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Research article

On degree theory for non-monotone type fractional order delay differential equations

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Abstract: In this paper, we establish a qualitative theory for implicit fractional order differential equations (IFODEs) with nonlocal initial condition (NIC) with delay term. Because area related to investigate existence and uniqueness of solution is important field in recent times. Also researchers are using existence theory to derive some prior results about a dynamical problem weather it exists or not in reality. In literature, we have different tools to study qualitative nature of a problem. On the same line the exact solution of every problem is difficult to determined. Therefore, we use technique of numerical analysis to approximate the solutions, where stability analysis is an important aspect. Therefore, we use a tool from non-linear analysis known as topological degree theory to develop sufficient conditions for existence and uniqueness of solution to the considered problem. Further, we also develop sufficient conditions for Hyers- Ulam type stability for the considered problem. To justify our results, we also give an illustrative example.

Keywords: arbitrary order differential equations; existence theory; topological degree theory
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1. Introduction

One of the emerging field of applied mathematics is fractional calculus, that particularly deals with real order derivatives and integrals. The important advantage of real order models in comparison with integer order models is that fractional integrals and derivatives are very useful in problems related to memory and hereditary properties of the materials. The rapid development in nano-technology, this branch of applied mathematics has attracted more attention of many researchers. By applying arbitrary order derivatives and integrals, one can study real world problems in more significant and accurate ways. For details study and applications, see [1–9] and reference there in. Here we remark that authors have investigated some interesting characteristics for dynamical systems of fractional differential equations (FDEs) like classifications of the equilibrium points and determination of topological properties through phase diagram in [10, 11].

The existence theory of fractional differential equations have been studied by using various methods of functional analysis [12–16]. For example in [17], authors have studied the following problem under Riemann-Liouville fractional derivative using Schaefer 's fixed point theorem with non-monotone term as

$$\begin{cases} \mathcal{D}_{0^+}^{\mu} u(t) = \Phi(t, u(t), \mathcal{D}_{0^+}^{\mu} u(t)), \quad t \in \mathrm{I} = (0, \mathbf{T}], \ \mathbf{T} < \infty, \\ t^{1-\mu} u(0) = u_0, \end{cases} \tag{1.1}$$

where $\mathcal{D}_{0^+}^{\mu}$ represent Riemann-Liouville fractional derivative, $0 < \mu \leq 1$ and $\Phi \in \mathbf{C}[\mathrm{I} \times \mathbb{R}^2, \mathbb{R}]$.

Since Riemann-Liouville fractional derivative has many applications in pure and applied mathematics. But it has used very less in applied sciences problems. Because FDEs involve such type of derivative need conditions for fractional order for clarifying the physical meaning in real world problems. For dynamical system, we need conditions in such away that it could explain the physical behavior of the system. So in this work, we turn our attention to Caputo fractional derivative, which uses initial or boundary conditions like ordinary differential equations.

Here we remark that delay differential equations have numerous applications in modeling real world problems related to their past history. One of the important type of delay differential equation is known as pantograph. These type differential equations are the special class of delay differential equations which involve proportional type delay factor. For the first time, such type of differential equations had been studied to improve the speed of electric train or buses. An important construction had been made by Ockendon and Taylor [18]. In recent times these type differential equations are increasingly used to model various real world problems. For instance the phenomenon related to lifting and pressing goods and materials by machine use this kind of model. Existence theory is important subject of differential equations. In last many years pantograph type differential equations have been considered for the existence and uniqueness of solution using different methods (see [19–22]).

Motivated from aforementioned work and literature about the importance of pantograph equations, in this research work we have considered problem (1.1) under delay differential equations involving Caputo fractional derivative and under nonlocal initial condition with non-monotone term as

$$\begin{cases} {}^{\mathrm{c}}\mathcal{D}_{0+}^{\mu} u(t) = \Phi(t, u(t), {}^{\mathrm{c}}\mathcal{D}_{0^+}^{\mu} u(\alpha t)), \quad t \in \mathrm{I}, \\ t^{1-\mu} u(0) = u_0 + \Psi(t, u), \end{cases} \tag{1.2}$$

where ${}^{\mathrm{c}}\mathcal{D}_{0+}^{\mu}$ represent Caputo fractional derivative, $0 < \mu \leq 1$, $0 < \alpha < 1$ and $\Phi \in \mathbf{C}[\mathrm{I} \times \mathbb{R}^2, \mathbb{R}]$, $\Psi \in \mathbf{C}[\mathrm{I} \times \mathbb{R}, \mathbb{R}]$.

To develop qualitative theory for the considered problem, we use prior estimate method or topological degree method of nonlinear analysis. The suggested method reduces strong compact conditions of the operator to weaker one. Further new assumptions on the nonlocal condition are provided to guarantee the equivalency between the solution of the considered nonlocal problem and its corresponding operator equation [23]. The said conditions are vary rarely considered in literature. We have also investigated various kinds of Hyers-Ulam stability including Ulam-Hyers (U-H), generalized Ulam-Hyers (g-U-H), Ulam-Hyers-Rassias (U-H-R) and generalized Ulam-Hyers-Rassias (g-U-H-R) stabilities. In recent past such type of stability have been studied very well for the aforementioned area, for example see some papers as [24–26]. The novelty of this work is the extension of topological degree theory for delay type problems involving non-monotone term as well as nonlocal condition. The proposed method has the ability to use measure of non-compactness to relax the strong compact condition to some weaker one. Here when the proposed problem is converted to operator form, we split the transformed problem into two parts. For which we prove one part just to satisfies Lipschtiz conditions to obtain contraction result, while for the second part we prove relatively compact condition. This procedure makes degree theory a powerful tool in recent times to investigate FDEs. Further the mentioned degree method has numerous applications in investigation of solutions to variety of boundary value problems (see [27–29]).

