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*Research article*

# Shifted-Legendre orthonormal method for high-dimensional heat conduction equations

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**Abstract:** In this paper, a numerical alogorthm for solving high-dimensional heat conduction equations is proposed. Based on Shifted-Legendre orthonormal polynomial and $\varepsilon-$best approximate solution, we extend the algorithm from low-dimensional space to high-dimensional space, and prove the convergence of the algorithm. Compared with other numerical methods, the proposed algorithm has the advantages of easy expansion and high convergence order, and we prove that the algorithm has $\alpha$-Order convergence. The validity and accuracy of this method are verified by some numerical experiments.

**Keywords:** shifted-Legendre polynomials; heat conduction equation; $\varepsilon-$best approximate solution; convergence order
**Mathematics Subject Classification:** 65M12, 65N12

## 1. Introduction

In this paper, we propose shifted-Legendre orthogonal function method for high-dimensional heat conduction equation [1]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = k(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}), \quad t \in [0,1], x \in [0,a], y \in [0,b], z \in [0,c], \\ u(0,x,y,z) = \phi(x,y,z), \\ u(t,0,y,z) = u(t,a,y,z) = 0, \\ u(t,x,0,z) = u(t,x,b,z) = 0, \\ u(t,x,y,0) = u(t,x,y,c) = 0. \end{cases} \tag{1.1}$$

Where $u(t,x,y,z)$ is the temperature field, $\phi(x,y,z)$ is a known function, $k$ is the thermal diffusion efficiency, and $a, b, c$ are constants that determine the size of the space.

Heat conduction system is a very common and important system in engineering problems, such as the heat transfer process of objects, the cooling system of electronic components and so on [1–4]. Generally, heat conduction is a complicated process, so we can't get the analytical solution of heat conduction equation. Therefore, many scholars proposed various numerical algorithms for heat conduction equation [5–8]. Reproducing kernel method is also an effective numerical algorithm for solving boundary value problems including heat conduction equation [9–14]. Galerkin schemes and Green's function are also used to construct numerical algorithms for solving one-dimensional and two-dimensional heat conduction equations [15–19]. Alternating direction implicit (ADI) method can be very effective in solving high-dimensional heat conduction equations [20, 21]. In addition, the novel local knot method and localized space time method are also used to solve convection-diffusion problems [22–25]. These methods play an important reference role in constructing new algorithms in this paper.

Legendre orthogonal function system is an important function sequence in the field of numerical analysis. Because its general term is polynomial, Legendre orthogonal function system has many advantages in the calculation process. Scholars use Legendre orthogonal function system to construct numerical algorithm of differential equations [26–28].

Based on the orthogonality of Legendre polynomials, we delicately construct a numerical algorithm that can be extended to high-dimensional heat conduction equation. The proposed algorithm has $\alpha$-Order convergence, and our algorithm can achieve higher accuracy compared with other algorithms.

The content of the paper is arranged like this: The properties of shifted Legendre polynomials, homogenization and spatial correlation are introduced in Section 2. In Section 3, we theoretically deduce the numerical algorithm methods of high-dimensional heat conduction equations. The convergence of the algorithm is proved in Section 4. Finally, three numerical examples and a brief summary are given at the end of this paper.

## 2. Preliminaries

In this section, the concept of shifted-Legendre polynomials and the space to solve Eq (1.1) are introduced. These knowledge will pave the way for describing the algorithm in this paper.

### 2.1. Shifted-Legendre polynomial

The traditional Legendre polynomial is the orthogonal function system on $[-1, 1]$. Since the variables $t, x, y, z$ to be analyzed for Eq (1.1) defined in different intervals, it is necessary to transform the Legendre polynomial on $[c_1, c_2]$, $c_1, c_2 \in \mathbb{R}$, and the shifted-Legendre polynomials after translation transformation and expansion transformation by Eq (2.1).

$$\begin{aligned} &p_0(x) = 1, \quad p_1(x) = \frac{2(x-c_1)}{c_2-c_1} - 1, \\ &p_{i+1}(x) = \frac{2i+1}{i+1}[\frac{2(x-c_1)}{c_2-c_1} - 1]p_i(x) - \frac{i}{i+1}p_{i-1}(x), \quad i = 1, 2, \cdots . \end{aligned} \tag{2.1}$$

Obviously, $\{p_i(x)\}_{i=0}^{\infty}$ is a system of orthogonal functions on $L^2[c_1, c_2]$, and

$$\int_{c_1}^{c_2} p_i(x)p_j(x)dx = \begin{cases} \dfrac{c_2-c_1}{2i+1}, & i = j, \\ 0, & i \neq j. \end{cases}$$

Let $L_i(x) = \sqrt{\frac{2i+1}{c_2-c_1}}p_i(x)$. Based on the knowledge of ref. [29], we begin to discuss the algorithm in this paper.

**Lemma 2.1.** *[29] $\{L_i(x)\}_{i=0}^{\infty}$ is a orthonormal basis in $L^2[c_1, c_2]$.*

*2.2. Homogenization of boundary value conditions*

Considering that the problem studied in this paper has a nonhomogeneous boundary value condition, the problem (1.1) can be homogenized by making a transformation as follows.

$$v(t, x, y, z) = u(t, x, y, z) - \phi(x, y, z).$$

Here, homogenization is necessary because we can easily construct functional spaces that meet the homogenization boundary value conditions. This makes us only need to pay attention to the operator equation itself in the next research, without considering the interference caused by boundary value conditions.

