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Research article

On Caputo fractional derivative inequalities by using strongly $(\alpha, h-m)$-convexity

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Abstract: In the literature of mathematical inequalities, one can have different variants of the well-known Hadamard inequality for CFD (Caputo fractional derivatives). These variants include generalizations, extensions and refinements which have been proved by defining new classes of functions. This paper aims to formulate inequalities of the Hadamard type which will simultaneously produce refinements and generalizations of many fractional versions of such inequalities that already exist in the literature. The error bounds of some existing inequalities are also proved by applying well-known identities. The results given in this paper are improvements of several well-known Hadamard type Caputo fractional derivative inequalities.

Keywords: Caputo fractional derivative; Hadamard inequality; strongly $(\alpha, h-m)$-convex function; convex function; strongly convex function
Mathematics Subject Classification: 26D10, 26A33, 31A10

1. Introduction

The Hadamard inequality was established by Charles Hermite (1883) and Jacques Hadamard (1893); they discovered it independently, see [1, 2]. In recent years along with other celebrated mathematical inequalities, various authors have studied the Hadamard inequality very frequently. Especially, it is analyzed for fractional integral and derivative operators at large number, see [3–18] and references therein.

A very useful notion of convex function was defined at the start of twentieth century due to the Hadamard inequality, it provides necessary and sufficient conditions for a function to be convex. A lot of variants of this inequality have been established by applying functions directly linked with convex function. The mathematical inequality (1.1) satisfied by convex functions, motivates to give new definitions which broaden the notion of convexity elegantly. For instance the notions of m-convex, h-convex, (α, m)-convex, (s, m)-convex, $(h-m)$-convex, p-convex, (p, h)-convex and strongly convex functions have been established in literature by modifying the inequality (1.1) conveniently. The variants of Hadamard inequality for these functions provide generalizations and refinements of the classical variant.

The main goal of this paper is to study the Caputo fractional derivative versions of the Hadamard inequality for strongly $(\alpha, h-m)$-convex functions which provide at the same time refinements as well as generalizations of various well-known inequalities exist in literature. The error bounds of some Caputo fractional derivative inequalities by applying well-known identities are given. Next, we give preliminary definitions which are useful for establishing the results of this paper.

Definition 1.1. [19] A function $\zeta : J \to \mathbb{R}$ is called convex if it satisfies the inequality:

$$\zeta(xu + (1-u)y) \leq u\zeta(x) + (1-u)\zeta(y), \tag{1.1}$$

where $u \in [0, 1]$ and $x, y \in J$, J is an interval in $\mathbb{R}$.

If (1.1) holds in the reverse order, then ζ is called concave function.

Convex functions can be visualized elegantly by the Hadamard inequality. It is stated as follows:

Theorem 1.2. *[20] Suppose that $\zeta : J \to \mathbb{R}$ is a convex function. For $a, b \in J$ with $a < b$, we have the following inequality*

$$\zeta\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \zeta(x)dx \leq \frac{\zeta(a)+\zeta(b)}{2}. \tag{1.2}$$

If orders in (1.2) *are reversed, then ζ must be concave function.*

Convex functions have been extended in many ways to obtain the generalizations and refinements of well-known inequalities. For example, operator m-convex functions are defined in [21], m-convex functions on set valued functions are defined in [22]. A recently introduced definition of $(\alpha, h-m)$ convex function unifies the well-known classes of (s, m)-convex, $(h-m)$-convex and (α, m)-convex functions along with many other kinds of convexities.

Definition 1.3. [23] Let $h : J \to \mathbb{R}$ be a non-negative function, where $J \subseteq \mathbb{R}$ is an interval that contains $(0, 1)$. A function $\zeta : [0, b] \to \mathbb{R}$ is called $(\alpha, h-m)$-convex function, if ζ is non-negative and satisfies the inequality:

$$\zeta(ux + m(1-u)y) \leq h(u^{\alpha})\zeta(x) + mh(1-u^{\alpha})\zeta(y) \tag{1.3}$$

for all $x, y \in [0, b]$, $u \in (0, 1)$ and $(\alpha, m) \in [0, 1]^2$.

Remark 1.4. The inequality (1.3) provides the definitions of (α, m) convex function for $h(u) = u$; $(h-m)$-convex function for $\alpha = 1$; (s, m)-convex function for $h(u) = u^s, \alpha = 1$; m-convex function for $h(u) = u, \alpha = 1$; s-convex function for $m = 1$, $\alpha = 1$, $h(u) = u^s$; h-convex function for $m = 1, \alpha = 1$; convex function for $h(u) = u, \alpha = 1, m = 1$; Godunova-Levin function for $h(u) = \frac{1}{u}, \alpha = 1, m = 1$ and s-Godunova-Levin function for $h(u) = \frac{1}{u^s}, \alpha = 1, m = 1$.

The definition of strongly convex function was introduced by Polyak as follows:

Definition 1.5. [24] Let $(\mathbb{X}, \|\cdot\|)$ be a normed space. A function $\zeta : D \subset \mathbb{X} \to \mathbb{R}$ is called strongly convex function with modulus $C \geq 0$, if it satisfies the inequality:

$$\zeta(xu + (1-u)y) \leq u\zeta(x) + (1-u)\zeta(y) - Cu(1-u)\|x-y\|^2 \tag{1.4}$$

for all $x, y \in D$, where D is a convex subset of $\mathbb{X}$ and $u \in [0,1]$.

For further properties, applications and utilization of strongly convex functions we refer the readers to [25–32]. We give the definition of strongly $(\alpha, h-m)$-convex function as follows:

Definition 1.6. [33] Let $h : J \to \mathbb{R}$ be a non-negative function, where $J \subseteq \mathbb{R}$ is an interval which contains $(0,1)$. A function $\zeta : [0,b] \to \mathbb{R}$ is called strongly $(\alpha, h-m)$-convex function with modulus $C \geq 0$, if ζ is non-negative and satisfies the inequality:

$$\zeta(ux + m(1-u)y) \leq h(u^\alpha)\zeta(x) + mh(1-u^\alpha)\zeta(y) - mCh(u^\alpha)h(1-u^\alpha)|y-x|^2 \tag{1.5}$$

for $(\alpha, m) \in [0,1]^2$, $x, y \in [0,b]$ and $u \in (0,1)$.

The inequality (1.5) provides the definitions of strongly convex and convex, strongly h-convex and h-convex, strongly m-convex and m-convex, strongly (s,m)-convex and (s,m)-convex, strongly (α,m)-convex and (α,m)-convex, strongly $(h-m)$-convex and $(h-m)$-convex functions in specific cases.

Next, we give definition of the CFD.

Definition 1.7. [34] Let $\zeta \in AC^n[a,b]$ and $n = [\Re(\gamma)] + 1$. Then CFD of order $\gamma \in \mathbb{C}$, $\Re(\gamma) > 0$ of the function ζ are defined by:

$$^{C}D^{\gamma}_{a+}\zeta(x) = \frac{1}{\Gamma(n-\gamma)}\int_a^x \frac{\zeta^{(n)}(u)}{(x-u)^{\gamma-n+1}}du, \quad x > a, \tag{1.6}$$

and

$$^{C}D^{\gamma}_{b-}\zeta(x) = \frac{(-1)^n}{\Gamma(n-\gamma)}\int_x^b \frac{\zeta^{(n)}(u)}{(u-x)^{\gamma-n+1}}du, \quad x < b. \tag{1.7}$$

If $\gamma = n \in \{1,2,3,\ldots\}$ and usual derivative of order n exists, then CFD $\left({}^{C}D^{\gamma}_{a+}\zeta\right)(x)$ coincides with $\zeta^{(n)}(x)$, whereas $\left({}^{C}D^{\gamma}_{b-}\zeta\right)(x)$ coincides with $\zeta^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$.

In particular we have

$$\left({}^{C}D^{0}_{a+}\zeta\right)(x) = \left({}^{C}D^{0}_{b-}\zeta\right)(x) = \zeta(x), \tag{1.8}$$

where $n = 1$ and $\gamma = 0$.

For establishing the Hadamard type inequalities, the definitions of Caputo derivative and strongly $(\alpha, h-m)$-convex function are the key factors. To study the error estimates of Caputo fractional Hadamard type inequalities the following two lemmas are very useful. We also utilize the well-known integral versions of the Hölder and power mean inequalities.

