AIMS Mathematics

AIMS Mathematics, 7(6): 9603–9626.
DOI: 10.3934/math.2022534
Received: 24 December 2021
Revised: 29 January 2022
Accepted: 08 February 2022
Published: 15 March 2022

http://www.aimspress.com/journal/Math

Research article

A new approach for operations on neutrosophic soft sets based on the novel norms for constructing topological structures

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Abstract: Neutrosophic sets have recently emerged as a tool for dealing with imprecise, indeterminate, inconsistent data, while soft sets may have the potential to deal with uncertainties that classical methods cannot control. Combining these two types of sets results in a unique hybrid structure, a neutrosophic soft set (NS-set), for working effectively in uncertain environments. This paper focuses on determining operations on NS-sets through two novel norms. Accordingly, the $\min - \text{norm}$ and $\max - \text{norm}$ are well-defined here for the first time to construct the intersection, union, difference, AND, OR operations. Then, the topology, open set, closed set, interior, closure, regularity concepts on NS-sets are introduced based on these just constructed operations. All the properties in the paper are stated in theorem form, which is proved convincingly and logically. In addition, we also elucidate the relationship between the topology on NS-sets and the fuzzy soft topologies generated by the truth, indeterminacy, falsity degrees by theorems and counterexamples.

Keywords: neutrosophic soft set; neutrosophic soft operations; neutrosophic soft topology; neutrosophic soft interior; neutrosophic soft closure
Mathematics Subject Classification: 54A40, 54E55, 54D10

1. Introduction

Data is a valuable source of knowledge that contains helpful information if exploited effectively [1]. One of the challenges facing data researchers is the ambiguity and uncertainty of the data they have access to, which makes it difficult for them to process information. But these challenges are, in a positive sense, opportunities for the development of new techniques and tools, such as they various approaches based on fuzzy set theory [2]. The advent of fuzzy theory has prompted extensive work on ideas such as fuzzy sets [3], vague sets [4], soft sets [5], and neutrosophic sets [6]. It was originally thought that the development of new theories would eclipse fuzzy theory, but that does not seem to be the case [7]. This research field is becoming more and more active, with a number of fundamental contributions to the rapid development of new theories [8,9]. One of the most prominent applications is the use of fuzzy set theory in emerging and vibrant fields like machine learning [10,11] or topological data analysis [12,13].

In recent years, the study of soft sets [5] and neutrosophic sets [14] has become an attractive research area. Neutrosophic sets recently emerged as a tool for dealing with imprecise, indeterminate, and inconsistent data [15]. In contrast, soft sets show potential for dealing with uncertainties that classical methods cannot control [16]. Combining these two types of sets results in a unique hybrid structure, a *neutrosophic soft set (NS-set)* [17], for working effectively in uncertain environments. Maji proposed this [17,18] in 2013 and it was modified by Deli and Broumi [19] in 2015. Furthermore, Karaaslan [20] redefined this concept and its operations to be more efficient and complete. Since then, this structure has proved to be quite effective when applied in real life in many fields, such as decision making [17], market prediction [21], and medical diagnosis [22,23].

The topology on NS-sets is one of the issues that needs more attention, alongside neutrosophic topology [24,25] and soft topology [26]. This issue has emerged recently to help complete the overall picture for NS and aid its practical applications based on topology [27,28]. In 2017, Bera and Mahapatra [29] gave general operations to construct a topology on NS-sets. They also presented concepts related to topological space such as interior, closure, neighborhood, boundary, regularity, base, subspace, separation axioms, along with specific illustrations and proofs. In 2018, these authors [30] continued to develop further studies on connectedness and compactness on NS-topological space. In 2019, Ozturk, Aras, and Bayramov [31] introduced a new approach to topology on NS-sets. This approach is quite different from the previous work [29], and was further developed by constructing separation axioms [32] in the same year, 2018. Recently, the continuum [33] or compactness [34] on the topological space generated on NS-sets has also been studied with the same properties as the normal space. Many variations [35] of the topological space on NS-sets have also attracted the attention of researchers, and most of the related works are inspired by topology on neutrosophic and soft sets with the idea of a hybrid structure [36,37].

In this work, we construct the topological space and related concepts on NS-sets through general operations in a way that is very different from the work of Bera and Mahapatra [29,30], but more general than the work of Ozturk, Aras, and Bayramov [31,32], with our operations based on the generality of min and max operations. This work begins by defining two new operations to create the relationships between NS-sets. These relations are then used as the kernel for forming topology and topological relations on NS-sets. One emphasis shown here is on elucidating the relationship between the topology on NS-sets and the component fuzzy soft topologies. All the ideas in this work are presented convincingly and clearly through definitions, theorems, and their consequences.

In summary, the significant contributions of this study are as follows:

(1) Defining two novel concepts, called $\min -\text{norm}$ and $\max -\text{norm}$, to provide a theoretical foundation for determining operations on NS-sets, including intersection, union, difference, AND, and OR.

(2) Constructing the topology, open set, closed set, interior, closure, and regularity concepts on NS-sets based on just determined operations.

(3) Elucidating the relationship between the topology on NS-sets and the fuzzy soft topologies generated by truth, indeterminacy, falsity degrees by the theorems and counterexamples.

(4) The concepts are well-defined, and the theorems are proved convincingly and logically.

This work is organized as follows: Section 1 presents the motivation and introduces the significant contributions. Section 2 briefly introduces NS-sets and related concepts. The two new ideas, $\min -\text{norm}$ and $\max -\text{norm}$, are provided in Section 3 as a theoretical foundation for determining operations on NS-sets, including intersection, union, difference, AND, and OR. In Section 4, the topology on NS-sets is defined with related concepts such as open set, closed set, interior, closure, and regularity. Furthermore, the relationship between the topology on NS-sets and the fuzzy soft topologies generated by truth, indeterminacy, and falsity functions by theorems and counterexamples in Section 5. The last section presents conclusions and future research trends in this area.

2. Preliminaries

This section recalls the NS-set proposed in 2013 by Maji [17,18], then modified and improved in 2015 by Deli and Broumi [19]. This concept is based on combining soft [5] and neutrosophic [6] sets. Some background related to NS-sets is briefly presented below so that readers can better understand the following sections.

Without loss of generality, we consider X to be a universal set, $\mathcal{E}$ to be a parameter set, and $\mathcal{N}(X)$ to denote the collection of all neutrosophic sets on X.

Definition 1. ([18,19])**.** The pair $(A,\mathcal{E})$ is a *NS-set* on X where $A:\mathcal{E}\longrightarrow\mathcal{N}(X)$ is a set valued function determined by $e\longmapsto A(e)\coloneqq A_e$ with

$$A_e: X \longrightarrow]^-0;1^+[\times]^-0;1^+[\times]^-0;1^+[$$

$$x\longmapsto A_e(x)\coloneqq\langle T_{A_e}(x);I_{A_e}(x);F_{A_e}(x)\rangle \tag{1}$$

for all $e\in\mathcal{E}$, and the real function triples $T_{A_e}, I_{A_e}, F_{A_e}: X\longrightarrow]^-0;1^+[$ indicate truth, indeterminacy, and falsity degrees, respectively, with no restriction on their sum.

In other words, the NS-set can be described as a set of ordered tuples as follows:

$$(A,\mathcal{E})=\big\{\big(e,A(e)\big): e\in\mathcal{E}, A(e)\in\mathcal{N}(X)\big\} \tag{2}$$

$$=\big\{\big(e,\langle x,T_{A_e}(x),I_{A_e}(x),F_{A_e}(x)\rangle\big): e\in\mathcal{E}, x\in X\big\} \tag{3}$$

$$\coloneqq\left\{\left(e,\frac{x}{T_{A_e}(x),I_{A_e}(x),F_{A_e}(x)}\right): e\in\mathcal{E}, x\in X\right\}. \tag{4}$$

If nothing changes, the symbol $\mathcal{NS}(X)$ indicates the collection of all NS-sets on X. Besides, if the NS-sets consider the same parameter set $\mathcal{E}$, then it is not mentioned repeatedly in order to simplify the notations. Moreover, because the values of T, I, F belong to the unit interval $[0;1]$, the integral part of the values is almost zero. Typically, it may occur that the integer part is omitted (for example, $.1$ instead of 0.1). Therefore, if it does not lead to confusion, this omitted format of a decimal is always used in all the tables used in this paper.

Definition 2. ([18,19]).
a. $\emptyset_{\mathcal{E}}$ is a *null NS-set* if

$$\forall e \in \mathcal{E}, \forall x \in X, \begin{cases} T_{\emptyset_{\mathcal{E}}}(x) = 0 \\ I_{\emptyset_{\mathcal{E}}}(x) = 0 \\ F_{\emptyset_{\mathcal{E}}}(x) = 1 \end{cases}. \tag{5}$$

b. $\emptyset_{\tilde{\mathcal{E}}}$ is a *semi-null NS-set* if

$$\exists e \in \mathcal{E}, \forall x \in X, \begin{cases} T_{\emptyset_{\tilde{\mathcal{E}}}}(x) = 0 \\ I_{\emptyset_{\tilde{\mathcal{E}}}}(x) = 0 \\ F_{\emptyset_{\tilde{\mathcal{E}}}}(x) = 1 \end{cases}. \tag{6}$$

c. $X_{\mathcal{E}}$ is an *absolute NS-set* if

$$\forall e \in \mathcal{E}, \forall x \in X, \begin{cases} T_{X_{\mathcal{E}}}(x) = 1 \\ I_{X_{\mathcal{E}}}(x) = 1 \\ F_{X_{\mathcal{E}}}(x) = 0 \end{cases}. \tag{7}$$

d. $X_{\tilde{\mathcal{E}}}$ is a *semi-absolute NS-set* if

$$\exists e \in \mathcal{E}, \forall x \in X, \begin{cases} T_{X_{\tilde{\mathcal{E}}}}(x) = 1 \\ I_{X_{\tilde{\mathcal{E}}}}(x) = 1 \\ F_{X_{\tilde{\mathcal{E}}}}(x) = 0 \end{cases}. \tag{8}$$

Definition 3. ([19,31]). Let A and B be two NS-sets on X.
a. A is a *NS-subset* of B, written as $A \subseteq B$, if

$$\forall e \in \mathcal{E}, \forall x \in X, \begin{cases} T_{A_e}(x) \le T_{B_e}(x) \\ I_{A_e}(x) \le I_{B_e}(x) \\ F_{A_e}(x) \ge F_{B_e}(x) \end{cases}. \tag{9}$$

b. A is a *NS-superset* of B, written as $A \supseteq B$, if B is a *NS-subset* of A.
c. $\bar{A}$ is the *complement* of A if

$$\forall e \in \mathcal{E}, \forall x \in X, \begin{cases} T_{\bar{A}_e}(x) = F_{A_e}(x) \\ I_{\bar{A}_e}(x) = 1 - I_{A_e}(x) \\ F_{\bar{A}_e}(x) = T_{A_e}(x) \end{cases}. \tag{10}$$

Example 1. Let two NS-sets M and N be represented in Table 1 as follows:

Table 1. NS-sets M and N.

