



Research article

Superconvergence for optimal control problems governed by semilinear parabolic equations

Chunjuan Hou¹, Zuliang Lu^{2,3,*}, Xuejiao Chen¹, Xiankui Wu² and Fei Cai²

¹ Department of Data Science, Guangzhou Huashang College, Guangzhou 511300, China

² Key Laboratory for Nonlinear Science and System Structure, Chongqing Three Gorges University, Chongqing, 404000, China

³ Research Center for Mathematics and Economics, Tianjin University of Finance and Economics, Tianjin, 300222, China

* **Correspondence:** Email: zulianglu@sanxiau.edu.cn.

Abstract: In this paper, we first investigate optimal control problem for semilinear parabolic and introduce the standard $L^2(\Omega)$ -orthogonal projection and the elliptic projection. Then we present some necessary intermediate variables and their error estimates. At last, we derive the error estimates between the finite element solutions and L^2 -orthogonal projection or the elliptic projection of the exact solutions.

Keywords: finite element approximation; semilinear parabolic equation; optimal control problem; the elliptic projection; L^2 -orthogonal projection; superconvergence

Mathematics Subject Classification: 49J20, 65N30

1. Introduction

As we know, optimal control theory is widely used in many subjects. In the past few decades, it has attracted the attention of more and more scholars, and is also related to some specific applications, from finance to aerospace industry, from biology to medicine and so on. For example, how a spacecraft to land on the moon surface at rest with minimal fuel consumption [1]? Under what circumstances the tumor can be eliminated [2]?

In fact, optimal control problem (OCP) for partial differential equations (PDEs) is a challenging research hotspot, and much has been done both on the mathematical analysis and on its numerical approximation. Among numerous numerical methods, finite element discretization of the state equation is widely applied. Finite element approximation of optimal control problems plays a great role in modern science, technology, engineering, etc. We can find systematic introduction of finite

element methods and optimal control problems governed by PDEs, for example [6, 7, 28]. There have been extensive studies in error estimates, convergence of finite element approximation for OCP. Casas, Mateos and Raymond [3, 4] have studied a priori error estimation of semilinear elliptic boundary control problems. Chen and Huang [5, 9] have gained a priori error estimates of stochastic elliptic PDEs, both a priori and a posteriori error estimates of Stokes equations with H^1 -norm state constraint. For a posteriori error estimates of quadratic OCP governed by linear parabolic equation, see Liu and Yan [24], for optimal rates of convergence with Ritz-Galerkin approximations and numerical approximation of a parabolic time OCP, see Lasiecka and Knowles [10, 11]. In particular, Liu and Yan studied the posteriori error estimates for control problems governed by elliptic equations [12], and extended it to the OCP dominated by parabolic equation [13] and Stokes equation [14].

Furthermore, the superconvergence properties of OCP is a research focus in the field of optimal control problem, because superconvergence has always been an important tool to obtain high performance finite element discretization, which can provide high-precision approximate solutions. The research on superconvergence began in the late 1970s, and obtained fruitful results, see, e.g. [8, 15, 21–23, 27].

When the objective function in the OCP contains the gradient of scalar function, the mixed finite element method is an effective numerical method. In recent ten years, for the OCP of PDEs by the mixed finite element method, professor Chen's team has studied this aspect deeply, and has made a series of research achievements, such as a priori error estimation, a posteriori error estimation, L^∞ -error estimates and superconvergence etc [16–20, 25, 26].

Among the numerous research, Chen and Dai in [27] showed the superconvergence for optimal control problems governed by semilinear elliptic equations. The purpose of this paper is to extend the superconvergence property of [27] to the semilinear parabolic control problems.

In this paper, given the state y and the co-state p variables together with their approximations y_h and p_h , we say that the approx super converges if the state and co-state variables are approximated by the piecewise linear functions, the control variable is approximated by the piecewise constant functions, we can get the superconvergence properties for both the control variable and the state variables. We are interested in the following optimal control problem

$$\min_{u \in K} \left\{ \frac{1}{2} \int_0^T \left(\|y(x, t) - y_d(x, t)\|_{L^2(\Omega)}^2 + \|u(x, t)\|_{L^2(\Omega)}^2 \right) dt \right\} \quad (1.1)$$

$$y_t(x, t) - \operatorname{div}(A(x)\nabla y(x, t)) + \phi(y(x, t)) = f(x, t) + u(x, t), \quad x \in \Omega, t \in J, \quad (1.2)$$

$$y(x, t) = 0, \quad x \in \partial\Omega, t \in J, \quad (1.3)$$

$$y(x, 0) = y_0(x), \quad x \in \Omega, \quad (1.4)$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$, $0 < T < +\infty$, $J = [0, T]$, $y_t(x, t)$ denotes the partial derivative of y in time, $A(x) = (a_{ij}(x))_{n \times n} \in (W^{1,\infty}(\bar{\Omega}))^{n \times n}$, such that $(A(x)\xi) \cdot \xi \geq c |\xi|^2$, $\forall \xi \in \mathbb{R}^n$, $c > 0$. We assume that the function $\phi(\cdot) \in W^{2,\infty}(-R, R)$ for any $R > 0$, $\phi'(y) \in L^2(\Omega)$ for any $y \in L^2(J; H^1(\Omega))$, and $\phi'(y) \geq 0$. Moreover, we assume that $y_d(x, t) \in C(J; L^2(\Omega))$, $y_0(x) \in H_0^1(\Omega)$ and K is a nonempty closed convex set in $L^2(J; L^2(\Omega))$, defined by

$$K = \left\{ v(x, t) \in L^2(J; L^2(\Omega)) : \int_\Omega \int_0^T v(x, t) dx dt \geq 0, \quad a.e. \quad x \in \Omega, t \in J \right\}.$$

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha| \leq m} \|D^\alpha v\|_{L^p(\Omega)}^p$, a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. We denote by $L^s(0, T; W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(J; W^{m,p}(\Omega))} = \left(\int_0^T \|v\|_{W^{m,p}(\Omega)}^s dt \right)^{\frac{1}{s}}$ for $s \in [1, \infty)$, and the standard modification for $s = \infty$. Similarly, one can define the spaces $H^k(0, T; W^{m,q}(\Omega))$ and $C^k(0, T; W^{m,q}(\Omega))$.

The paper is organized as follows: in section 2, we briefly review the finite element method, and then the approximation schemes for the model optimal control problem will be constructed. In section 3, some intermediate error estimates which is the base of the result will be gained. In section 4, superconvergence properties for both control and state variables are derived.

2. Finite element method for optimal control problems

In this section, we will discuss the finite element approximation of the quadratic optimal control problem governed by semilinear parabolic equations (1.1)–(1.4). We set $W = L^2(J; V)$ with $V = H_0^1(\Omega)$, $X = L^2(0, T; U)$ with $U = L^2(\Omega)$, $\|\cdot\|_V = \|\cdot\|_{H_0^1(\Omega)}$ and $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. Let

$$\begin{aligned} a(v, w) &= \int_{\Omega} (A(x) \nabla v) \cdot \nabla w dx, & \forall v, w \in V, \\ (f_1, f_2) &= \int_{\Omega} f_1 \cdot f_2 dx, & \forall f_1, f_2 \in L^2(\Omega). \end{aligned}$$

It follows from Friedrichs' inequality that

$$\begin{aligned} a(v, v) &\geq c \|v\|_V^2, & \forall v \in V, \\ |a(v, w)| &\leq C \|v\|_V \|w\|_V, & \forall v, w \in V. \end{aligned}$$

We denote by $H^{-1}(\Omega)$ the dual space to $H_0^1(\Omega)$. If $f \in H^{-1}(\Omega)$, we note

$$\|\cdot\|_{H^{-1}(\Omega)} = \|\cdot\|_{-1}, \quad \|f\|_{-1} = \sup_{u \in H_0^1(\Omega), \|u\|_{H_0^1(\Omega)} \leq 1} (f, u). \quad (2.1)$$

Then the standard weak formula for the state equation reads: find $y(u)$ such that

$$(y_t, w) + a(y(u), w) + (\phi(y(u)), w) = (f + u, w), \quad \forall w \in V.$$

Thus the above equation has a solution.