Lemma 1.8. *[8] Let $0 \leq a < b$ and $\zeta \in C^{n+1}[a,b]$. Then one can have the following identity for CFD*

$$\begin{aligned} &\frac{\zeta^{(n)}(a) + \zeta^{(n)}(b)}{2} - \frac{\Gamma(n-\gamma+1)}{2(b-a)^{n-\gamma}}\left[\left({}^{C}D^{\gamma}_{a^+}\zeta\right)(b) + (-1)^n\left({}^{C}D^{\gamma}_{b\text{-}}\zeta\right)(a)\right] \\ &= \frac{b-a}{2}\int_0^1 \left[(1-u)^{n-\gamma} - u^{n-\gamma}\right]\zeta^{(n+1)}(ua + (1-u)b)\,du. \end{aligned} \tag{1.9}$$

Lemma 1.9. *[9] Let $0 \leq a < mb$, $m \in (0,1]$ and $\zeta \in C^{n+1}[a,b]$. Then one can have the following identity for CFD*

$$\begin{aligned}
&\frac{2^{n-\frac{\gamma}{k}-1}k\Gamma_k(n-\frac{\gamma}{k}+k)}{(mb-a)^{n-\frac{\gamma}{k}}}\left[\left({}^{C}D^{\gamma,k}_{(\frac{a+bm}{2})^+}\zeta\right)(mb)+m^{n-\frac{\gamma}{k}+1}(-1)^n\left({}^{C}D^{\gamma,k}_{(\frac{a+mb}{2m})^-}\zeta\right)\left(\frac{a}{m}\right)\right]\\
&-\frac{1}{2}\left[\zeta^{(n)}\left(\frac{a+mb}{2}\right)+m\zeta^{(n)}\left(\frac{a+mb}{2m}\right)\right]\\
&=\frac{mb-a}{4}\left[\int_0^1 u^{n-\frac{\gamma}{k}}\zeta^{(n+1)}\left(\frac{u}{2}a+m\left(\frac{2-u}{2}\right)b\right)-\int_0^1 u^{n-\frac{\gamma}{k}}\zeta^{(n+1)}\left(\frac{2-u}{2m}a+\frac{u}{2}b\right)du\right] \qquad (1.10)
\end{aligned}$$

with $\gamma > 0$.

The rest of paper is composed of the following sections: In Section 2, we give refinements of two Hadamard type inequalities for strongly $(\alpha, h-m)$-convex functions via CFD. It is noted that the particular cases of the main results are connected with classical results already published in the near past publish articles. In Section 3, by applying the identities (1.9) and (1.10), we find improvements of error bounds of various inequalities.

2. Main results

The following result provides Hadamard inequality for strongly $(\alpha, h-m)$-convex function for CFD. It will reproduce many CFD inequalities for various kinds of convex functions.

Theorem 2.1. *Suppose that $\zeta \in C^n[a,b]$ and $\zeta^{(n)}$ satisfies the inequality (1.5). Then it must satisfy the CFD inequality:*

$$\begin{aligned}
&\zeta^{(n)}\left(\frac{bm+a}{2}\right)+\frac{mC(n-\gamma)}{(n-\gamma+2)}h\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)\left\{(b-a)^2+\frac{2(b-a)(\frac{a}{m}-mb)}{(n-\gamma+1)}+\frac{2\left(\frac{a}{m}-mb\right)^2}{(n-\gamma)(n-\gamma+1)}\right\}\\
&\leq\frac{\Gamma(n-\gamma+1)}{(bm-a)^{n-\gamma}}\left[h\left(1-\frac{1}{2^\alpha}\right)m^{n-\gamma+1}(-1)^n\left({}^{C}D^{\gamma}_{b^-}\zeta\right)\left(\frac{a}{m}\right)+h\left(\frac{1}{2^\alpha}\right)\left({}^{C}D^{\gamma}_{a^+}\zeta\right)(mb)\right]\\
&\leq(n-\gamma)\left\{\left[h\left(1-\frac{1}{2^\alpha}\right)m\zeta^{(n)}(b)+h\left(\frac{1}{2^\alpha}\right)\zeta^{(n)}(a)\right]\int_0^1 h(u^\alpha)u^{n-\gamma-1}du\right.\\
&+\left[h\left(1-\frac{1}{2^\alpha}\right)m^2\zeta^{(n)}\left(\frac{a}{m^2}\right)+h\left(\frac{1}{2^\alpha}\right)m\zeta^{(n)}(b)\right]\int_0^1 h(1-u^\alpha)u^{n-\gamma-1}du\\
&\left.-Cm\left[(b-a)^2+m\left(b-\frac{a}{m^2}\right)^2\right]\int_0^1 h(u^\alpha)h(1-u^\alpha)u^{n-\gamma-1}du\right\}, \qquad (2.1)
\end{aligned}$$

where $m \in (0,1]$, $0 \leq a < mb$ and $\gamma > 0$.

Proof. As $\zeta^{(n)}$ is strongly $(\alpha, h-m)$-convex function, from (1.5) one can easily have

$$\zeta^{(n)}\left(\frac{mx+y}{2}\right)\leq h\left(1-\frac{1}{2^\alpha}\right)m\zeta^{(n)}(x)+h\left(\frac{1}{2^\alpha}\right)\zeta^{(n)}(y)-mCh\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)|x-y|^2. \qquad (2.2)$$

Let $x=(1-u)\frac{a}{m}+ub\leq b$ and $y=m(1-u)b+ua\geq a$, where $u\in[0,1]$. Then one can have

$$\zeta^{(n)}\left(\frac{bm+a}{2}\right)\leq h\left(1-\frac{1}{2^{\alpha}}\right)m\zeta^{(n)}\left((1-u)\frac{a}{m}+ub\right)+h\left(\frac{1}{2^{\alpha}}\right)\zeta^{(n)}(m(1-u)b+ua)$$
$$-mCh\left(\frac{1}{2^{\alpha}}\right)h\left(1-\frac{1}{2^{\alpha}}\right)\left((1-u)\frac{a}{m}+ub-(m(1-u)b+ua)\right)^{2}. \tag{2.3}$$

By multiplying (2.3) with $u^{n-\gamma-1}$ and integrating over $[0,1]$, we obtain

$$\zeta^{(n)}\left(\frac{bm+a}{2}\right)\int_0^1 u^{n-\gamma-1}du$$
$$\leq h\left(1-\frac{1}{2^{\alpha}}\right)\left[\int_0^1 m\zeta^{(n)}((1-u)\frac{a}{m}+ub)u^{n-\gamma-1}du+h\left(\frac{1}{2^{\alpha}}\right)\int_0^1\zeta^{(n)}(m(1-u)b+ua)u^{n-\gamma-1}du\right]$$
$$-mCh\left(\frac{1}{2^{\alpha}}\right)h\left(1-\frac{1}{2^{\alpha}}\right)\int_0^1\left(u^{n-\gamma-1}\left((1-u)\frac{a}{m}+ub-(m(1-u)b+ua)\right)^{2}\right)du. \tag{2.4}$$

By applying Definition 1.7 and after a little computation, we have

$$\zeta^{(n)}\left(\frac{bm+a}{2}\right)\leq\frac{\Gamma(n-\gamma+1)}{(bm-a)^{n-\gamma}}\left[h\left(1-\frac{1}{2^{\alpha}}\right)m^{n-\gamma+1}(-1)^n\left({}^{C}D^{\gamma}_{b^-}\zeta\right)\left(\frac{a}{m}\right)+h\left(\frac{1}{2^{\alpha}}\right)\left({}^{C}D^{\gamma}_{a^+}\zeta\right)(mb)\right]$$
$$-mC(n-\gamma)h\left(\frac{1}{2^{\alpha}}\right)h\left(1-\frac{1}{2^{\alpha}}\right)$$
$$\times\left\{\frac{(b-a)^2}{n-\gamma+2}+\frac{2(b-a)(\frac{a}{m}-mb)}{(n-\gamma+1)(n-\gamma+2)}+\frac{2\left(\frac{a}{m}-mb\right)^2}{(n-\gamma)(n-\gamma+1)(n-\gamma+2)}\right\}. \tag{2.5}$$