In this paper, in order to avoid the disadvantages of too many symbols, the homogeneous heat conduction system is still represented by $u$, the thermal diffusion efficiency $k = 1$, and the homogeneous system of heat conduction equation is simplified as follows:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} - \dfrac{\partial u}{\partial t} = f(x, y, z), \quad t \in [0, 1], x \in [0, a], y \in [0, b], z \in [0, c], \\ u(0, x, y, z) = 0, \\ u(t, 0, y, z) = u(t, a, y, z) = 0, \\ u(t, x, 0, z) = u(t, x, b, z) = 0, \\ u(t, x, y, 0) = u(t, x, y, c) = 0. \end{cases} \tag{2.2}$$

*2.3. Introduction of the space*

The solution space of Eq (2.2) is a high-dimensional space, which can be generated by some one-dimensional spaces. Therefore, this section first defines the following one-dimensional space.

Remember $AC$ represents the space of absolutely continuous functions.

**Definition 2.1.** $W_1[0, 1] = \{u(t)|u \in AC, u(0) = 0, u' \in L^2[0, 1]\}$, *and*

$$\langle u, v\rangle_{W_1} = \int_0^1 u'v'dt, \quad u, v \in W_1.$$

Let $c_1 = 0, c_2 = 1$, so $\{T_i(t)\}_{i=0}^{\infty}$ is the orthonormal basis in $L^2[0, 1]$, where $T_i(t) = L_i(t)$, note $T_n(t) = \sum\limits_{i=0}^{n} c_i t^i$. And $\{JT_n(t)\}_{n=0}^{\infty}$ is the orthonormal basis of $W_1[0, 1]$, where

$$JT_n(t) = \sum_{i=0}^{n} c_i \frac{t^{i+1}}{i+1}.$$

**Definition 2.2.** $W_2[0, a] = \{u(x)|u' \in AC, u(0) = u(a) = 0, u'' \in L^2[0, a]\}$, *and*

$$\langle u, v\rangle_{W_2} = \int_0^a u''v''dx, \quad u, v \in W_2.$$

Similarly, $\{P_n(x)\}_{n=0}^{\infty}$ is the orthonormal basis in $L^2[0,a]$, and denote $P_n(x) = \sum_{j=0}^{n} d_j x^j$, where $d_j \in \mathbb{R}$.

Let

$$JP_n(x) = \sum_{j=0}^{n} d_j \frac{x^{j+2} - a^{j+1}x}{(j+1)(j+2)},$$

obviously, $\{JP_n(x)\}_{n=0}^{\infty}$ is the orthonormal basis of $W_2[0,a]$.

## 3. Numerical method and analysis

We start with solving one-dimensional heat conduction equation, and then extend the algorithm to high-dimensional heat conduction equations.

### *3.1. The scheme of one-dimensional model*

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial t} = f(x), \quad t \in [0,1], x \in [0,a], \\ u(0,x) = 0, \\ u(t,0) = u(t,a) = 0. \end{cases} \tag{3.1}$$

Let $D = [0,1] \times [0,a]$, $CC$ represents the space of completely continuous functions, and $\mathbb{N}_n$ represents a set of natural numbers not exceeding $n$.

**Definition 3.1.** $W(D) = \{u(t,x) | \frac{\partial u}{\partial x} \in CC, (t,x) \in D, u(0,x) = 0, u(t,0) = u(t,a) = 0, \frac{\partial^3 u}{\partial t \partial x^2} \in L^2(D)\}$, *and*

$$\langle u, v \rangle_{W(D)} = \iint\limits_D \frac{\partial^3 u}{\partial t \partial x^2} \frac{\partial^3 v}{\partial t \partial x^2} d\sigma.$$

**Theorem 3.1.** *$W(D)$ is an inner product space.*

*Proof.* $\forall u(t,x) \in W(D)$, if $\langle u, u \rangle_{W(D)} = 0$, means

$$\iint\limits_D [\frac{\partial^3 u(t,x)}{\partial t \partial x^2}]^2 d\sigma = 0,$$

and it implies

$$\frac{\partial^3 u(t,x)}{\partial t \partial x^2} = \frac{\partial}{\partial t}(\frac{\partial^2 u(t,x)}{\partial x^2}) = 0.$$

Combined with the conditions of $W(D)$, we can get $u = 0$.

Obviously, $W(D)$ satisfies other conditions of inner product space. □

**Theorem 3.2.** $\forall u \in W(D), v_1(t)v_2(x) \in W(D)$*, then*

$$\langle u(t,x), v_1(t)v_2(x) \rangle_{W(D)} = \langle \langle u(t,x), v_1(t) \rangle_{W_1}, v_2(x) \rangle_{W_2}.$$

*Proof.* $\langle u(t,x), v_1(t)v_2(x)\rangle_{W(D)} = \iint\limits_D \frac{\partial^3 u(t,x)}{\partial t\partial x^2}\frac{\partial^3[v_1(t)v_2(x)]}{\partial t\partial x^2}d\sigma$

$$= \iint\limits_D \frac{\partial^2}{\partial x^2}[\frac{\partial u(t,x)}{\partial t}]\frac{\partial v_1(t)}{\partial t}\frac{\partial^2 v_2(x)}{\partial x^2}d\sigma$$

$$= \int_0^a \frac{\partial^2}{\partial x^2}\langle u(t,x), v_1(t)\rangle_{W_1}\frac{\partial^2 v_2(x)}{\partial x^2}dx$$

$$= \langle\langle u(t,x), v_1(t)\rangle_{W_1}, v_2(x)\rangle_{W_2}.$$

□

**Corollary 3.1.** *$\forall u_1(t)u_2(x) \in W(D), v_1(t)v_2(x) \in W(D)$, then*

$$\langle u_1(t)u_2(x), v_1(t)v_2(x)\rangle_{W(D)} = \langle u_1(t), v_1(t)\rangle_{W_1}\langle u_2(x), v_2(x)\rangle_{W_2}.$$