We recast (1.1)–(1.4) in the following weak form: find (y, u) such that

$$\min_{u \in K} \left\{ \frac{1}{2} \int_0^T (\|y - y_d\|^2 + \|u\|^2) dt \right\} \quad (2.2)$$

$$(y_t, w) + a(y, w) + (\phi(y), w) = (f + u, w), \quad \forall w \in V = H_0^1(\Omega). \quad (2.3)$$

It is well known (see, e.g., [28]) that the control problem (2.2)–(2.3) has a solution (y, u) , and that if a pair (y, u) is the solution of (2.2)–(2.3), then there is a co-state $p \in H^1(J; L^2(\Omega)) \cap W$ such that the triplet (y, p, u) satisfies optimality conditions as follows:

$$(y_t, w) + a(y, w) + (\phi(y), w) = (f + u, w), \quad \forall w \in V, \quad (2.4)$$

$$-(p_t, q) + a(q, p) + (\phi'(y)p, q) = (y - y_d, q), \quad \forall q \in V, \quad (2.5)$$

$$\int_0^T (u + p, v - u) dt \geq 0, \quad \forall v \in K, \quad (2.6)$$

$$y(u)(x, 0) = y_0(x), \quad p(u)(x, T) = 0 \quad \forall x \in \Omega. \quad (2.7)$$

Now we introduce the following significant result (see [29]).

Lemma 2.1. [29] *A necessary and sufficient condition for the optimality of a control $u \in K$ with corresponding state $y(u)$ and co-state $p(u)$, respectively, is the following relation:*

$$u = \max(0, \bar{p}) - p, \quad (2.8)$$

where $\bar{p} = \frac{\int_0^T \int_\Omega p dx dt}{\int_0^T \int_\Omega 1 dx dt}$ denotes the integral average on $\Omega \times J$ of the function p .

In the following, we will consider the semi-discrete finite element for the problem.

Let $\mathcal{T}^h$ be regular triangulations of Ω , such that $\bar{\Omega} = \bigcup_{\tau \in \mathcal{T}^h} \bar{\tau}$. Let $h = \max_{\tau \in \mathcal{T}^h} \{h_\tau\}$, where h_τ denotes the diameter of the element τ . Note two spaces as follows:

$$U^h = \{u_h \in U : u_h|_\tau = \text{constant}, \tau \in \mathcal{T}^h\}, \quad (2.9)$$

$$V^h = \{v_h \in C(\bar{\Omega}) : v_h|_\tau \in \mathcal{P}_1, \tau \in \mathcal{T}^h, y_h|_{\partial\Omega} = 0\} \quad (2.10)$$

$$K^h := L^2(J; U^h) \cap K, \quad (2.11)$$

where $\mathcal{P}_1$ is the space of polynomials of degree less than or equal to 1. In addition, c or C denotes a general positive constant independent of h .

Now, the finite element approximation of the optimal control problem (2.2)–(2.3) is as follows:

$$\min_{u_h \in K^h} \left\{ \frac{1}{2} \int_0^T (\|y_h - y_d\|^2 + \|u_h\|^2) dt \right\} \quad (2.12)$$

$$(y_{h,t}, w_h) + a(y_h, w_h) + (\phi(y_h), w_h) = (f + u_h, w_h), \quad \forall w_h \in V^h. \quad (2.13)$$

The optimal control problem (2.12)–(2.13) has a solution (y_h, u_h) , and that if a pair (y_h, u_h) is the solution of (2.12)–(2.13), then there is a co-state p_h such that the triplet (y_h, p_h, u_h) satisfying the following optimal conditions:

$$(y_{h,t}, w_h) + a(y_h, w_h) + (\phi(y_h), w_h) = (f + u_h, w_h), \quad \forall w_h \in V^h, \quad (2.14)$$

$$-(p_{h,t}, q_h) + a(q_h, p_h) + (\phi'(y_h)p_h, q_h) = (y_h - y_d, q_h), \quad \forall q_h \in V^h, \quad (2.15)$$

$$\int_0^T (u_h + p_h, v_h - u_h) dt \geq 0, \quad \forall v_h \in K^h, \quad (2.16)$$

$$y_h(u_h)(x, 0) = y_0^h(x), \quad p_h(u_h)(x, T) = 0, \quad \forall x \in \Omega. \quad (2.17)$$

Similar to Lemma 2.1, we can get the relationship between the control approximation u_h and the co-state approximation p_h , which satisfies

$$u_h = \max(0, \bar{p}_h) - p_h, \quad (2.18)$$

where $\bar{p}_h = \frac{\int_0^T \int_{\Omega} p_h dx dt}{\int_0^T \int_{\Omega} 1 dx dt}$ denotes the integral average on $\Omega \times J$ of the function p_h .

3. Intermediate error estimates

First of all, we will introduce some intermediate variables. For any $\tilde{u} \in K$, let $(y(\tilde{u}), p(\tilde{u}))$ be the solution of the following equations:

$$(y_t(\tilde{u}), w) + a(y(\tilde{u}), w) + (\phi(y(\tilde{u})), w) = (f + \tilde{u}, w), \quad \forall w \in V, \quad (3.1)$$

$$-(p_t(\tilde{u}), q) + a(q, p(\tilde{u})) + (\phi'(y(\tilde{u}))p(\tilde{u}), q) = (y(\tilde{u}) - y_d, q), \quad \forall q \in V. \quad (3.2)$$

Then, for any $\tilde{u} \in K$, let $(y_h(\tilde{u}), p_h(\tilde{u}))$ be the solution of the following equations:

$$(y_{h,t}(\tilde{u}), w_h) + a(y_h(\tilde{u}), w_h) + (\phi(y_h(\tilde{u})), w_h) = (f + \tilde{u}, w_h), \quad \forall w_h \in V^h, \quad (3.3)$$

$$-(p_{h,t}(\tilde{u}), q_h) + a(q_h, p_h(\tilde{u})) + (\phi'(y_h(\tilde{u}))p_h(\tilde{u}), q_h) = (y_h(\tilde{u}) - y_d, q_h), \quad \forall q_h \in V^h. \quad (3.4)$$

Note that $(y, p) = (y(u), p(u))$, $(y_h, p_h) = (y_h(u_h), p_h(u_h))$.

Now we give the standard $L^2(\Omega)$ -orthogonal projection $Q_h : U \rightarrow U^h$, for $U = L^2(\Omega)$, which satisfies: for any $\psi \in U$

$$(\psi - Q_h\psi, u_h) = 0, \quad \forall u_h \in U^h, \quad (3.5)$$

and the elliptic projection $R_h : V \rightarrow V^h$, which satisfies: for all $v \in V$

$$a(v - R_h v, v_h) = 0, \quad v_h \in V^h. \quad (3.6)$$

We have the following approximation properties (see e.g., [27] and [30]):

$$\|\psi - Q_h\psi\|_{-s} \leq Ch^{1+s}|\psi|_1, \quad s = 0, 1, \quad (3.7)$$

$$\|w - R_h w\| \leq Ch^2\|w\|_2, \quad \text{for } w \in H^2(\Omega). \quad (3.8)$$

Lemma 3.1. *Let $u \in L^2(J; H^1(\Omega))$, for h sufficiently small, there exists a positive constant C such that*

$$\|y(Q_h u) - y(u)\|_{L^2(J; H^1(\Omega))} \leq Ch^2, \quad (3.9)$$

$$\|p(Q_h u) - p(u)\|_{L^2(J; H^1(\Omega))} \leq Ch^2. \quad (3.10)$$

Proof. Choose $\tilde{u} = Q_h u$ and $\tilde{u} = u$ in (3.1)–(3.2), respectively, then we have the following error equations

$$\begin{aligned} (y_t(Q_h u) - y_t(u), w) + a(y(Q_h u) - y(u), w) + (\phi(y(Q_h u)) - \phi(y(u)), w) \\ = ((Q_h u - u), w), \end{aligned} \quad (3.11)$$

$$\begin{aligned}
& -(p_t(Q_h u) - p_t(u), q) + a(q, p(Q_h u) - p(u)) + (\phi'(y(Q_h u))p(Q_h u) - \phi'(y(u))p(u), q) \\
& = (y(Q_h u) - y(u), q),
\end{aligned} \tag{3.12}$$

for any $w \in V$ and $q \in V$.