As $\zeta^{(n)}$ is strongly $(\alpha,h-m)$-convex function, from (1.5) one can easily have

$$m\zeta^{(n)}\left((1-u)\frac{a}{m}+ub\right)\leq mh(u^{\alpha})\zeta^{(n)}(b)+m^2h(1-u^{\alpha})\zeta^{(n)}\left(\frac{a}{m^2}\right)-Cm^2h(u^{\alpha})h(1-u^{\alpha})\left(b-\frac{a}{m^2}\right)^2. \tag{2.6}$$

By multiplying (2.6) with $h\left(1-\frac{1}{2^{\alpha}}\right)u^{n-\gamma-1}$ and integrating over $[0,1]$ we obtain

$$h\left(1-\frac{1}{2^{\alpha}}\right)\int_0^1 m\zeta^{(n)}\left((1-u)\frac{a}{m}+ub\right)u^{n-\gamma-1}du$$
$$\leq h\left(1-\frac{1}{2^{\alpha}}\right)\left\{\int_0^1 m\zeta^{(n)}(b)h(u^{\alpha})u^{n-\gamma-1}du+\int_0^1 m^2\zeta^{(n)}\left(\frac{a}{m^2}\right)h(1-u^{\alpha})u^{n-\gamma-1}du\right.$$
$$\left.-Cm^2\left(b-\frac{a}{m^2}\right)^2\int_0^1 h(u^{\alpha})h(1-u^{\alpha})u^{n-\gamma-1}du\right\}. \tag{2.7}$$

By using Definition 1.7, one can have

$$\frac{\Gamma(n-\gamma+1)}{(bm-a)^{n-\gamma}}\left[h\left(1-\frac{1}{2^{\alpha}}\right)m^{n-\gamma+1}(-1)^n\left({}^{C}D^{\gamma}_{b^-}\zeta\right)\left(\frac{a}{m}\right)\right]$$
$$\leq h\left(1-\frac{1}{2^{\alpha}}\right)(n-\gamma)\left\{m\zeta^{(n)}(b)\int_0^1 h(u^{\alpha})u^{n-\gamma-1}du+m^2\zeta^{(n)}\left(\frac{a}{m^2}\right)\int_0^1 h(1-u^{\alpha})u^{n-\gamma-1}du\right.$$

$$-Cm^2\left(b-\frac{a}{m^2}\right)^2\int_0^1 h(u^\alpha)h(1-u^\alpha)u^{n-\gamma-1}du\Bigg\}. \tag{2.8}$$

Again by using strongly $(\alpha, h-m)$-convexity for the function $\zeta^{(n)}$ and adopting the same pattern as we did for inequality (2.6), we have the following inequality

$$\begin{aligned}&\frac{\Gamma(n-\gamma+1)}{(bm-a)^{n-\gamma}}\left[h\left(\frac{1}{2^\alpha}\right)\left({}^CD^\gamma_{a^+}\zeta\right)(mb)\right]\\&\leq(n-\gamma)h\left(\frac{1}{2^\alpha}\right)\Bigg\{\zeta^{(n)}(a)\int_0^1 h(u^\alpha)u^{n-\gamma-1}du+m\zeta^{(n)}(b)\int_0^1 h(1-u^\alpha)u^{n-\gamma-1}du\\&\quad-Cm(b-a)^2\int_0^1 h(u^\alpha)h(1-u^\alpha)u^{n-\gamma-1}du\Bigg\}.\end{aligned} \tag{2.9}$$

By adding (2.8) and (2.9), we have

$$\begin{aligned}&\frac{\Gamma(n-\gamma+1)}{(bm-a)^{n-\gamma}}\left[h\left(1-\frac{1}{2^\alpha}\right)m^{n-\gamma+1}(-1)^n\left({}^CD^\gamma_{b^-}\zeta\right)\left(\frac{a}{m}\right)+h\left(\frac{1}{2^\alpha}\right)\left({}^CD^\gamma_{a^+}\zeta\right)(mb)\right]\\&\leq(n-\gamma)\left\{\left[h\left(1-\frac{1}{2^\alpha}\right)m\zeta^{(n)}(b)+h\left(\frac{1}{2^\alpha}\right)\zeta^{(n)}(a)\right]\int_0^1 h(u^\alpha)u^{n-\gamma-1}du\right.\\&\quad+\left[h\left(1-\frac{1}{2^\alpha}\right)m^2\zeta^{(n)}\left(\frac{a}{m^2}\right)+h\left(\frac{1}{2^\alpha}\right)m\zeta^{(n)}(b)\right]\int_0^1 h(1-u^\alpha)u^{n-\gamma-1}du\\&\quad\left.-Cm\left[(b-a)^2+m\left(b-\frac{a}{m^2}\right)^2\right]\int_0^1 h(u^\alpha)h(1-u^\alpha)u^{n-\gamma-1}du\right\}.\end{aligned} \tag{2.10}$$

Inequalities (2.5) and (2.10) constituted the required inequality. □

The following result gives another variant of Hadamard inequality for strongly $(\alpha, h-m)$-convex function for CFD. It will reproduce many CFD inequalities for various kinds of convex functions.

Theorem 2.2. *Under the supposition of Theorem 2.1, the following inequality holds:*

$$\begin{aligned}&\zeta^{(n)}\left(\frac{bm+a}{2}\right)+\frac{mC(n-\gamma)}{2(n-\gamma+2)}h\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)\\&\quad\times\left\{\frac{(b-a)^2}{2}+\frac{(b-a)(\frac{a}{m}-mb)(n-\gamma+3)}{(n-\gamma+1)}+\frac{(\frac{a}{m}-mb)^2[(n-\gamma)^2+5n-5\gamma+8]}{2(n-\gamma)(n-\gamma+1)}\right\}\\&\leq\frac{2^{n-\gamma}\Gamma(n-\gamma+1)}{(bm-a)^{n-\gamma}}\left[h\left(1-\frac{1}{2^\alpha}\right)m^{n-\gamma+1}(-1)^n\left({}^CD^\gamma_{(\frac{a+bm}{2m})^-}\zeta\right)\left(\frac{a}{m}\right)+h\left(\frac{1}{2^\alpha}\right)\left({}^CD^\gamma_{(\frac{a+bm}{2})^+}\zeta\right)(mb)\right]\\&\leq(n-\gamma)\left\{\left[h\left(1-\frac{1}{2^\alpha}\right)m^2\,\zeta^{(n)}\left(\frac{a}{m^2}\right)+h\left(\frac{1}{2^\alpha}\right)m\zeta^{(n)}(b)\right]\int_0^1 h\left(1-\left(\frac{u}{2}\right)^\alpha\right)u^{n-\gamma-1}du\right.\\&\quad+\left[h\left(1-\frac{1}{2^\alpha}\right)m\zeta^{(n)}(b)+h\left(\frac{1}{2^\alpha}\right)\zeta^{(n)}(a)\right]\int_0^1 h\left(\frac{u}{2}\right)^\alpha u^{n-\gamma-1}du\\&\quad\left.-mC\left((b-a)^2+m\left(b-\frac{a}{m^2}\right)^2\right)\int_0^1 h\left(\frac{u}{2}\right)^\alpha h\left(1-\left(\frac{u}{2}\right)^\alpha\right)u^{n-\gamma-1}du\right\}\end{aligned} \tag{2.11}$$

with $\gamma > 0$.