M	e_1	e_2	N	e_1	e_2
x_1	⟨.2,.3,.4⟩	⟨.3,.5,.5⟩	x_1	⟨.3,.6,.1⟩	⟨.4,.5,.4⟩
x_2	⟨.3,.4,.3⟩	⟨.6,.2,.4⟩	x_2	⟨.6,.5,.2⟩	⟨.7,.3,.2⟩
x_3	⟨.3,.5,.2⟩	⟨.4,.4,.3⟩	x_3	⟨.4,.5,.3⟩	⟨.6,.3,.3⟩
x_4	⟨.2,.7,.6⟩	⟨.3,.4,.3⟩	x_4	⟨.9,1,.4⟩	⟨.4,.5,.1⟩

Based on Eq (9) of Definition 3, $M \subseteq N$.

Example 2. The NS-set P and its complement $\bar{P}$ are represented according to Eq (10) in Table 2 as follows:

Table 2. NS-sets P and $\bar{P}$.

P	e_1	e_2	e_3	$\bar{P}$	e_1	e_2	e_3
x_1	⟨.9,.8,.2⟩	⟨.3,.7,.2⟩	⟨.4,.6,.3⟩	x_1	⟨.2,.2,.9⟩	⟨.2,.3,.3⟩	⟨.3,.4,.4⟩
x_2	⟨.7,.6,.2⟩	⟨.3,.4,.6⟩	⟨.4,.3,.2⟩	x_2	⟨.2,.4,.7⟩	⟨.6,.6,.3⟩	⟨.2,.7,.4⟩
x_3	⟨.3,.3,.5⟩	⟨.1,.2,.3⟩	⟨.9,.5,.8⟩	x_3	⟨.5,.7,.3⟩	⟨.3,.8,.1⟩	⟨.8,.5,.9⟩

Theorem 1. *If* $A \in \mathcal{NS}(X)$,

(1) $\bar{\bar{A}} = A$,

(2) $\overline{\emptyset_{\mathcal{E}}} = X_{\mathcal{E}}$,

(3) $\overline{\emptyset_{\tilde{\mathcal{E}}}} = X_{\tilde{\mathcal{E}}}$,

(4) $\overline{X_{\mathcal{E}}} = \emptyset_{\mathcal{E}}$,

(5) $\overline{X_{\tilde{\mathcal{E}}}} = \emptyset_{\tilde{\mathcal{E}}}$.

Proof. These properties are directly inferred from the definitions of the null, semi-null, absolute, semi-absolute NS-sets and the complement operation.

3. Another novel approach for operations on NS-sets

In this section, we focus on defining two novel norms, called $min-norm$ and $max-norm$, as the foundations for determining operations on NS-sets in general. Each operation is well-defined along with its well-proven properties.

3.1. $min-norm$ and $max-norm$

Definition 4. A $\min -\mathrm{norm}$ is the binary operation $\bullet: [0;1] \times [0;1] \rightarrow [0,1]$ that obeys the conditions as follows:

(a) $\bullet$ has the commutative and associative properties,

(b) $\forall x \in [0,1], x\bullet 1 = 1\bullet x = x$,

(c) $\forall x \in [0,1], x\bullet 0 = 0\bullet x = 0$,

(d) $\forall x, y \in [0,1], x \geq x\bullet y$.

Definition 5. A $\max -\mathrm{norm}$ is the binary operation $\circ: [0;1] \times [0;1] \rightarrow [0,1]$ that obeys the following conditions:

(a) $\circ$ has the commutative and associative properties,
(b) $\forall x \in [0,1], x \circ 1 = 1 \circ x = 1$,
(c) $\forall x \in [0,1], y \circ 0 = 0 \circ x = x$,
(d) $\forall x, y \in [0,1], x \leq x \circ y$.

Definition 6. The $\min -\text{norm}$ $\bullet$ and $\max -\text{norm}$ $\circ$ satisfy *De Morgan's law* if they obey the following conditions:

$$\forall x, y \in [0,1], (1-x) \circ (1-y) = 1 - x \bullet y, \tag{11}$$

$$\forall x, y \in [0,1], (1-x) \bullet (1-y) = 1 - x \circ y. \tag{12}$$

Some commonly used $\min -\text{norm}$ and $\max -\text{norm}$ are shown in Table 3. On the other hand, all of these norms satisfy De Morgan's law in pairs.

Table 3. Some commonly used $\min -\text{norm}$ $\bullet$ and $\max -\text{norm}$ $\circ$ satisfying the De Morgan's law.

	$\min -\text{norms}$	$\max -\text{norms}$
1	$\forall x, y \in [0,1], x \bullet y = xy$	$\forall x, y \in [0,1], x \circ y = x + y - xy$
2	$\forall x, y \in [0,1], x \bullet y = \min\{x, y\}$	$\forall x, y \in [0,1], x \circ y = \max\{x, y\}$
3	$\forall x, y \in [0,1], x \bullet y = \max\{x + y - 1, 0\}$	$\forall x, y \in [0,1], x \circ y = \min\{x + y, 1\}$

3.2. Operations on NS-sets

3.2.1. Intersection

Definition 7. The *intersection* of the two NS-sets A and B, written as $A \cap B$, is determined by

$$\forall e \in \mathcal{E}, \forall x \in X, \begin{cases} T_{A \cap B_e}(x) = T_{A_e}(x) \bullet T_{B_e}(x) \\ I_{A \cap B_e}(x) = I_{A_e}(x) \bullet I_{B_e}(x) \\ F_{A \cap B_e}(x) = F_{A_e}(x) \circ F_{B_e}(x) \end{cases}. \tag{13}$$

Example 3. Let two NS-sets A and B be represented in Table 4 as follows:

Table 4. NS-sets A and B.

A	e_1	e_2	B	e_1	e_2
x_1	$\langle .2,.4,.5 \rangle$	$\langle .5,.2,.8 \rangle$	x_1	$\langle .7,.2,.7 \rangle$	$\langle .6,0,.5 \rangle$
x_2	$\langle .1,.4,.3 \rangle$	$\langle .8,.9,.4 \rangle$	x_2	$\langle .3,.9,.1 \rangle$	$\langle .7,.7,.9 \rangle$
x_3	$\langle .1,.2,.7 \rangle$	$\langle .8,.9,.4 \rangle$	x_3	$\langle .6,.4,.7 \rangle$	$\langle .8,.1,0 \rangle$

If using $\min -\text{norms}$ $x \bullet y = \max\{x + y - 1, 0\}$ and $\max -\text{norms}$ $x \circ y = \min\{x + y, 1\}$, the intersection $A \cap B$ of the two above NS-sets is described according to Eq (13) in Table 5 as follows:

Table 5. NS-sets $A \cap B$.

$A \cap B$	e_1	e_2
x_1	$\langle 0,0,1 \rangle$	$\langle .1,0,1 \rangle$
x_2	$\langle 0,.3,.4 \rangle$	$\langle .5,.6,1 \rangle$
x_3	$\langle 0,0,1 \rangle$	$\langle .6,0,.4 \rangle$

Theorem 2. *If* $A, B, C \in \mathcal{NS}(X)$,

(1) $A \cap A = A$,
(2) $A \cap \emptyset_{\mathcal{E}} = \emptyset_{\mathcal{E}}$,
(3) $A \cap \emptyset_{\tilde{\mathcal{E}}} = \emptyset_{\tilde{\mathcal{E}}}$,
(4) $A \cap X_{\mathcal{E}} = A$,
(5) $A \cap X_{\tilde{\mathcal{E}}} = A$,
(6) $A \cap (B \cap C) = (A \cap B) \cap C$,
(7) $A \cap B = B \cap A$.

Proof. These properties are directly inferred from the definitions of norms and intersection operation.

Definition 8. Let $(A_i)_{i \in I}$ be a collection of NS-sets on X. The *intersection* of the collection of NS-sets $(A_i)_{i \in I}$, written as $\bigcap_{i \in I} A_i$, is determined by

$$\forall e \in \mathcal{E}, \forall x \in X, \begin{cases} T_{\bigcap_{i \in I} A_{i_e}}(x) = \underset{i \in I}{\bullet} \left\{ T_{A_{i_e}}(x) \right\} \\ I_{\bigcap_{i \in I} A_{i_e}}(x) = \underset{i \in I}{\bullet} \left\{ I_{A_{i_e}}(x) \right\}. \\ F_{\bigcap_{i \in I} A_{i_e}}(x) = \underset{i \in I}{\circ} \left\{ F_{A_{i_e}}(x) \right\} \end{cases} \tag{14}$$

3.2.2. Union

Definition 9. The *union* of the two NS-sets A and B, written as $A \cup B$, is determined by

$$\forall e \in \mathcal{E}, \forall x \in X, \begin{cases} T_{A \cup B_e}(x) = T_{A_e}(x) \circ T_{B_e}(x) \\ I_{A \cup B_e}(x) = I_{A_e}(x) \circ I_{B_e}(x) . \\ F_{A \cup B_e}(x) = F_{A_e}(x) \bullet F_{B_e}(x) \end{cases} \tag{15}$$

Example 4. If using min $-$norms $x \bullet y = \max\{x + y - 1, 0\}$ and max $-$norms $x \circ y = \min\{x + y, 1\}$, the union $A \cup B$ of the two above NS-sets A and B in Example 3 is described according to Eq (14) in Table 6 as follows:

Table 6. NS-sets $A \cup B$.

$A \cup B$	e_1	e_2
x_1	$\langle .9,.6,.2 \rangle$	$\langle 1,.2,.3 \rangle$
x_2	$\langle .4,1,1 \rangle$	$\langle 1,1,.3 \rangle$
x_3	$\langle .7,.6,.4 \rangle$	$\langle 1,1,0 \rangle$

Theorem 3. *If* $A, B, C \in \mathcal{NS}(X)$,

(1) $A \cup A = A$,

(2) $A \cup \emptyset_{\mathcal{E}} = A$,
(3) $A \cup \emptyset_{\tilde{\mathcal{E}}} = A$,
(4) $A \cup X_{\mathcal{E}} = X_{\mathcal{E}}$,
(5) $A \cup X_{\tilde{\mathcal{E}}} = X_{\tilde{\mathcal{E}}}$,
(6) $A \cup (B \cup C) = (A \cup B) \cup C$,
(7) $A \cup B = B \cup A$.