First, choose $w = y(Q_h u) - y(u)$ in (3.11), we have

$$\begin{aligned}
& (y_t(Q_h u) - y_t(u), y(Q_h u) - y(u)) + a(y(Q_h u) - y(u), y(Q_h u) - y(u)) \\
& + (\phi(y(Q_h u)) - \phi(y(u)), y(Q_h u) - y(u)) = (Q_h u - u, y(Q_h u) - y(u)).
\end{aligned} \tag{3.13}$$

Now, we estimate the right hand side of (3.13). Using (3.7), we have

$$\begin{aligned}
(Q_h u - u, y(Q_h u) - y(u)) &= (Q_h u - u, y(Q_h u) - y(u)) \\
&\leq C \|y(Q_h u) - y(u)\|_1 \cdot \|Q_h u - u\|_{-1} \\
&\leq Ch^2 \|u\|_1 \cdot \|y(Q_h u) - y(u)\|_1.
\end{aligned} \tag{3.14}$$

From (3.13) and (3.14), using ϵ -Cauchy inequality and the assumption of A and $\phi(\cdot)$, we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|y(Q_h u) - y(u)\|^2 + c \|y(Q_h u) - y(u)\|_1^2 \\
& \leq (y_t(Q_h u) - y_t(u), y(Q_h u) - y(u)) + a(y(Q_h u) - y(u), y(Q_h u) - y(u)) \\
& \quad + (\phi(y(Q_h u)) - \phi(y(u)), y(Q_h u) - y(u)) \\
& = (Q_h u - u, y(Q_h u) - y(u)) \\
& \leq Ch^2 \|y(Q_h u) - y(u)\|_1 \\
& \leq Ch^4 + \epsilon \|y(Q_h u) - y(u)\|_1^2.
\end{aligned} \tag{3.15}$$

Note that

$$y(Q_h u)(x, 0) - y(u)(x, 0) = 0,$$

next, integrating the both sides of (3.15) in time from 0 to t , we get

$$\|y(Q_h u) - y(u)\|_{L^\infty(J; L^2(\Omega))}^2 + c \|y(Q_h u) - y(u)\|_{L^2(J; H^1(\Omega))}^2 \leq Ch^4,$$

which implies (3.9).

Choose $q = p(Q_h u) - p(u)$ in (3.11), we have

$$\begin{aligned}
& -(p_t(Q_h u) - p_t(u), p(Q_h u) - p(u)) + a(p(Q_h u) - p(u), p(Q_h u) - p(u)) \\
& + (\phi'(y(Q_h u))p(Q_h u) - \phi'(y(u))p(u), p(Q_h u) - p(u)) \\
& = (y(Q_h u) - y(u), p(Q_h u) - p(u)),
\end{aligned} \tag{3.16}$$

namely,

$$\begin{aligned}
& -(p_t(Q_h u) - p_t(u), p(Q_h u) - p(u)) + a(p(Q_h u) - p(u), p(Q_h u) - p(u)) \\
& + (\phi'(y(Q_h u))(p(Q_h u) - p(u)), p(Q_h u) - p(u)) \\
& = (y(Q_h u) - y(u), p(Q_h u) - p(u))
\end{aligned}$$

$$+(p(u)(\phi'(y(u)) - \phi'(y(Q_h u))), p(Q_h u) - p(u)). \quad (3.17)$$

Notice that

$$\begin{aligned} (y(Q_h u) - y(u), p(Q_h u) - p(u)) &\leq C \|y(Q_h u) - y(u)\| \cdot \|p(Q_h u) - p(u)\| \\ &\leq Ch^2 \|p(Q_h u) - p(u)\|_1 \\ &\leq Ch^4 + \|p(Q_h u) - p(u)\|_1^2. \end{aligned} \quad (3.18)$$

Using the assumption for $\phi(\cdot)$ and (3.9), we have

$$\begin{aligned} &(p(u)(\phi'(y(u)) - \phi'(y(Q_h u))), p(Q_h u) - p(u)) \\ &\leq C \|p(u)\|_{0,4} \|\phi'(y(u)) - \phi'(y(Q_h u))\| \cdot \|p(Q_h u) - p(u)\|_{0,4} \\ &\leq C \|p(u)\|_1 \|\phi\|_{W^{2,\infty}} \|y(u) - y(Q_h u)\| \cdot \|p(Q_h u) - p(u)\|_1 \\ &\leq Ch^2 \|p(Q_h u) - p(u)\|_1 \\ &\leq Ch^4 + \|p(Q_h u) - p(u)\|_1^2, \end{aligned} \quad (3.19)$$

where we used the embedding $\|v\|_{0,4} \leq C \|v\|_1$. Then, using (3.17), (3.18), (3.19) and the assumption for $\phi(\cdot)$, we have

$$\begin{aligned} &-\frac{1}{2} \frac{d}{dt} \|p(Q_h u) - p(u)\|^2 + c \|p(Q_h u) - p(u)\|_1^2 \\ &\leq -(p_t(Q_h u) - p_t(u), p(Q_h u) - p(u)) + a(p(Q_h u) - p_h(u), p(Q_h u) - p(u)) \\ &\quad + (\phi'(y(Q_h u))(p(Q_h u) - p(u)), p(Q_h u) - p(u)) \\ &= (y(Q_h u) - y(u), p(Q_h u) - p(u)) \\ &\quad + (p(u)(\phi'(y(u)) - \phi'(y(Q_h u))), p(Q_h u) - p(u)) \\ &\leq Ch^2 \|p(Q_h u) - p(u)\|_1 \end{aligned} \quad (3.20)$$

$$\leq Ch^4 + \frac{c}{2} \|p(Q_h u) - p(u)\|_1^2. \quad (3.21)$$

Next, we consider the given condition

$$p(Q_h u)(x, T) - p(u)(x, T) = 0, \quad (3.22)$$

then, we integrate in time from t to T in (3.11) and use Gronwall's inequality, we have

$$\|p(Q_h u) - p(u)\|_{L^\infty(J; L^2(\Omega))}^2 + \|p(Q_h u) - p(u)\|_{L^2(J; H^1(\Omega))}^2 \leq Ch^4, \quad (3.23)$$

which implies (3.10). $\square$

Lemma 3.2. For any $\tilde{u} \in K$, if the intermediate solution satisfies

$$y(\tilde{u}), y_t(\tilde{u}), p(\tilde{u}), p_t(\tilde{u}) \in L^2(J; H^1(\Omega)) \cap L^2(J; H^2(\Omega)),$$

and Ω is convex, then we have

$$\|y_h(\tilde{u}) - R_h y(\tilde{u})\|_{L^2(J; H^1(\Omega))} \leq Ch^2, \quad (3.24)$$

$$\|p_h(\tilde{u}) - R_h p(\tilde{u})\|_{L^2(J; H^1(\Omega))} \leq Ch^2. \quad (3.25)$$

Proof. From (3.1)–(3.2) and (3.3)–(3.4), we have the following error equations:

$$(y_{h,t}(\tilde{u}) - y_t(\tilde{u}), w_h) + a(y_h(\tilde{u}) - y(\tilde{u}), w_h) + (\phi(y_h(\tilde{u})) - \phi(y(\tilde{u})), w_h) = 0, \quad (3.26)$$

$$\begin{aligned} & - (p_{h,t}(\tilde{u}) - p_t(\tilde{u}), q_h) + a(q_h, p_h(\tilde{u}) - p(\tilde{u})) + (\phi'(y_h(\tilde{u}))p_h(\tilde{u}) - \phi'(y(\tilde{u}))p(\tilde{u}), q_h) \\ & = (y_h(\tilde{u}) - y(\tilde{u}), q_h), \end{aligned} \quad (3.27)$$

for any $w_h \in V_h$ and $q_h \in V_h$. Using the definition of R_h , the above equation can be restated as

$$\begin{aligned} & (y_{h,t}(\tilde{u}) - R_h y_t(\tilde{u}), w_h) + a(y_h(\tilde{u}) - R_h y(\tilde{u}), w_h) + (\phi(y_h(\tilde{u})) - \phi(R_h y(\tilde{u})), w_h) \\ & = (y_t(\tilde{u}) - R_h y_t(\tilde{u}), w_h) + (\phi(y(\tilde{u})) - \phi(R_h y(\tilde{u})), w_h), \end{aligned} \quad (3.28)$$

$$\begin{aligned} & - (p_{h,t}(\tilde{u}) - R_h p_t(\tilde{u}), q_h) + a(q_h, p_h(\tilde{u}) - R_h p(\tilde{u})) + (\phi'(y_h(\tilde{u}))(p_h(\tilde{u}) - R_h p(\tilde{u})), q_h) \\ & = (y_h(\tilde{u}) - y(\tilde{u}), q_h) + (R_h p_t(\tilde{u}) - p_t(\tilde{u}), q_h) \\ & \quad + (p(\tilde{u})(\phi'(y(\tilde{u})) - \phi'(y_h(\tilde{u}))), q_h) + (\phi'(y_h(\tilde{u}))(p(\tilde{u}) - R_h p(\tilde{u})), q_h). \end{aligned} \quad (3.29)$$

