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Research article

Statistical solution and piecewise Liouville theorem for the impulsive discrete Zakharov equations

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Abstract: This article studies the discrete Zakharov equations with impulsive effect. The authors first prove that the problem is global well-posed and that the process formed by the solution operators possesses a pullback attractor. Then they establish that there is a family of invariant Borel probability measures contained in the pullback attractor, and that this family of measures satisfies the Liouville type theorem piecewise and is a statistical solution of the impulsive discrete Zakharov equations.

Keywords: statistical solution; impulsive discrete Zakharov equation; Liouville type theorem; pullback attractor
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1. Introduction

In this paper, we consider the following impulsive discrete Zakharov equations:

$$i\dot{\psi}_j + (A\psi)_j - h^2(D\psi)_j - u_j\psi_j + i\gamma\psi_j = g_j(t), \quad t > s,\ t \neq t_k,\ j,k \in \mathbb{Z}, \tag{1.1}$$

$$\ddot{u}_j - (Au)_j + h^2(Du)_j - (A(|\psi|^2))_j + \alpha\dot{u}_j + \mu u_j = f_j(t), \quad t > s,\ t \neq t_k,\ j,k \in \mathbb{Z}, \tag{1.2}$$

with the impulsive and initial conditions

$$\psi_j(t_k^+) - \psi_j(t_k) = I_{jk}^{\psi}(\psi_j(t_k)),\ \ \dot{u}_j(t_k^+) - \dot{u}_j(t_k) = I_{jk}^{u}(\dot{u}_j(t_k)), \quad j,k \in \mathbb{Z},\ t_k \in \mathbb{R}, \tag{1.3}$$

$$\psi_j(s^+) = \lim_{\theta\to s^+}\psi_j(\theta) = \psi_{j,s^+},\ \ u_j(s) = u_{j,s},\ \dot{u}_j(s^+) = \lim_{\theta\to s^+}\dot{u}_j(\theta) = u_{1j,s^+},\ \ j \in \mathbb{Z}, \tag{1.4}$$

where $s \in \mathbb{R}$ is the initial time, the unknown functions $\psi_j(\cdot) \in \mathbb{C}$ and $u_j(\cdot) \in \mathbb{R}$ denote respectively the envelope of the high-frequency electric filed and the plasmas density, $g_j(\cdot)$ and $f_j(\cdot)$ are given complex and real functions, respectively, $I_{jk}^{u}(\cdot)$ and $I_{jk}^{\psi}(\cdot)$ are given and assumed to satisfy some conditions. In addition, A and D are both linear operators defined as

$$(Au)_j = u_{j+1} - 2u_j + u_{j-1}, \quad \forall\, u = (u_j)_{j\in\mathbb{Z}},$$

$$(Du)_j = u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2}, \ \ \forall\, u = (u_j)_{j\in\mathbb{Z}},$$

$\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of integers, real numbers and complex numbers, respectively, i is the unit of imaginary numbers such that $i^2 = -1$, h, γ, α and μ are all positive constants, and $\{t_k\}_{k\in\mathbb{Z}}$ is a given real sequence of impulsive points satisfying

$$t_{k+1} - t_k \geqslant \eta, \ \ k \in \mathbb{Z}, \ \text{ and } \lim_{k\to+\infty} t_k = +\infty, \ \lim_{k\to-\infty} t_k = -\infty. \tag{1.5}$$

Equations (1.1) and (1.2) can be regarded as the discrete approximation with respect to the spatial variable $x \in \mathbb{R}$ of the following Zakharov equations

$$\begin{aligned} &i\psi_t + \psi_{xx} - h^2\psi_{xxxx} - \psi u + i\gamma\psi = g(x,t), \\ &u_{tt} - u_{xx} + h^2 u_{xxxx} - (|\psi|^2)_{xx} + \alpha u_t + \mu u = f(x,t), \end{aligned}$$

which describes the interaction between high frequency Langmuir waves and low frequency ion-acoustic waves. For the discrete Zakharov Eqs (1.1) and (1.2) without impulses, reference [14] investigates the finite dimensionality and upper semicontinuity of the kernel sections, and reference [31] studied its pullback attractor and invariant measures. Here we take the impulsive effect into account in the discrete Zakharov equations and consider its statistical solution.