Proof. These properties are directly inferred from the definitions of norms and union operation.

Theorem 4. *If the* $\min -norm$ *and* $\max -norm$ *satisfy De Morgan's law, for all* $A, B \in \mathcal{NS}(X)$,
(1) $\overline{A \cap B} = \bar{A} \cup \bar{B}$,
(2) $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

Proof.
(1) $\forall e \in \mathcal{E}, \forall x \in X$,

$$\begin{cases} T_{\overline{A \cap B}_e}(x) = F_{A \cap B_e} = F_{A_e}(x) \circ F_{B_e}(x) \\ I_{\overline{A \cap B}_e}(x) = 1 - I_{A \cap B_e} = 1 - I_{A_e}(x) \bullet I_{B_e}(x), \\ F_{\overline{A \cap B}_e}(x) = T_{A \cap B_e} = T_{A_e}(x) \bullet T_{B_e}(x) \end{cases} \tag{16}$$

and

$$\begin{cases} T_{\bar{A} \cup \bar{B}_e}(x) = T_{\bar{A}_e}(x) \circ T_{\bar{B}_e}(x) = F_{A_e}(x) \circ F_{B_e}(x) \\ I_{\bar{A} \cup \bar{B}_e}(x) = I_{\bar{A}_e}(x) \circ I_{\bar{B}_e}(x) = \left(1 - I_{A_e}(x)\right) \circ \left(1 - I_{B_e}(x)\right). \\ F_{\bar{A} \cup \bar{B}_e}(x) = F_{\bar{A}_e}(x) \bullet F_{\bar{B}_e}(x) = T_{A_e}(x) \bullet T_{B_e}(x) \end{cases} \tag{17}$$

Moreover,

$$\left(1 - I_{A_e}(x)\right) \circ \left(1 - I_{B_e}(x)\right) = 1 - I_{A_e}(x) \bullet I_{B_e}(x), \tag{18}$$

due to De Morgan's law of the $\min -\text{norm}$ and $\max -\text{norm}$. Therefore,

$$\forall e \in \mathcal{E}, \forall x \in X, \begin{cases} T_{\overline{A \cap B}_e}(x) = T_{\bar{A} \cup \bar{B}_e}(x) \\ I_{\overline{A \cap B}_e}(x) = I_{\bar{A} \cup \bar{B}_e}(x) \ . \\ F_{\overline{A \cap B}_e}(x) = F_{\bar{A} \cup \bar{B}_e}(x) \end{cases} \tag{19}$$

(2) $\forall e \in \mathcal{E}, \forall x \in X$,

$$\begin{cases} T_{\overline{A \cup B}_e}(x) = F_{A \cup B_e} = F_{A_e}(x) \bullet F_{B_e}(x) \\ I_{\overline{A \cup B}_e}(x) = 1 - I_{A \cup B_e} = 1 - I_{A_e}(x) \circ I_{B_e}(x), \\ F_{\overline{A \cup B}_e}(x) = T_{A \cup B_e} = T_{A_e}(x) \circ T_{B_e}(x) \end{cases} \tag{20}$$

and

$$\begin{cases} T_{\bar{A} \cap \bar{B}_e}(x) = T_{\bar{A}_e}(x) \bullet T_{\bar{B}_e}(x) = F_{A_e}(x) \bullet F_{B_e}(x) \\ I_{\bar{A} \cap \bar{B}_e}(x) = I_{\bar{A}_e}(x) \bullet I_{\bar{B}_e}(x) = \left(1 - I_{A_e}(x)\right) \bullet \left(1 - I_{B_e}(x)\right). \\ F_{\bar{A} \cap \bar{B}_e}(x) = F_{\bar{A}_e}(x) \circ F_{\bar{B}_e}(x) = T_{A_e}(x) \circ T_{B_e}(x) \end{cases} \tag{21}$$

Moreover,

$$\left(1 - I_{A_e}(x)\right) \bullet \left(1 - I_{B_e}(x)\right) = 1 - I_{A_e}(x) \circ I_{B_e}(x), \tag{22}$$

due to De Morgan's law of the $\min -$norm and $\max -$norm. Therefore,

$$\forall e \in \mathcal{E}, \forall x \in X, \begin{cases} T_{\overline{A \cup B}_e}(x) = T_{\bar{A} \cap \bar{B}_e}(x) \\ I_{\overline{A \cup B}_e}(x) = I_{\bar{A} \cap \bar{B}_e}(x) \\ F_{\overline{A \cup B}_e}(x) = F_{\bar{A} \cap \bar{B}_e}(x) \end{cases}. \tag{23}$$

The distributive properties between intersection and union operations are not satisfied in the case of these general operations. Counterexamples are shown in Example 5.

Example 5. Let the NS-set C be represented in Table 7 as follows:

Table 7. NS-sets C.

C	e_1	e_2
x_1	⟨.2,.1,.9⟩	⟨.3,.2,.6⟩
x_2	⟨.3,.7,.6⟩	⟨.8,.2,.5⟩
x_3	⟨.2,.1,.4⟩	⟨.3,.2,.5⟩

If using $\min -$norm $x \bullet y = \max\{x + y - 1, 0\}$ and $\max -$norm $x \circ y = \min\{x + y, 1\}$ with the two above NS-sets A and B in Example 3, the two NS-sets $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ can be described in Table 8 as follows:

Table 8. NS-sets $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$.

$A \cap (B \cup C)$	e_1	e_2	$(A \cap B) \cup (A \cap C)$	e_1	e_2
x_1	⟨.1,0,1⟩	⟨.4,0,.9⟩	x_1	⟨0,0,1⟩	⟨.1,0,1⟩
x_2	⟨0,1,.3⟩	⟨.8,.8,.8⟩	x_2	⟨0,.4,.3⟩	⟨.6,.1,.9⟩
x_3	⟨0,0,.8⟩	⟨.8,.2,.4⟩	x_3	⟨0,0,1⟩	⟨.1,.1,.9⟩

Therefore, $A \cap (B \cup C) \neq (A \cap B) \cup (A \cap C)$. Similarly, see Table 9:

Table 9. NS-sets $A \cap (B \cup C) \neq (A \cap B) \cup (A \cap C)$.

$A \cup (B \cap C)$	e_1	e_2	$(A \cup B) \cap (A \cup C)$	e_1	e_2
x_1	⟨.2,.4,.5⟩	⟨.5,.2,.8⟩	x_1	⟨.3,.1,.6⟩	⟨.8,0,.7⟩
x_2	⟨.1,1,0⟩	⟨1,.9,.4⟩	x_2	⟨0,1,1⟩	⟨1,1,.3⟩
x_3	⟨.1,.2,.7⟩	⟨.9,.9,0⟩	x_3	⟨0,0,.5⟩	⟨1,1,0⟩

Therefore, $A \cup (B \cap C) \neq (A \cup B) \cap (A \cup C)$.

Definition 10. Let $(A_i)_{i \in I}$ be a collection of NS-sets on X. The *union* of the collection of NS-sets $(A_i)_{i \in I}$, written as $\bigcup_{i \in I} A_i$, is determined by

$$\forall e\in\mathcal{E},\forall x\in X,\begin{cases}T_{\bigcup_{i\in I}A_{i_e}}(x)=\underset{i\in I}{\circ}\left\{T_{A_{i_e}}(x)\right\}\\ I_{\bigcup_{i\in I}A_{i_e}}(x)=\underset{i\in I}{\circ}\left\{I_{A_{i_e}}(x)\right\}\\ F_{\bigcup_{i\in I}A_{i_e}}(x)=\underset{i\in I}{\bullet}\left\{F_{A_{i_e}}(x)\right\}\end{cases}. \tag{24}$$

3.2.3. Difference

Definition 11. The *difference* of the two NS-sets A and B, written as $A\backslash B$, is determined by $A\backslash B = A\cap\bar{B}$, i.e.,

$$\forall e\in\mathcal{E},\forall x\in X,\begin{cases}T_{A\backslash B_e}(x)=T_{A_e}(x)\bullet F_{B_e}(x)\\ I_{A\backslash B_e}(x)=I_{A_e}(x)\bullet\left(1-I_{B_e}(x)\right)\\ F_{A\backslash B_e}(x)=F_{A_e}(x)\circ T_{B_e}(x)\end{cases}. \tag{25}$$

Example 6. If using $\min-$norm $x\bullet y=\max\{x+y-1,0\}$ and $\max-$norm $x\circ y=\min\{x+y,1\}$, the difference $A\backslash B$ of the two above NS-sets A and B in Example 3 is described according to Eq (25), see Table 10:

Table 10. NS-sets $A\backslash B$.

$A\backslash B$	e_1	e_2
x_1	⟨0,.2,1⟩	⟨0,.2,1⟩
x_2	⟨0,0,.6⟩	⟨.7,.2,1⟩
x_3	⟨0,0,1⟩	⟨0,.8,1⟩

Theorem 5. *If the* $\min-norm$ *and* $\max-norm$ *satisfy De Morgan's law, for all* $A,B,C\in\mathcal{NS}(X)$,

(1) $A\backslash B\subseteq A$,
(2) $\overline{A\backslash B}=\bar{A}\cup B$,
(3) $\bar{A}\backslash\bar{B}=B\backslash A$,
(4) $A\backslash(B\cup C)=(A\backslash B)\cap(A\backslash C)$,
(5) $(A\cap B)\backslash C=(A\backslash C)\cap(B\backslash C)$,
(6) $(A\backslash B)\cap(C\backslash D)=(C\backslash B)\cap(A\backslash D)=(A\cap C)\backslash(B\cup D)$.

Proof.

(1) $\forall e\in\mathcal{E},\forall x\in X,\begin{cases}T_{A\backslash B_e}(x)=T_{A_e}(x)\bullet T_{B_e}(x)\le T_{A_e}(x)\\ I_{A\backslash B_e}(x)=I_{A_e}(x)\bullet\left(1-I_{B_e}(x)\right)\le I_{A_e}(x)\\ F_{A\backslash B_e}(x)=F_{A_e}(x)\circ F_{B_e}(x)\ge F_{A_e}(x)\end{cases}$. This implies that $A\backslash B\subseteq A$.