We have organized our paper as: Section one is devoted to introduction and literature overview. In the Section two, we have recollected basic definitions results that we need. In the Section 3, we give our main results. Section four is related to stability analysis. Section five is devoted to an illustrative example. Last section is concluded with brief remark.

2. Preliminaries

Here we recall some fundamental results from [30].

Definition 2.1. *Fractional integral of any real order μ of a function $u(t) \in \mathrm{L}^1([\mathrm{x}_1, \mathrm{x}_2], \mathbb{R})$ where* $\mathrm{x}_1, \mathrm{x}_2 \in \mathbb{R}$*, is define by*

$$\mathrm{I}^{\mu}_{\mathrm{x}_1+}u(t) = \frac{1}{\Gamma(\mu)}\int_{\mathrm{x}_1}^{t}(t-\eta)^{\mu-1}u(\eta)\,\mathrm{d}\eta. \tag{2.1}$$

Definition 2.2. *Caputo fractional order derivative of any real order μ of a function $u(t)$ on* $[\mathrm{x}_1, \mathrm{x}_2]$ *where* $\mathrm{x}_1, \mathrm{x}_2 \in \mathbb{R}$ *is given by*

$$^{\mathrm{c}}\mathcal{D}^{\mu}_{\mathrm{x}_1+}\mathrm{u}(\mathrm{t}) = \frac{1}{\Gamma(\mathrm{j}-\mu)}\int_{\mathrm{x}_1}^{\mathrm{t}}(\mathrm{t}-\eta)^{\mathrm{j}-\mu-1}\mathrm{u}^{(\mathrm{j})}(\eta)\,\mathrm{d}\eta, \quad j-1<\mu\le j.$$

Lemma 2.2.1. *The unique solution of FDEs of the form*

$$^{c}\mathcal{D}^{\mu}_{0+}u(t) = 0,\ \mathrm{j}-1<\mu\le \mathrm{j} \tag{2.2}$$

is given by:

$$u(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{\mathrm{j}-1}t^{\mathrm{j}-1},$$

for $\mathrm{c_i} \in \mathbb{R}$, $\mathrm{i} = 0, 1, 2, ..., \mathrm{j}-1$.

Lemma 2.2.2. *The unique solution of FDEs of the form*

$$^{c}\mathcal{D}^{\mu}_{0+}u(t) = \mathrm{f}(t) \tag{2.3}$$

is given by:

$$u(t) = \mathrm{I}^{\mu}_{+0}[\mathrm{f}(t)] + \sum_{\mathrm{i}=0}^{\mathrm{j}-1} \mathrm{c_i} t^{\mathrm{i}},$$

for arbitrary $\mathrm{c_i} \in \mathbb{R}, \quad \mathrm{i} = 0, 1, 2, ..., \mathrm{j} - 1.$

Onward, the set $\mathbf{X}$ will represent Banach space on $\mathbf{C}[\mathrm{I}, \mathbb{R}]$. The norm on $\mathbf{X}$ is define as $\|u\| = \sup\{|u(t)|, \quad t \in \mathrm{I}\}$ and $\Sigma \subset \mathrm{P}(\mathbf{X})$ represent family of all bounded sub sets of $\mathbf{X}$.

Definition 2.3. *[30] The Kuratowski measure of non-compactness* $\xi : \Sigma \to \mathbb{R}_+$ *is defined as*

$$\xi(\sigma) = inf\{\epsilon > 0\},$$

where $\sigma \in \Sigma$ *admits a finite cover by sets of diameter* $\leq \epsilon$*.*

For further properties and definitions of Kuratowski measure, one may see [30].

Definition 2.4. *Let* $\mathbf{T} : \mathbf{G} \to \mathbf{X}$ *be a continuous function, where* $\mathbf{G} \subset \mathbf{X}$*. We say* $\mathbf{T}$ *is* ξ*-Lipschitz (* ξ*-contraction for* $\mathrm{Q} < 1$*) if constant* $\mathrm{Q} \geq 0$ *exist such that*

$$\xi(\mathbf{T}(\mathrm{G}')) \leq \mathrm{Q}\, \xi(\mathrm{G}'), \forall \quad \mathrm{G}' \subset \mathbf{G}.$$

Definition 2.5. $\mathbf{T}$ *is* ξ*-condensing if*

$$\xi(\mathbf{T}(\mathrm{G}')) < \xi(\mathrm{G}'), \forall \quad \mathrm{G}' \subset \mathbf{G} \quad with \quad \xi(\mathrm{G}') > 0.$$

or, $\xi(\mathbf{T}(\mathrm{G}')) \geq \xi(\mathrm{G}')$ *implies* $\xi(\mathrm{G}') = 0$*,*

Definition 2.6. *The map* $\mathbf{T} : \mathbf{G} \to \mathbf{X}$ *be Lipschitz (contraction for* $\mathrm{Q}' < 1$*) if there exist* $\mathrm{Q}' > 0$*, such that*

$$\|\mathbf{T}(u) - \mathbf{T}(v)\| \leq \mathrm{Q}'\|u - v\|, \quad \forall \quad u, v \in \mathbf{G}. \tag{2.4}$$