Let's recall some relevant results about the statistical solution and invariant measure. Firstly, the abstract framework for the theory of statistical solutions for general evolution equations was presented in [1, 2]. The statistical solution for the 3D and 2D incompressible Navier-Stokes equations was investigated in [6, 7, 12, 26]. The invariant measure and statistical solution for the 3D globally modified Navier-Stokes equations were investigated in [4, 17, 19, 21, 30]. The statistical solution for the non-autonomous magneto-micropolar fluids and Klein-Gordon-Schrödinger equations was verified in [23, 27]. Some sufficient conditions ensuring the existence of trajectory statistical solutions for autonomous evolution equations were formulated in [22]. Later, the approach of [22] was applied in [10, 24, 25]. The invariant sample measure and random Liouville type theorem for the two-dimensional stochastic Navier-Stokes equations were studied in [29]. Very recently, the invariant measure for discrete long-wave-short-wave resonance equations and discrete Zakharov equations was studied in [16,31]. Especially, the statistical solution and piecewise Liouville theorem for the impulsive discrete reaction-diffusion equations and impulsive discrete Klein-Gordon-Schrödinger-type equations were investigated in [11,28].

The long-term behavior of the impulsive equations was studied in [8, 9, 15] and the references therein. Especially, Yan, Wu and Zhong in [18] studied the impulsive reaction-diffusion equations and proved the existence of the uniform attractors in $L^p(\Omega)$, $L^{2p-2}(\Omega)$ and $H_0^1(\Omega)$.

The goal of this article is to investigate the existence and piecewise Liouville type theory of the statistical solution for the impulsive discrete Zakharov equations. The impulses will naturally lead to the discontinuity of the solutions, and this discontinuity produces difficulties when we estimate the solutions since the Gronwall's inequality is no longer valid on the interval containing any impulsive point. In addition, compared with the impulsive reaction-diffusion equations on infinite lattices discussed in [28], the impulsive discrete Zakharov equations addressed here contain the second order derivative and nonlinear term $(A(|\psi|^2))_j$. These facts need us to do some technical analysis and meticulous estimations.

The article is arranged as follows. In the next section, we investigate the global well-posedness of the impulsive problems (1.1)–(1.4). In Section 3, we show that the solution operator of problems (1.1)–(1.4) generates a continuous process $\{S(t,s)\}_{t\geqslant s}$ on the selected phase space and possesses a pullback attractor. In Section 4, we first prove some type of piecewise continuity of $\{S(t,s)\}_{t\geqslant s}$ with respect to s and construct the invariant measure $\{m_\theta\}_{\theta\in\mathbb{R}}$ for $\{S(t,s)\}_{t\geqslant s}$. Then we establish that the invariant measure satisfies piecewise the Liouville type theorem and is a statistical solution of the impulsive problems (1.1)–(1.4).

2. Global well-posedness

In this section, we first introduce some notations and operators. Then we prove the global well-posedness of problems (1.1)–(1.4).

Besides the given positive constants h, γ, α, μ and η in Eqs (1.1), (1.2) and (1.5), we will use the following constants throughout this article:

$$\lambda = \frac{\mu\alpha}{4\mu+\alpha^2} \in (0, \frac{\alpha}{4}), \quad \delta = \mu\alpha(\sqrt{\alpha^2+4\mu}(\sqrt{\alpha^2+a\mu}+\alpha))^{-1} > 0, \quad \beta = \min\{2\delta, \gamma\},$$
$$c_0 = \max\{\frac{1}{\gamma}, \ \frac{32}{\alpha}\}, \quad \sigma = \beta - \frac{1}{\eta}\ln(4+4L^2+\frac{4+\mu+2\lambda^2}{\mu}),$$

where $L > 0$ is a constant which will be specified in assumption (H1). Also, we use $\mathbb{R}_+$ and $\mathbb{Z}_+$ to denote the sets of positive real numbers and positive integers respectively. Set

$$\ell^2 = \Big\{u = (u_j)_{j\in\mathbb{Z}} : u_j \in \mathbb{C}, \ \sum_{j\in\mathbb{Z}} |u_j|^2 < +\infty\Big\}, \tag{2.1}$$
$$l^2 = \Big\{u = (u_j)_{j\in\mathbb{Z}} : u_j \in \mathbb{R}, \ \sum_{j\in\mathbb{Z}} u_j^2 < +\infty\Big\}. \tag{2.2}$$