(2) $\overline{A\backslash B}=\overline{A\cap\bar{B}}=\bar{A}\cup\bar{\bar{B}}=\bar{A}\cup B$ due to Theorem 1.

(3) $\bar{A}\backslash\bar{B}=\bar{A}\cap\bar{\bar{B}}=\bar{A}\cap B=B\cap\bar{A}=B\backslash A$.

(4) $A\backslash(B\cup C)=A\cap\overline{B\cup C}=A\cap(\bar{B}\cap\bar{C})=(A\cap\bar{B})\cap(A\cap\bar{C})=(A\backslash B)\cap(A\backslash C)$ due to Theorems 3 and 4.

(5) $(A\cap B)\backslash C=(A\cap B)\cap\bar{C}=(A\cap\bar{C})\cap(B\cap\bar{C})=(A\backslash C)\cap(B\backslash C)$ due to Theorem 3.

(6) $(A\backslash B)\cap(C\backslash D)=(A\cap\bar{B})\cap(C\cap\bar{D})=(C\cap\bar{B})\cap(A\cap\bar{D})=(C\backslash B)\cap(A\backslash D)$ due to Theorem 3.

(7) $(A\backslash B) \cap (C\backslash D) = (A \cap \bar{B}) \cap (C \cap \bar{D}) = (A \cap C) \cap (\bar{B} \cap \bar{D}) = (A \cap C) \cap \overline{B \cup D} = (A \cap C)\backslash(B \cup D)$ due to Theorems 3 and 4.

3.2.4. AND and OR

Definition 12. The *AND operation* of the two NS-sets A and B with the same parameter set $\mathcal{E}$, written as $A \wedge B$, is determined over the same parameter set $\mathcal{E} \times \mathcal{E}$ by

$$\forall(e_1, e_2) \in \mathcal{E} \times \mathcal{E}, \forall x \in X, \begin{cases} T_{A\wedge B_{(e_1,e_2)}}(x) = T_{A_{e_1}}(x) \bullet T_{B_{e_2}}(x) \\ I_{A\wedge B_{(e_1,e_2)}}(x) = I_{A_{e_1}}(x) \bullet I_{B_{e_2}}(x) \\ F_{A\wedge B_{(e_1,e_2)}}(x) = F_{A_{e_1}}(x) \circ F_{B_{e_2}}(x) \end{cases}. \tag{26}$$

Definition 13. The *OR operation* of the two NS-sets A and B with the same parameter set $\mathcal{E}$, written as $A \wedge B$, is determined over the same parameter set $\mathcal{E} \times \mathcal{E}$ by

$$\forall(e_1, e_2) \in \mathcal{E} \times \mathcal{E}, \forall x \in X, \begin{cases} T_{A\vee B_{(e_1,e_2)}}(x) = T_{A_{e_1}}(x) \circ T_{B_{e_2}}(x) \\ I_{A\vee B_{(e_1,e_2)}}(x) = I_{A_{e_1}}(x) \circ I_{B_{e_2}}(x) \\ F_{A\vee B_{(e_1,e_2)}}(x) = F_{A_{e_1}}(x) \bullet F_{B_{e_2}}(x) \end{cases}. \tag{27}$$

Example 7. If using $\min -\text{norm}$ $x \bullet y = \max\{x + y - 1, 0\}$ and $\max -\text{norm}$ $x \circ y = \min\{x + y, 1\}$, the AND $A \wedge B$ and OR $A \vee B$ operations of the two above NS-sets A and B in Example 3 is described according to Eqs (26) and (27) in Table 11 as follows:

Table 11. NS-sets $A \wedge B$ and $A \vee B$.

$A \wedge B$	(e_1, e_1)	(e_1, e_2)	(e_2, e_1)	(e_2, e_2)	$A \vee B$	(e_1, e_1)	(e_1, e_2)	(e_2, e_1)	(e_2, e_2)
x_1	⟨0,0,1⟩	⟨0,0,1⟩	⟨.2,0,1⟩	⟨.1,0,1⟩	x_1	⟨.9,.6,.2⟩	⟨.8,.4,1⟩	⟨1,.4,.5⟩	⟨1,.2,.3⟩
x_2	⟨0,.3,.4⟩	⟨0,.1,1⟩	⟨.1,.8,.5⟩	⟨.5,.6,1⟩	x_2	⟨.4,1,1⟩	⟨.8,1,.2⟩	⟨1,1,0⟩	⟨1,1,.3⟩
x_3	⟨0,0,1⟩	⟨0,0,.7⟩	⟨.4,.3,.1⟩	⟨.6,0,.4⟩	x_3	⟨.7,.6,.4⟩	⟨.9,.3,0⟩	⟨1,1,.1⟩	⟨1,1,0⟩

Theorem 6. *If the* $\min -norm$ *and* $\max -norm$ *satisfy De Morgan's law, for all* $A, B \in \mathcal{NS}(X)$,

(1) $\overline{A \wedge B} = \bar{A} \vee \bar{B}$,

(2) $\overline{A \vee B} = \bar{A} \wedge \bar{B}$.

Proof.

(1) $\forall(e_1, e_2) \in \mathcal{E} \times \mathcal{E}, \forall x \in X$,

$$\begin{cases} T_{\overline{A\wedge B}_{(e_1,e_2)}}(x) = F_{A\wedge B_{(e_1,e_2)}} = F_{A_{e_1}}(x) \circ F_{B_{e_2}}(x) \\ I_{\overline{A\wedge B}_{(e_1,e_2)}}(x) = 1 - I_{A\wedge B_{(e_1,e_2)}} = 1 - I_{A_{e_1}}(x) \bullet I_{B_{e_2}}(x), \\ F_{\overline{A\wedge B}_{(e_1,e_2)}}(x) = T_{A\wedge B_{(e_1,e_2)}} = T_{A_{e_1}}(x) \bullet T_{B_{e_2}}(x) \end{cases} \tag{28}$$

and

$$\begin{cases} T_{\bar{A}\vee\bar{B}_{(e_1,e_2)}}(x) = T_{\bar{A}_{e_1}}(x) \circ T_{\bar{B}_{e_2}}(x) = F_{A_{e_1}}(x) \circ F_{B_{e_2}}(x) \\ I_{\bar{A}\vee\bar{B}_{(e_1,e_2)}}(x) = I_{\bar{A}_{e_1}}(x) \circ I_{\bar{B}_{e_2}}(x) = \left(1 - I_{A_{e_1}}(x)\right) \circ \left(1 - I_{B_{e_2}}(x)\right). \\ F_{\bar{A}\vee\bar{B}_{(e_1,e_2)}}(x) = F_{\bar{A}_{e_1}}(x) \bullet F_{\bar{B}_{e_2}}(x) = T_{A_{e_1}}(x) \bullet T_{B_{e_2}}(x) \end{cases} \tag{29}$$

Moreover,

$$\left(1 - I_{A_{e_1}}(x)\right) \circ \left(1 - I_{B_{e_2}}(x)\right) = 1 - I_{A_e}(x) \bullet I_{B_e}(x), \tag{30}$$

due to De Morgan's law of the $\min -\text{norm}$ and $\max -\text{norm}$. Therefore,

$$\forall (e_1, e_2) \in \mathcal{E} \times \mathcal{E}, \forall x \in X, \begin{cases} T_{\overline{A \wedge B}_{(e_1,e_2)}}(x) = T_{\bar{A}\vee\bar{B}_{(e_1,e_2)}}(x) \\ I_{\overline{A \wedge B}_{(e_1,e_2)}}(x) = I_{\bar{A}\vee\bar{B}_{(e_1,e_2)}}(x) \, . \\ F_{\overline{A \wedge B}_{(e_1,e_2)}}(x) = F_{\bar{A}\vee\bar{B}_{(e_1,e_2)}}(x) \end{cases} \tag{31}$$

(2) $\forall (e_1, e_2) \in \mathcal{E} \times \mathcal{E}, \forall x \in X$,

$$\begin{cases} T_{\overline{A \vee B}_{(e_1,e_2)}}(x) = F_{A \vee B_{(e_1,e_2)}} = F_{A_{e_1}}(x) \bullet F_{B_{e_2}}(x) \\ I_{\overline{A \vee B}_{(e_1,e_2)}}(x) = 1 - I_{A \vee B_{(e_1,e_2)}} = 1 - I_{A_{e_1}}(x) \circ I_{B_{e_2}}(x), \\ F_{\overline{A \vee B}_{(e_1,e_2)}}(x) = T_{A \vee B_{(e_1,e_2)}} = T_{A_{e_1}}(x) \circ T_{B_{e_2}}(x) \end{cases} \tag{32}$$

and

$$\begin{cases} T_{\bar{A}\wedge\bar{B}_{(e_1,e_2)}}(x) = T_{\bar{A}_{e_1}}(x) \bullet T_{\bar{B}_{e_2}}(x) = F_{A_{e_1}}(x) \bullet F_{B_{e_2}}(x) \\ I_{\bar{A}\wedge\bar{B}_{(e_1,e_2)}}(x) = I_{\bar{A}_{e_1}}(x) \bullet I_{\bar{B}_{e_2}}(x) = \left(1 - I_{A_{e_1}}(x)\right) \bullet \left(1 - I_{B_{e_2}}(x)\right). \\ F_{\bar{A}\wedge\bar{B}_{(e_1,e_2)}}(x) = F_{\bar{A}_{e_1}}(x) \circ F_{\bar{B}_{e_2}}(x) = T_{A_{e_1}}(x) \circ T_{B_{e_2}}(x) \end{cases} \tag{33}$$

Moreover,

$$\left(1 - I_{A_{e_1}}(x)\right) \bullet \left(1 - I_{B_{e_2}}(x)\right) = 1 - I_{A_{e_1}}(x) \circ I_{B_{e_2}}(x), \tag{34}$$

due to De Morgan's law of the $\min -\text{norm}$ and $\max -\text{norm}$. Therefore,

$$\forall (e_1, e_2) \in \mathcal{E} \times \mathcal{E}, \forall x \in X, \begin{cases} T_{\overline{A \vee B}_{(e_1,e_2)}}(x) = T_{\bar{A}\wedge\bar{B}_{(e_1,e_2)}}(x) \\ I_{\overline{A \vee B}_{(e_1,e_2)}}(x) = I_{\bar{A}\wedge\bar{B}_{(e_1,e_2)}}(x) \, . \\ F_{\overline{A \vee B}_{(e_1,e_2)}}(x) = F_{\bar{A}\wedge\bar{B}_{(e_1,e_2)}}(x) \end{cases} \tag{35}$$

4. Topology on NS-sets

This section uses the operations just constructed above as the core to build the topology and related concepts on NS-sets. It is important to note that the norms used must satisfy De Morgan's law.