Proposition 2.1. *If* $\mathbf{T}, \mathbf{T}' : \mathbf{G} \to \mathbf{X}$ *are* ξ*-Lipschitz maps with constants* Q *and* $\mathrm{Q}^{'}$ *respectively, then* $\mathbf{T} + \mathbf{T}' : \mathbf{G} \to \mathbf{X}$ *are* ξ*-Lipschitz with constants* $\mathrm{Q} + \mathrm{Q}^{'}$*.*

Proposition 2.2. *If* $\mathbf{T} : \mathbf{G} \to \mathbf{X}$ *is compact, then* $\mathbf{T}$ *is* ξ*-Lipschitz with constant* $\mathrm{Q} = 0$*.*

Proposition 2.3. *If* $\mathbf{T} : \mathbf{G} \to \mathbf{X}$ *is Lipschitz with constant* Q*, then* $\mathbf{T}$ *is* ξ*-Lipschitz with the same constant* Q*.*

Theorem 2.7. *[30] Let* $\Lambda : \mathbf{X} \to \mathbf{X}$ *be a* ξ*-condensing map and the set*

$$\Theta = \{\mathrm{x} \in \mathbf{X} : there\ exist \quad \vartheta \in [0, 1] \quad such\ that \quad \mathrm{x} = \vartheta\Lambda\mathrm{x}\}.$$

If Θ *is a bounded set in* $\mathbf{X}$*, so there exist* $\mathrm{r} > 0$ *such that* $\Theta \subset \mathbf{G}_\mathrm{r}(0)$*, then the degree*

$$\deg(I - \vartheta\Lambda\mathrm{x}, \mathbf{G}_\mathrm{r}(0), 0) = 1, \quad for\ all \quad \vartheta \in [0, 1].$$

Consequently, Λ *ha at least one fixed point and the set of the fixed points of* Λ *lies in* $\mathbf{G}_\mathrm{r}(0)$*.*

3. Main results

In the current section, we apply our proposed method to derive conditions for the existence of at least one solution to IFODEs (1.2) subject to nonlocal initial condition, let

$$\begin{cases} {}^{c}\mathcal{D}_{0+}^{\mu}u(t) = \Phi(t, u(t), {}^{c}\mathcal{D}_{0^+}^{\mu}u(\alpha t)), \quad 0 < \mu \leq 1, \quad t \in \mathrm{I}, \\ t^{1-\mu}u(0) = u_0 + \Psi(t, u), \end{cases} \tag{3.1}$$

where $0 < \alpha < 1$ and $\Psi \in \mathbf{C}[\mathrm{I} \times \mathbb{R}, \mathbb{R}]$, $\Phi \in \mathbf{C}[\mathrm{I} \times \mathbb{R}^2, \mathbb{R}]$.
Using Lemma 2.2.2, we have

$$u(t) = t^{\mu-1}\left(u_0 + \Psi(t, u)\right) + \int_0^t \frac{(t-\eta)^{\mu-1}}{\Gamma(\mu)}\Phi(\eta, u(\eta), {}^{c}\mathcal{D}_{0^+}^{\mu}u(\alpha\eta))\mathrm{d}\eta, \tag{3.2}$$

where $t \in \mathrm{I}$, $\Psi \in \mathbf{C}[\mathrm{I} \times \mathbb{R}, \mathbb{R}]$ and $\Phi \in \mathbf{C}[\mathrm{I}{\times}\mathbb{R}^2, \mathbb{R}]$.
Let

$$\mathrm{h}(t) = \Phi(t, u(t), \mathrm{h}(\alpha t)), \tag{3.3}$$

then Eq (3.2), becomes

$$u(t) = t^{\mu-1}\left(u_0 + \Psi(t, u)\right) + \int_0^t \frac{(t-\eta)^{\mu-1}}{\Gamma(\mu)}\mathrm{h}(\eta)\mathrm{d}\eta, \quad t \in \mathrm{I}, \tag{3.4}$$

where $\mathrm{h} : (0, \mathrm{T}] \to \mathbb{R}$ is continuous. Next, assume the following conditions:

(A_1) There exist $\mathbf{a}$, $\mathbf{b} \geq 0$, and $\mathrm{q}_1 \in [0, 1)$, such that

$$\|\Psi(t, u)\| \leq \mathbf{a}\|u\|^{\mathrm{q}_1} + \mathbf{b},$$

for every $(t, u) \in \mathrm{I} \times \mathbb{R}$.

(A_2) There exist $\mathrm{Q}_1 \in [0, 1)$, such that

$$\|\Psi(t, u_1) - \Psi(t, u_2)\| \leq \mathrm{Q}_1\|u_1 - u_2\|,$$

for every $(t, u_1), (t, u_2) \in \mathrm{I} \times \mathbb{R}$.

(A_3) There exist $\mathbf{c}$, $\mathbf{d} \geq 0$ and $\mathrm{q}_2 \in [0, 1)$, such that

$$\|\mathrm{h}(t)\| \leq \mathbf{c}\|u\|^{\mathrm{q}_2} + \mathbf{d},$$

for every $t \in \mathrm{I}$.