For brevity, we use X to denote ℓ^2 or l^2, and equip X with the inner product and norm as

$$(u,v) = \sum_{j\in\mathbb{Z}} u_j\bar{v}_j, \ \ \|u\|^2 = (u,u), \ \ \forall u = (u_j)_{j\in\mathbb{Z}}, \ \ v = (v_j)_{j\in\mathbb{Z}} \in X,$$

where $\bar{v}_j$ is the conjugate of v_j. For two elements $u, v \in X$, we define a bilinear form on X via

$$(u,v)_\mu = (Bu, Bv) + \mu(u,v), \tag{2.3}$$

where B is a linear operator defined as $(Bu)_j = u_{j+1} - u_j, \ \ j \in \mathbb{Z}, \ \ u = (u_j)_{j\in\mathbb{Z}} \in X$. We also define a linear operator B^* from X to X via

$$(B^*u)_j = u_{j-1} - u_j, \ \ j \in \mathbb{Z}, \ \ u = (u_j)_{j\in\mathbb{Z}} \in X.$$

Direct computations give

$$\begin{aligned}(Bu,v) &= \sum_{j\in\mathbb{Z}}(u_{j+1}-u_j)\bar{v}_j = \sum_{j\in\mathbb{Z}} u_{j+1}\bar{v}_j - \sum_{j\in\mathbb{Z}} u_j\bar{v}_j \\ &= \sum_{j\in\mathbb{Z}} u_j\bar{v}_{j-1} - \sum_{j\in\mathbb{Z}} u_j\bar{v}_j = \sum_{j\in\mathbb{Z}} u_j(\bar{v}_{j-1}-\bar{v}_j) = (u, B^*v), \ \ \forall u, \ \ v \in X,\end{aligned}$$

$$(Au, v) = \sum_{j\in\mathbb{Z}}(u_{j+1} - 2u_j + u_{j-1})\bar{v}_j = \sum_{j\in\mathbb{Z}} u_{j+1}\bar{v}_j - 2\sum_{j\in\mathbb{Z}} u_j\bar{v}_j + \sum_{j\in\mathbb{Z}} u_{j-1}\bar{v}_j, \quad \forall u, \quad v \in X,$$

and

$$\begin{aligned}(Bu, Bv) &= \sum_{j\in\mathbb{Z}}(u_{j+1} - u_j)(\bar{v}_{j+1} - \bar{v}_j)\\ &= \sum_{j\in\mathbb{Z}} u_{j+1}\bar{v}_{j+1} - \sum_{j\in\mathbb{Z}} u_{j+1}\bar{v}_j - \sum_{j\in\mathbb{Z}} u_j\bar{v}_{j+1} + \sum_{j\in\mathbb{Z}} u_j\bar{v}_j\\ &=2\sum_{j\in\mathbb{Z}} u_j\bar{v}_j - \sum_{j\in\mathbb{Z}} u_{j+1}\bar{v}_j - \sum_{j\in\mathbb{Z}} u_{j-1}\bar{v}_j = -(Au, v), \quad \forall u, \quad v \in X.\end{aligned}$$

Similarly, we have $(Du, v) = (Au, Av), \ \ \forall\, u, v \in X$. At the same time, we have

$$\begin{aligned}\|Bu\|^2 =&(Bu, Bu) = \sum_{j\in\mathbb{Z}} u_{j+1}\bar{u}_{j+1} - \sum_{j\in\mathbb{Z}} u_{j+1}\bar{u}_j - \sum_{j\in\mathbb{Z}} u_j\bar{u}_{j+1} + \sum_{j\in\mathbb{Z}} u_j\bar{u}_j\\ \leqslant&2\|u\|^2 + \frac{1}{2}(\sum_{j\in\mathbb{Z}} |u_{j+1}|^2 + |\bar{u}_j|^2) + \frac{1}{2}(\sum_{j\in\mathbb{Z}} |u_j|^2 + |\bar{u}_{j-1}|^2) = 4\|u\|^2.\end{aligned}$$

Combining the above facts, we conclude that B^* is the adjoint operator of B, and

$$\begin{cases} (Bu, v) = (u, B^*v),\ (Au, v) = -(Bu, Bv),\ (Du, v) = (Au, Av), \ \ \forall\, u, v \in X,\\ \|B^*u\|^2 = \|Bu\|^2 \leqslant 4\|u\|^2,\ \|Au\|^2 \leqslant 16\|u\|^2,\ \|Du\|^2 \leqslant 256\|u\|^2, \ \ \forall\, u \in X. \end{cases} \tag{2.4}$$