First, let $w_h = y_h(\tilde{u}) - R_h y(\tilde{u})$ in (3.28), using the ϵ -Cauchy inequality and the assumptions for A and $\phi(\cdot)$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|y_h(\tilde{u}) - R_h y(\tilde{u})\|^2 + c \|y_h(\tilde{u}) - R_h y(\tilde{u})\|_1^2 \\ & \leq (y_t(\tilde{u}) - R_h y_t(\tilde{u}), y_h(\tilde{u}) - R_h y(\tilde{u})) \\ & \quad + a(y_h(\tilde{u}) - R_h y(\tilde{u}), y_h(\tilde{u}) - R_h y(\tilde{u})) + (\phi(y_h(\tilde{u})) - \phi(R_h y(\tilde{u})), y_h(\tilde{u}) - R_h y(\tilde{u})) \\ & = (y_{h,t}(\tilde{u}) - R_h y_t(\tilde{u}), y_h(\tilde{u}) - R_h y(\tilde{u})) + (\phi(y(\tilde{u})) - \phi(R_h y(\tilde{u})), y_h(\tilde{u}) - R_h y(\tilde{u})) \\ & \leq Ch^2 \|y_t(\tilde{u})\|_2 \|y_h(\tilde{u}) - R_h y(\tilde{u})\| + C \|\phi\|_{W^{1,\infty}} \|y(\tilde{u})\|_2 \cdot \|y_h(\tilde{u}) - R_h y(\tilde{u})\| \\ & \leq Ch^2 \|y_h(\tilde{u}) - R_h y(\tilde{u})\|_1 \\ & \leq Ch^4 + \frac{C}{2} \|y_h(\tilde{u}) - R_h y(\tilde{u})\|_1^2. \end{aligned} \quad (3.30)$$

It is known that

$$y_h(\tilde{u})(x, 0) - R_h y(\tilde{u})(x, 0) = y_0^h - R_h y_0 = 0,$$

then integrating in time for (3.30) and using Gronwall's inequality, we have

$$\|y_h(\tilde{u}) - R_h y(\tilde{u})\|_{L^\infty(J; L^2(\Omega))} + \|y_h(\tilde{u}) - R_h y(\tilde{u})\|_{L^2(J; H^1(\Omega))} \leq Ch^2, \quad (3.31)$$

which implies (3.24).

Then, let $q_h = p_h(\tilde{u}) - R_h p(\tilde{u})$ in (3.29). Note that

$$\begin{aligned} (y_h(\tilde{u}) - y(\tilde{u}), p_h(\tilde{u}) - R_h p(\tilde{u})) & \leq \|y_h(\tilde{u}) - y(\tilde{u})\| \cdot \|p_h(\tilde{u}) - R_h p(\tilde{u})\| \\ & \leq Ch^2 \|y(\tilde{u})\|_2 \cdot \|p_h(\tilde{u}) - R_h p(\tilde{u})\| \\ & \leq Ch^2 \|p_h(\tilde{u}) - R_h p(\tilde{u})\|_1, \end{aligned} \quad (3.32)$$

and

$$(R_h p_t(\tilde{u}) - p_t(\tilde{u}), p_h(\tilde{u}) - R_h p(\tilde{u})) \leq C \|R_h p_t(\tilde{u}) - p_t(\tilde{u})\| \cdot \|p_h(\tilde{u}) - R_h p(\tilde{u})\|$$

$$\leq Ch^2 \|p_t(\tilde{u})\|_2 \cdot \|p_h(\tilde{u}) - R_h p(\tilde{u})\|. \quad (3.33)$$

Using the assumption for $\phi(\cdot)$, we get

$$\begin{aligned} & (p(\tilde{u})(\phi'(y(\tilde{u})) - \phi'(y_h(\tilde{u}))), p_h(\tilde{u}) - R_h p(\tilde{u})) \\ & \leq C \|p(\tilde{u})\|_{0,4} \|\phi'(y(\tilde{u})) - \phi'(y_h(\tilde{u}))\| \cdot \|p_h(\tilde{u}) - R_h p(\tilde{u})\|_{0,4} \\ & \leq Ch^2 \|p(\tilde{u})\|_1 \cdot \|\phi\|_{W^{2,\infty}} \|y(\tilde{u})\|_2 \cdot \|p_h(\tilde{u}) - R_h p(\tilde{u})\|_1 \\ & \leq Ch^2 \|p_h(\tilde{u}) - R_h p(\tilde{u})\|_1, \end{aligned} \quad (3.34)$$

where we used the embedding $\|v\|_{0,4} \leq C \|v\|_1$. Then, using the definition of R_h and the assumption for $\phi(\cdot)$, we get

$$\begin{aligned} & (\phi'(y_h(\tilde{u}))(p(\tilde{u}) - R_h p(\tilde{u})), p_h(\tilde{u}) - R_h p(\tilde{u})) \\ & \leq C \|\phi\|_{W^{1,\infty}} \|p(\tilde{u}) - R_h p(\tilde{u})\| \cdot \|p_h(\tilde{u}) - R_h p(\tilde{u})\| \\ & \leq Ch^2 \|\phi\|_{W^{1,\infty}} \|p(\tilde{u})\|_2 \cdot \|p_h(\tilde{u}) - R_h p(\tilde{u})\| \\ & \leq Ch^2 \|p_h(\tilde{u}) - R_h p(\tilde{u})\|_1. \end{aligned} \quad (3.35)$$

From (3.29) and (3.32)–(3.35), we have

$$\begin{aligned} & c \|p_h(\tilde{u}) - R_h p(\tilde{u})\|_1^2 \\ & \leq a(p_h(\tilde{u}) - R_h p(\tilde{u}), p_h(\tilde{u}) - R_h p(\tilde{u})) + (\phi'(y_h(\tilde{u}))(p_h(\tilde{u}) - R_h p(\tilde{u})), p_h(\tilde{u}) - R_h p(\tilde{u})) \\ & = (y_h(\tilde{u}) - y(\tilde{u}), p_h(\tilde{u}) - R_h p(\tilde{u})) + (p(\tilde{u})(\phi'(y(\tilde{u})) - \phi'(y_h(\tilde{u}))), p_h(\tilde{u}) - R_h p(\tilde{u})) \\ & \quad + (\phi'(y_h(\tilde{u}))(p(\tilde{u}) - R_h p(\tilde{u})), p_h(\tilde{u}) - R_h p(\tilde{u})) \\ & \leq Ch^2 \|p_h(\tilde{u}) - R_h p(\tilde{u})\|_1. \end{aligned} \quad (3.36)$$

Note that

$$p_h(\tilde{u})(x, T) - R_h p(\tilde{u})(x, T) = 0,$$

then combining (3.32)–(3.36), and using the ϵ -Cauchy inequality and the assumptions for A and $\phi(\cdot)$, (3.29) can be rewritten as

$$-\frac{1}{2} \frac{d}{dt} \|p_h(\tilde{u}) - R_h p(\tilde{u})\|^2 + c \|p_h(\tilde{u}) - R_h p(\tilde{u})\|_1^2 \leq Ch^4 + \frac{1}{2} \|p_h(\tilde{u}) - R_h p(\tilde{u})\|^2. \quad (3.37)$$

Integrating the above inequality in time and using Gronwall's inequality, we have

$$\|p_h(\tilde{u}) - R_h p(\tilde{u})\|_{L^\infty(J; W)} + \|p_h(\tilde{u}) - R_h p(\tilde{u})\|_{L^2(J; H^1(\Omega))} \leq Ch^2, \quad (3.38)$$

which implies (3.25). $\square$

Lemma 3.3. For $\tilde{u} \in L^2(J; H^1(\Omega))$, assume $p(\tilde{u}), p_t(\tilde{u}), y(\tilde{u}), y_t(\tilde{u}) \in L^2(J; H^1(\Omega)) \cap L^2(J; H^2(\Omega))$, then we have the estimate

$$\|p(\tilde{u}) - p_h(\tilde{u})\|_{L^2(J; H^1(\Omega))} \leq Ch^2. \quad (3.39)$$