(A_4) There exist a constant $\mathrm{L}_\Phi > 0$, such that

$$\|\Phi(t, \bar{u}, u) - \Phi(t, \bar{v}, v)\| \quad \leq \quad \mathrm{L}_\Phi\left[\|\bar{u} - \bar{v}\| + \|u - v\|\right],$$

for each $\quad t \in \mathrm{I}, \quad$ and $\quad \forall\ \bar{u},\ u, \bar{v},\ v \in \mathbb{R}$.
Set $\mathrm{p}(t) = \Phi(t, \bar{u}(t), \mathrm{p}(\alpha t))$ and $\mathrm{q}(t) = \Phi(t, u(t), \mathrm{q}(\alpha t))$, then

$$\begin{aligned} \|\mathrm{p}(t) - \mathrm{q}(t)\| &= \|\Phi(t, \bar{u}(t), \mathrm{p}(\alpha t)) - \Phi(t, u(t), \mathrm{q}(\alpha t))\|, \\ &\leq \mathrm{L}_\Phi\left[\|\bar{u} - u\| + \|\mathrm{p}(\alpha t) - \mathrm{q}(\alpha t)\|\right], \\ &\leq \mathrm{L}_\Phi\|\bar{u} - u\| + \mathrm{L}_\Phi\|\mathrm{p}(t) - \mathrm{q}(t)\|, \\ &\leq \frac{\mathrm{L}_\Phi}{1 - \mathrm{L}_\Phi}\|\bar{u} - u\|. \end{aligned}$$

The above assumptions are key to our main results.
Let, $\Omega = \mathbf{C}(\mathbf{I})$, define the operators:

$$\begin{cases} \mathbf{f} : \Omega \to \Omega, (\mathbf{f}u)(t) = & t^{\mu-1}\left(u_0 + \Psi(t,u)\right), \\ \mathbf{g} : \Omega \to \Omega, (\mathbf{g}u)(t) = & \displaystyle\int_0^t \frac{(t-\eta)^{\mu-1}}{\Gamma(\mu)} \mathrm{h}(\eta) \mathrm{d}\eta, \end{cases}$$

and

$$\mathbf{F} : \Omega \to \Omega, \quad \mathbf{F}u = \mathbf{f}u + \mathbf{g}u.$$

From the above relations, Eq (3.4), can be written as:

$$u = \mathbf{F}u. \tag{3.5}$$

The relation (3.5) is actually a fixed point problem, so the fixed point of $\mathbf{F}$ is actually the solution of our proposed model (3.1).

Lemma 3.0.1. *The map* $\mathbf{f} : \Omega \to \Omega$ *is Lipschitz.*

Proof. Let, $u, \bar{u} \in \Omega$, then

$$\begin{aligned} \|\mathbf{f}\bar{u} - \mathbf{f}u\| &= \sup_{t\in \mathrm{I}} \left\{ |(\mathbf{f}\bar{u})(t) - (\mathbf{f}u)(t)| \right\}, \\ &\leq \sup_{t\in \mathrm{I}} \left\{ \left| t^{\mu-1} \right| |\Psi(t,\bar{u}) - \Psi(t,u)| \right\}, \\ &= \mathbf{L}\, \|\bar{u} - u\|, \text{ where } \mathbf{L} = \mathrm{Q}_1 \mathrm{T}^{\mu-1}, \text{ for every } u, \bar{u} \in \Omega. \end{aligned}$$

Proposition 2.3 implies that, $\mathbf{f}$ is ξ-Lipschitz.
Using assumption (A_1), $\mathbf{f}$, satisfies the following growth condition:

$$\|\mathbf{f}u\| \leq \mathbf{a}' \|u\|^{\mathrm{q}_1} + \mathbf{b}', \text{ where } \mathbf{a}' = \mathbf{a}\mathrm{T}^{\mu-1} \text{ and } \mathbf{b}' = (|u_0| + \mathbf{b})\mathrm{T}^{\mu-1}, \text{ for every } u \in \Omega. \tag{3.6}$$

□

Lemma 3.0.2. *The operator* $\mathbf{g} : \Omega \to \Omega$ *is compact.*

Proof. For continuity of $\mathbf{g}$, let $\{u_{\mathrm{j}}\} \subset \Omega, u \in \Omega$ be such that $\|u_{\mathrm{j}} - u\| \to 0$ as $\mathrm{j} \to \infty$. We have to show that $\|\mathbf{g}u_{\mathrm{j}} - \mathbf{g}u\| \to 0$ as $\mathrm{j} \to \infty$. For any $\epsilon > 0$, there exist $\mathrm{Q} \geq 0$, such that

$$\begin{aligned} \|u_{\mathrm{j}}\| &\leq \mathrm{Q}, \quad \forall\, \mathrm{n} \in \mathrm{N}, \\ \|u\| &\leq \mathrm{Q}. \end{aligned}$$