4.1. NS-topological space

Definition 14. A collection $\tau \subseteq \mathcal{NS}(X)$ is *NS-topology* on X if it obeys the following properties:

(a) $\emptyset_{\mathcal{E}}$ and $X_{\mathcal{E}}$ belongs to τ,

(b) The intersection of any finite collection of τ's elements belongs to τ,

(c) The union of any collection of τ's elements belongs to τ.

Then, the pair (X, τ) is a *NS-topological space* and each element of τ is a *NS-open set*.

Example 8. Let three NS-sets K_1, K_2, K_3 be represented in Table 12 as follows:

Table 12. NS-sets K_1, K_2, K_3.

K_1	e_1	e_2	K_2	e_1	e_2	K_3	e_1	e_2
x_1	$\langle .2,.2,1\rangle$	$\langle .5,.5,.7\rangle$	x_1	$\langle .3,.3,.9\rangle$	$\langle .6,.6,.6\rangle$	x_1	$\langle .4,.4,.8\rangle$	$\langle .7,.7,.5\rangle$
x_2	$\langle .3,.3,.9\rangle$	$\langle .6,.6,.6\rangle$	x_2	$\langle .4,.4,.8\rangle$	$\langle .7,.7,.5\rangle$	x_2	$\langle .5,.5,.7\rangle$	$\langle .8,.8,.4\rangle$
x_3	$\langle .4,.4,.8\rangle$	$\langle 0,.9,1\rangle$	x_3	$\langle .5,.5,.7\rangle$	$\langle .8,.8,.4\rangle$	x_3	$\langle .6,.6,.6\rangle$	$\langle .9,.9,.3\rangle$

If using the $\min -$norm $x \bullet y = \min\{x, y\}$, $\max -$norm $x \circ y = \max\{x, y\}$, the collection $\tau = \{\emptyset_{\mathcal{E}}, X_{\mathcal{E}}, K_1, K_2, K_3\}$ is a NS-topology.

Theorem 7.

(1) $\tau_0 = \{\emptyset_{\mathcal{E}}, X_{\mathcal{E}}\}$ *is a NS-topology (anti-discrete).*

(2) $\tau_\infty = \mathcal{NS}(X)$ *is a NS-topology (discrete).*

(3) *If* τ_1 *and* τ_2 *are two NS-topologies,* $\tau_1 \cap \tau_2$ *is a NS-topology.*

Proof. This proof focuses on the proof of Property 3 because Properties 1 and 2 are directly inferred.

- $\emptyset_{\mathcal{E}}, X_{\mathcal{E}} \in \tau_1; \emptyset_{\mathcal{E}}, X_{\mathcal{E}} \in \tau_2 \Rightarrow \emptyset_{\mathcal{E}}, X_{\mathcal{E}} \in \tau_1 \cap \tau_2$.
- If $\{K_j\}_1^n$ is a finite family of NS-sets in $\tau_1 \cap \tau_2$, $K_i \in \tau_1$ and $K_i \in \tau_2$ for all i. So $\cap \{K_j\}_1^n \in \tau_1$ and $\{K_j\}_1^n \in \tau_2$. Thus $\cap \{K_j\}_1^n \in \tau_1 \cap \tau_2$.
- If letting $\{K_i | i \in I\}$ be a family of NS-sets in $\tau_1 \cap \tau_2$, $K_i \in \tau_1$ and $K_i \in \tau_2$ for all $i \in I$. So $\cup_{i \in I} K_i \in \tau_1$ and $\cup_{i \in I} K_i \in \tau_2$. Therefore, $\cup_{i \in I} K_i \in \tau_1 \cap \tau_2$.

It should be noted that if τ_1 and τ_2 are two NS-topologies, $\tau_1 \cup \tau_2$ cannot be a NS-topology. Counterexamples are shown in Example 9.

Example 9. Let three NS-sets H_1, H_2, H_3 be represented in Table 13 as follows:

Table 13. NS-sets K_1, K_2, K_3.

H_1	e_1	e_2	H_2	e_1	e_2	H_3	e_1	e_2
x_1	$\langle 1,0,1\rangle$	$\langle 0,1,0\rangle$	x_1	$\langle 0,1,0\rangle$	$\langle 1,0,1\rangle$	x_1	$\langle 1,0,1\rangle$	$\langle 1,0,1\rangle$
x_2	$\langle 1,0,1\rangle$	$\langle 0,1,0\rangle$	x_2	$\langle 0,1,0\rangle$	$\langle 1,0,1\rangle$	x_2	$\langle 1,0,1\rangle$	$\langle 1,0,1\rangle$

If using the $\min -$norm $x \bullet y = xy$, $\max -$norm $x \circ y = x + y - xy$ and letting $\tau_1 = \{\emptyset_{\mathcal{E}}, X_{\mathcal{E}}, H_1, H_2\}$ and $\tau_2 = \{\emptyset_{\mathcal{E}}, X_{\mathcal{E}}, H_3\}$ be two NS-topologies, the collection $\tau_1 \cup \tau_2 = \{\emptyset_{\mathcal{E}}, X_{\mathcal{E}}, H_1, H_2, H_3\}$ is not a NS-topology due to $H_1 \cup H_2 \notin \tau_1 \cup \tau_2$, see Table 14.

Table 14. NS-sets $H_1 \cup H_2$.

$H_1 \cup H_2$	e_1	e_2
x_1	$\langle 1,0,1\rangle$	$\langle 1,1,0\rangle$
x_2	$\langle 1,0,1\rangle$	$\langle 1,1,0\rangle$

Definition 15. A NS-set $A \in \mathcal{NS}(X)$ is *NS-closed set* if it has the complement $\bar{A}$ is a NS-open set. The symbol $\bar{\tau}$ is denoted as the collection of all NS-closed sets.

Theorem 8.

(1) $\emptyset_\mathcal{E}$ *and* $X_\mathcal{E}$ belongs to $\bar{\tau}$.

(2) The union of any finite collection of $\bar{\tau}$*'s elements belongs to* $\bar{\tau}$.

(3) The intersection of any collection of $\bar{\tau}$*'s elements belongs to* $\bar{\tau}$.

Proof. These properties are directly inferred from the definitions of a NS-closed set and De Morgan's law for intersection and union.

4.2. NS-interior

Definition 16. The *NS-interior* of a NS-set A, written as $\operatorname{int}(A)$, is the union of all NS-open subsets of A. It is considered the biggest NS-open set which is contained by A.

Example 10. Let three NS-sets L_1, L_2, K be represented as follows:

Table 15. NS-sets L_1, L_2, K.

L_1	e_1	e_2	L_2	e_1	e_2	K	e_1	e_2
x_1	$\langle .7,.8,.3\rangle$	$\langle .4,.5,.8\rangle$	x_1	$\langle .3,.2,.7\rangle$	$\langle .6,.5,.2\rangle$	x_1	$\langle .8,.8,.3\rangle$	$\langle .8,.8,.3\rangle$
x_2	$\langle .5,.2,.6\rangle$	$\langle .3,.4,.2\rangle$	x_2	$\langle .5,.8,.4\rangle$	$\langle .7,.6,.8\rangle$	x_2	$\langle .4,.6,.5\rangle$	$\langle .4,.6,.5\rangle$

If using the $\min-\text{norm}$ $x \bullet y = \max\{x + y - 1, 0\}$, $\max-\text{norm} = \min\{x + y, 1\}$, the collection $\tau = \{\emptyset_\mathcal{E}, X_\mathcal{E}, L_1, L_2\}$ is the NS-topology. It is easy to see that $\emptyset_\mathcal{E}, L_1 \subseteq K$ and $\emptyset_\mathcal{E} \cup L_1 = L_1 \subseteq K$. Therefore, $\operatorname{int}(K) = A$.

Theorem 9. A NS-set A is a NS-open set if and only if $A = \operatorname{int}(A)$.

Proof. If $A \in \tau$ then A is the biggest NS-open set that is contained by A. So $A = \operatorname{int}(A)$. Conversely, $A = \operatorname{int}(A) \in \tau$.

Theorem 10. *If* $A, B \in \mathcal{NS}(X)$,

(1) $\operatorname{int}(\operatorname{int}(A)) = \operatorname{int}(A)$,

(2) $\operatorname{int}(\emptyset_\mathcal{E}) = \emptyset_\mathcal{E}$ and $\operatorname{int}(X_\mathcal{E}) = X_\mathcal{E}$,

(3) $A \subseteq B \Rightarrow \operatorname{int}(A) \subseteq \operatorname{int}(B)$,

(4) $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$,

(5) $\operatorname{int}(A) \cup \operatorname{int}(B) \subseteq \operatorname{int}(A \cup B)$.

Proof.

(1) Due to $\operatorname{int}(A) \in \tau$, $\operatorname{int}(\operatorname{int}(A)) = \operatorname{int}(A)$.

(2) $\emptyset_\mathcal{E} \in \tau \Longrightarrow \operatorname{int}(\emptyset_\mathcal{E}) = \emptyset_\mathcal{E}$ and $X_\mathcal{E} \in \tau \Longrightarrow \operatorname{int}(X_\mathcal{E}) = X_\mathcal{E}$.

(3) Due to $A \subseteq B$, $\text{int}(A) \subseteq A \subseteq B$ and $\text{int}(B) \subseteq B$. Because $\text{int}(B)$ is the biggest NS-open set contained in B, $\text{int}(A) \subseteq \text{int}(B)$.

(4) Since $\text{int}(A) \in \tau$ and $\text{int}(B) \in \tau$, then $\text{int}(A) \cup \text{int}(B) \in \tau$. It is known that $\text{int}(A) \subseteq A$ and $\text{int}(B) \subseteq B$, so $\text{int}(A) \cup \text{int}(B) \subseteq A \cup B$. Moreover, $\text{int}(A \cup B)$ is the biggest NS-open set contained in $A \cup B$. Therefore, $\text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B)$.

(5) Since $\text{int}(A \cap B) \subseteq A \cap B$, so $\text{int}(A \cap B) \subseteq A$ and $\text{int}(A \cap B) \subseteq B$. Therefore, $\text{int}(A \cap B) \subseteq \text{int}(A)$ and $\text{int}(A \cap B) \subseteq \text{int}(B)$ or $\text{int}(A \cap B) \subseteq \text{int}(A) \cap \text{int}(B)$.