Proof. After rewriting

$$\begin{aligned} p(\tilde{u}) - p_h(\tilde{u}) &= p(\tilde{u}) - R_h p(\tilde{u}) + R_h p(\tilde{u}) - R_h p(Q_h \tilde{u}) \\ &\quad + R_h p(Q_h \tilde{u}) - p_h(Q_h \tilde{u}) + p_h(Q_h \tilde{u}) - p_h(\tilde{u}), \end{aligned}$$

from Lemma 3.1 and assumption of p , it is known that

$$\|R_h p(\tilde{u}) - R_h p(Q_h \tilde{u})\|_{L^2(J; H^1(\Omega))} \leq Ch^2,$$

and from Lemma 3.2, we get

$$\|p_h(Q_h \tilde{u}) - R_h p(Q_h \tilde{u})\|_{L^2(J; H^1(\Omega))} \leq Ch^2, \quad (3.40)$$

so we have

$$\begin{aligned} \|p(\tilde{u}) - p_h(\tilde{u})\|_{L^2(J; H^1(\Omega))} &\leq \|p(\tilde{u}) - R_h p(\tilde{u})\|_{L^2(J; H^1(\Omega))} + \|R_h p(\tilde{u}) - R_h p(Q_h \tilde{u})\|_{L^2(J; H^1(\Omega))} \\ &\quad + \|R_h p(Q_h \tilde{u}) - p_h(Q_h \tilde{u})\|_{L^2(J; H^1(\Omega))} + \|p_h(Q_h \tilde{u}) - p_h(\tilde{u})\|_{L^2(J; H^1(\Omega))} \\ &\leq Ch^2 + \|p_h(Q_h \tilde{u}) - p_h(\tilde{u})\|_{L^2(J; H^1(\Omega))}. \end{aligned}$$

Choose $\tilde{u} = Q_h \tilde{u}$, $w_h = y_h(Q_h \tilde{u}) - y_h(\tilde{u})$ in (3.3), and let $q_h = p_h(Q_h \tilde{u}) - p_h(\tilde{u})$ in (3.4), then we obtain the following error equations

$$\begin{aligned} (y_{h,t}(Q_h \tilde{u}) - y_{h,t}(\tilde{u}), y_h(Q_h \tilde{u}) - y_h(\tilde{u})) &+ (\phi(y_h(Q_h \tilde{u})) - \phi(y_h \tilde{u}), y_h(Q_h \tilde{u}) - y_h(\tilde{u})) \\ &+ a(y_h(Q_h \tilde{u}) - y_h(\tilde{u}), y_h(Q_h \tilde{u}) - y_h(\tilde{u})) = (B(Q_h \tilde{u} - \tilde{u}), y_h(Q_h \tilde{u}) - y_h(\tilde{u})), \end{aligned} \quad (3.41)$$

$$\begin{aligned} &- (p_{h,t}(Q_h \tilde{u}) - p_{h,t}(\tilde{u}), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) \\ &+ (\phi'(y_h(Q_h \tilde{u}))p_h(Q_h \tilde{u}) - \phi'(y_h \tilde{u})p_h(\tilde{u}), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) \\ &+ a(p_h(Q_h \tilde{u}) - p_h(\tilde{u}), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) = (y_h(Q_h \tilde{u}) - y_h(\tilde{u}), p_h(Q_h \tilde{u}) - p_h(\tilde{u})). \end{aligned} \quad (3.42)$$

Then from equality (3.41), using ϵ -Cauchy inequality and (3.7), we derive

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|y_h(Q_h \tilde{u}) - y_h(\tilde{u})\|^2 &+ c \|y_h(Q_h \tilde{u}) - y_h(\tilde{u})\|_1^2 \\ &\leq (B(Q_h \tilde{u} - \tilde{u}), y_h(Q_h \tilde{u}) - y_h(\tilde{u})) \\ &= (Q_h \tilde{u} - \tilde{u}, B^*(y_h(Q_h \tilde{u}) - y_h(\tilde{u}))) \\ &\leq C \|\tilde{u} - Q_h \tilde{u}\|_{-1} \|B^*(y_h(Q_h \tilde{u}) - y_h(\tilde{u}))\|_1 \\ &\leq Ch^2 \|y_h(Q_h \tilde{u}) - y_h(\tilde{u})\|_1 \\ &\leq Ch^4 + \epsilon \|y_h(Q_h \tilde{u}) - y_h(\tilde{u})\|_1^2. \end{aligned} \quad (3.43)$$

Note that

$$y_h(Q_h \tilde{u})(x, 0) - Q_h(\tilde{u})(x, 0) = 0,$$

next, integrating both sides of (3.43) in time, we obtain

$$\|y_h(Q_h \tilde{u}) - y_h(\tilde{u})\|_{L^\infty(J; U)}^2 + c \|y_h(Q_h \tilde{u}) - y_h(\tilde{u})\|_{L^2(J; H^1(\Omega))}^2 \leq Ch^4,$$

and obviously

$$\|y_h(Q_h \tilde{u}) - y_h(\tilde{u})\|_{L^2(J; H^1(\Omega))} \leq Ch^2. \quad (3.44)$$

Next, we consider the equality (3.42) similar to the above idea.

$$\begin{aligned} & - (p_{h,t}(Q_h \tilde{u}) - p_{h,t}(\tilde{u}), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) + (\phi'(y_h(Q_h \tilde{u}))(p_h(Q_h \tilde{u}) - p_h(\tilde{u})), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) \\ & + a(p_h(Q_h \tilde{u}) - p_h(\tilde{u}), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) \\ & = (y_h(Q_h \tilde{u}) - y_h(\tilde{u}), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) \\ & + (p_h(\tilde{u})(\phi'(y_h(\tilde{u})) - \phi'(y_h(Q_h \tilde{u}))), p_h(Q_h \tilde{u}) - p_h(\tilde{u})). \end{aligned} \quad (3.45)$$

Note that

$$\begin{aligned} (y_h(Q_h \tilde{u}) - y_h(\tilde{u}), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) & \leq C \|y_h(Q_h \tilde{u}) - y_h(\tilde{u})\| \cdot \|p_h(Q_h \tilde{u}) - p_h(\tilde{u})\| \\ & \leq C \|y_h(Q_h \tilde{u}) - y_h(\tilde{u})\|_1^2 + \|p_h(Q_h \tilde{u}) - p_h(\tilde{u})\|_1^2. \end{aligned} \quad (3.46)$$

Using the assumption for $\phi(\cdot)$ and (3.9), we get

$$\begin{aligned} & (p_h(\tilde{u})(\phi'(y_h(\tilde{u})) - \phi'(y_h(Q_h \tilde{u}))), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) \\ & \leq C \|p_h(\tilde{u})\|_{0,4} \|\phi'(y_h(\tilde{u})) - \phi'(y_h(Q_h \tilde{u}))\| \cdot \|p_h(Q_h \tilde{u}) - p_h(\tilde{u})\|_{0,4} \\ & \leq C \|p_h(\tilde{u})\|_1 \|\phi\|_{W^{2,\infty}} \|y_h(\tilde{u}) - y_h(Q_h \tilde{u})\| \cdot \|p_h(Q_h \tilde{u}) - p_h(\tilde{u})\|_1 \\ & \leq C \|y_h(Q_h \tilde{u}) - y_h(\tilde{u})\|_1^2 + \|p_h(Q_h \tilde{u}) - p_h(\tilde{u})\|_1^2, \end{aligned} \quad (3.47)$$

where we used the embedding $\|v\|_{0,4} \leq C\|v\|_1$. Then, using (3.45), (3.46), (3.47) and the assumption for $\phi(\cdot)$, we have

$$\begin{aligned} & -\frac{1}{2} \frac{d}{dt} \|p_h(Q_h \tilde{u}) - p_h(\tilde{u})\|^2 + c \|p_h(Q_h \tilde{u}) - p_h(\tilde{u})\|_1^2 \\ & \leq - (p_{h,t}(Q_h \tilde{u}) - p_{h,t}(\tilde{u}), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) + a(p_h(Q_h \tilde{u}) - p_h(\tilde{u}), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) \\ & \quad + (\phi'(y_h(Q_h \tilde{u}))(p_h(Q_h \tilde{u}) - p_h(\tilde{u})), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) \\ & = (y_h(Q_h \tilde{u}) - y_h(\tilde{u}), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) \\ & \quad + (p_h(\tilde{u})(\phi'(y_h(\tilde{u})) - \phi'(y_h(Q_h \tilde{u}))), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) \\ & \leq C \|y_h(Q_h \tilde{u}) - y_h(\tilde{u})\|_1^2 + \frac{C}{2} \|p_h(Q_h \tilde{u}) - p_h(\tilde{u})\|_1^2. \end{aligned} \quad (3.48)$$