Clearly, the bilinear form $(\cdot, \cdot)_\mu$ defined by (2.3) is also a linear product in X. Since

$$\mu\|u\|^2 \leqslant \mu\|u\|^2 + \|Bu\|^2 = \|u\|_\mu^2 \leqslant (\mu + 4)\|u\|^2, \quad u \in X,$$

the norm $\|\cdot\|_\mu$ induced by $(\cdot, \cdot)_\mu$ is equivalent to the norm $\|\cdot\|$. Write

$$\ell^2 = (\ell^2, (\cdot, \cdot), \|\cdot\|), \ \ l_\mu^2 = (l^2, (\cdot, \cdot)_\mu, \|\cdot\|_\mu), \ \ l^2 = (l^2, (\cdot, \cdot), \|\cdot\|),$$

then ℓ^2, l_μ^2, and l^2 are all Hilbert spaces. Set $E_\mu = \ell^2 \times l_\mu^2 \times l^2$. For $z^{(m)} = (u^{(m)}, v^{(m)}, \varphi^{(m)})^T \in E_\mu$ $(m = 1, 2)$, the inner product and norm of E_μ are defined as

$$\begin{aligned}(z^{(1)}, z^{(2)})_{E_\mu} &= (u^{(1)}, u^{(2)}) + \mu(v^{(1)}, v^{(2)}) + (\varphi^{(1)}, \varphi^{(2)})\\ &= \sum_{j\in\mathbb{Z}} (u_j^{(1)}\bar{u}_j^{(2)} + (Bv^{(1)})_j(Bv^{(2)})_j + \mu v_j^{(1)} v_j^{(1)} + \varphi_j^{(1)}\varphi_j^{(2)}),\\ \|z\|_{E_\mu}^2 =&(z, z)_{E_\mu} = \sum_{j\in\mathbb{Z}} |z_j|_{E_\mu}^2 = \sum_{j\in\mathbb{Z}} (|\psi_j|^2 + |(Bu)_j|^2 + \mu u_j^2 + \varphi_j^2), \ \ z \in E_\mu,\end{aligned}$$

where $\bar{u}_j^{(2)}$ stands for the conjugate of $u_j^{(2)}$.

At the same time, to describe some type of continuity of the solutions for the impulsive problems (1.1)–(1.4), we introduce, for the given impulsive points $\{t_k\}_{k\in\mathbb{Z}}$ satisfying (1.5), two sets $PC(I; \mathbb{R})$ and $PC(I; X)$ of piecewise continuous functions from interval $I \subset \mathbb{R}$ to $\mathbb{R}$ and to ℓ^2 respectively as follows.

$PC(I; \mathbb{R}) =\{y(\cdot) \in \mathbb{R} :$ is continuous for $t \in I, t \neq t_k,$ is left continuous for $t \in I$ and

$$\begin{aligned} &\text{has discontinuities of the first kind at the impulsive points } t_k \in I, k \in \mathbb{Z}\},\\ PC(I;X) = &\{u(\cdot) \in X: \text{ is continuous for } t \in I, t \neq t_k, \text{is left continuous for } t \in I \text{ and}\\ &\text{has discontinuities of the first kind at the impulsive points } t_k \in I, k \in \mathbb{Z}\}. \end{aligned}$$

In addition, $PC^1(I;\mathbb{R})$ and $PC^1(I;X)$ denote the set of functions whose first derivatives belong to $PC(I;\mathbb{R})$ and $PC(I;X)$, respectively.

For convenience, we shall express problems (1.1)–(1.4) as a non-autonomous first-order ODE with respect to time t in E_μ. To this end, we put

$$\begin{aligned} &\psi = (\psi_j)_{j\in\mathbb{Z}}, \quad u = (u_j)_{j\in\mathbb{Z}}, \quad u\psi = (u_j\psi_j)_{j\in\mathbb{Z}}, \quad A|\psi|^2 = ((A|\psi|^2)_j)_{j\in\mathbb{Z}},\\ &\psi_{s^+} = (\psi_{j,s^+})_{j\in\mathbb{Z}}, \quad u_s = (u_{j,s})_{j\in\mathbb{Z}}, \quad u_{1,s^+} = (u_{1j,s^+})_{j\in\mathbb{Z}}, \quad f(t) = (f_j(t))_{j\in\mathbb{Z}}, \quad g(t) = (g_j(t))_{j\in\mathbb{Z}}. \end{aligned}$$

