximate the solution at various scale level and different fractional order by using the proposed method. In Figures 1 and 2, we give the detail graphically.
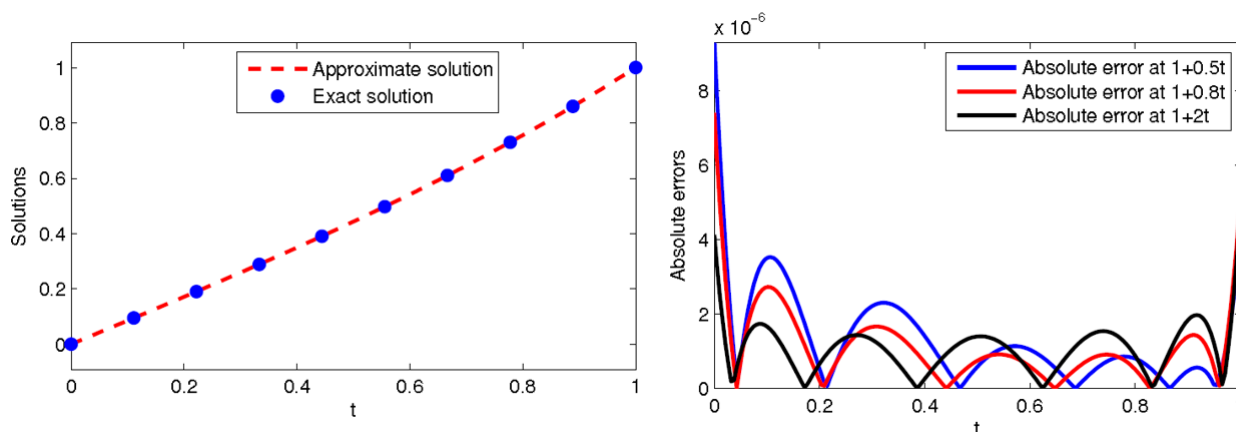
Figure 1. Comparison between exact and numerical solution and absolute errors at various variable different fractional order for Example 4.1.
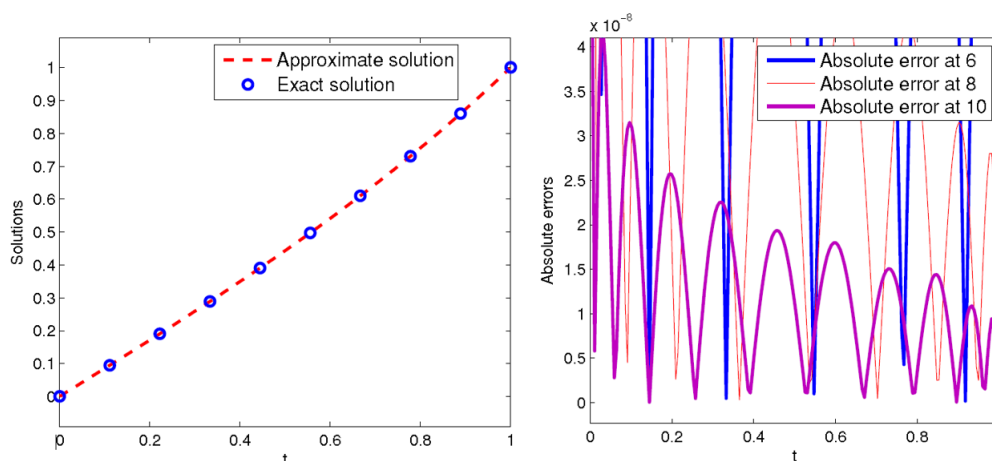
Figure 2. Comparison between exact and numerical solution and absolute errors at different scale level Example 4.1.

Example 4.2. Consider the given problem with boundary conditions

$$
\begin{cases}
D^{\varphi(t)}v(t) = 4v(t) + \dfrac{720t^{6-\varphi(t)}}{\Gamma(7-\varphi(t))} + \dfrac{24t^{4-\varphi(t)}}{\Gamma(5-\varphi(t))} + \dfrac{2t^{2-\varphi(t)}}{\Gamma(3-\varphi(t))} \\
\quad - 4(t^6 + t^4 + t^2), \ t \in [0,1], \ 1 < \varphi(t) \le 2, \\
v(0) = 0, \ v(1) = 3.
\end{cases}
\tag{4.2}
$$