Next, we consider the given condition

$$p_h(Q_h \tilde{u})(x, T) - p_h(\tilde{u})(x, T) = 0, \quad (3.49)$$

then, we integrate in time for (3.48) and use Gronwall's inequality and (3.44), we have

$$\|p_h(Q_h \tilde{u}) - p_h(\tilde{u})\|_{L^\infty(J; L^2(\Omega))}^2 + \|p_h(Q_h \tilde{u}) - p_h(\tilde{u})\|_{L^2(J; H^1(\Omega))}^2 \leq Ch^4, \quad (3.50)$$

which implies (3.39). $\square$

Let $y(u)$ and $y_h(u_h)$ are the solutions of (2.3) and (2.13), respectively. Let

$$J(u) = \left\{ \frac{1}{2} \int_0^T (\|p - p_d\|^2 + \|y - y_d\|^2 + \|u\|^2) dt \right\},$$

$$J_h(u_h) = \left\{ \frac{1}{2} \int_0^T (\|p_h(u_h) - p_d\|^2 + \|y_h(u_h) - y_d\|^2 + \|u_h\|^2) dt \right\}.$$

Then, the simplified problems of (2.2) and (2.12) read as

$$\min_{u \in K} \{J(u)\}, \quad (3.51)$$

and

$$\min_{u_h \in K_h} \{J_h(u_h)\}, \quad (3.52)$$

respectively. It can be shown that

$$(J'(u), v) = \int_0^T (u + p, v) dt,$$

$$(J'(u_h), v) = \int_0^T (u_h + p(u_h), v) dt,$$

$$(J'(Q_h u), v) = \int_0^T (Q_h u + p(Q_h u), v) dt,$$

$$(J'_h(u_h), v) = \int_0^T (u_h + p_h, v) dt,$$

where $p(u_h)$ and $p(Q_h u)$ are solutions of (3.1)–(3.2) for $\tilde{u} = u_h$ and $\tilde{u} = Q_h u$, respectively.

In many application, $J(\cdot)$ is uniform convex near the solution u . The convexity of $J(\cdot)$ is bound up with the second order sufficient conditions of the control problem, which are supposed in many studies on numerical methods of the problem. Next, there is a constant $c > 0$, independent of h , such that

$$(J'(Q_h u) - J'(u_h), Q_h u - u_h) \geq c \|Q_h u - u_h\|_{L^2(J; U)}^2, \quad (3.53)$$

where u and u_h are solutions of (3.51) and (3.52) respectively, $Q_h u$ is the orthogonal projection of u which is introduced in (3.5). From beginning to end, we will use the above inequality in this paper. More discussion of this can be found in [3, 4].

4. Superconvergence

In this section, superconvergence for both the control variable and the state variables will be discussed. Let π^c defined in [31] is the average operator such that $\pi^c u = Q_h u$. Let

$$\Omega^+ = \{\cup_\tau : \tau \subset \Omega, u|_\tau > 0\},$$

$$\Omega^0 = \{\cup_\tau : \tau \subset \Omega, u|_\tau = 0\},$$

$$\Omega^- = \Omega \setminus (\Omega^+ \cup \Omega^0).$$

In this paper, we assume that u and T_h are regular such that $\text{meas}(\Omega^-) = \text{meas}(\Omega^b) \leq Ch$.

Theorem 4.1. Let u be the solution of (2.4)–(2.6) and u_h be the solution of (2.14)–(2.16). We assume that the exact control and state solution satisfy

$$u, u + p \in L^2(J; W^{1,\infty}(\Omega)),$$

and

$$y(u), p(u) \in (L^2(J; H^2(\Omega))).$$

Then, we have

$$\|Q_h u - u_h\|_{L^2(J; U)} \leq Ch^{\frac{3}{2}}. \quad (4.1)$$

Proof. Set $v = u_h$ in (2.6) and $v_h = Q_h u$ in (2.16), and add the two inequalities, then we get

$$\int_0^T \{(u_h + p_h - u - p, Q_h u - u_h) + (u + p, Q_h u - u)\} dt \geq 0. \quad (4.2)$$

By using the definition of Q_h and (4.2), we get

$$\begin{aligned} & \int_0^T (Q_h u - u_h, Q_h u - u_h) dt \\ &= \int_0^T (u - u_h, Q_h u - u_h) dt \\ &\leq \int_0^T \{(p_h - p, Q_h u - u_h) + (u + p, Q_h u - u)\} dt. \end{aligned} \quad (4.3)$$

For the first term of (4.3), we separate it into three parts,

$$\begin{aligned} & \int_0^T (p_h - p, Q_h u - u_h) dt \\ &= \int_0^T (p_h - p(u_h), Q_h u - u_h) dt + \int_0^T (p(u_h) - p(Q_h u), Q_h u - u_h) dt \\ & \quad + \int_0^T (p(Q_h u) - p(u), Q_h u - u_h) dt, \end{aligned} \quad (4.4)$$

from (4.3)–(4.4), we get that

$$\begin{aligned} & \int_0^T \{(Q_h u - u_h, Q_h u - u_h) - (p(u_h) - p(Q_h u), Q_h u - u_h)\} dt \\ &\leq \int_0^T (p_h - p(u_h), Q_h u - u_h) dt + \int_0^T (p(Q_h u) - p(u), Q_h u - u_h) dt \\ & \quad + \int_0^T (u + p, Q_h u - u) dt. \end{aligned} \quad (4.5)$$

We can estimate the following by ϵ -Cauchy inequality

$$\int_0^T (p(u_h) - p_h, Q_h u - u_h) dt \leq C \int_0^T \|p(u_h) - p_h\| \cdot \|Q_h u - u_h\| dt$$

$$\begin{aligned}
&\leq \int_0^T \|p(u_h) - p_h\|_1^2 dt + \epsilon \int_0^T \|Q_h u - u_h\|^2 dt \\
&= \|p(u_h) - p_h\|_{L^2(J; H^1 \Omega)}^2 + \epsilon \|Q_h u - u_h\|_{L^2(J; U)}^2
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
\int_0^T (p(Q_h u) - p(u), Q_h u - u_h) dt &\leq C \int_0^T \|p(Q_h u) - p(u)\| \cdot \|Q_h u - u_h\| dt \\
&\leq \int_0^T \|p(Q_h u) - p(u)\|^2 dt + \epsilon \int_0^T \|Q_h u - u_h\|^2 dt \\
&= \|p(Q_h u) - p(u)\|_{L^2(J; U)}^2 + \|Q_h u - u_h\|_{L^2(J; U)}^2.
\end{aligned} \tag{4.7}$$

For the second term of (4.3)

$$\int_0^T (u + p, Q_h u - u) dt = \int_0^T \left\{ \int_{\Omega^+} + \int_{\Omega^0} + \int_{\Omega^b} (u + p, Q_h u - u) dx \right\} dt.$$

Obviously, $(Q_h u - u)|_{\Omega^0} = 0$. From (2.6), we have pointwise a.e. $(u + p) \geq 0$, we set $\tilde{u}|_{\Omega^+} = 0$ and $\tilde{u}|_{\Omega \setminus \Omega^+} = u$, so that $(u + p, u)|_{\Omega^+} \leq 0$. So, $(u + p)|_{\Omega^+} = 0$. Then

$$\begin{aligned}
\int_0^T (u + p, Q_h u - u) dt &= \int_0^T (u + p, Q_h u - u)_{\Omega^b} dt \\
&\leq \int_0^T (u + p - \pi^c(u + p), Q_h u - u)_{\Omega^b} dt \\
&\leq Ch^2 \int_0^T \|u + p\|_{1, \Omega^b} \|u\|_{1, \Omega^b} dt \\
&\leq Ch^2 \int_0^T \|u + p\|_{1, \infty} \|u\|_{1, \infty} \cdot \text{meas}(\Omega^b) dt \\
&\leq Ch^3.
\end{aligned} \tag{4.8}$$