With the previous notations and operators, Eqs (1.1)–(1.3) can be written as

$$i\dot{\psi} + A\psi - h^2D\psi - u\psi + i\gamma\psi = g(t), \quad t > s,\ t \neq t_k,\ k \in \mathbb{Z}, \tag{2.5}$$

$$\ddot{u} - Au + h^2Du - A|\psi|^2 + \alpha\dot{u} + \mu u = f(t), \quad t > s,\ t \neq t_k,\ k \in \mathbb{Z}, \tag{2.6}$$

$$I_k^{\psi}(\psi(t_k)) = \left(I_{jk}^{\psi}(\psi_j(t_k))\right)_{j\in\mathbb{Z}}, \quad I_k^u(\dot{u}(t_k)) = \left(I_{jk}^u(\dot{u}_j(t_k))\right)_{j\in\mathbb{Z}},\ k \in \mathbb{Z}. \tag{2.7}$$

We further set

$$\varphi = \dot{u} + \lambda u, \quad z = (\psi, u, \varphi)^T, \quad F(z,t) = (-i\psi u - ig(t), 0, f(t) + A|\psi|^2)^T,$$

$$\Theta = \begin{pmatrix} \gamma I - iA + ih^2D & 0 & 0\\ 0 & \lambda I & -I\\ 0 & \lambda(\lambda-\alpha)I + \mu I - A + h^2D & (\alpha-\lambda)I \end{pmatrix}, \tag{2.8}$$

where I is the identity operator. Then the impulsive problems (1.1)–(1.4) can be written as

$$\dot{z} + \Theta z = F(z,t), \quad t > s, \quad t \neq t_k, \quad k \in \mathbb{Z}, \tag{2.9}$$

$$z(t_k^+) - z(t_k) = \begin{pmatrix} \psi(t_k^+) - \psi(t_k)\\ u(t_k^+) - u(t_k)\\ \varphi(t_k^+) - \varphi(t_k) \end{pmatrix} = \begin{pmatrix} I_k^{\psi}(\psi(t_k))\\ 0\\ I_k^u(\dot{u}(t_k)) \end{pmatrix}, \tag{2.10}$$

$$z(s^+) = (\psi(s^+), u(s), \varphi(s^+))^T = (\psi(s^+), u(s), \dot{u}(s^+) + \lambda u(s))^T, \quad s \in \mathbb{R}. \tag{2.11}$$

To ensure the global well-posedness of problems (2.9)–(2.11), we suppose that the functions $f(\cdot)$, $g(\cdot)$, $I_k^{\psi}(\cdot) = (I_{jk}^{\psi}(\cdot))_{j\in\mathbb{Z}}$ and $I_k^u(\cdot) = (I_{jk}^u(\cdot))_{j\in\mathbb{Z}}$ satisfy the following conditions.

(H1) *For any $j, k \in \mathbb{Z}$, $I_{jk}^{\psi}(0) = I_{jk}^u(0) = 0$ and there exists a constant $L > 0$ such that*

$$|I_{jk}^{\psi}(z') - I_{jk}^{\psi}(z'')| \leqslant L|z' - z''|; \quad |I_{jk}^u(z') - I_{jk}^u(z'')| \leqslant L|z' - z''|, \quad \forall\, z', z'' \in \mathbb{R}, \tag{2.12}$$

$$\sigma = \beta - \frac{1}{\eta}\ln(4 + 4L^2 + \frac{4 + \mu + 2\lambda^2}{\mu}) > 0. \tag{2.13}$$

(H2) *Assume that $f(t) = (f_j(t))_{j\in\mathbb{Z}} \in C(R, l^2)$ and $g(t) = (g_j(t))_{j\in\mathbb{Z}} \in C(\mathbb{R}, \ell^2)$, and that for each $s \in \mathbb{R}$,*

$$e^{\sigma s}\int_{-\infty}^{s} e^{\sigma\eta}(\|f(\eta)\|^2 + \|g(\eta)\|^2)d\eta < +\infty, \quad \int_{-\infty}^{s} e^{\sigma\eta}\|g(\eta)\|^2 d\eta < e^{(\frac{\sigma}{2}+\omega)s}K(s), \tag{2.14}$$

for some continuous function $K(\cdot)$ on the real line, bounded on intervals of the form $(-\infty, t)$, with $0 < \omega < \frac{\sigma}{2}$.