Moreover,

$$\begin{cases} \text{int}(A) \cap \text{int}(B) \subseteq \text{int}(A) \subseteq A \\ \text{int}(A) \cap \text{int}(B) \subseteq \text{int}(B) \subseteq B \end{cases} \Rightarrow \text{int}(A) \cap \text{int}(B) \subseteq A \cap B$$

and $\text{int}(A \cap B)$ is the biggest NS-open set contained in $A \cap B$, so

$$\text{int}(A) \cap \text{int}(B) \subseteq \text{int}(A \cap B).$$

Thus, $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$.

4.3. NS-closure

Definition 17. The *NS-closure* of a NS-set A, written as $\text{cl}(A)$, is the intersection of all NS-closed supersets of A. The $\text{cl}(A)$ is the smallest NS-closed set which contains A.

Example 11. For the NS-topology τ given in Example 10, let NS-set H be represented in Table 16 as follows:

Table 16. NS-sets H.

H	e_1	e_2	$\overline{L_1}$	e_1	e_2	e_1	e_2
x_1	⟨.2,.2,.8⟩	⟨.2,.2,.8⟩	x_1	⟨.3,.2,.7⟩	⟨.8,.5,.4⟩	⟨.7,.8,.3⟩	⟨.2,.5,.6⟩
x_2	⟨.3,.4,.8⟩	⟨.3,.4,.8⟩	x_2	⟨.6,.8,.5⟩	⟨.2,.6,.3⟩	⟨.4,.2,.5⟩	⟨.8,.4,.7⟩

It is easy to see that $\overline{\emptyset_\mathcal{E}} = X_\mathcal{E}$, $\overline{X_\mathcal{E}} = \emptyset_\mathcal{E}$. So $\emptyset_\mathcal{E}$, $X_\mathcal{E}, \overline{L_1}$, $\overline{L_2}$ are all NS-closed sets. Since $H \subseteq X_\mathcal{E}$, $\text{cl}(H) = L_2$.

Theorem 11. A NS-set A is a NS-closed set if and only if $A = \text{cl}(A)$.

Proof. Let A be a NS-closed set. Because $A \subseteq A$ and $\text{cl}(A)$ is the smallest NS-closed set that contains A, $\text{cl}(A) \subseteq A$. Therefore, $A = \text{cl}(A)$. Conversely, if $A = \text{cl}(A)$ then A is a NS-closed set.

Theorem 12. *If* $A, B \in \mathcal{NS}(X)$,

(1) $\text{cl}\big(\text{cl}(A)\big) = \text{cl}(A)$,

(2) $\text{cl}(\emptyset_\mathcal{E}) = \emptyset_\mathcal{E}$ and $\text{cl}(X_\mathcal{E}) = X_\mathcal{E}$,

(3) $A \subseteq B \Rightarrow \text{cl}(A) \subseteq \text{cl}(B)$,

(4) $\text{cl}(A \cap B) \subseteq \text{cl}(A) \cap \text{cl}(B)$,

(5) $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.

Proof.

(1) Directly inferring from Theorem 9.

(2) Directly inferring from Definition 14 and Theorem 9.

(3) Since $A \subseteq B \subseteq \mathrm{cl}(B)$ and $\mathrm{cl}(A)$ is the smallest NS-closed set containing A, $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.

(4) Since $A \cap B \subseteq A \subseteq \mathrm{cl}(A)$ and $A \cap B \subseteq B \subseteq \mathrm{cl}(B)$, $A \cap B \subseteq \mathrm{cl}(A) \cap \mathrm{cl}(B)$. Therefore, $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A) \cap \mathrm{cl}(B)$.

(5) It is easy to see that $A \subseteq A \cup B \subseteq \mathrm{cl}(A \cup B)$, $B \subseteq A \cup B \subseteq \mathrm{cl}(A \cup B)$, $\mathrm{cl}(A)$ is the smallest NS-closed set that contains A, and $\mathrm{cl}(B)$ is the smallest NS-closed set that containing B. So $\mathrm{cl}(A) \subseteq \mathrm{cl}(A \cup B)$ and $\mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B)$. Therefore, $\mathrm{cl}(A) \cup \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B)$.

Moreover, since $A \subseteq \mathrm{cl}(A)$ and $B \subseteq \mathrm{cl}(B)$, $A \cup B \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$. Therefore, $\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

Thus, $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

Theorem 13. *If* $A, B \in \mathcal{NS}(X)$,

(1) $\overline{\mathrm{int}(A)} = \mathrm{cl}(\bar{A})$,

(2) $\overline{\mathrm{cl}(A)} = \mathrm{int}(\bar{A})$.

Proof.

(1) Because

$$\mathrm{int}(A) = \cup_{i \in I} \{H_i \in \tau : H_i \subseteq A\},$$

$$\overline{\mathrm{int}(A)} = \overline{\cup_{i \in I} \{H_i \in \tau : H_i \subseteq A\}} = \cap_{i \in I} \{\overline{H_i} \in \bar{\tau} : \overline{H_i} \supseteq \bar{A}\} = \mathrm{cl}(\bar{A}). \tag{36}$$

(2) Because

$$\mathrm{cl}(A) = \cup_{i \in I} \{H_i \in \bar{\tau} : H_i \supseteq A\},$$

$$\overline{\mathrm{cl}(A)} = \overline{[\cap_{i \in I} \{H_i \in \bar{\tau} : H_i \supseteq A\}]} = \cup_{i \in I} \{\overline{H_i} \in \tau : \overline{H_i} \subseteq \bar{A}\} = \mathrm{int}(\bar{A}). \tag{37}$$

4.4. NS-boundary

Definition 18. The *NS-boundary* of a NS-set A, written as ∂A, is the intersection of the NS-closure of A and the NS-closure of $\bar{A}$.

Example 12. For the NS-topology τ given in Example 10 and the NS-set H given in Example 11, the complement of H is represented in Table 17 as follows:

Table 17. NS-sets $\overline{H}$.

$\overline{H}$	e_1	e_2
x_1	⟨.8,.8,.2⟩	⟨.8,.8,.2⟩
x_2	⟨.8,.6,.3⟩	⟨.8,.6,.3⟩

It is easy to see that $\mathrm{cl}(H) = X_{\mathcal{E}}$ and $\mathrm{cl}(\overline{H}) = X_{\mathcal{E}}$. So $\partial H = X_{\mathcal{E}} \cap X_{\mathcal{E}} = X_{\mathcal{E}}$.

Theorem 14. *If* $A \in \mathcal{NS}(X)$,

(1) $\partial A = \mathrm{cl}(A) \cap \overline{\mathrm{int}(A)}$,

(2) $\mathrm{int}(A) \cap \partial A = \emptyset_{\mathcal{E}}$,

(3) $\partial A = \emptyset_{\mathcal{E}}$ if and only if A is a NS-open and NS-closed set.

Proof.

(1) $\partial A = \mathrm{cl}(A) \cap \mathrm{cl}(\bar{A})$=$\mathrm{cl}(A) \cap \overline{\mathrm{int}(A)}$ due to Theorem 13.

(2) It is easy to see that

$$\mathrm{int}(A) \cap \partial(A) = \mathrm{int}(A) \cap \mathrm{cl}(A) \cap \mathrm{cl}(\bar{A}) = \mathrm{int}(A) \cap \mathrm{cl}(A) \cap \overline{\mathrm{int}(A)}$$
$$= \mathrm{int}(A) \cap \overline{\mathrm{int}(A)} \cap \mathrm{cl}(A) = \emptyset_{\mathcal{E}}$$

due to Theorem 13.

(3) Since

$$\partial(A) = \mathrm{cl}(A) \cap \mathrm{cl}(\bar{A}) = \mathrm{cl}(A) \cap \overline{\mathrm{int}(A)} = \emptyset_{\mathcal{E}},$$

$\mathrm{cl}(A) \cap \mathrm{int}(A) \neq \emptyset_{\mathcal{E}}$. So $A \subseteq \mathrm{cl}(A) \subseteq \mathrm{int}(A) \subseteq A$. Therefore, $A = \mathrm{cl}(A) = \mathrm{int}(A)$ or A is a NS-open and NS-closed set.

Conversely, if A is a NS-open and NS-closed set, $A = \mathrm{int}(A)$ and $A = \mathrm{cl}(A)$. Therefore,

$$\partial(A) = \mathrm{cl}(A) \cap \mathrm{cl}(\bar{A}) = \mathrm{cl}(A) \cap \overline{\mathrm{int}(A)} = \mathrm{cl}(A) \cap \overline{\mathrm{cl}(A)} = \emptyset_{\mathcal{E}}.$$

4.5. Regular property

Definition 19.

a. The NS-open set M is *regular* if $M = \mathrm{int}(\mathrm{cl}(M))$.

b. The NS-closed set M is *regular* if $M = \mathrm{cl}(\mathrm{int}(M))$.

Theorem 15. *If* $M, N \in \mathcal{NS}(X)$,

(1) If M *is a NS-closed set,* $\mathrm{int}(M)$ *is a regular NS-open set.*

(2) If M *is a NS-open set,* $\mathrm{cl}(M)$ *is a regular NS-closed set.*

(3) If M *and* N *are two regular NS-open sets,* $M \subseteq N \Leftrightarrow \mathrm{cl}(M) \subseteq \mathrm{cl}(N)$.

(4) If M *and* N *are two regular NS-closed sets,* $M \subseteq N \Leftrightarrow \mathrm{int}(M) \subseteq \mathrm{int}(N)$.

(5) If M *is a regular NS-closed set,* $\overline{M}$ *is a regular NS-open set.*

(6) If M *is a regular NS-open set,* $\overline{M}$ *is a regular NS-closed set.*

Proof.

(1) If M is a NS-closed set,

$$\mathrm{int}(M) \subseteq M \Rightarrow \mathrm{cl}[\mathrm{int}(M)] \subseteq \mathrm{cl}(M) = M \Rightarrow \mathrm{int}[\mathrm{cl}(\mathrm{int}(M))] \subseteq \mathrm{int}(M). \quad (38)$$

$$\mathrm{int}(M) \subseteq \mathrm{cl}(\mathrm{int}(M)) \Rightarrow \mathrm{int}(\mathrm{int}(M)) = \mathrm{int}(M) \subseteq \mathrm{int}[\mathrm{cl}(\mathrm{int}(M))]. \quad (39)$$

Therefore, $\mathrm{int}(M) = \mathrm{int}[\mathrm{cl}(\mathrm{int}(M))]$ or $\mathrm{int}(M)$ is regular.