Let

$$\rho_{ij}(t,x) = JT_i(t)JP_j(x), \quad i,j \in \mathbb{N}.$$

**Theorem 3.3.** *$\{\rho_{ij}(t,x)\}_{i,j=0}^{\infty}$is an orthonormal basis in $W(D)$.*

*Proof.* $\forall \rho_{ij}(t,x), \rho_{lm}(t,x) \in W(D), \quad i,j,l,m \in \mathbb{N}$,

$$\begin{aligned}\langle\rho_{ij}(t,x), \rho_{lm}(t,x)\rangle_{W(D)} &= \langle JT_i(t)JP_j(x), JT_l(t)JP_m(x)\rangle_{W(D)}\\ &= \langle JT_i(t), JT_l(t)\rangle_{W_1}\langle JP_j(x), JP_m(x)\rangle_{W_2}.\end{aligned}$$

So

$$\langle\rho_{ij}(t,x), \rho_{lm}(t,x)\rangle_{W(D)} = \begin{cases}1, & i=l, j=m,\\ 0, & others.\end{cases}$$

In addition, $\forall u \in W(D)$, if $\langle u, \rho_{ij}\rangle_{W(D)} = 0$, means

$$\langle u(t,x), JT_i(t)JP_j(x)\rangle_{W(D)} = \langle\langle u(t,x), JT_i(t)\rangle_{W_1}, JP_j(x)\rangle_{W_2} = 0.$$

Note that $\{JP_j(x)\}_{j=0}^{\infty}$ is the complete system of $W_2$, so $\langle u(t,x), JT_i(t)\rangle_{W_1} = 0$.

Similarly, we can get $u(t,x) = 0$. □

Let $\mathcal{L} : W(D) \to L^2(D)$,

$$\mathcal{L}u = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}.$$

So, Eq (3.1) can be simplified as

$$\mathcal{L}u = f. \tag{3.2}$$

**Definition 3.2.** *$\forall \varepsilon > 0$, if $u \in W(D)$ and*

$$\|\mathcal{L}u - f\|^2_{L(D)} < \varepsilon, \tag{3.3}$$

*then $u$ is called the $\varepsilon-$best approximate solution for $\mathcal{L}u = f$.*

**Theorem 3.4.** *Any $\varepsilon > 0$, there is $N \in \mathbb{N}$, when $n > N$, then*

$$u_n(t,x) = \sum_{i=0}^{n}\sum_{j=0}^{n}\eta^*_{ij}\rho_{ij}(t,x) \tag{3.4}$$

*is the $\varepsilon$−best approximate solution for $\mathcal{L}u = f$, where $\eta^*_{ij}$ satisfies*

$$\|\sum_{i=0}^{n}\sum_{j=0}^{n}\eta^*_{ij}\mathcal{L}\rho_{ij} - f\|^2_{L^2(D)} = \min_{d_{ij}}\|\sum_{i=0}^{n}\sum_{j=0}^{n}d_{ij}\mathcal{L}\rho_{ij} - f\|^2_{L^2(D)}, \quad d_{ij} \in \mathbb{R}, i, j \in \mathbb{N}_n.$$

*Proof.* According to the Theorem 3.3, if $u$ satisfies Eq (3.2), then $u(t,x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\eta_{ij}\rho_{ij}(t,x)$, where $\eta_{ij}$ is the Fourier coefficient of $u$.

Note that $\mathcal{L}$ is a bounded operator [30], hence, any $\varepsilon > 0$, there is $N \in \mathbb{N}$, when $n > N$, then

$$\|\sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty}\eta_{ij}\rho_{ij}\|^2_{W(D)} < \frac{\varepsilon}{\|\mathcal{L}\|^2}.$$

So,

$$\begin{aligned}
\|\sum_{i=0}^{n}\sum_{j=0}^{n}\eta^*_{ij}\mathcal{L}\rho_{ij} - f\|^2_{L^2(D)} &= \min_{d_{ij}}\|\sum_{i=0}^{n}\sum_{j=0}^{n}d_{ij}\mathcal{L}\rho_{ij} - f\|^2_{L^2(D)} \\
&\leq \|\sum_{i=0}^{n}\sum_{j=0}^{n}\eta_{ij}\mathcal{L}\rho_{ij} - f\|^2_{L^2(D)} \\
&= \|\sum_{i=0}^{n}\sum_{j=0}^{n}\eta_{ij}\mathcal{L}\rho_{ij} - \mathcal{L}u\|^2_{L^2(D)} \\
&= \|\sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty}\eta_{ij}\mathcal{L}\rho_{ij}\|^2_{L^2(D)} \\
&\leq \|\mathcal{L}\|^2\|\sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty}\eta_{ij}\rho_{ij}\|^2_{W(D)} \\
&< \varepsilon.
\end{aligned}$$

□

For obtain $u_n(t,x)$, we need to find the coefficients $\eta^*_{ij}$ by solving Eq (3.5).

$$\min_{\{\eta_{ij}\}_{i,j=0}^{n}} J = \|\mathcal{L}u_n - f\|^2_{L^2(D)} \tag{3.5}$$