Some examples illustrating the existence of functions f and g satisfying (H2) can be found in [31, Example 3.1]. It is also not difficult to see the existence of the functions I_k^{ψ} and I_k^u satisfying (H1). At the same time, we will assume

$$4h(4\mu + \alpha^2) \leqslant \mu\alpha. \tag{2.15}$$

We next show the local existence and uniqueness of solution to problems (2.9)–(2.11).

Lemma 2.1. *Let h, γ, α, μ be all positive constants satisfying* (2.15)*, and assumptions* (H1) and (H2) *hold. Then for each given initial time s and initial value $z(s^+) = (\psi(s^+), u(s), \varphi(s^+)) \in E_\mu$, there exists a unique solution z to problems* (2.9)–(2.11) *such that*

$$z(\cdot) \in PC([s, T_*); E_\mu) \cap PC^1((s, T_*); E_\mu),$$

for some $T_* > s$. Moreover, if $T_* < +\infty$, then $\lim\limits_{t\to T_*^-} \|z(t)\|_{E_\mu} = +\infty$.

Proof. Obviously, we can see that Θ maps E_μ into itself and $F(\cdot,\cdot)$ maps $E_\mu \times \mathbb{R}$ into E_μ, respectively. Now let $\mathcal{B}$ be a bounded set in E_μ, then for any $t \in \mathbb{R}$, we have, after some computations,

$$\begin{aligned}\|F(z^{(1)}, t) - F(z^{(2)}, t)\|_{E_\mu}^2 =& \|(-i(\psi^{(1)}u^{(1)} - \psi^{(2)}u^{(2)}), 0, A|\psi^{(1)}|^2 - A|\psi^{(2)}|^2)^T\|_{E_\mu}^2 \\ \leqslant& (\frac{2}{\mu} + 34)L(\mathcal{B})\|z^{(1)} - z^{(2)}\|_{E_\mu}^2,\end{aligned} \tag{2.16}$$

where $L(\mathcal{B})$ is a positive constant depending on $\mathcal{B}$. Inequality (2.16) implies that $F(z, t)$ is locally (with respect to $z \in E_\mu$) Lipschitz from $E_\mu \times \mathbb{R}$ to E_μ and the Lipschitz constant $(\frac{2}{\mu} + 34)L(\mathcal{B})$ is independent of time t. Therefore, by the classical theory (see e.g. [3, Theorems 2.3 and 2.6]) of the impulsive differential equations, we obtain the desired results. □

Lemma 2.2. *Assume function $y(\cdot) \in PC^1(\mathbb{R}; \mathbb{R})$ satisfies*

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} + ay(t) \leqslant q(t), & t \neq t_k, k \in \mathbb{Z}, \\ y(t_k^+) \leqslant by(t_k), & k \in \mathbb{Z}, \\ y(s^+) \leqslant y_0, & s \in \mathbb{R}, \end{cases} \tag{2.17}$$

where $q(\cdot) \in PC(\mathbb{R}; \mathbb{R})$, $a > 0$, $b > 0$ and y_0 are constants. Then

$$y(t) \leqslant y_0 b^{n(s,t)} e^{-a(t-s)} + \int_s^t b^{n[\vartheta,t)} e^{-a(t-\vartheta)} q(\vartheta)\mathrm{d}\vartheta, \quad \forall\, t > s, \tag{2.18}$$

hereinafter $n(s, t)$ and $n[\vartheta, t)$ denote the number of members of the impulsive points $\{t_k\}_{k\in\mathbb{Z}}$ lying in the intervals (s, t) and $[\vartheta, t)$, respectively.

Proof. This lemma is an extension of [3, Lemma 2.2]. Here we consider the function $y(\cdot) \in PC^1(\mathbb{R}; \mathbb{R})$ since we investigate the pullback asymptotic behavior as $s \to -\infty$. We can prove Lemma 2.2 via an induction argument. In fact, for a given $s \in \mathbb{R}$, there exists some $k_0 \in \mathbb{Z}$ such that $s \in (t_{k_0}, t_{k_0+1}]$. Then, by assumption, $y(t)$ satisfies the differential inequality in (2.17) on (s, t_{k_0+1}). Applying Gronwall's inequality on (s, t_{k_0+1}) and using the left-continuity of $y(t)$, we see that $y(t)$ satisfies (2.18) for $t \in (s, t_{k_0+1}]$. Then we consider (2.17) on the interval $(t_{k_0+1}, t_{k_0+2}]$, with the initial value $y(t_{k_0+1}^+) \leqslant d_{t_{k_0+1}} y(t_{k_0+1})$. Also applying Gronwall's inequality on (t_{k_0+1}, t_{k_0+2}) and using the left-continuity of $y(t)$, we see that $y(t)$ also satisfies (2.18) for $t \in (t_{k_0+1}, t_{k_0+2}]$. Analogously, we can prove that $y(t)$ satisfies (2.18) for $t \in (t_{k_0+m}, t_{k_0+m+1}]$ for any $m \in \mathbb{Z}_+$. The detailed proof is omitted here. □