(2) If M is a NS-open set,

$$\mathrm{int}(\mathrm{cl}(M)) \subseteq \mathrm{cl}(M) \Rightarrow \mathrm{cl}\left(\mathrm{int}(\mathrm{cl}(M))\right) \subseteq \mathrm{cl}(\mathrm{cl}(M)) = \mathrm{cl}(M). \quad (40)$$

Because $\mathrm{int}(M) = M$,

$$M \subseteq \mathrm{cl}(M) \Rightarrow \mathrm{int}(M) = M \subseteq \mathrm{int}(\mathrm{cl}(M)) \Rightarrow \mathrm{cl}(M) \subseteq \mathrm{cl}\left(\mathrm{int}(\mathrm{cl}(M))\right). \quad (41)$$

Therefore, $\mathrm{cl}(M) = \mathrm{cl}\left(\mathrm{int}(\mathrm{cl}(M))\right)$ or $\mathrm{cl}(M)$ is regular.

(3) Clearly, $M \subseteq N \Rightarrow \mathrm{cl}(M) \subseteq \mathrm{cl}(N)$ and $\mathrm{int}(\mathrm{cl}(M)) = M$, $\mathrm{int}(\mathrm{cl}(N)) = N$ due to M, N are regular. Conversely,

$$\mathrm{cl}(M) \subseteq \mathrm{cl}(N) \Longrightarrow \mathrm{int}\big(\mathrm{cl}(M)\big) = M \subseteq \mathrm{int}\big(\mathrm{cl}(N)\big) = N \Longrightarrow M \subseteq N.$$

(4) Clearly, $M \subseteq N \Longrightarrow \mathrm{int}(M) \subseteq \mathrm{int}(N)$ and $M = \mathrm{cl}\big(\mathrm{int}(M)\big)$, $N = \mathrm{cl}\big(\mathrm{int}(N)\big)$ due to M, N are regular. Conversely,

$$\mathrm{int}(M) \subseteq \mathrm{int}(N) \Longrightarrow \mathrm{cl}\big(\mathrm{int}(M)\big) = M \subseteq \mathrm{cl}\big(\mathrm{int}(N)\big) = N \Longrightarrow M \subseteq N.$$

(5) If M is a regular NS-open set, $M = \mathrm{int}\big(\mathrm{cl}(M)\big)$. So

$$\mathrm{cl}\big(\mathrm{int}(\bar{M})\big) = \mathrm{cl}\big(\overline{\mathrm{cl}(M)}\big) = \overline{\mathrm{int}\big(\mathrm{cl}(M)\big)} = \bar{M}.$$

Therefore, $\bar{M}$ is a regular NS-closed set.

(6) Similarly, if M is a regular NS-closed set, $\mathrm{int}\big(\mathrm{cl}(\bar{M})\big) = \bar{M}$. So $\bar{M}$ is a regular NS-open set.

5. The relationship between NS-topology and fuzzy soft topology

Theorem 16. Let $\tau = \{K_i : i \in I\}$ be NS-topology on X where

$$K_i = \left\{\left(e, \frac{x}{T_{K_e^i}(x), I_{K_e^i}(x), F_{K_e^i}(x)}\right) : e \in \mathcal{E}, x \in X\right\}. \tag{42}$$

Three collections

$$\mathcal{T} = (\mathcal{T}_i)_{i \in I} = \left\{\left(e, \langle x, T_{K_e^i}(x)\rangle\right) : e \in \mathcal{E}, x \in X\right\}, \tag{43}$$

$$\mathcal{I} = (\mathcal{I}_i)_{i \in I} = \left\{\left(e, \langle x, I_{K_e^i}(x)\rangle\right) : e \in \mathcal{E}, x \in X\right\}, \tag{44}$$

$$\mathcal{F} = (\mathcal{F}_i)_{i \in I} = \left\{\left(e, \langle x, 1 - F_{K_e^i}(x)\rangle\right) : e \in \mathcal{E}, x \in X\right\}, \tag{45}$$

are the fuzzy soft topologies on X.

Proof.

- $\emptyset_{\mathcal{E}} \in \tau \Rightarrow \widetilde{\emptyset} \in \mathcal{T}, \widetilde{\emptyset} \in \mathcal{I}, \widetilde{\emptyset} \in \mathcal{F}$.
- $X_{\mathcal{E}} \in \tau \Rightarrow \tilde{X} \in \mathcal{T}; \tilde{X} \in \mathcal{I}; \tilde{X} \in \mathcal{F}$.
- Let $(\mathcal{T}_i)_{i \in I}$ be a family of fuzzy soft sets in $\mathcal{T}$, $(\mathcal{I}_i)_{i \in I}$ be a family of fuzzy soft sets in $\mathcal{I}$, and $(\mathcal{F}_i)_{i \in I}$ be a family of fuzzy soft sets in $\mathcal{F}$. They make a family of NS-sets $\{K_i : i \in I\}$ where

$$K_i = \left\{\left(e, \frac{x}{T_{K_e^i}(x), I_{K_e^i}(x), F_{K_e^i}(x)}\right) : e \in \mathcal{E}, x \in X\right\} \in \tau. \tag{46}$$

So $\cup_{i \in I} K_i \in \tau$ or

$$\cup_{i \in I} K_i = \left\{\left(e, \frac{x}{\underset{i \in I}{\circ}\{T_{K_{i_e}}(x)\}, \underset{i \in I}{\circ}\{I_{K_{i_e}}(x)\}, \underset{i \in I}{\bullet}\{F_{K_{i_e}}(x)\}}\right) : e \in \mathcal{E}, x \in X\right\} \in \tau. \tag{47}$$

Therefore,

$$\left\{\left[\langle \underset{i\in I}{\circ}\left\{T_{K_{i_e}}(x)\right\}\rangle : x \in X\right]_{e\in\mathcal{E}}\right\} = \widetilde{\bigcup_{\iota\in I}}\left\{T_{K_{i_e}}(X) : e \in \mathcal{E}\right\} \in \mathcal{T}, \tag{48}$$

$$\left\{\left[\langle \underset{i\in I}{\circ}\left\{T_{K_{i_e}}(x)\right\}\rangle : x \in X\right]_{e\in\mathcal{E}}\right\} = \widetilde{\bigcup_{\iota\in I}}\left\{I_{K_{i_e}}(X) : e \in \mathcal{E}\right\} \in \mathcal{I}, \tag{49}$$

$$\begin{aligned}
&\left\{\left[\langle \underset{i\in I}{\bullet}\left\{F_{K_{i_e}}(x)\right\}\rangle : x \in X\right]_{e\in\mathcal{E}}\right\}^C \\
&= \left\{\left[\langle 1 - \underset{i\in I}{\bullet}\left\{F_{K_{i_e}}(x)\right\}\rangle : x \in X\right]_{e\in\mathcal{E}}\right\} \\
&= \left\{\left[\langle \underset{i\in I}{\circ}\left\{\left(1 - F_{K_{i(e)}}(a)\right)\right\}\rangle : a \in X\right]_{e\in E}\right\} \\
&= \widetilde{\bigcup_{\iota\in I}}\left\{\left(I_{K_{i(e)}}(X)\right)^C_{e\in\mathcal{E}}\right\} \in \mathcal{F}.
\end{aligned} \tag{50}$$

- Let $\left\{\mathcal{T}_j \in \mathcal{T}\right\}_1^n, \left\{\mathcal{I}_j \in \mathcal{I}\right\}_1^n, \left\{\mathcal{F}_j \in \mathcal{F}\right\}_1^n$ be finite families of fuzzy soft sets on X and satisfy

$$K_j = \left\{\left(e, \frac{x}{T_{K_e^j}(x), I_{K_e^j}(x), F_{K_e^j}(x)}\right) : e \in \mathcal{E}, x \in X\right\} \in \tau. \tag{51}$$

So, we have $\cap_1^n K_j \in \tau$, i.e.,

$$\cap_1^n K_j = \left\{\left(e, \frac{x}{\left\{\bullet T_{K_{i_e}}(x)\right\}_1^n, \left\{\bullet I_{K_{i_e}}(x)\right\}_1^n, \left\{\circ F_{K_{i_e}}(x)\right\}_1^n}\right) : e \in \mathcal{E}, x \in X\right\}. \tag{52}$$

Hence,

$$\left[\left\{\left\{\bullet T_{K_{i_e}}(x)\right\}_1^n : x \in X\right\}_{e\in\mathcal{E}}\right] = \widetilde{\cap}_1^n \left\{\left[T_{K_{i_e}}(X)\right]_{e\in\mathcal{E}}\right\} \in \mathcal{T}, \tag{53}$$

$$\left[\left\{\left\{\bullet I_{K_{i_e}}(x)\right\}_1^n : x \in X\right\}_{e\in\mathcal{E}}\right] = \widetilde{\cap}_1^n \left\{\left[I_{K_{i_e}}(X)\right]_{e\in\mathcal{E}}\right\} \in \mathcal{I}, \tag{54}$$

$$\begin{aligned}
&\left\{\left[\left\{\circ\, F_{K_{i_e}}(x)\right\}_1^n : x \in X\right]_{e\in\mathcal{E}}\right\}^C \\
&= \left\{\left[1 - \left\{\circ\, F_{K_{i_e}}(x)\right\}_1^n : x \in X\right]_{e\in\mathcal{E}}\right\} \\
&= \left\{\left[\bullet\left\{1 - F_{K_{i_e}}(x)\right\}_1^n : x \in X\right]_{e\in\mathcal{E}}\right\} \\
&= \widetilde{\cap}_1^n \left\{\left[I_{K_{i_e}}(X)\right]_{e\in\mathcal{E}}\right\} \in \mathcal{F}.
\end{aligned} \tag{55}$$

In the general case, the opposite of Theorem 16 is not true. This is demonstrated through a counterexample, as shown in Example 13.

Example 13. Let four NS-sets H_1, H_2, and H_3 be represented in Table 18 as follows:

Table 18. NS-sets H_1, H_2, and H_3.

		e_1	e_2
H_1	x_1	$\langle .25, .25, .75\rangle$	$\langle .25, .25, .75\rangle$
	x_2	$\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\rangle$	$\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\rangle$
H_2	x_1	$\langle .5, .75, .5\rangle$	$\langle .5, .75, .5\rangle$
	x_2	$\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle$	$\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle$
H_3	x_1	$\langle .75, .5, .25\rangle$	$\langle .75, .5, .25\rangle$
	x_2	$\langle \frac{2}{3}, \frac{1}{3}, \frac{1}{3}\rangle$	$\langle \frac{2}{3}, \frac{1}{3}, \frac{1}{3}\rangle$
$H_1 \cup H_2$	x_1	$\langle .75, 1, .25\rangle$	$\langle .75, 1, .25\rangle$
	x_2	$\langle \frac{2}{3}, 1, 0\rangle$	$\langle \frac{2}{3}, 1, 0\rangle$

If using the $\min-\text{norm}$ $x\bullet y = \max\{x + y - 1, 0\}$, $\max-\text{norm} = \min\{x + y, 1\}$, three collections defined in Table 19 as follows are the fuzzy soft topologies on X.