In addition,

$$\begin{aligned}
J &= \|\mathcal{L}u_n - f\|^2_{L^2(D)} \\
&= \langle \mathcal{L}u_n - f, \mathcal{L}u_n - f\rangle_{L^2(D)} \\
&= \langle \mathcal{L}u_n, \mathcal{L}u_n\rangle_{L^2(D)} - 2\langle \mathcal{L}u_n, f\rangle_{L^2(D)} + \langle f, f\rangle_{L^2(D)} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{l=0}^{n}\sum_{m=0}^{n}\eta_{ij}\eta_{lm}\langle \mathcal{L}\rho_{ij}, \mathcal{L}\rho_{lm}\rangle_{L^2(D)} - 2\sum_{i=0}^{n}\sum_{j=0}^{n}\eta_{ij}\langle \mathcal{L}\rho_{ij}, f\rangle_{L^2(D)} + \langle f, f\rangle_{L^2(D)}.
\end{aligned}$$

So,

$$\frac{\partial J}{\partial \eta_{ij}} = 2\sum_{l=0}^{n}\sum_{m=0}^{n}\eta_{lm}\langle \mathcal{L}\rho_{ij}, \mathcal{L}\rho_{lm}\rangle_{L^2(D)} - 2\eta_{ij}\langle \mathcal{L}\rho_{ij}, f\rangle_{L^2(D)}, \quad i, j \in \mathbb{N}_n$$

and the equations $\frac{\partial J}{\partial \eta_{ij}} = 0, \quad i, j \in \mathbb{N}_n$ can be simplified to

$$\mathbf{A}\eta = \mathbf{B}, \tag{3.6}$$

where

$$\mathbf{A} = (\langle \mathcal{L}\rho_{ij}, \mathcal{L}\rho_{lm}\rangle_{L^2(D)})_{N\times N}, \quad N = (n+1)^2,$$
$$\eta = (\eta_{ij})_{N\times 1}, \mathbf{B} = (\langle \mathcal{L}\rho_{ij}, f\rangle_{L^2(D)})_{N\times 1}.$$

**Theorem 3.5.** *$A\eta = \boldsymbol{B}$ has a unique solution.*

*Proof.* It can be proved that $\mathbf{A}$ is nonsingular. Let $\eta$ satisfy $\mathbf{A}\eta = 0$, that is,

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\langle \mathcal{L}\rho_{ij}, \mathcal{L}\rho_{lm}\rangle_{L^2(D)}\eta_{ij} = 0, \quad l, m \in \mathbb{N}_n.$$

So, we can get the following equations:

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\langle \eta_{ij}\mathcal{L}\rho_{ij}, \eta_{lm}\mathcal{L}\rho_{lm}\rangle_{L^2(D)} = 0, \quad l, m \in \mathbb{N}_n.$$

By adding the above $(n+1)^2$ equations, we can get

$$\langle \sum_{i=0}^{n}\sum_{j=0}^{n}\eta_{ij}\mathcal{L}\rho_{ij}, \sum_{l=0}^{n}\sum_{m=0}^{n}\eta_{lm}\mathcal{L}\rho_{lm}\rangle_{L^2(D)} = \|\sum_{i=0}^{n}\sum_{j=0}^{n}\eta_{ij}\mathcal{L}\rho_{ij}\|^2_{L^2(D)} = 0.$$

So,

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\eta_{ij}\mathcal{L}\rho_{ij} = 0.$$

Note that $\mathcal{L}$ is reversible. Therefore, $\eta_{ij} = 0, \quad i, j \in \mathbb{N}_n$. □

According to Theorem 3.5, $u_n(t,x)$ can be obtained by substituting $\eta = A^{-1}B$ into $u_n = \sum_{i=0}^{n}\sum_{j=0}^{n}\eta_{ij}\rho_{ij}(t,x)$.

### *3.2. Two-dimensional heat conduction equation*

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} - \dfrac{\partial u}{\partial t} = f(x,y), \quad t \in [0,1], x \in [0,a], y \in [0,b], \\ u(0,x,y) = 0, \\ u(t,0,y) = u(t,a,y) = 0, \\ u(t,x,0) = u(t,x,b) = 0. \end{cases} \tag{3.7}$$

Similar to definition 2.2, we can give the definition of linear space $W_3[0,b]$ as follows:

$$W_3[0,b] = \{u(y) | u' \in AC, y \in [0,b], u(0) = u(b) = 0, u'' \in L^2[0,b]\}.$$

Similarly, let $\{Q_n(y)\}_{n=0}^{\infty}$ is the orthonormal basis in $L^2[0,b]$, and denote $Q_n(y) = \sum_{k=0}^{n} q_k y^k$.

Let

$$JQ_n(y) = \sum_{k=0}^{n} q_k \frac{y^{k+2} - b^{k+1}y}{(k+1)(k+2)},$$

it is easy to prove that $\{JQ_n(y)\}_{n=0}^{\infty}$ is the orthonormal basis of $W_3[0,b]$.

Let $\Omega = [0,1] \times [0,a] \times [0,b]$. Now we define a three-dimensional space.

**Definition 3.3.** $W(\Omega) = \{u(t,x,y)|\frac{\partial^2 u}{\partial x\partial y} \in CC, (t,x,y) \in \Omega, u(0,x,y) = 0, u(t,0,y) = u(t,a,y) = 0, u(t,x,0) = u(t,x,b) = 0, \frac{\partial^5 u}{\partial t\partial x^2\partial y^2} \in L^2(\Omega)\}$, *and*

$$\langle u,v\rangle_{W(\Omega)} = \iiint\limits_{\Omega} \frac{\partial^5 u}{\partial t\partial x^2\partial y^2}\frac{\partial^5 v}{\partial t\partial x^2\partial y^2}d\Omega, \quad u,v \in W(\Omega).$$

Similarly, we give the following theorem without proof.