According to (3.53), the left hand of (4.5) can be restated as:

$$\begin{aligned}
&\int_0^T \{(Q_h u - u_h, Q_h u - u_h) - (p(u_h) - p(Q_h u), Q_h u - u_h)\} dt \\
&= \int_0^T \{(Q_h u + p(Q_h u), Q_h u - u_h) - (u_h + p(u_h), Q_h u - u_h)\} dt \\
&= \int_0^T (J'(Q_h u) - J'(u_h), Q_h u - u_h) dt \\
&\geq c \|Q_h u - u_h\|_{L^2(J; U)}^2.
\end{aligned} \tag{4.9}$$

Then, combining (3.10), (3.39) and (4.5)–(4.9), we have

$$\|Q_h u - u_h\|_{L^2(J; U)} \leq Ch^{\frac{3}{2}},$$

which completes the proof of Theorem 4.1. $\square$

Theorem 4.2. Let u be the solution of (2.4)–(2.6), u_h be the solution of (2.14)–(2.16) and Ω is convex. We assume that the exact control and state solution satisfy

$$u, u + p \in L^2(J; W^{1,\infty}(\Omega)),$$

and

$$y(u), p(u) \in L^2(J; H^1(\Omega)) \cap L^2(J; H^2(\Omega)).$$

Then, we have

$$\|y_h - R_h y\|_{L^2(J; H^1(\Omega))} \leq Ch^{\frac{3}{2}}, \quad (4.10)$$

$$\|p_h - R_h p\|_{L^2(J; H^1(\Omega))} \leq Ch^{\frac{3}{2}}. \quad (4.11)$$

Proof. From (2.4)–(2.5) and (2.14)–(2.15), We have the following error equations

$$(y_{h,t} - y_t, w_h) + a(y_h - y, w_h) + (\phi(y_h) - \phi(y), w_h) = (u_h - u, w_h), \quad \forall w_h \in V_h, \quad (4.12)$$

$$-(p_{h,t} - p_t, q_h) + a(q_h, p_h - p) + (\phi'(y_h)p_h - \phi'(y)p, q_h) = (y_h - y, q_h), \quad \forall q_h \in V_h. \quad (4.13)$$

Using the definition of R_h , we have

$$\begin{aligned} & (y_{h,t} - R_h y_t, w_h) + a(y_h - R_h y, w_h) + (\phi(y_h) - \phi(R_h y), w_h) \\ &= (y_t - R_h y_t, w_h) + (u_h - u, w_h) + (\phi(y) - \phi(R_h y), w_h), \end{aligned} \quad (4.14)$$

$$\begin{aligned} & -(p_{h,t} - R_h p_t, q_h) + a(q_h, p_h - R_h p) + (\phi'(y_h)(p_h - R_h p), q_h) \\ &= (R_h p_t - p_t, q_h) + (y_h - y, q_h) + (\phi'(y_h)(p - R_h p), q_h) + (p(\phi'(y) - \phi'(y_h)), q_h), \end{aligned} \quad (4.15)$$

for any w_h and $q_h \in V_h$.

First, taking $w_h = y_h - R_h y$ in (4.14) and using the assumption of $\phi(\cdot)$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|y_h - R_h y\|^2 + c \|y_h - R_h y\|_1^2 \\ & \leq (y_{ht} - R_h y_t, y_h - R_h y) + a(y_h - R_h y, y_h - R_h y) + (\phi(y_h) - \phi(R_h y), y_h - R_h y) \\ & = (y_t - R_h y_t, y_h - R_h y) + (u_h - Q_h u, y_h - R_h y) \\ & \quad + (Q_h u - u, y_h - R_h y) + (\phi(y) - \phi(R_h y), y_h - R_h y) \\ & \leq C(\|y_t - R_h y_t\| \cdot \|y_h - R_h y\| + \|u_h - Q_h u\| \cdot \|y_h - R_h y\| \\ & \quad + \|Q_h u - u\|_{-1} \cdot \|y_h - R_h y\|_1 + \|\phi\|_{1,\infty} \cdot \|y - R_h y\| \cdot \|y_h - R_h y\|) \\ & \leq C(h^2 \|y_h - R_h y\| + \|u_h - Q_h u\| \cdot \|y_h - R_h y\|_1 + h^2 \|u\|_1 \cdot \|y_h - R_h y\|_1 \\ & \quad + h^2 \|\phi\|_{1,\infty} \|y\|_2 \|y_h - R_h y\|) \\ & \leq C(h^4 + h^3 + \|y_h - R_h y\|_1^2 + \|y_h - R_h y\|_1^2) \\ & \leq Ch^3 + \epsilon \|y_h - R_h y\|_1^2. \end{aligned} \quad (4.16)$$

Note that

$$y_h(x, 0) - R_h y(x, 0) = 0,$$

integrating in time and using Gronwall's inequality, we estimate

$$\|y_h - R_h y\|_{L^\infty(J;U)}^2 + \|y_h - R_h y\|_{L^2(J;H^1(\Omega))}^2 \leq Ch^3, \quad (4.17)$$

which implies (4.10).

Then, we take $q_h = p_h - R_h p$ in (4.15). Notice that

$$(R_h p_t - p_t, p_h - R_h p) \leq Ch^2 \|p_h - R_h p\|_1,$$

and

$$\begin{aligned} (y_h - y, p_h - R_h p) &= (y_h - R_h y, p_h - R_h p) + (R_h y - y, p_h - R_h p) \\ &\leq C \left(h^4 + \|y_h - R_h y\|_1^2 + \|p_h - R_h p\|_1^2 \right). \end{aligned} \quad (4.18)$$

Using the definition of R_h and the assumption for $\phi(\cdot)$, we have

$$\begin{aligned} (\phi'(y_h)(p - R_h p), p_h - R_h p) &\leq Ch^2 \|\phi\|_{1,\infty} \|p\|_2 \|p_h - R_h p\| \\ &\leq Ch^2 \|p_h - R_h p\|_1 \\ &\leq C \left(h^4 + \|p_h - R_h p\|_1^2 \right), \end{aligned} \quad (4.19)$$

and

$$\begin{aligned} &(p(\phi'(y) - \phi'(y_h)), p_h - R_h p) \\ &\leq C \|\phi\|_{2,\infty} (p(y - y_h), p_h - R_h p) \\ &\leq C \|\phi\|_{2,\infty} \|y - y_h\| \cdot \|p\|_{0,4} \|p_h - R_h p\|_{0,4} \\ &\leq C \|\phi\|_{2,\infty} (\|y - R_h y\| + \|R_h y - y_h\|) \|p\|_1 \|p_h - R_h p\|_1 \\ &\leq C \left(h^4 + \|R_h y - y_h\|_1^2 + \|p_h - R_h p\|_1^2 \right). \end{aligned} \quad (4.20)$$

From (4.15) and (4.18)–(4.20), we have

$$\begin{aligned} &-\frac{1}{2} \frac{d}{dt} \|p_h - R_h p\|^2 + c \|p_h - R_h p\|_1^2 \\ &\leq -(p_{h,t} - R_h p_t, q_h) + a(p_h - R_h p, p_h - R_h p) + (\phi'(y_h)(p_h - R_h p), p_h - R_h p) \\ &= (R_h p_t - p_t, p_h - R_h p) + (y_h - y, p_h - R_h p) + (\phi'(y_h)(p - R_h p), p_h - R_h p) \\ &\quad + (p(\phi'(y) - \phi'(y_h)), p_h - R_h p) \\ &\leq C \left(h^4 + \|y_h - R_h y\|_1^2 + \|p_h - R_h p\|_1^2 \right). \end{aligned} \quad (4.21)$$

Note that

$$p_h(x, T) - R_h p(x, T) = 0,$$

integrating in time and using Gronwall's inequality and (4.10), we estimate

$$\|p_h - R_h p\|_{L^\infty(J;U)}^2 + \|p_h - R_h p\|_{L^2(J;H^1(\Omega))}^2 \leq Ch^3, \quad (4.22)$$

which implies (4.11). $\square$

5. Conclusions

In this paper, we present finite element approximation method for solving semilinear parabolic OCP. When the state and co-state variables are approximated by the piecewise linear functions, the control variable is approximated by the piecewise constant functions, superconvergence properties for both the control variable and the state variables are discussed. In our future work, we shall use this method to deal with hyperbolic optimal control problems, including linear and nonlinear styles.