The exact solution of the problem at fixed order $\varphi(t) = 2$ is given by

$$
v(t) = t^6 + t^4 + t^2.
$$

We approximate the solution at various scale level and different variable fractional order by using the proposed method in Figures 3 and 4 respectively as:
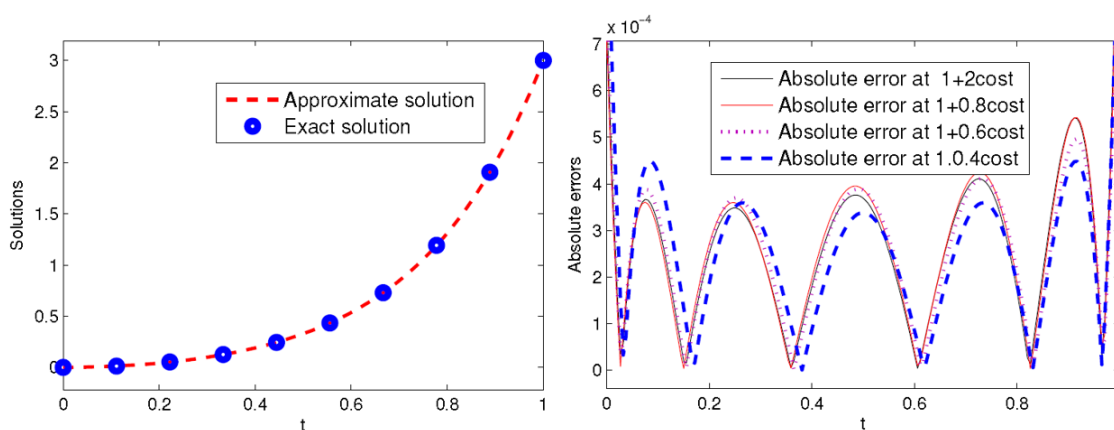

Figure 3. Comparison between exact and numerical solution and absolute errors at various variable fractional orders Example 4.2.
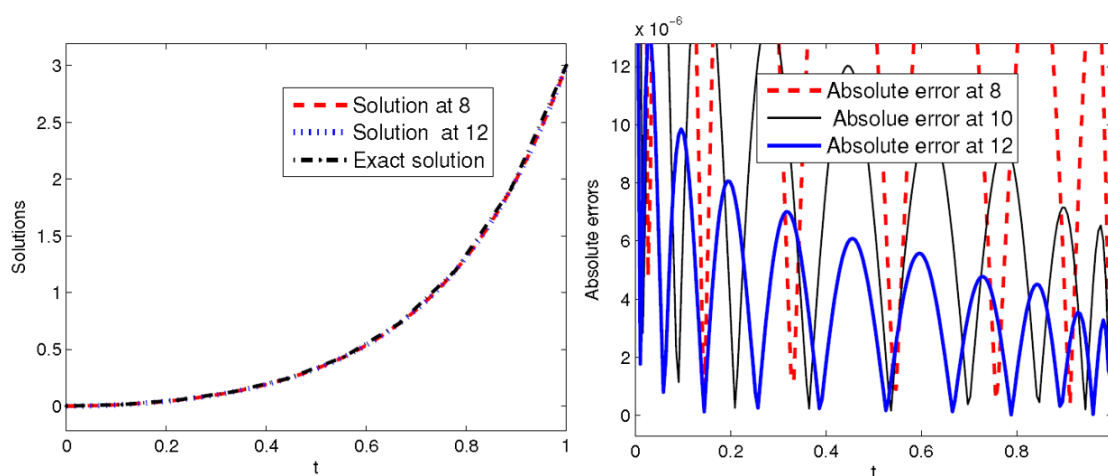

Figure 4. Comparison between exact and numerical solution and absolute errors at various scale level for Example 4.2.

From Figure 3, we see as $\varphi(t) \to 2$ the solution converges to exact solution at order 2. Also as we increase scale level the absolute error is decreasing and greater the scale level higher be the accuracy and vice versa in Figure 4.

Example 4.3. Consider the given problem with boundary conditions

$$\begin{cases} D^{\varphi(t)}v(t) = 3v(t) + f(t), \ t \in [0, 1], \ 1 < \varphi(t) \le 2, \\ v(0) = 2, \ v(1) = 3.71. \end{cases} \tag{4.3}$$

The exact solution of the problem at $\varphi(t)$ is given by

$$v(t) = \sin(2t), \ \text{where} \ f(t) = -7\sin(2t).$$

We approximate the solution at various scale level and different variable fractional order by using the proposed method.
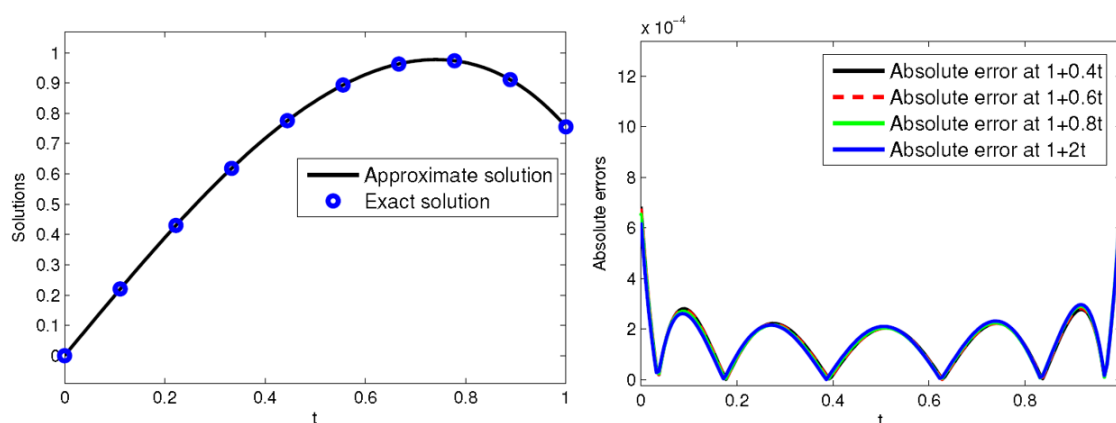