As $\mathrm{h} \in \mathbf{C}[\mathrm{I}, \mathbb{R}]$, so $\mathrm{h}_{\mathrm{j}} \to \mathrm{h}$ as $\mathrm{j} \to \infty$, where $\mathrm{h}_{\mathrm{j}} = \Phi(t, u_{\mathrm{j}}(t), \mathrm{h}_{\mathrm{j}}(\alpha t))$ and $\mathrm{h} = \Phi(t, u(t), \mathrm{h}(\alpha t))$.
Consider

$$\begin{aligned} \|\mathbf{g}u_{\mathrm{j}} - \mathbf{g}u\| &= \sup_{t\in \mathrm{I}} \left| \int_0^t \frac{(t-\eta)^{\mu-1}}{\Gamma(\mu)} \Phi(t, u_{\mathrm{j}}(t), \mathrm{h}_{\mathrm{j}}(\alpha t))(\eta) \mathrm{d}\eta - \int_0^t \frac{(t-\eta)^{\mu-1}}{\Gamma(\mu)} \Phi(t, u(t), \mathrm{h}(\alpha t)) \mathrm{d}\eta \right|, \\ &\leq \sup_{t\in \mathrm{I}} \int_0^t \left| \frac{(t-\eta)^{\mu-1}}{\Gamma(\mu)} \right| \left| \Phi(t, u_{\mathrm{j}}(t), \mathrm{h}_{\mathrm{j}}(\alpha t)) - \Phi(t, u(t), \mathrm{h}(\alpha t)) \right| \mathrm{d}\eta \to 0 \text{ as } \mathrm{j} \to \infty. \end{aligned}$$

Thus $\mathbf{g}$ is continuous.
The operator $\mathbf{g}$ is compact, for this take $\mathrm{S} \subset \Omega$, where S is a bounded subset of Ω.
For $\mathrm{Q} \geq 0$, be such that

$$\|u\| \leq \mathrm{Q}, \ \forall \ u \in \mathrm{S}.$$

The operator $\mathbf{g}$ satisfy the following growth condition

$$\|\mathbf{g}u\| \leq \mathbf{c}'\|u\| + \mathbf{d}', \text{ where } \mathbf{c}' = \frac{\mathbf{c}\mathrm{T}^{\mu}}{\Gamma(\mu+1)} \text{ and } \mathbf{d}' = \frac{\mathbf{d}\mathrm{T}^{\mu}}{\Gamma(\mu+1)}, \text{ for all } u \in \mathrm{S}. \tag{3.7}$$

The above relation (3.7) implies that

$$\|\mathbf{g}u\| \leq \mathbf{c}'\mathrm{Q}^{\mathrm{q}_2} + \mathbf{d}'.$$

So $\mathbf{g}(\mathrm{S})$ is bounded in Ω.
Let $\mathrm{t}_1, \mathrm{t}_2 \in [0, \mathrm{T}]$ such that $t_1 \geq t_2$, then

$$\begin{aligned} |(\mathbf{g}u)(t_1) - (\mathbf{g}u)(t_2)| &\leq \int_0^{t_1} \Big|\frac{(t_1-\eta)^{\mu-1}}{\Gamma(\mu)} - \int_0^{t_2} \Big|\frac{(t_2-\eta)^{\mu-1}}{\Gamma(\mu)}\Big||\mathrm{h}(\eta)|\mathrm{d}\eta, \\ &\leq \frac{[\mathbf{c}\mathrm{Q}^{\mathrm{q}_2} + \mathbf{d}]}{\Gamma(\mu+1)}(t_1^{\mu} - t_2^{\mu}) \to 0 \text{ as } t_1 \to t_2, \text{ for every } u \in \mathrm{S}. \end{aligned}$$

The set $\mathbf{g}(\mathrm{S})$, satisfies the hypothesis of Arzelá-Ascoli theorem, so $\mathbf{g}(\mathrm{S})$ is relatively compact in Ω. As a result, $\mathbf{g}$ is ξ-Lipschitz with zero constant. □

Here we present main theorem of existence.

Theorem 3.1. *The problem* (3.1) *has at least one solution if* Φ *and* Ψ *satisfies conditions* (A_1) *and* (A_3).

Proof. Take

$$\nabla = \{u \in \Omega : \text{ there exist } \vartheta \in [0,1) \text{ such that } u = \vartheta \mathbf{F}u\}.$$

The set ∇ in Ω is a bounded set. For this let, $u \in \nabla$ and $\vartheta \in [0,1)$, such that $u = \vartheta\mathbf{F}u$, we have

$$\begin{aligned} \|u\| &= \vartheta\|\mathbf{F}u\| \leq \vartheta\left(\|\mathbf{f}u\| + \|\mathbf{g}u\|\right), \\ &\leq \vartheta\left[\mathbf{a}'\|u\|_1^{\mathrm{q}} + \mathbf{b}' + \mathbf{c}'\|u\|_2^{\mathrm{q}} + \mathbf{d}'\right], \end{aligned}$$

which means ∇ in Ω is bonded.
Hence, Theorem 2.7 guaranty that the operator $\mathbf{F}$ possess at least one fixed point and the set of fixed points is bounded in Ω. □

Remark 1. *In assumption* (A_1) *if* $\mathrm{q}_1 = 1$, *then Theorem 3.1 steal hold if* $\mathbf{a}' < 1$.

Remark 2. *In assumption* (A_2) *if* $\mathrm{q}_2 = 1$, *then Theorem 3.1 hold if* $\mathbf{c}' < 1$.

Remark 3. *In assumptions* (A_1) *and* (A_2) *if* $\mathrm{q}_1 = \mathrm{q}_2 = 1$, *then Theorem 3.1 hold if* $\mathbf{a}' < 1$ *and* $\mathbf{c}' < 1$.