Proof. Let $x = \frac{a}{m}\left(\frac{2-u}{2}\right) + b\frac{u}{2}$ and $y = a\frac{u}{2} + m\left(\frac{2-u}{2}\right)b$, where $u \in [0,1]$ in (2.2). Then one can have the inequality in the following form

$$\zeta^{(n)}\left(\frac{bm+a}{2}\right) \le h\left(1-\frac{1}{2^\alpha}\right)m\zeta^{(n)}\left(\frac{a}{m}\left(\frac{2-u}{2}\right)+b\frac{u}{2}\right) + h\left(\frac{1}{2^\alpha}\right)\zeta^{(n)}\left(a\frac{u}{2}+m\left(\frac{2-u}{2}\right)b\right) - mCh\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)\left(\frac{a}{m}\left(\frac{2-u}{2}\right)+b\frac{u}{2}-a\frac{u}{2}-m\left(\frac{2-u}{2}\right)b\right)^2. \tag{2.12}$$

By multiplying (2.12) with $u^{n-\gamma-1}$ and then integrating over $[0,1]$, one can have

$$\begin{aligned}&\zeta^{(n)}\left(\frac{bm+a}{2}\right)\int_0^1 u^{n-\gamma-1}du \\ &\le \int_0^1\left(h\left(1-\frac{1}{2^\alpha}\right)m\zeta^{(n)}\left(\frac{a}{m}\left(\frac{2-u}{2}\right)+b\frac{u}{2}\right)+h\left(\frac{1}{2^\alpha}\right)\zeta^{(n)}\left(a\frac{u}{2}+m\left(\frac{2-u}{2}\right)b\right)\right)u^{n-\gamma-1}du \\ &\quad - mCh\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)\int_0^1\left(\frac{a}{m}\left(\frac{2-u}{2}\right)+b\frac{u}{2}-a\frac{u}{2}-m\left(\frac{2-u}{2}\right)b\right)^2u^{n-\gamma-1}du.\end{aligned} \tag{2.13}$$

By using Definition 1.7, we have

$$\begin{aligned}\zeta^{(n)}\left(\frac{bm+a}{2}\right) \le{}& \frac{2^{n-\gamma}\Gamma(n-\gamma+1)}{(bm-a)^{n-\gamma}}\left[h\left(1-\frac{1}{2^\alpha}\right)m^{n-\gamma+1}(-1)^n\left({}^CD^\gamma_{(\frac{a+bm}{2m})^-}\zeta\right)\left(\frac{a}{m}\right)+h\left(\frac{1}{2^\alpha}\right)\left({}^CD^\gamma_{(\frac{a+bm}{2})^+}\zeta\right)(mb)\right] \\ &-\frac{mC(n-\gamma)}{2(n-\gamma+2)}h\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)\left\{\frac{(b-a)^2}{2}+\frac{(b-a)(\frac{a}{m}-mb)(n-\gamma+3)}{(n-\gamma+1)}\right. \\ &\left.+\frac{(\frac{a}{m}-mb)^2[(n-\gamma)^2+5n-5\gamma+8]}{2(n-\gamma)(n-\gamma+1)}\right\}.\end{aligned} \tag{2.14}$$

As $\zeta^{(n)}$ is strongly $(\alpha, h-m)$-convex function, from (1.5) one can easily have

$$m\zeta^{(n)}\left(\frac{a}{m}\left(\frac{2-u}{2}\right)+b\frac{u}{2}\right) \le m^2h\left(1-\left(\frac{u}{2}\right)^\alpha\right)\zeta^{(n)}\left(\frac{a}{m^2}\right)+mh\left(\frac{u}{2}\right)^\alpha\zeta^{(n)}(b) - m^2Ch\left(\frac{u}{2}\right)^\alpha h\left(1-\left(\frac{u}{2}\right)^\alpha\right)\left(b-\frac{a}{m^2}\right)^2. \tag{2.15}$$

By multiplying (2.15) with $h\left(1-\frac{1}{2^\alpha}\right)u^{n-\gamma-1}$ and integrating over $[0,1]$, one can have

$$\begin{aligned}&h\left(1-\frac{1}{2^\alpha}\right)\int_0^1 m\zeta^{(n)}\left(\frac{a}{m}\left(\frac{2-u}{2}\right)+b\frac{u}{2}\right)u^{n-\gamma-1}du \\ &\le h\left(1-\frac{1}{2^\alpha}\right)\left(\int_0^1 m^2h\left(1-\left(\frac{u}{2}\right)^\alpha\right)\zeta^{(n)}\left(\frac{a}{m^2}\right)u^{n-\gamma-1}du+\int_0^1 mh\left(\frac{u}{2}\right)^\alpha\zeta^{(n)}(b)u^{n-\gamma-1}du\right) \\ &\quad - h\left(1-\frac{1}{2^\alpha}\right)\int_0^1 mCh\left(\frac{u}{2}\right)^\alpha h\left(1-\left(\frac{u}{2}\right)^\alpha\right)\left[m\left(b-\frac{a}{m^2}\right)^2\right]u^{n-\gamma-1}du.\end{aligned} \tag{2.16}$$

With the help of Definition 1.7, one can see that

$$\frac{2^{n-\gamma}\Gamma(n-\gamma+1)}{(bm-a)^{n-\gamma}}\left[h\left(1-\frac{1}{2^\alpha}\right)m^{n-\gamma+1}(-1)^n\left({}^CD^\gamma_{(\frac{a+bm}{2m})^-}\zeta\right)\left(\frac{a}{m}\right)\right]$$

$$\leq (n-\gamma)h\left(1-\frac{1}{2^{\alpha}}\right)\left\{m^2\zeta^{(n)}\left(\frac{a}{m^2}\right)\int_0^1 h\left(1-\left(\frac{u}{2}\right)^{\alpha}\right)u^{n-\gamma-1}du+m\zeta^{(n)}(b)\int_0^1 h\left(\frac{u}{2}\right)^{\alpha}u^{n-\gamma-1}du\right.$$
$$\left.-mC\left(m\left(b-\frac{a}{m^2}\right)^2\right)\int_0^1 h\left(\frac{u}{2}\right)^{\alpha}h\left(1-\left(\frac{u}{2}\right)^{\alpha}\right)u^{n-\gamma-1}du\right\}. \tag{2.17}$$

Also, using the definition of strongly $(\alpha, h-m)$-convex function $\zeta^{(n)}$ and working on the same lines as we did for the inequality (2.15), one can have the following inequality

$$\frac{2^{n-\gamma}\Gamma(n-\gamma+1)}{(bm-a)^{n-\gamma}}\left[h\left(\frac{1}{2^{\alpha}}\right)\left({}^{C}D^{\gamma}_{(\frac{a+bm}{2})^{+}}\zeta\right)(mb)\right]$$
$$\leq h\left(\frac{1}{2^{\alpha}}\right)(n-\gamma)\left\{\zeta^{(n)}(a)\int_0^1 h\left(\frac{u}{2}\right)^{\alpha}u^{n-\gamma-1}du+m\zeta^{(n)}(b)\int_0^1 h\left(1-\left(\frac{u}{2}\right)^{\alpha}\right)u^{n-\gamma-1}du\right.$$
$$\left.-mC(b-a)^2\int_0^1 h\left(\frac{u}{2}\right)^{\alpha}h\left(1-\left(\frac{u}{2}\right)^{\alpha}\right)u^{n-\gamma-1}du\right\}. \tag{2.18}$$

By adding (2.17) and (2.18), we get

$$\frac{2^{n-\gamma}\Gamma(n-\gamma+1)}{(bm-a)^{n-\gamma}}\left[h\left(1-\frac{1}{2^{\alpha}}\right)m^{n-\gamma+1}(-1)^n\left({}^{C}D^{\gamma}_{(\frac{a+bm}{2m})^{-}}\zeta\right)\left(\frac{a}{m}\right)+h\left(\frac{1}{2^{\alpha}}\right)\left({}^{C}D^{\gamma}_{(\frac{a+bm}{2})^{+}}\zeta\right)(mb)\right]$$
$$\leq (n-\gamma)\left\{\left[h\left(1-\frac{1}{2^{\alpha}}\right)m^2\zeta^{(n)}\left(\frac{a}{m^2}\right)+h\left(\frac{1}{2^{\alpha}}\right)m\zeta^{(n)}(b)\right]\int_0^1 h\left(1-\left(\frac{u}{2}\right)^{\alpha}\right)u^{n-\gamma-1}du\right.$$
$$+\left[h\left(1-\frac{1}{2^{\alpha}}\right)m\zeta^{(n)}(b)+h\left(\frac{1}{2^{\alpha}}\right)\zeta^{(n)}(a)\right]\int_0^1 h\left(\frac{u}{2}\right)^{\alpha}u^{n-\gamma-1}du$$
$$\left.-mC\left((b-a)^2+m\left(b-\frac{a}{m^2}\right)^2\right)\int_0^1 h\left(\frac{u}{2}\right)^{\alpha}h\left(1-\left(\frac{u}{2}\right)^{\alpha}\right)u^{n-\gamma-1}du\right\}. \tag{2.19}$$

From (2.14) and (2.19), (2.11) can be obtained. □

Theorems 2.1 and 2.2 are connected with many results, which are explained in the following remark.