Figure 5. Comparison between exact and numerical solution and absolute errors at various variable fractional order for Example 4.3.
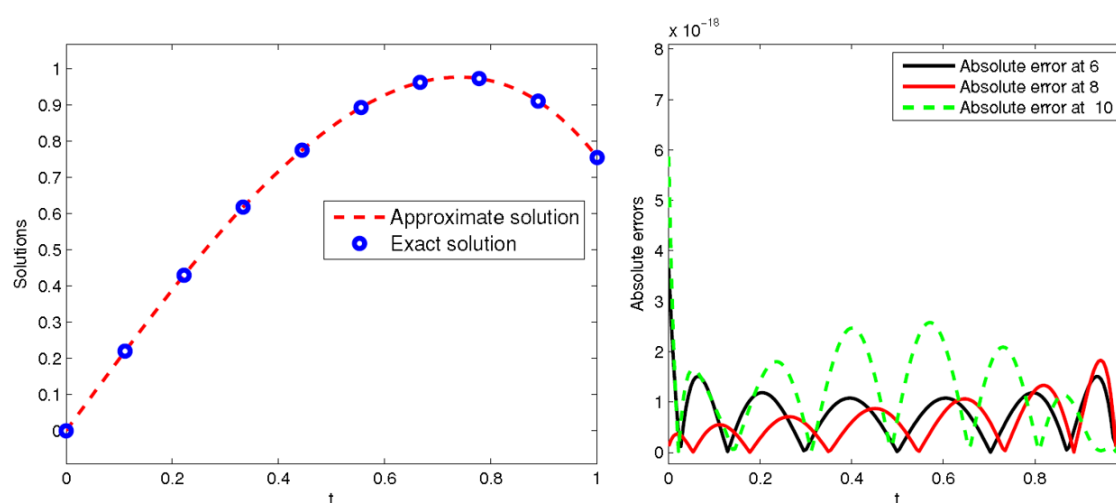

Figure 6. Comparison between exact and numerical solution and absolute errors at various scale level for Example 4.3.

From Figure 5, we see as $\varphi(t) \to 2$ the solution converges to exact solution at order 2. Also as we increase the scale level, the absolute error is decreasing and greater the scale level higher be the accuracy and vice versa. This process can be seen in Figure 6.

Remark 4.4. Here we tabulate the absolute error in above three examples 4.1–4.3 for different scale level and fixing the order in Table 1. The CPU time complexity has been recorded in seconds.

Table 1. Absolute errors at different Scale level with fractional order $\varphi(t) = 1.8$ in given three Examples 4.1–4.3.

Scale Level	Example 4.1	CPU time	Example 4.2	CPU time	Example 4.3	CPU time
6	5.0×10^{-6}	40.5	1.6×10^{-5}	48.7	8.0×10^{-12}	44.3
8	4.0×10^{-6}	50.8	1.4×10^{-5}	70.4	7.0×10^{-13}	57.6
10	3.5×10^{-6}	70.9	1.3×10^{-5}	80.6	5.0×10^{-14}	75.3
12	2×10^{-6}	90.2	1.2×10^{-5}	95.9	8.0×10^{-15}	88.8
14	1.0×10^{-6}	110.6	1.0×10^{-5}	112.4	5.0×10^{-16}	105.6

5. Conclusions and Discussion

We have established an algorithm for VOFDEs including both initial and BVPs. BPs have been applied to construct operational matrices for variable order derivative and integrations. Based on these matrices we have converted our considered problems to some algebraic type equations. By using Matlab we have solved the concerned algebraic equation to get the required numerical solution. VOFDEs have greater degree of freedom in choosing of order for more accurate results. Hence these can be used as a powerful tool to describe various real world problems more accurately as compared to constant fractional order. By graphical presentation we have shown the validity of our results. Further we see that our method is scaling based method greater the scale more will be the accuracy and vice versa. Further our method has been omitted discretization and collocation which help in saving memory from wasting. We have also computed some theoretical analysis of time complexity in Table 1 which indicates that as the scale level is enlarging, the corresponding time is also increase. This is the weakness of the method but not a major drawback as the precision and accuracy is increasing. In future the above technique will be used for numerical treatment of various dynamical systems under different kinds of fractional order operators.

Conflict of interest

The authors declare no conflict of interest.

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