**Theorem 3.6.** $\{\rho_{ijk}(t,x,y)\}_{i,j,k=0}^{\infty}$*is an orthonormal basis of* $W(\Omega)$*, where*

$$\rho_{ijk}(t,x,y) = JT_i(t)JP_j(x)JQ_k(y), \;\; i,j,k \in \mathbb{N}_n.$$

Therefore, we can get $u_n$ as

$$u_n(t,x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}\eta_{ijk}\rho_{ijk}(t,x,y), \tag{3.8}$$

according to the theory in Section 3.1, we can find all $\eta_{ijk}, \;\; i,j,k \in \mathbb{N}_n$.

*3.3. Three-dimensional heat conduction equation*

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} - \dfrac{\partial u}{\partial t} = f(x,y,z), \;\; t \in [0,1], x \in [0,a], y \in [0,b], z \in [0,c], \\ u(0,x,y,z) = 0, \\ u(t,0,y,z) = u(t,a,y,z) = 0, \\ u(t,x,0,z) = u(t,x,b,z) = 0, \\ u(t,x,y,0) = u(t,x,y,c) = 0. \end{cases} \tag{3.9}$$

By Lemma 2.1, note that the orthonormal basis of $L_2[0,c]$ is $\{R_n(z)\}_{n=0}^{\infty}$, and denote $R_n(z) = \sum\limits_{m=0}^{n} r_m z^m$, where $r_m$ is the coefficient of polynomial $R_n(z)$.

We can further obtain the orthonormal basis $JR_n(z) = \sum\limits_{m=0}^{n} r_m \frac{z^{m+2}-c^{m+1}z}{(m+1)(m+2)}$ of $W_4[0,c]$, where

$$JR_n(z) = \sum_{m=0}^{n} r_m \frac{z^{m+2}-c^{m+1}z}{(m+1)(m+2)},$$

and

$$W_4[0,c] = \{u(z)|u' \in AC, z \in [0,c], u(0) = u(c) = 0, u'' \in L^2[0,c]\}.$$

Let $G = [0,1]\times[0,a]\times[0,b]\times[0,c]$. Now we define a four-dimensional space.

**Definition 3.4.** $W(G) = \{u(t,x,y,z)|\frac{\partial^3 u}{\partial x\partial y\partial z} \in CC, (t,x,y,z) \in G, u(0,x,y,z) = 0, u(t,0,y,z) = u(t,a,y,z) = 0, u(t,x,0,z) = u(t,x,b,z) = 0, u(t,x,y,0) = u(t,x,y,c) = 0, \frac{\partial^7 u}{\partial t\partial x^2\partial y^2\partial z^2} \in L^2(G)\}$, *and*

$$\langle u,v\rangle_{W(G)} = \iiiint\limits_{G} \frac{\partial^7 u}{\partial t\partial x^2\partial y^2\partial z^2}\frac{\partial^7 v}{\partial t\partial x^2\partial y^2\partial z^2}dG, \quad u,v \in W(G),$$

*where* $dG$=$dtdxdydz$.

Similarly, we give the following theorem without proof.

**Theorem 3.7.** $\{\rho_{ijk}(t,x,y,z)\}_{i,j,k,m=0}^{\infty}$*is an orthonormal basis of* $W(G)$*, where*

$$\rho_{ijkm}(t,x,y,z) = JT_i(t)JP_j(x)JQ_k(y)JR_m(z), \quad i,j,k,m \in \mathbb{N}.$$

Therefore, we can get $u_n$ as

$$u_n(t,x,y,z) = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}\sum_{m=0}^{n}\eta_{ijkm}\rho_{ijkm}(t,x,y,z), \tag{3.10}$$

according to the theory in Section 3.1, we can find all $\eta_{ijkm}, \quad i,j,k,m \in \mathbb{N}_n$.

## 4. Convergence analysis

Suppose $u(t,x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\eta_{ij}\rho_{ij}(t,x)$ is the exact solution of Eq (3.5). Let $P_{N_1,N_2}u(t,x) = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\eta_{ij}T_i(t)P_j(x)$ is the projection of $u$ in $L(D)$.

**Theorem 4.1.** *Suppose* $\frac{\partial^{r+l}u(t,x)}{\partial t^r \partial x^l} \in L^2(D)$*, and* $N_1 > r, N_2 > l$*, then, the error estimate of* $P_{N_1,N_2}u(t,x)$ *is*

$$\|u - P_{N_1,N_2}u\|_{L^2(D)}^2 \leq CN^{-\alpha},$$

*where* $C$ *is a constant,* $N = min\{N_1,N_2\}, \alpha = min\{r,l\}$.

*Proof.* According to the lemma in ref. [29], it follows that

$$\|u - u_{N_1}\|_{L_t^2[0,1]}^2 = \|u - P_{t,N_1}u\|_{L_t^2[0,1]}^2 \leq C_1N_1^{-r}\|\frac{\partial^r}{\partial t^r}u(t,x)\|_{L_t^2[0,1]}^2,$$

where $u_{N_1} = P_{t,N_1}u$ represents the projection of $u$ on variable $t$ in $L^2[0,1]$, and $\|\cdot\|_{L_t^2[0,1]}$ represents the norm of $(\cdot)$ with respect to variable $t$ in $L^2[0,1]$.