Remark 2.3. The inequalities (2.1) and (2.11) provide the fractional Hadamard inequalities for strongly (α, m) convex function for $h(u) = u$; strongly $(h-m)$-convex function for $\alpha = 1$; strongly (s, m)-convex function for $h(u) = u^s, \alpha = 1$; strongly m-convex function for $h(u) = u, \alpha = 1$; strongly convex function for $h(u) = u, \alpha = 1, m = 1$. For $C = 0$ the inequalities (2.1) and (2.11) give the Hadamard inequalities for $(\alpha, h-m)$-convex function which further gives the Hadamard inequality for: Convex, s-convex, h-convex, m-convex, (α, m)-convex, $(h-m)$-convex and (s, m)-convex functions respectively.

3. More fractional Hadamard type inequalities for CFD

Here we give Hadamard type inequalities by using identities for CFD, which are stated in Lemmas 1.8 and 1.9.

Theorem 3.1. *Suppose that $\zeta \in C^{n+1}[a,b]$ and $|\zeta^{(n+1)}|$ satisfies the inequality* (1.5) *and $h(x+y) \leq h(x)h(y)$. Then it satisfies the CFD inequality:*

$$\left|\frac{\zeta^{(n)}(a)+\zeta^{(n)}(b)}{2} - \frac{\Gamma(n-\gamma+1)}{2(b-a)^{n-\gamma}}\left[\left({}^{C}D^{\gamma}_{a^+}\zeta\right)(b) + (-1)^n\left({}^{C}D^{\gamma}_{b^-}\zeta\right)(a)\right]\right|$$
$$\leq \frac{b-a}{2}\left\{\left[\frac{(2^{np-\gamma p+1}-1)^{\frac{1}{p}}-1}{2^{n-\gamma+\frac{1}{p}}(np-\gamma p+1)^{\frac{1}{p}}}\right]\left[\left|\zeta^{(n+1)}(a)\right|\left(\left(\int_0^{\frac{1}{2}}(h(u^\alpha))^q du\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 (h(u^\alpha))^q du\right)^{\frac{1}{q}}\right)\right.\right.$$
$$\left.\left. + m\left|\zeta^{(n+1)}\left(\frac{b}{m}\right)\right|\left(\left(\int_0^{\frac{1}{2}}(h(1-u^\alpha))^q du\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 (h(1-u^\alpha))^q du\right)^{\frac{1}{q}}\right)\right] - \frac{Cmh(1)\left(\frac{b}{m}-a\right)^2(2^{n-\gamma}-1)}{2^{n-\gamma}(n-\gamma+1)}\right\} \tag{3.1}$$

with $0 \leq a < mb, \gamma > 0$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. As $|\zeta^{(n+1)}|$ is strongly $(\alpha, h-m)$-convex function, one can have

$$|\zeta^{(n+1)}(ua+(1-u)b)| = \left|\zeta^{(n+1)}\left(ua+m(1-u)\frac{b}{m}\right)\right|$$
$$\leq h(u^\alpha)|\zeta^{(n+1)}(a)| + mh(1-u^\alpha)\left|\zeta^{(n+1)}\left(\frac{b}{m}\right)\right| - Cmh(u^\alpha)h(1-u^\alpha)\left(\frac{b}{m}-a\right)^2.$$

From the definition of strongly $(\alpha, h-m)$-convexity of $|\zeta^{(n+1)}|$ and Lemma 1.8, one can have

$$\left|\frac{\zeta^{(n)}(a)+\zeta^{(n)}(b)}{2} - \frac{\Gamma(n-\gamma+1)}{2(b-a)^{n-\gamma}}\left[\left({}^{C}D^{\gamma}_{a^+}\zeta\right)(b) + (-1)^n\left({}^{C}D^{\gamma}_{b^-}\zeta\right)(a)\right]\right|$$
$$\leq \frac{b-a}{2}\int_0^1 \left|(1-u)^{n-\gamma}-u^{n-\gamma}\right|\left|\zeta^{(n+1)}\left(ua+m(1-u)\frac{b}{m}\right)\right| du$$
$$\leq \frac{b-a}{2}\int_0^1 \left|(1-u)^{n-\gamma}-u^{n-\gamma}\right|\left(h(u^\alpha)|\zeta^{(n+1)}(a)| + mh(1-u^\alpha)\left|\zeta^{(n+1)}\left(\frac{b}{m}\right)\right|\right.$$
$$\left. - Cmh(u^\alpha)h(1-u^\alpha)\left(\frac{b}{m}-a\right)^2\right) du$$
$$\leq \frac{b-a}{2}\left[\int_0^{1/2}((1-u)^{n-\gamma}-u^{n-\gamma})\left(h(u^\alpha)|\zeta^{(n+1)}(a)| + mh(1-u^\alpha)\left|\zeta^{(n+1)}\left(\frac{b}{m}\right)\right|\right.\right.$$
$$\left. - Cmh(u^\alpha)h(1-u^\alpha)\left(\frac{b}{m}-a\right)^2\right) du + \int_{1/2}^1 (u^{n-\gamma}-(1-u)^{n-\gamma})$$
$$\left. \times\left(h(u^\alpha)|\zeta^{(n+1)}(a)| + mh(1-u^\alpha)\left|\zeta^{(n+1)}\left(\frac{b}{m}\right)\right| - Cmh(u^\alpha)h(1-u^\alpha)\left(\frac{b}{m}-a\right)^2\right) du\right]. \tag{3.2}$$

Now, by applying Hölder's inequality on the right side of inequality (3.2), we have

$$\int_0^{1/2}((1-u)^{n-\gamma}-u^{n-\gamma})\left(h(u^\alpha)|\zeta^{(n+1)}(a)| + mh(1-u^\alpha)\left|\zeta^{(n+1)}\left(\frac{b}{m}\right)\right| - Cmh(u^\alpha)h(1-u^\alpha)\left(\frac{b}{m}-a\right)^2\right) du$$

$$\leq \int_0^{1/2} ((1-u)^{n-\gamma} - u^{n-\gamma}) \left(h(u^\alpha)|\zeta^{(n+1)}(a)| + mh(1-u^\alpha) \left| \zeta^{(n+1)}\left(\frac{b}{m}\right) \right| - Cmh(1)\left(\frac{b}{m} - a\right)^2 \right) du$$

$$= |\zeta^{(n+1)}(a)| \left[\left(\frac{1-(1/2)^{np-\gamma p+1}}{np-\gamma p+1} \right)^{\frac{1}{p}} - \left(\frac{1}{2^{np-\gamma p+1}(np-\gamma p+1)} \right)^{\frac{1}{p}} \right] \left(\int_0^{\frac{1}{2}} (h(u^\alpha))^q du \right)^{\frac{1}{q}}$$

$$+ m \left| \zeta^{(n+1)}\left(\frac{b}{m}\right) \right| \left[\left(\frac{1-(1/2)^{np-\gamma p+1}}{np-\gamma p+1} \right)^{\frac{1}{p}} - \left(\frac{1}{2^{np-\gamma p+1}(np-\gamma p+1)} \right)^{\frac{1}{p}} \right]$$

$$\times \left(\int_0^{\frac{1}{2}} (h(1-u^\alpha))^q du \right)^{\frac{1}{q}} - \frac{Cmh(1)\left(\frac{b}{m}-a\right)^2 (2^{n-\gamma}-1)}{2^{n-\gamma}(n-\gamma+1)} \tag{3.3}$$

and

$$\int_{1/2}^1 (u^{n-\gamma} - (1-u)^{n-\gamma}) \left(h(u^\alpha)|\zeta^{(n+1)}(a)| + mh(1-u^\alpha) \left| \zeta^{(n+1)}\left(\frac{b}{m}\right) \right| - Cmh(u^\alpha)h(1-u^\alpha)\left(\frac{b}{m} - a\right)^2 \right) du$$