Table 19. NS-sets $\mathcal{T}$, $\mathcal{I}$, and $\mathcal{F}$.

		e_1	e_2
$\mathcal{T}$	$\tilde{\emptyset}$	$\langle 0,0\rangle$	$\langle 0,0\rangle$
	$\tilde{X}$	$\langle 1,1\rangle$	$\langle 1,1\rangle$
	$\mathcal{T}_1$	$\langle .25, \frac{1}{3}\rangle$	$\langle .25, \frac{1}{3}\rangle$
	$\mathcal{T}_2$	$\langle .5, \frac{1}{3}\rangle$	$\langle .5, \frac{1}{3}\rangle$
	$\mathcal{T}_3$	$\langle .75, \frac{2}{3}\rangle$	$\langle .75, \frac{2}{3}\rangle$
$\mathcal{I}$	$\tilde{\emptyset}$	$\langle 0,0\rangle$	$\langle 0,0\rangle$
	$\tilde{X}$	$\langle 1,1\rangle$	$\langle 1,1\rangle$
	$\mathcal{I}_1$	$\langle .25, \frac{1}{3}\rangle$	$\langle .25, \frac{1}{3}\rangle$
	$\mathcal{I}_2$	$\langle .75, \frac{2}{3}\rangle$	$\langle .75, \frac{2}{3}\rangle$
	$\mathcal{I}_3$	$\langle .5, \frac{1}{3}\rangle$	$\langle .5, \frac{1}{3}\rangle$
$\mathcal{F}$	$\tilde{\emptyset}$	$\langle 0,0\rangle$	$\langle 0,0\rangle$
	$\tilde{X}$	$\langle 1,1\rangle$	$\langle 1,1\rangle$
	$\mathcal{F}_1$	$\langle .25, \frac{1}{3}\rangle$	$\langle .25, \frac{1}{3}\rangle$
	$\mathcal{F}_2$	$\langle .5, \frac{1}{3}\rangle$	$\langle .5, \frac{1}{3}\rangle$
	$\mathcal{F}_3$	$\langle .75, \frac{2}{3}\rangle$	$\langle .75, \frac{2}{3}\rangle$

The $\mathcal{T}, \mathcal{I}, \mathcal{F}$ are fuzzy soft topologies on X, but $\tau = \{\emptyset_{\mathcal{E}}, X_{\mathcal{E}}, H_1, H_2, H_3\}$ is not a NS- topology on X because $H_1 \cup H_2 \notin \tau$.

Theorem 17. Let three collections

$$\mathcal{T} = (\mathcal{T}_i)_{i\in I} = \left\{\left(e, \langle x, T_{K_e^i}(x)\rangle\right) : e \in \mathcal{E}, x \in X\right\}, \tag{56}$$

$$\mathcal{I} = (I_i)_{i\in I} = \left\{\left(e, \langle x, I_{K_e^i}(x)\rangle\right) : e \in \mathcal{E}, x \in X\right\}, \tag{57}$$

$$\mathcal{F} = (\mathcal{F}_i)_{i\in I} = \left\{\left(e, \langle x, 1 - F_{K_e^i}(x)\rangle\right) : e \in \mathcal{E}, x \in X\right\}, \tag{58}$$

be the fuzzy soft topologies on X. Let $\tau = \{K_i : i \in I\}$ where

$$K_i = \left\{\left(e, \frac{x}{T_{K_e^i}(x), I_{K_e^i}(x), F_{K_e^i}(x)}\right) : e \in \mathcal{E}, x \in X\right\}. \tag{59}$$

If for all l, m, n, we have

$$\mathcal{T}_l \cap \mathcal{T}_m = \mathcal{T}_n \Longrightarrow \begin{cases} \mathcal{I}_l \cap \mathcal{I}_m = \mathcal{I}_n \\ \mathcal{F}_l \cap \mathcal{F}_m = \mathcal{F}_n \end{cases}, \tag{60}$$

$$\mathcal{T}_l \cup \mathcal{T}_m = \mathcal{T}_n \Longrightarrow \begin{cases} \mathcal{I}_l \cap \mathcal{I}_m = \mathcal{I}_n \\ \mathcal{F}_l \cap \mathcal{F}_m = \mathcal{F}_n \end{cases}. \tag{61}$$

Then τ is the NS-topology on X.

Proof.

- Obviously, $\emptyset_{\mathcal{E}}, X_{\mathcal{E}} \in \tau$.
- Let $\{K_i : i \in I\} \subset \tau$ be a family of NS-sets on X. We have $\{\mathcal{T}_i\}$, $\{\mathcal{I}_i\}$, $\{\mathcal{F}_i\}$ are families of fuzzy soft sets on X. So,

$$\exists n_0 \in I, \mathcal{T}_{n_0} = \cup_{i\in I} \mathcal{T}_i \in \mathcal{T}, \mathcal{I}_{n_0} = \cup_{i\in I} \mathcal{I}_i \in \mathcal{I}, \mathcal{F}_{n_0} = \cup_{i\in I} \mathcal{F}_i \in \mathcal{F}. \tag{62}$$

Thus, $\cup_{i\in I} K_i = \mathcal{T}_{n_0} \in \tau$.

- Let $\{K_j \in \tau\}_1^n$ be a finite family of NS-sets on X. We have $\{\mathcal{T}_j\}_1^n$, $\{\mathcal{I}_j\}_1^n$, $\{\mathcal{F}_j\}_1^n$ as finite families of fuzzy soft sets on X. So,

$$\exists m_0 \in I, \mathcal{T}_{m_0} = \cap_1^n \mathcal{T}_j \in \mathcal{T}, \mathcal{I}_{m_0} = \cap_1^n \mathcal{I}_j \in \mathcal{I}, \mathcal{F}_{m_0} = \cap_1^n \mathcal{F}_j \in \mathcal{F}, \tag{63}$$

Thus, $\cap_1^n K_j \in \tau$.

Theorem 18. Let $\tau = \{K_i : i \in I\}$ be the NS-topology on X where

$$K_i = \left\{\left(e, \frac{x}{T_{K_e^i}(x), I_{K_e^i}(x), F_{K_e^i}(x)}\right) : e \in \mathcal{E}, x \in X\right\}. \tag{64}$$

For each $e \in E$, three collections

$$\mathcal{T}_e = \left(\mathcal{T}_{e_i}\right)_{i\in I} = \left\{\langle x, T_{K_e^i}(x)\rangle : x \in X\right\}, \tag{65}$$

$$\mathcal{I}_e = \left(I_{e_i}\right)_{i\in I} = \left\{\langle x, I_{K_e^i}(x)\rangle : x \in X\right\}, \tag{66}$$

$$\mathcal{F}_e = \left(\mathcal{F}_{e_i}\right)_{i\in I} = \left\{\langle x, 1 - F_{K_e^i}(x)\rangle, x \in X\right\}, \tag{67}$$

are the fuzzy topologies on X.

Proof. It can be implied from Theorem 17.

In the general case, the opposite of Theorem 18 is not true. This is demonstrated through the counterexample shown in Example 14.

Example 14. We return to Example 12 with the same hypothesis. Then,

$$\mathcal{T}_{e_1} = \left\{(0,0), (1,1), \left(.25, \tfrac{1}{3}\right), \left(.5, \tfrac{1}{3}\right), \left(.75, \tfrac{2}{3}\right)\right\}, \tag{68}$$

$$\mathcal{I}_{e_1} = \left\{(0,0), (1,1), \left(.25, \tfrac{1}{3}\right), \left(.75, \tfrac{2}{3}\right), \left(.5, \tfrac{1}{3}\right)\right\}, \tag{69}$$

$$\mathcal{F}_{e_1} = \left\{(0,0), (1,1), \left(.25, \tfrac{1}{3}\right), \left(.5, \tfrac{1}{3}\right), \left(.75, \tfrac{2}{3}\right)\right\}, \tag{70}$$

are fuzzy topologies on X. Similarly,

$$\mathcal{T}_{e_2} = \left\{(0,0), (1,1), \left(.25, \tfrac{1}{3}\right), \left(.5, \tfrac{1}{3}\right), \left(.75, \tfrac{2}{3}\right)\right\}, \tag{71}$$

$$\mathcal{I}_{e_2} = \left\{(0,0), (1,1), \left(.25, \tfrac{1}{3}\right), \left(.5, \tfrac{1}{3}\right), \left(.75, \tfrac{2}{3}\right)\right\}, \tag{72}$$

$$\mathcal{F}_{e_2} = \left\{(0,0), (1,1), \left(.25, \tfrac{1}{3}\right), \left(.5, \tfrac{1}{3}\right), \left(.75, \tfrac{2}{3}\right)\right\}, \tag{73}$$

are also fuzzy topologies, but $\tau = \{\emptyset_{\mathcal{E}}, X_{\mathcal{E}}, H_1, H_2, H_3\}$ is not a NS-topology on X because $K_1 \cup K_2 \notin \tau$.

6. Conclusions

In this paper, two novel norms are proposed to serve as the core for determining operations on NS-sets. These operations are used to construct the topology and related concepts such as open set, closed set, interior, closure, and regularity. Another highlight of this work is demonstrating the relationship between the topologies on NS-sets and fuzzy soft sets. The topology on NS-sets can parameterize the topologies on fuzzy soft sets, but the reverse is not guaranteed. This work's advantage is the structure's logic is presented with well-defined concepts and convincingly proven theorems.

Determining these concepts in a novel way enables a variety of methods for studying NS-sets, and offers a unique opportunity for future research and development in this field. Such research could focus on separation axioms, continuity, compactness, and paracompactness on NS-sets. Moreover, the relationship between topology on hybrid structure, NS-sets, and component structures, neutrosophic sets and soft sets, is also of research interest. In addition, applications of neutrosophic soft topological spaces can be investigated to handle decision-making problems.

Furthermore, we are also turning our interests to building topology on a new type of set, neutrosophic fuzzy sets. We believe these results will be helpful for future studies on neutrosophic fuzzy topology to develop a general framework for practical applications. These issues present opportunities but also challenges for researchers interested in the field of fuzzy theory.

Conflict of interest

The authors declare that they have no competing interests in this paper.

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