Theorem 3.2. *Under assumptions* $(A_1) - (A_4)$, *if a real constant* $\ell > 0$, *exists such that*

$$\ell = \left(\mathrm{Q}_1\mathrm{T}^{\mu-1} + \frac{\mathrm{L}_{\Phi}\mathrm{T}^{\mu}}{(1-\mathrm{L}_{\Phi})\Gamma(\mu+1)}\right) < 1, \tag{3.8}$$

then the solution of Eq (3.1) *is unique.*

Proof. Thank to Banach contraction principle, let $\bar{u}$, $u \in \Omega$, then

$$\begin{aligned}\|\mathbf{F}\bar{u} - \mathbf{F}u\| &\leq \|\mathbf{f}\bar{u} - \mathbf{f}u\| + \|\mathbf{g}\bar{u} - \mathbf{g}u\| \\ &\leq \left(\mathrm{Q}_1\mathrm{T}^{\mu-1} + \frac{\mathrm{L}_\Phi\mathrm{T}^\mu}{(1-\mathrm{L}_\Phi)\Gamma(\mu+1)}\right)\|\bar{u} - u\| \\ &= \ell\|\bar{u} - u\|\end{aligned}$$

□

Hence, the solution of problem (3.1) is unique.

4. Stability analysis

In this section, we will study U-H type stability, which was introduced by Ulam [31] in 1940 and further generalized by Hyers [32] and Rassias [33]. Following are some important definitions, which are recall from [34].

Consider the operator $\mathbf{Y} : \Omega \to \Omega$ satisfying:

$$\mathbf{Y}(u) = u, \text{ for } u \in \Omega. \tag{4.1}$$

Definition 4.1. *The Eq* (4.1) *is U-H type stable if for every* $\epsilon > 0$ *and let* $u \in \Omega$*, be any solution of inequality*

$$\|u - \mathbf{Y}u\| \leq \epsilon, \text{ for } t \in \mathrm{I}, \tag{4.2}$$

there exist a unique solution $\bar{u}$ *of Eq* (4.1) *with constant* $C_q > 0$*, satisfying the following inequality*

$$\|\bar{u} - u\| \leq C_q\epsilon, \quad t \in \mathrm{I}. \tag{4.3}$$

Definition 4.2. *Further, if there exist function* $\psi \in \mathbf{C}(\mathbb{R}, \mathbb{R})$ *with* $\psi(0) = 0$*, for any solution* u *of inequality* (4.2) *and a unique solution* $\bar{u}$ *of Eq* (4.1)*, such that*

$$\|\bar{u} - u\| \leq \psi(\epsilon), \tag{4.4}$$

then Eq (4.1) *is g-U-H type stable.*

Remark 4. *If there exist a function* $\alpha(t) \in \mathbf{C}(\mathrm{I}; \mathbb{R})$*, then* $\bar{u} \in \Omega$ *will be the solution of inequality* (4.2) *if*
(*i*) $|\alpha(t)| \leq \epsilon, \quad \forall \quad t \in \mathrm{I}$,
(*ii*) $\mathbf{Y}\bar{u}(t) = \bar{u}(t) + \alpha(t), \quad \forall \quad t \in \mathrm{I}$.

Consider the corresponding perturb problem of Eq (3.1) as follow:

$$\begin{cases} {}^{\mathrm{c}}\mathcal{D}^\mu_{0+}u(t) = \Phi(t, u(t), {}^{\mathrm{c}}\mathcal{D}^\mu_{0+}u(\alpha t)) + \alpha(t), & t \in \mathrm{I}, \ 0 < \mu \leq 1, \\ t^{1-\mu}u(0) = u_0 + \Psi(t, u). \end{cases} \tag{4.5}$$

Lemma 4.2.1. *The following inequality hold for perturb problem* (4.5).

$$|u(t) - \mathbf{F}u| \leq \frac{\mathrm{T}^\mu\epsilon}{\Gamma(\mu+1)}, \quad t \in \mathrm{I}.$$

Proof. Using Lemma 2.2.2, we get the solution of Eq (4.5), as follow

$$u(t) = \mathbf{F}u + \int_0^t \frac{(t-\eta)^{\mu-1}}{\Gamma(\mu)}\alpha(\eta)\mathrm{d}\eta.$$

By Remark 4, we have

$$\left|u(t) - \mathbf{F}u\right| \leq \int_0^t \left|\frac{(t-\eta)^{\mu-1}}{\Gamma(\mu)}\right||\alpha(\eta)|\mathrm{d}\eta \leq \frac{\mathrm{T}^{\mu}\epsilon}{\Gamma(\mu+1)}.$$

□

Theorem 4.3. *Under assumption* (*iv*) *and Lemma 4.2.1, problem* (3.1)*, is U-H and g-U-H stable if* $(1-\mathrm{L}_{\Phi})\Gamma(\mu+1) \neq \left(\mathrm{Q}_1(1-\mathrm{L}_{\Phi})\Gamma(\mu+1)\mathrm{T}^{\mu-1} + \mathrm{L}_{\Phi}\mathrm{T}^{\mu}\right)$ *hold.*

Proof. For unique solution u of problem (3.1), let $\bar{u}$ is any other solution of problem (3.1), then

$$\begin{aligned} \|\bar{u} - u\| &= \|\bar{u} - \mathbf{F}u\| \leq \|\bar{u} - \mathbf{F}\bar{u}\| + \|\mathbf{F}\bar{u} - \mathbf{F}u\|, \\ &\leq \frac{\mathrm{T}^{\mu}\epsilon}{\Gamma(\mu+1)} + \left(\mathrm{Q}_1\mathrm{T}^{\mu-1} + \frac{\mathrm{L}_{\Phi}\mathrm{T}^{\mu}}{(1-\mathrm{L}_{\Phi})\Gamma(\mu+1)}\right)\|\bar{u} - u\|, \\ &= \mathbf{C}_{\mathrm{q}}\epsilon, \quad \text{where} \quad \mathbf{C}_q = \frac{\mathrm{T}^{\mu}(1-\mathrm{L}_{\Phi})}{(1-\mathrm{L}_{\Phi})\Gamma(\mu+1) - (\mathrm{Q}_1(1-\mathrm{L}_{\Phi})\Gamma(\mu+1)\mathrm{T}^{\mu-1} + \mathrm{L}_{\Phi}\mathrm{T}^{\mu})}. \end{aligned}$$

Hence, the consider problem (3.1), is U-H stable.