$$\leq \int_{1/2}^1 (u^{n-\gamma} - (1-u)^{n-\gamma}) \left(h(u^\alpha)|\zeta^{(n+1)}(a)| + mh(1-u^\alpha) \left| \zeta^{(n+1)}\left(\frac{b}{m}\right) \right| - Cmh(1)\left(\frac{b}{m} - a\right)^2 \right) du$$

$$= |\zeta^{(n+1)}(a)| \left[\left(\frac{1-(1/2)^{np-\gamma p+1}}{np-\gamma p+1} \right)^{\frac{1}{p}} - \left(\frac{1}{2^{np-\gamma p+1}(np-\gamma p+1)} \right)^{\frac{1}{p}} \right] \left(\int_{\frac{1}{2}}^1 (h(u^\alpha))^q du \right)^{\frac{1}{q}}$$

$$+ m \left| \zeta^{(n+1)}\left(\frac{b}{m}\right) \right| \left[\left(\frac{1-(1/2)^{np-\gamma p+1}}{np-\gamma p+1} \right)^{\frac{1}{p}} - \left(\frac{1}{2^{np-\gamma p+1}(np-\gamma p+1)} \right)^{\frac{1}{p}} \right]$$

$$\times \left(\int_{\frac{1}{2}}^1 (h(1-u^\alpha))^q du \right)^{\frac{1}{q}} - \frac{Cmh(1)\left(\frac{b}{m}-a\right)^2 (2^{n-\gamma}-1)}{2^{n-\gamma}(n-\gamma+1)}. \tag{3.4}$$

By using values of integrals from (3.3) and (3.4) in (3.2), we obtain the inequality (3.1). □

Corollary 3.2. *The class of* (α, m)*-convex functions satisfies the CFD inequality as follows:*

$$\left| \frac{\zeta^{(n)}(a) + \zeta^{(n)}(b)}{2} - \frac{\Gamma(n-\gamma+1)}{2(b-a)^{n-\gamma}} \left[\left({}^C D^\gamma_{a^+} \zeta \right)(b) + (-1)^n \left({}^C D^\gamma_{b^-} \zeta \right)(a) \right] \right|$$

$$\leq \frac{b-a}{2} \left\{ \left[\frac{(2^{np-\gamma p+1}-1)^{\frac{1}{p}} - 1}{2^{n-\gamma+\frac{1}{p}}(np-\gamma p+1)^{\frac{1}{p}}} \right] \left[|\zeta^{(n+1)}(a)| \left(\frac{1+(2^{\alpha q+1}-1)^{\frac{1}{q}}}{(2^{\alpha q+1}(\alpha q+1))^{\frac{1}{q}}} \right) \right. \right.$$

$$\left. \left. + m \left| \zeta^{(n+1)}\left(\frac{b}{m}\right) \right| \left(\left(\int_0^{\frac{1}{2}} ((1-u^\alpha))^q du \right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 ((1-u^\alpha))^q du \right)^{\frac{1}{q}} \right) \right] \right\}. \tag{3.5}$$

Proof. In the inequality (3.1) of Theorem 3.1 if one set $C = 0$ and $h(u) = u$, we obtain the inequality (3.5). □

Corollary 3.3. *The class of m-convex functions satisfies CFD inequality as follows:*

$$\left| \frac{\zeta^{(n)}(a) + \zeta^{(n)}(b)}{2} - \frac{\Gamma(n-\gamma+1)}{2(b-a)^{n-\gamma}} \left[\left({}^C D^\gamma_{a^+} \zeta \right)(b) + (-1)^n \left({}^C D^\gamma_{b^-} \zeta \right)(a) \right] \right|$$

$$\leq \frac{b-a}{2}\left\{\left[\frac{(2^{np-\gamma p+1}-1)^{\frac{1}{p}}-1}{2^{n-\gamma+\frac{1}{p}}(np-\gamma p+1)^{\frac{1}{p}}}\right]\left[\left|\zeta^{(n+1)}(a)\right|\left(\frac{1+(2^{q+1}-1)^{\frac{1}{q}}}{(2^{q+1}(q+1))^{\frac{1}{q}}}\right)+m\left|\zeta^{(n+1)}\left(\frac{b}{m}\right)\right|\left(\frac{(2^{q+1}-1)^{\frac{1}{q}}+1}{(2^{q+1}(q+1))^{\frac{1}{q}}}\right)\right]\right\}. \tag{3.6}$$

Proof. In the inequality (3.1) of Theorem 3.1, if one set $C = 0, \alpha = 1$ and $h(u) = u$, we obtain the inequality (3.6). □

In the following theorem, we give an error bound of inequality (2.11) with the help of Lemma 1.9.

Theorem 3.4. *Suppose that $\zeta \in C^{n+1}[a,b]$ and $|\zeta^{(n+1)}|^q$ with $q > 1$ satisfies the inequality* (1.4). *Then it satisfies the CFD inequality:*

$$\begin{aligned}
&\left|\frac{2^{n-\gamma-1}\Gamma(n-\gamma+1)}{(mb-a)^{n-\gamma}}\left[\left({}^{C}D^{\gamma}_{(\frac{a+bm}{2})^+}\zeta\right)(mb)+m^{n-\gamma+1}(-1)^n\left({}^{C}D^{\gamma}_{(\frac{a+mb}{2m})^-}\zeta\right)\left(\frac{a}{m}\right)\right]\right.\\
&\left.-\frac{1}{2}\left[\zeta^{(n)}\left(\frac{a+mb}{2}\right)+m\zeta^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|\\
&\leq \frac{mb-a}{4(n-\gamma+1)^{\frac{1}{p}}}\left[\left(|\zeta^{(n+1)}(a)|^q\int_0^1 h\left(\frac{u}{2}\right)^{\alpha}u^{n-\gamma}du+m|\zeta^{(n+1)}(b)|^q\int_0^1 h\left(1-\left(\frac{u}{2}\right)^{\alpha}\right)u^{n-\gamma}du\right.\right.\\
&\left.-Cm(b-a)^2\int_0^1 h\left(\frac{u}{2}\right)^{\alpha}h\left(1-\left(\frac{u}{2}\right)^{\alpha}\right)u^{n-\gamma}du\right)^{\frac{1}{q}}+\left(m\left|\zeta^{(n+1)}\left(\frac{a}{m^2}\right)\right|^q\int_0^1 h\left(1-\left(\frac{u}{2}\right)^{\alpha}\right)u^{n-\gamma}du\right.\\
&\left.\left.+|\zeta^{(n+1)}(b)|^q\int_0^1 h\left(\frac{u}{2}\right)^{\alpha}u^{n-\gamma}du-Cm\left(b-\frac{a}{m^2}\right)^2\int_0^1 h\left(\frac{u}{2}\right)^{\alpha}h\left(1-\left(\frac{u}{2}\right)^{\alpha}\right)u^{n-\gamma}du\right)^{\frac{1}{q}}\right]
\end{aligned} \tag{3.7}$$

with $0 \leq a < mb$.

Proof. By using Lemma 1.9 for $k = 1$. Then applying modulus property and using power mean inequality and the strongly $(\alpha, h-m)$-convexity of $|\zeta^{(n+1)}|^q$, we have

$$\begin{aligned}
&\left|\frac{2^{n-\gamma-1}\Gamma(n-\gamma+1)}{(mb-a)^{n-\gamma}}\left[\left({}^{C}D^{\gamma}_{(\frac{a+bm}{2})^+}\zeta\right)(mb)+m^{n-\gamma+1}(-1)^n\left({}^{C}D^{\gamma}_{(\frac{a+mb}{2m})^-}\zeta\right)\left(\frac{a}{m}\right)\right]\right.\\
&\left.-\frac{1}{2}\left[\zeta^{(n)}\left(\frac{a+mb}{2}\right)+m\zeta^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|\\
&\leq \frac{mb-a}{4}\left[\int_0^1 u^{n-\gamma}\left(\left|\zeta^{(n+1)}\left(\frac{u}{2}a+m\left(\frac{2-u}{2}\right)b\right)\right|+\left|\zeta^{(n+1)}\left(\frac{2-u}{2m}a+\frac{u}{2}b\right)\right|\right)du\right]\\
&\leq \frac{mb-a}{4}\left(\int_0^1 u^{n-\gamma}du\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 u^{n-\gamma}\left|\zeta^{(n+1)}\left(a\frac{u}{2}+m\left(\frac{2-u}{2}\right)b\right)\right|^q du\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 u^{n-\gamma}\left|\zeta^{(n+1)}\left(a\left(\frac{2-u}{2m}\right)+b\frac{u}{2}\right)\right|^q du\right)^{\frac{1}{q}}\right]\\
&\leq \frac{mb-a}{4(n-\gamma+1)^{\frac{1}{p}}}\left[\left(|\zeta^{(n+1)}(a)|^q\int_0^1 h\left(\frac{u}{2}\right)^{\alpha}u^{n-\gamma}du+m|\zeta^{(n+1)}(b)|^q\int_0^1 h\left(1-\left(\frac{u}{2}\right)^{\alpha}\right)u^{n-\gamma}du\right.\right.
\end{aligned}$$