Acknowledgments

This work is supported by Guangdong Basic and Applied Basic Research Foundation of Joint Fund Project (2021A1515111048), Science Research Team Project in Guangzhou Huashang College (2021HSKT01), National Science Foundation of China (11201510), National Social Science Fund of China (19BGL190), China Postdoctoral Science Foundation (2017T100155, 2015M580197), Innovation Team Building at Institutions of Higher Education in Chongqing (CXTDX201601035), and Chongqing Research Program of Basic Research and Frontier Technology (cstc2019jcyj-msxmX0280) and Scientific and Technological Research Program of Chongqing Municipal Education Commission (KJZD-K202001201).

Conflict of interest

The authors declare no conflict of interest.

References

1. F. Gazzola, E. M. Marchini, The moon lander optimal control problem revisited, *Math. Eng.*, **3** (2021), 1–14. <https://doi.org/10.3934/mine.2021040>
2. T. Burden, J. Ernstberger, K. R. Fister, Optimal control applied to immunotherapy, *Discrete Cont. Dyn.-B*, **4** (2004), 135–146. <https://doi.org/10.3934/dcdsb.2004.4.135>
3. E. Casas, M. Mateos, Second order optimality conditions for semilinear elliptic control problems with finitely many state constraints, *J. Control Optim.*, **40** (2002), 1431–1454. <https://doi.org/10.1137/S0363012900382011>
4. E. Casas, J. P. Raymond, Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations, *SIAM J. Control Optim.*, **45** (2006), 1586–1611. <https://doi.org/10.1137/050626600>
5. Y. Chen, F. Huang, Spectral method approximation of flow optimal control problems with H^1 -norm state constraint, *Numer. Math.: Theory, Methods Appl.*, **10** (2017), 614–638. <https://doi.org/10.4208/nmtma.2017.m1419>
6. P. Neittaanmaki, D. Tiba, *Optimal control of nonlinear parabolic systems: theory, Algorithms and Applications*, CRC Press, 1994.
7. D. Tiba, *Lectures on the optimal control of elliptic equations*, University of Jyväskylä, Department of Mathematics, 1995.

8. Y. Tang, Y. Hua, Convergence and superconvergence of variational discretization for parabolic bilinear optimization problems, *J. Ineq. Appl.*, **1** (2019), 1–13. <https://doi.org/10.1186/s13660-019-2195-3>
9. F. Huang, Y. Chen, Y. Huang, A priori error estimates of a meshless method for optimal control problems of stochastic elliptic PDEs, *Int. J. Comp. Math.*, **96** (2019), 1048–1065. <https://doi.org/10.1080/00207160.2018.1483022>
10. I. Lasiecka, Ritz-Galerkin approximation of the time optimal boundary control problem for parabolic systems with Dirichlet boundary conditions, *SIAM J. Control Optim.*, **22** (1984), 477–500. <https://doi.org/10.1137/0322029>
11. G. Knowles, Finite element approximation of parabolic time optimal control problems, *SIAM J. Control Optim.*, **20** (1982), 414–427. <https://doi.org/10.1137/0320032>
12. W. Liu, N. Yan, A posteriori error estimates for optimal boundary control, *SIAM J. Numer. Anal.*, **39** (2001), 73–99. <https://doi.org/10.1137/S0036142999352187>
13. W. Liu, H. Ma, T. Tang, N. Yan, A posteriori error estimates for Discontinuous Galerkin time-stepping method for optimal control problems governed by parabolic equations, *SIAM J. Numer. Anal.*, **42** (2006), 1032–1061. <https://doi.org/10.1137/S0036142902397090>
14. W. Liu, N. Yan, A posteriori error estimates for control problems governed by Stokes equations, *SIAM J. Numer. Anal.*, **40** (2002), 1850–1869. <https://doi.org/10.1137/S0036142901384009>
15. T. Hou, Y. Chen, Superconvergence of RT1 mixed finite element approximations for elliptic control problems, *Sci. China Math.*, **56** (2013), 267–281. <https://doi.org/10.1007/s11425-012-4461-4>
16. Z. Lu, Y. Chen, A posteriori error estimates of triangular mixed finite element methods for semilinear optimal control problems, *Adv. Appl. Math. Mech.*, **1** (2009), 242–256.
17. Y. Chen, W. Liu, A posteriori error estimates for mixed finite element solutions of convex optimal control problems, *J. Compu. Appl. Math.*, **211** (2008), 76–89. <https://doi.org/10.1016/j.cam.2006.11.015>
18. Z. Lu, Y. Chen, L^∞ -error estimates of triangular mixed finite element methods for optimal control problems governed by semilinear elliptic equations, *Sibirskii Zhurnal Vychislitel'noi Matematiki*, **12** (2009), 91–105.
19. X. Xing, Y. Chen, L^∞ -error estimates for general optimal control problem by mixed finite element methods, *Int J Numer Anal Model*, **5** (2008), 441–456. <https://doi.org/10.1038/456441a>
20. Y. Chen, H. Leng, W. Yang, Error estimates of pseudostress-velocity MFEM for optimal control problems governed by stokes equations, *Appl. Numer. Math.*, **135** (2019), 407–422. <https://doi.org/10.1016/j.apnum.2018.09.009>
21. T. Hou, H. Leng, Superconvergence analysis and two-grid algorithms of pseudostress-velocity MFEM for optimal control problems governed by Stokes equations, *Appl. Numer. Math.*, **138** (2019), 78–93. <https://doi.org/10.1016/j.apnum.2018.12.008>
22. Y. Dai, Y. Chen, Superconvergence for general convex optimal control problems governed by semilinear parabolic equations, *Int. Scholarly Research Notices*, (2014), 364–375. <https://doi.org/10.1155/2014/579047>

23. C. Hou, Y. Chen, Z. Lu, Superconvergence property of finite element methods for parabolic optimal control problems, *J. Indus. Manag. Optim.*, **7** (2011), 927–945. <https://doi.org/10.3934/jimo.2011.7.927>
24. W. Liu, N. Yan, A posteriori error estimates for optimal control problems governed by parabolic equations, *Numer. Math.*, **93** (2003), 497–521. <https://doi.org/10.1007/s002110100380>
25. X. Xing, Y. Chen, Superconvergence of mixed methods for optimal control problems governed by parabolic equations, *Adva. Appl. Math. Mech.*, **3** (2011), 401–419. <https://doi.org/10.4208/aamm.10-m1006>
26. Y. Chen, Y. Wang, Y. Huang et al, Two-grid methods of expanded mixed finite-element solutions for nonlinear parabolic problems, *Appl. Numer. Math.*, **144** (2019), 204–222. <https://doi.org/10.1016/j.apnum.2019.04.015>
27. Y. Chen, Y. Dai, Superconvergence for optimal control problems governed by semi-linear elliptic equations, *J. Sci. Comp.*, **39** (2019), 206–221. <https://doi.org/10.1007/s10915-008-9258-9>
28. J. L. Lions, *Optimal control of systems governed by partial differential equations*, Springer Verlag, Berlin, 1971. <https://doi.org/10.1007/978-3-642-65024-6>
29. Y. Chen, N. Yi, W. Liu, A Legendre-Galerkin spectral method for optimal control problems governed by elliptic equations, *SIAM J. Numer. Anal.*, **46** (2008), 2254–2275. <https://doi.org/10.1137/070679703>
30. Y. Kwon, F. A. Milner, L^∞ -Error Estimates for Mixed Methods for Semilinear Second-Order Elliptic Equations, *J. Numer. Anal.*, **25** (1988), 46–53. <https://doi.org/10.1137/0725005>
31. R. Li, W. Liu, N. Yan, A posteriori error estimates of recovery type for distributed convex optimal control problems, *J. Sci. Comp.*, **33** (2007), 155–182. <https://doi.org/10.1007/s10915-007-9147-7>



AIMS Press

© 2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)