Let $\psi : (0,1) \to (0,\infty)$ be a non-decreasing function, such that $\psi(\epsilon) = \epsilon$ with $\psi(0) = 0$, then

$$\|\bar{u} - u\| \leq \mathbf{C}_{\mathrm{q}}\psi(\epsilon),$$

this show that the problem (3.1) is g-U-H stable. □

Definition 4.4. *The Eq* (4.1) *is U-H-R type stable for a function* $\phi \in \mathbf{C}[\mathrm{I}, \mathbb{R}]$*, if for any* $\epsilon > 0$ *and let* $u \in \Omega$ *be any other solution of inequality:*

$$\|u - \mathbf{Y}u\| \leq \phi(t)\epsilon, \ \ for \ \ t \in \mathrm{I}, \tag{4.6}$$

there exist a unique solution $\bar{u}$ *of Eq* (4.1) *with a constant* $C_q > 0$*, satisfying*

$$\|\bar{u} - u\| \leq C_q\phi(t)\epsilon, \ \forall \ t \in \mathrm{I}. \tag{4.7}$$

Definition 4.5. *For a function* $\chi \in \mathbb{C}[\mathrm{I}, \mathbb{R}]$*, if there exists a constant* $\mathbb{C}_{q,\chi}$ *and for any* $\epsilon > 0$*, let* u *be any solution of inequality* (4.6) *and* $\bar{u}$ *be a unique solution of Eq* (4.1)*, such that*

$$\|\bar{u} - u\| \leq \mathbb{C}_{q,\chi}\chi(t), \ \ \forall \ t \in \mathrm{I}, \tag{4.8}$$

then Eq (4.1) *is g-U-H-R stable.*

Remark 5. *If there exist a function* $\alpha(t) \in \mathbb{C}(\mathrm{I}; \mathbb{R})$*, then* $\bar{u} \in \Omega$ *is the solution of inequality* (4.6) *if:*
(*i*) $|\alpha(t)| \leq \epsilon\chi(t), \quad \forall \quad t \in \mathrm{I},$
(*ii*) $\mathbf{Y}\bar{u}(t) = \bar{u} + \alpha(t), \quad \forall \quad t \in \mathrm{I}.$

Lemma 4.5.1. *The following inequality hold for the perturb Eq* (4.5).

$$\|u - \mathbf{F}u\| \leq \frac{\mathrm{T}^{\mu}\chi\epsilon}{\Gamma(\mu+1)}, \quad t \in \mathrm{I}.$$

Proof. The proof is similar to Lemma 4.2.1, so we left for the reader. □

Theorem 4.6. *Under condition* (*iv*) *and Lemma 4.5.1, problem* (3.1) *is U-H-R and g-U-H-R stable if,*

$$(1 - \mathrm{L}_{\Phi})\Gamma(\mu+1) \neq \left(\mathrm{Q}_1(1 - \mathrm{L}_{\Phi})\Gamma(\mu+1)\mathrm{T}^{\mu-1} + \mathrm{L}_{\Phi}\mathrm{T}^{\mu}\right), \text{ hold.}$$

Proof. For a unique solution u of Eq (3.1), let $\bar{u}$ be any other solution of Eq (3.1), then

$$\begin{aligned}
\|\bar{u} - u\| &= \|\bar{u} - \mathbf{F}u\| \leq \|\bar{u} - \mathbf{F}\bar{u}\| + \|\mathbf{F}\bar{u} - \mathbf{F}u\| \\
&\leq \frac{\mathrm{T}^{\mu}\chi\epsilon}{\Gamma(\mu+1)} + \left(\mathrm{Q}_1\mathrm{T}^{\mu-1} + \frac{\mathrm{L}_{\Phi}\mathrm{T}^{\mu}}{(1-\mathrm{L}_{\Phi})\Gamma(\mu+1)}\right)\|\bar{u} - u\| \\
&= \mathbb{C}_{\mathrm{q},\chi}\chi(t)\epsilon,
\end{aligned}$$

where $\mathbb{C}_{q,\chi} = \frac{\mathrm{T}^{\mu}(1-\mathrm{L}_{\Phi})}{(1-\mathrm{L}_{\Phi})\Gamma(\mu+1)-\left(\mathrm{Q}_1(1-\mathrm{L}_{\Phi})\Gamma(\mu+1)\mathrm{T}^{\mu-1}+\mathrm{L}_{\Phi}\mathrm{T}^{\mu}\right)}$. Which implies that the proposed problem (3.1), is U-H-R stable.
Put $\mathbb{C}_{q,\chi} = \frac{\mathrm{T}^{\mu}(1-\mathrm{L}_{\Phi})\epsilon}{(1-\mathrm{L}_{\Phi})\Gamma(\mu+1)-\left(\mathrm{Q}_1(1-\mathrm{L}_{\Phi})\Gamma(\mu+1)\mathrm{T}^{\mu-1}+L_{\Phi}\mathrm{T}^{\mu}\right)}$ then, we have