$$-Cm(b-a)^2\int_0^1 h\left(\frac{u}{2}\right)^\alpha h\left(1-\left(\frac{u}{2}\right)^\alpha\right)u^{n-\gamma}du\Bigg)^{\frac{1}{q}} + \left(m\left|\zeta^{(n+1)}\left(\frac{a}{m^2}\right)\right|^q\int_0^1 h\left(1-\left(\frac{u}{2}\right)^\alpha\right)u^{n-\gamma}du\right.$$

$$\left.+|\zeta^{(n+1)}(b)|^q\int_0^1 h\left(\frac{u}{2}\right)^\alpha u^{n-\gamma}du - Cm\left(b-\frac{a}{m^2}\right)^2\int_0^1 h\left(\frac{u}{2}\right)^\alpha h\left(1-\left(\frac{u}{2}\right)^\alpha\right)u^{n-\gamma}du\right)^{\frac{1}{q}}\Bigg].$$

The proof is completed. □

Theorem 3.5. *Under the assumptions of Theorem 3.4, the following inequality holds:*

$$\Bigg|\frac{2^{n-\gamma-1}\Gamma(n-\gamma+1)}{(mb-a)^{n-\gamma}}\left[\left({}^CD^\gamma_{(\frac{a+bm}{2})^+}\zeta\right)(mb)+m^{n-\gamma+1}(-1)^n\left({}^CD^\gamma_{(\frac{a+mb}{2m})^-}\zeta\right)\left(\frac{a}{m}\right)\right]$$
$$-\frac{1}{2}\left[\zeta^{(n)}\left(\frac{a+mb}{2}\right)+m\zeta^{(n)}\left(\frac{a+mb}{2m}\right)\right]\Bigg|$$
$$\leq\frac{mb-a}{4(np-\gamma p+1)^{\frac{1}{p}}}\Bigg[\Bigg(|\zeta^{(n+1)}(a)|^q\int_0^1 h\left(\frac{u}{2}\right)^\alpha du+m|\zeta^{(n+1)}(b)|^q\int_0^1 h\left(1-\left(\frac{u}{2}\right)^\alpha\right)du$$
$$-Cm(b-a)^2\int_0^1 h\left(\frac{u}{2}\right)^\alpha h\left(1-\left(\frac{u}{2}\right)^\alpha\right)du\Bigg)^{\frac{1}{q}}+\Bigg(m\left|\zeta^{(n+1)}\left(\frac{a}{m^2}\right)\right|^q\int_0^1 h\left(1-\left(\frac{u}{2}\right)^\alpha\right)du$$
$$+|\zeta^{(n+1)}(b)|^q\int_0^1 h\left(\frac{u}{2}\right)^\alpha du-Cm\left(b-\frac{a}{m^2}\right)^2\int_0^1 h\left(\frac{u}{2}\right)^\alpha h\left(1-\left(\frac{u}{2}\right)^\alpha\right)du\Bigg)^{\frac{1}{q}}\Bigg]. \tag{3.8}$$

Proof. By using Lemma 1.9 for $k = 1$. Then applying modulus inequality and Holder's inequality, we have

$$\Bigg|\frac{2^{n-\gamma-1}\Gamma(n-\gamma+1)}{(mb-a)^{n-\gamma}}\left[\left({}^CD^\gamma_{(\frac{a+bm}{2})^+}\zeta\right)(mb)+m^{n-\gamma+1}(-1)^n\left({}^CD^\gamma_{(\frac{a+mb}{2m})^-}\zeta\right)\left(\frac{a}{m}\right)\right]$$
$$-\frac{1}{2}\left[\zeta^{(n)}\left(\frac{a+mb}{2}\right)+m\zeta^{(n)}\left(\frac{a+mb}{2m}\right)\right]\Bigg|$$
$$\leq\frac{mb-a}{4}\left[\int_0^1 u^{n-\gamma}\left|\zeta^{(n+1)}\left(\frac{u}{2}a+m\frac{(2-u)}{2}b\right)\right|du+\int_0^1 u^{n-\gamma}\left|\zeta^{(n+1)}\left(\frac{2-u}{2m}a+\frac{u}{2}b\right)\right|du\right].$$

By applying Holder's inequality, we have

$$\Bigg|\frac{2^{n-\gamma-1}\Gamma(n-\gamma+1)}{(mb-a)^{n-\gamma}}\left[\left({}^CD^\gamma_{(\frac{a+bm}{2})^+}\zeta\right)(mb)+m^{n-\gamma+1}(-1)^n\left({}^CD^\gamma_{(\frac{a+mb}{2m})^-}\zeta\right)\left(\frac{a}{m}\right)\right]$$
$$-\frac{1}{2}\left[\zeta^{(n)}\left(\frac{a+mb}{2}\right)+m\zeta^{(n)}\left(\frac{a+mb}{2m}\right)\right]\Bigg|$$
$$\leq\frac{mb-a}{4(np-\gamma p+1)^{\frac{1}{p}}}\left[\left(\int_0^1\left|\zeta^{(n+1)}\left(\frac{u}{2}a+m\frac{(2-u)b}{2}\right)\right|^q du\right)^{\frac{1}{q}}+\left(\int_0^1\left|\zeta^{(n+1)}\left(\frac{2-u}{2m}a+\frac{u}{2}b\right)\right|^q du\right)^{\frac{1}{q}}\right].$$

By applying strongly $(\alpha, h-m)$-convexity of $|\zeta^{(n+1)}|^q$, we have

$$\Bigg|\frac{2^{n-\gamma-1}\Gamma(n-\gamma+1)}{(mb-a)^{n-\gamma}}\left[\left({}^CD^\gamma_{(\frac{a+bm}{2})^+}\zeta\right)(mb)+m^{n-\gamma+1}(-1)^n\left({}^CD^\gamma_{(\frac{a+mb}{2m})^-}\zeta\right)\left(\frac{a}{m}\right)\right]$$

$$-\frac{1}{2}\left[\zeta^{(n)}\left(\frac{a+mb}{2}\right)+m\zeta^{(n)}\left(\frac{a+mb}{2m}\right)\right]\Bigg|$$

$$\leq \frac{mb-a}{4(np-\gamma p+1)^{\frac{1}{p}}}\Bigg[\Bigg(|\zeta^{(n+1)}(a)|^q\int_0^1 h\left(\frac{u}{2}\right)^\alpha du+m|\zeta^{(n+1)}(b)|^q\int_0^1 h\left(1-\left(\frac{u}{2}\right)^\alpha\right)du$$

$$-Cm(b-a)^2\int_0^1 h\left(\frac{u}{2}\right)^\alpha h\left(1-\left(\frac{u}{2}\right)^\alpha\right)du\Bigg)^{\frac{1}{q}}+\Bigg(m\left|\zeta^{(n+1)}\left(\frac{a}{m^2}\right)\right|^q\int_0^1 h\left(1-\left(\frac{u}{2}\right)^\alpha\right)du$$

$$+|\zeta^{(n+1)}(b)|^q\int_0^1 h\left(\frac{u}{2}\right)^\alpha du-Cm\left(b-\frac{a}{m^2}\right)^2\int_0^1 h\left(\frac{u}{2}\right)^\alpha h\left(1-\left(\frac{u}{2}\right)^\alpha\right)du\Bigg)^{\frac{1}{q}}\Bigg].$$

The proof is completed. □

Remark 3.6. The inequalities (3.7) and (3.8) provide the fractional Hadamard inequalities for strongly (α, m) convex function for $h(u) = u$; strongly $(h-m)$-convex function for $\alpha = 1$; strongly (s, m)-convex function for $h(u) = u^s$, $\alpha = 1$; strongly m-convex function for $h(u) = u$, $\alpha = 1$; strongly convex function for $h(u) = u$, $\alpha = 1$, $m = 1$. For $C = 0$ the inequalities (3.7) and (3.8) give the Hadamard inequalities for $(\alpha, h-m)$-convex function which further give the Hadamard inequalities for $(h-m)$-convex, (α, m)-convex, (s, m)-convex, m-convex, h-convex, s-convex and convex functions respectively.