By integrating both sides of the above formula with respect to $x$, we can get

$$\begin{aligned}\|u - u_{N_1}\|_{L^2(D)}^2 &\leq C_1N_1^{-r}\int_0^a \|\frac{\partial^r}{\partial t^r}u\|_{L_t^2[0,1]}^2 dx \\ &= C_1N_1^{-r}\|\frac{\partial^r}{\partial t^r}u\|_{L^2(D)}^2.\end{aligned}$$

Moreover,

$$\begin{aligned}u(t,x) - u_{N_1}(t,x) &= \sum_{i=N_1+1}^{\infty}\langle u, T_i\rangle_{L_t^2[0,1]}T_i(t) \\ &= \sum_{i=N_1+1}^{\infty}\sum_{j=0}^{\infty}\langle\langle u, T_i\rangle_{L_t^2[0,1]}, P_j\rangle_{L_x^2[0,a]}P_j(x)T_i(t).\end{aligned}$$

According to the knowledge in Section 3,

$$\|u - u_{N_1}\|_{L^2(D)}^2 = \sum_{i=N_1+1}^{\infty}\sum_{j=0}^{\infty}c_{ij}^2,$$

where $c_{ij} = \langle\langle u, T_i\rangle_{L_t^2[0,1]}, P_j\rangle_{L_x^2[0,a]}$.

Therefore,

$$\sum_{i=N_1+1}^{\infty}\sum_{j=0}^{\infty} c_{ij}^2 \leq C_1 N_1^{-r} \|\frac{\partial^r}{\partial t^r} u\|_{L^2(D)}^2.$$

Similarly,

$$\sum_{i=0}^{\infty}\sum_{j=N_2+1}^{\infty} c_{ij}^2 \leq C_2 N_2^{-l} \|\frac{\partial^l}{\partial x^l} u\|_{L^2(D)}^2.$$

In conclusion,

$$\begin{aligned}
\|u - P_{N_1,N_2}u\|_{L^2(D)}^2 &= \|(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} - \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}) c_{ij}^2 T_i(t) P_j(x)\|_{L^2(D)}^2 \\
&\leq \sum_{i=N_1+1}^{\infty}\sum_{j=0}^{N_2} c_{ij}^2 + \sum_{i=0}^{\infty}\sum_{j=N2+1}^{\infty} c_{ij}^2 \\
&\leq \sum_{i=N_1+1}^{\infty}\sum_{j=0}^{\infty} c_{ij}^2 + \sum_{i=0}^{\infty}\sum_{j=N_2+1}^{\infty} c_{ij}^2 \\
&\leq C_1 N_1^{-r} \|\frac{\partial^r}{\partial t^r} u\|_{L^2(D)}^2 + C_2 N_2^{-l} \|\frac{\partial^l}{\partial x^l} u\|_{L^2(D)}^2 \\
&\leq C N^{-\alpha}.
\end{aligned}$$

□

**Theorem 4.2.** *Suppose* $\frac{\partial^{r+l} u(t,x)}{\partial t^r \partial x^l} \in L^2(D)$, *$u_n(t,x)$ is the $\varepsilon-$best approximate solution of Eq (3.2), and $n > max\{r, l\}$, then,*

$$\|u - u_n\|_{W(D)}^2 \leq C n^{-\alpha}.$$

*where C is a constant,* $\alpha = min\{r, l\}$.

*Proof.* According to Theorem 3.4 and Theorem 4.1, the following formula holds.

$$\|u - u_n\|_{W(D)}^2 \leq \|u - P_{N_1,N_2}u\|_{L^2(D)}^2 \leq C n^{-\alpha}.$$

□

So, the $\varepsilon-$approximate solution has $\alpha$ convergence order, and the convergence rate is related to $n$, where represents the number of bases, and the convergence order can calculate as follows.

$$C.R. = log_{\frac{n_2}{n_1}} \frac{max|e_{n_1}|}{max|e_{n_2}|}. \tag{4.1}$$

Where $n_i, i = 1, 2$ represents the number of orthonormal base elements.

## 5. Numerical examples

Here, three examples are compared with other algorithms. $N$ represents the number of orthonormal base elements. For example, $N = 10 \times 10$, which means that we use the orthonormal system $\{\rho_{ij}\}_{i,j=0}^{10}$ of $W(D)$ for approximate calculation, that is, we take the orthonormal system $\{JT_i(t)\}_{i=0}^{10}$ and $\{JP_j(x)\}_{j=0}^{10}$ to construct the $\varepsilon-$best approximate solution.

**Example 5.1.** *Consider the following one-demensional heat conduction system [7, 20]*

$$\begin{cases} u_t = u_{xx}, \quad (t, x) \in [0, 1] \times [0, 2\pi], \\ u(0, x) = \sin(x), \\ u(t, 0) = u(t, 2\pi) = 0. \end{cases}$$

The exact solution of Ex. 5.1 is $e^{-t}\sin x$.

In Table 1, *C.R.* is calculated according to Eq (4.2). The errors in Tables 1 and 2 show that the proposed algorithm is very effective. In Figures 1 and 2, the blue surface represents the surface of the real solution, and the yellow surface represents the surface of $u_n$. With the increase of $N$, the errors between the two surfaces will be smaller.