$$\|\bar{u} - u\| \leq \mathbb{C}_{\mathrm{q},\chi}\chi(t).$$

Which shows that the considered problem (3.1), is g-U-H-R stable. □

5. Illustrative example

Example 1. *Consider the following pantograph equation under nonlocal condition with monotone term as*

$$\begin{cases}
{}^{c}\mathcal{D}_{0^+}^{\frac{999}{1000}}u(t) = \dfrac{1}{(t+9)^2}\left(\dfrac{|u(t)|}{1 + |u(t)|^{\frac{1}{2}} + |{}^{c}\mathcal{D}_{0^+}^{\frac{999}{1000}}u(\alpha t)|}\right), \\
t^{1-\mu}u(0) = 1 + \dfrac{1}{e^t}\dfrac{\sin(|u(t)|)}{5}, \quad t \in \mathbf{I} = (0, 1].
\end{cases} \tag{5.1}$$

Here

$$\mathrm{h(t)} = \Phi(t, u(t), \mathrm{h}(\alpha t)) = \Phi(t, u(t), {}^{c}\mathcal{D}_{0^+}^{\mu}u(\alpha t)) = \frac{1}{(t+9)^2}\left(\frac{|u(t)|}{1 + |u(t)|^{\frac{1}{2}} + |{}^{c}\mathcal{D}_{0^+}^{\frac{999}{1000}}u(\alpha t)|}\right)$$

and

$$\Psi(\mathrm{t}, \mathrm{u}) = \frac{1}{e^t}\frac{\sin(|u(t)|)}{5}.$$

Now

$$\|\mathrm{h}(t)\| = \sup_{t\in\mathrm{I}}\left\{\left|\frac{1}{(t+9)^2}\left(\frac{|u(t)|}{1 + |u(t)|^{\frac{1}{2}} + |{}^{c}\mathcal{D}_{0^+}^{\frac{999}{1000}}u(\alpha t)|}\right)\right|\right\} \leq \frac{1}{81}\|u\|^{\frac{1}{2}}$$

and

$$\|\Psi\| \leq \frac{1}{5e}\|u\|^1.$$

Thus, Φ *and* Ψ *satisfies conditions* (A_1) *and* (A_3) *for*

$$\mu = \frac{999}{1000},\ \mathrm{q}_1 = \frac{1}{2},\ \mathrm{q}_2 = 1,\ \mathrm{I} = (0,1],\ \mathbf{a} = \frac{1}{81},\ \mathbf{c} = \frac{1}{5e}, \mathbf{d} = \mathbf{b} = 0.$$

Consider the set

$$\nabla = \{u \in \Omega : \textit{there exist } \vartheta \in [0,1] \textit{ such that } u = \vartheta \mathbf{F} u\}. \tag{5.2}$$

Let, $u \in \nabla$ *and* $\mu \in [0,1]$*, such that* $u = \mu\mathbf{F}u$*. By using the growth condition given in relations* (3.6) *and* (3.7)*, one can easily shows that the set* ∇ *in* Ω *is bounded, as a result of Theorem 3.1, the problem* (5.1) *has at least one solution. For uniqueness, we have*

$$\|\Phi(t,\bar{\mathrm{x}}(t),\bar{\mathrm{h}}(\alpha t)) - \Phi(t,u(t),\mathrm{h}(\alpha t))\| \leq \frac{1}{81}\|\bar{u} - u\|$$

and

$$\|\Psi(\bar{\mathrm{u}}) - \Psi(\mathrm{u})\| \leq \frac{1}{5e}\|\bar{\mathrm{u}} - u\|.$$

Thus Φ *and* Ψ *satisfies condition* (A_2) *and* (A_4) *with* $\mathrm{L}_\Phi = \frac{1}{81}$ *and* $\mathrm{Q}_1 = \frac{1}{5e}$*, then*

$$\ell \approx 0.0859164 < 1. \tag{5.3}$$

Hence, Theorem 3.2 guaranty that the solution of problem (5.1) *is unique.*
The problem (5.1) *is U-H and g-U-H stable, since*

$$(1 - \mathrm{L}_\Phi)\Gamma(\mu + 1) \neq \left(\mathrm{Q}_1(1 - \mathrm{L}_\Phi)\Gamma(\mu + 1)\mathrm{T}^{\mu-1} + L_\Phi \mathrm{T}^\mu\right),$$

for the given constants. Further, the problem (5.1) *is U-H-R and g-U-H-R stable with* $\chi(\mathrm{t}) = \mathrm{t}$ *for* $\mathrm{t} \in (0,1)$.

6. Conclusions

Some useful results for IFODEs devoted to existence and uniqueness by means of the topological degree theory under nonlocal condition of nonlinear type with non-monotone term have been established. We have used measure of non compactness for the construction of our results. Further, different kinds of Ulam type stability for the considered problem has been investigated. The analysis has been verified by providing an example at the end. From the whole analysis we have concluded that topological degree theory as more powerful and relax tool than traditional fixed point theory to study qualitative theory of FDEs. For future, we invite the young researchers to apply the same analysis to system of IFODEs and to some non-local problems involving fractal-fractional differential operators which have numerous applications in fractal geometry.

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Conflict of interest

There is no competing interest regarding this work.

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