4. Conclusions

In this article CFD inequalities of Hadamard type are combined in the form of compact results. These results generate many already published inequalities by selecting specific functions and parameters that appear in the inequality (1.5). Refinements of many Hadamard type inequalities for Caputo fractional derivatives are deducible. The connections of established results with known published results are shown in form of remarks and corollaries.

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Conflict of interest

The authors declare no conflict of interest.

References

1. J. Hadamard, Etude sur les proprietes des fonctions entieres e.t en particulier dune fonction consideree par Riemann, *J. Math. Pure Appl.,* **38** (1893), 171–215.

2. C. Hermite, Sur deux limites d'une intgrale dfinie, *Mathesis,* **3** (1883), 1–82.

3. P. Agarwal, M. Jleli, M. Tomar, Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals, *J. Inequal. Appl.,* **2017** (2017), 55. https://doi.org/10.1186/s13660-017-1318-y

4. S. I. Butt, A. Kashuri, M. Tariq, J. Nasir, A. Aslam, W. Gao, Hermite-Hadamard type inequalities vi n-polynomial exponential-type convexity and their applications, *Adv. Differ. Equ.,* **1** (2020), 1–25. https://doi.org/10.1186/s13662-020-02967-5

5. S. I. Butt, M. Tariq, A. Aslam, H. Ahmad, T. A. Nofal, Hermite-Hadamard type inequalities via generalized harmonic exponential convexity and applications, *J. Funct. Space.,* **2021** (2021), 1–12. https://doi.org/10.1155/2021/5533491

6. S. I. Butt, M. K. Bakula, D. Pečarić, J. Pečarić, Jensen Grüss inequality and its applications for the Zipf-Miandelbrot law, *Math. Method. Appl. Sci.,* **44** (2021), 1664–1673. https://doi.org/10.1002/mma.6869

7. F. Chen, On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity, *Chinese J. Math.,* **2014** (2014), 1–8. https://doi.org/10.1155/2014/173293

8. G. Farid, A. Javed, S. Naqvi, Hadamard and Fejer Hadamard inequalities and related results via Caputo fractional derivatives, *Bull. Math. Anal. Appl.,* **9** (2017), 16–30.

9. G. Farid, A. Javed, A. U. Rehman, Fractional integral inequalities of Hadamard type for m-convex functions via Caputo k-fractional derivatives, *J. Fract. Calc. Appl.,* **10** (2019), 120–134.

10. M. E. Özdemir, A. O. Akdemri, E. Set, On $(h - m)$-convexity and Hadamard-type inequalities, *Transylv. J. Math. Mech.,* **8** (2016), 51–58.

11. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.,* **57** (2013), 2403–2407. https://doi.org/10.1016/j.mcm.2011.12.048

12. A. Waheed, A. U. Rehman, M. I. Qureshi, F. A. Shah, K. A. Khan, G. Farid, On Caputo k-fractional derivatives and associated inequalities, *IEEE Access,* **7** (2019), 32137–32145. https://doi.org/10.1109/ACCESS.2019.2902317

13. N. Mehreen, M. Anwar, Hermite Hadamard type inequalities for exponentially p-convex functions and exponentially s-convex functions in the second sense with applications, *J. Inequal. Appl.,* **2019** (2019), 92. https://doi.org/10.1186/s13660-019-2047-1

14. X. Feng, B. Feng, G. Farid, S. Bibi, Q. Xiaoyan, Z. Wu, Caputo fractional derivative Hadamard inequalities for strongly m-convex functions, *J. Funct. Space.,* **2021** (2021), 1–11. https://doi.org/10.1155/2021/6642655

15. G. Farid, A. U. Rehman, S. Bibi, Y. M. Chu, Refinements of two fractional versions of Hadamard inequalities for Caputo fractional derivatives and related results, *Open J. Math. Sci.,* **5** (2021), 1–10. https://doi.org/10.30538/oms2021.0139

16. G. A. Anastassiou, Generalized fractional Hermite Hadamard inequalities involving m-convexity and (s, m)-convexity, *Facta Univ-Ser. Math.,* **28** (2013), 107–126.

17. C. Wang, H. Zhang, H. Zhang, W. Zhang, Globally projective synchronization for Caputo fractional quaternion-valued neural networks with discrete and distributed delays, *AIMS Math.,* **6** (2021), 14000–14012. https://doi.org/10.3934/math.2021809

18. H. Zhang, J. Cheng, H. Zhang, W. Zhang, J. Cao, Quasi-uniform synchronization of Caputo type fractional neural networks with leakage and discrete delays, *Chaos Soliton. Fract.,* **152** (2021), 111432. https://doi.org/10.1016/j.chaos.2021.111432

19. A. W. Roberts, D. E. Varberg, *Convex functions*, Academic Press: New York, 1973.
20. S. S. Dragomir, C. E. M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, *Sci. Direct Work. Pap.,* **1** (2003), 463–817.
21. J. Rooin, A. Alikhani, M. S. Moslehian, Operator m-convex functions, *Georgian Math. J.,* **25** (2018), 93–107. https://doi.org/10.1515/gmj-2017-0045
22. T. Lara, N. Merentes, R. Quintero, E. Rosales, On m-convexity of set-valued functions, *Adv. Oper. Theory,* **4** (2019), 767–783. https://doi.org/10.15352/aot.1810-1429
23. G. Farid, A. U. Rehman, Q. U. Ain, k-fractional integral inequalities of Hadamard type for $(h-m)$-convex functions, *Comput. Methods. Differ. Equ.,* **7** (2019), 1–22.
24. B. T. Polyak, Existence theorems and convergence of minimizing sequences in extremum problems with restrictions, *Sov. Math. Dokl.,* **7** (1966), 72–75.
25. K. Nikodem, Z. Páles, Characterizations of inner product spaces by strongly convex functions, *Banach J. Math. Anal.,* **5** (2011), 83–87. https://doi.org/10.15352/bjma/1313362982
26. J. Makò, A. Hézy, On strongly convex functions, *Carpathian J. Math.,* **32** (2016), 87–95. https://doi.org/10.37193/CJM.2016.01.09
27. N. Merentes, K. Nikodem, Remarks on strongly convex functions, *Aequationes Math.,* **80** (2010), 193–199. https://doi.org/10.1007/s00010-010-0043-0
28. K. Nikodem, *On strongly convex functions and related classes of functions*, Springer: New York, 2014, 365–405. https://doi.org/10.1007/978-1-4939-1246-9_16
29. S. Z. Ullah, M. A. Khan, Y. M. Chu, A note on generalized convex functions, *J. Inequal. Appl.,* **2019** (2019), 291. https://doi.org/10.1186/s13660-019-2242-0
30. S. Z. Ullah, M. A. Khan, Y. M. Chu, Majorizations theorems for strongly convex functions, *J. Inequal. Appl.,* **2019** (2019), 58. https://doi.org/10.1186/s13660-019-2007-9
31. S. Z. Ullah, M. A. Khan, Y. M. Chu, Integral majorization type inequalities for the functions in the sense of strong convexity, *J. Funct. Space.,* **2019** (2019), 11. https://doi.org/10.1155/2019/9487823
32. J. P. Vial, Strong convexity of sets and functions, *J. Math. Econ.,* **9** (1982), 187–205. https://doi.org/10.1016/0304-4068(82)90026-X
33. Z. Zhang, G. Farid, K. Mahreen, Inequalities for unified integral operators via strongly $(\alpha, h-m)$-convexity, *J. Funct. Space.,* **2021** (2021), 1–11. https://doi.org/10.1155/2021/6675826
34. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations,* Elsevier: Amsterdam, 2006.