**Table 1.** $\max |u - u_n|$ for Ex. 5.1.

| $N$ | HOC-ADI Method [20] | FVM [7] | Present method | C.R. |
|---|---|---|---|---|
| 4×4 | 6.12E-3 | 4.92E-2 | 9.892E-3 | – |
| 6×6 | 1.68E-3 | 2.05E-2 | 4.319E-4 | 3.8613 |
| 8×8 | 7.69E-4 | 1.27E-2 | 9.758E-6 | 6.5873 |
| 10×10 | 4.40E-4 | 9.20E-3 | 1.577E-7 | 9.2432 |

**Table 2.** $|u - u_n|$ for Ex. 5.1 ($n = 9$).

| $\lvert u - u_n\rvert$ | $t = 0.1$ | $t = 0.3$ | $t = 0.5$ | $t = 0.7$ | $t = 0.9$ |
|---|---|---|---|---|---|
| $x = \frac{\pi}{5}$ | 1.195E-8 | 3.269E-8 | 5.009E-8 | 6.473E-8 | 8.127E-8 |
| $x = \frac{3\pi}{5}$ | 2.583E-8 | 7.130E-8 | 1.088E-7 | 1.390E-7 | 1.577E-7 |
| $x = \frac{7\pi}{5}$ | 2.583E-8 | 7.130E-8 | 1.088E-7 | 1.390E-7 | 1.577E-7 |
| $x = \frac{9\pi}{5}$ | 1.195E-8 | 3.269E-8 | 5.009E-8 | 6.473E-8 | 8.127E-8 |

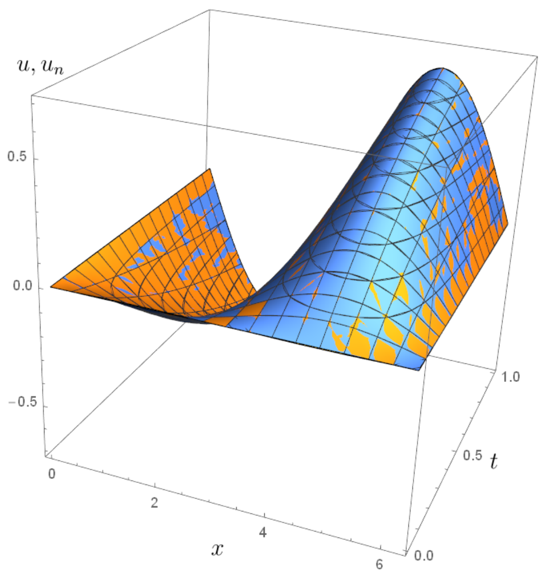


**Figure 1.** $u$ and $u_n$ in Example 5.1($n = 9$).

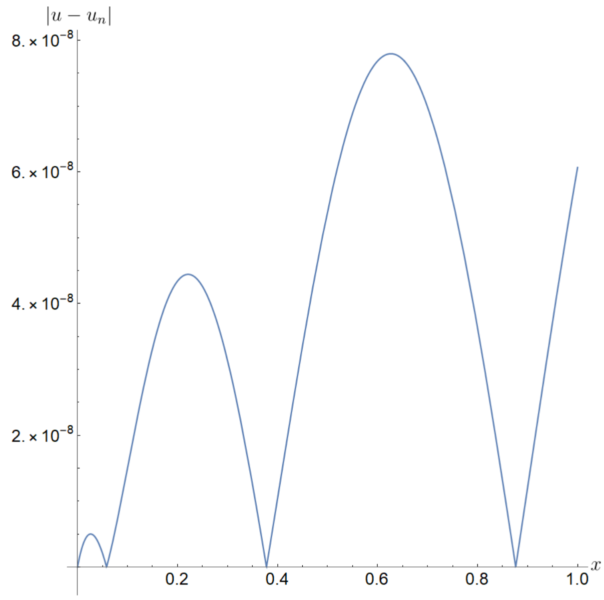


**Figure 2.** $|u(1,x) - u_n(1,x)|$ in Example 5.1($n = 9$).

**Example 5.2.** *Consider the following two-demensional heat conduction system [20, 21]*

$$\begin{cases} u_t = u_{xx} + u_{yy}, \quad (t,x,y) \in [0,1] \times [0,1] \times [0,1], \\ u(0,x,y) = \sin(\pi x)\sin(\pi y), \\ u(t,0,y) = u(t,1,y) = u(t,x,0) = u(t,x,1) = 0. \end{cases}$$

The exact solution of Ex. 5.2 is $u = e^{-2\pi^2 t}\sin(\pi x)\sin(\pi y)$.

Example 5.2 is a two-dimensional heat conduction equation. Table 3 shows the errors comparison with other algorithms. Table 4 lists the errors variation law in the $x-$axis direction. Figures 3 and 4 show the convergence effect of the scheme more vividly.

**Table 3.** The absolute errors $\max|u - u_n|$ for Ex. 5.2 ($t = 1, (x,y) \in [0,1] \times [0,1]$).

| $N$ | CCD-ADI Method [21] | RHOC-ADI Method [20] | Present method | C.R. |
|---|---|---|---|---|
| 4×4×4 | 8.820E-3 | 3.225E-2 | 5.986E-3 | – |
| 8×8×8 | 6.787E-5 | 1.969E-3 | 3.126E-5 | 2.52704 |

**Table 4.** The absolute errors $|u - u_n|$ for Ex. 5.2 ($t = 1, n = 7$).

| $\|u - u_n\|$ | $y = 0.1$ | $y = 0.3$ | $y = 0.5$ | $y = 0.7$ | $y = 0.9$ |
|---|---|---|---|---|---|
| $x = 0.1$ | 7.414E-6 | 1.963E-5 | 2.421E-5 | 1.963E-5 | 7.414E-6 |
| $x = 0.3$ | 1.963E-5 | 5.130E-5 | 6.347E-5 | 5.130E-5 | 1.963E-5 |
| $x = 0.5$ | 2.421E-5 | 6.347E-5 | 7.839E-5 | 6.347E-5 | 2.421E-5 |
| $x = 0.7$ | 1.963E-5 | 5.130E-5 | 6.347E-5 | 5.130E-5 | 1.963E-5 |
| $x = 0.9$ | 7.414E-6 | 1.963E-5 | 2.421E-5 | 1.963E-5 | 7.414E-6 |

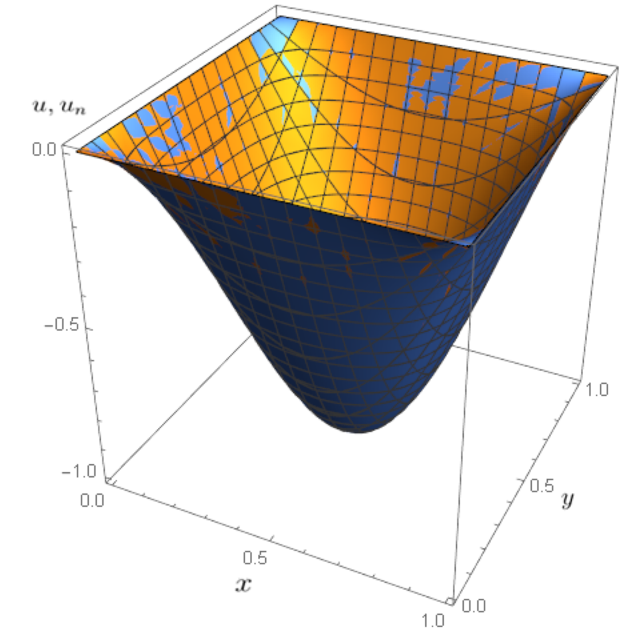


**Figure 3.** $u$ and $u_n$ in Example 5.2($n = 7$).

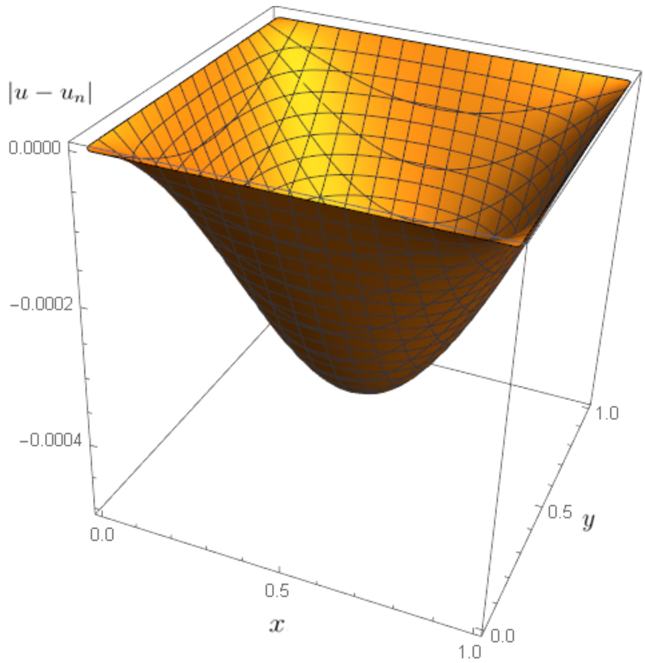


**Figure 4.** $u - u_n$ in Example 5.2($n = 7$).

**Example 5.3.** *Consider the three-demensional problem as following:*

$$\begin{cases} (\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2})u_t = u_{xx} + u_{yy} + u_{zz}, \quad (t, x, y, z) \in [0, 1] \times [0, a] \times [0, b] \times [0, c], \\ u(0, x, y) = \sin(\frac{\pi x}{a}) \sin(\frac{\pi y}{b}) \sin(\frac{\pi z}{c}), \\ u(t, 0, y) = u(t, 1, y) = u(t, x, 0) = u(t, x, 1) = 0. \end{cases}$$

The exact solution of Ex. 5.3 is $u = e^{-\pi^2 t}\sin(\frac{\pi x}{a})\sin(\frac{\pi y}{b})\sin(\frac{\pi z}{c})$.

Example 5.3 is a three-dimensional heat conduction equation, this kind of heat conduction system is also the most common case in the industrial field. Table 5 lists the approximation degree between the $\varepsilon-$best approximate solution and the real solution when the boundary time $t = 1$.

**Table 5.** The absolute errors $|u - u_n|$ for Ex. 5.3 ($t = 1, z = 0.1, n = 2$).

| $\lvert u - u_n\rvert$ | $y = 0.2$ | $y = 0.6$ | $y = 1.0$ | $y = 1.4$ | $y = 1.8$ |
|---|---|---|---|---|---|
| $x = 0.1$ | 1.130E-3 | 2.873E-3 | 3.451E-3 | 2.873E-3 | 1.130E-3 |
| $x = 0.3$ | 2.893E-3 | 7.350E-3 | 8.820E-3 | 7.350E-3 | 2.893E-3 |
| $x = 0.5$ | 3.482E-3 | 8.838E-3 | 1.059E-2 | 8.838E-3 | 3.482E-3 |
| $x = 0.7$ | 2.893E-3 | 7.350E-3 | 8.820E-3 | 7.735E-3 | 2.893E-3 |
| $x = 0.9$ | 1.130E-3 | 2.873E-3 | 3.451E-3 | 2.873E-3 | 1.130E-3 |

## 6. Conclusions

The Shifted-Legendre orthonormal scheme is applied to high-dimensional heat conduction equations. The algorithm proposed in this paper has some advantages. On the one hand, the algorithm is evolved from the algorithm for solving one-dimensional heat conduction equation, which is easy to be understood and expanded. On the other hand, the standard orthogonal basis proposed in this paper is a polynomial structure, which has the characteristics of convergence order.

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## Conflict of interest

The authors declare no conflict of interest.

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