We next use Lemma 2.2 to estimate the solutions.

Lemma 2.3. *Suppose assumptions* (H1) *and* (H2) *hold. Let* $z(t) = (\psi(t), u(t), \varphi(t))^T \in E_\mu$ *be the solution of problems* (2.9)–(2.11) *corresponding to the initial value* $z(s^+) = (\psi(s^+), u(s), \varphi(s^+))^T \in E_\mu$. *Then*

$$\|\psi(t)\|^2 \leqslant \|\psi(s^+)\|^2 e^{-\sigma(t-s)} + \frac{e^{-\sigma t}}{\gamma}\int_s^t e^{\sigma\theta}\|g(\theta)\|^2 \mathrm{d}\theta, \quad \forall\, t > s. \tag{2.19}$$

Proof. Taking the imaginary part of the inner product $(\ell^2, (\cdot,\cdot))$ of (2.5) with ψ, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t)\|^2 + \gamma\|\psi(t)\|^2 \leqslant \frac{1}{\gamma}\|g(t)\|^2, \quad \forall\, t > s,\ t \neq t_k,\ k \in \mathbb{Z}. \tag{2.20}$$

Now, for the impulsive condition, we have by (H1) that

$$\begin{aligned}\|\psi(t_k^+)\|^2 = \sum_{j\in\mathbb{Z}} |\psi_j(t_k^+)|^2 &= \sum_{j\in\mathbb{Z}} |\psi_j(t_k) + I_{jk}^{\psi}(\psi(t_k))|^2 \leqslant 2\sum_{j\in\mathbb{Z}} |\psi_j(t_k)|^2 + 2\sum_{j\in\mathbb{Z}} |I_{jk}^{\psi}(\psi_j(t_k))|^2 \\ &\leqslant 2\sum_{j\in\mathbb{Z}} |\psi_j(t_k)|^2 + 2\sum_{j\in\mathbb{Z}} |I_{jk}^{\psi}(\psi_j(t_k)) - I_{jk}^{\psi}(0)|^2 \\ &\leqslant (2+2L^2)\sum_{j\in\mathbb{Z}} |\psi_j(t_k)|^2 = (2+2L^2)\|\psi(t_k)\|^2.\end{aligned} \tag{2.21}$$

Applying Lemma 2.2 to (2.20) and (2.21) for $y(t) = \|\psi(t)\|^2$, we get

$$\|\psi(t)\|^2 \leqslant \|\psi_{s^+}\|^2 (2+2L^2)^{n(s,t)} e^{-\gamma(t-s)} + \frac{1}{\gamma}\int_s^t (2+2L^2)^{n[\theta,t)} e^{-\gamma(t-\theta)}\|g(\theta)\|^2 \mathrm{d}\theta, \quad \forall\, t > s. \tag{2.22}$$

Now (1.5) implies that

$$n(s,t) \leqslant \frac{t-s}{\eta} \ \text{ and } \ n[\theta,t) \leqslant \frac{t-\theta}{\eta}. \tag{2.23}$$

Thus, we have by (2.13) that

$$(2+2L^2)^{n(s,t)} e^{-\gamma(t-s)} \leqslant e^{-\sigma(t-s)} \ \text{ and } \ (2+2L^2)^{n[\theta,t)} e^{-\gamma(t-\theta)} \leqslant e^{-\sigma(t-\theta)}. \tag{2.24}$$

Inserting (2.24) into (2.22) yields (2.19). This ends the proof. □

Lemma 2.4. *Let* h, γ, α, μ *be positive constants satisfying* (2.15), *and assumptions* (H1) and (H2) *hold. Then for every given* $s \in \mathbb{R}$ *and* $z(s^+) \in E_\mu$ *the corresponding solution (guaranteed by Lemma 2.1) of problems* (2.9)–(2.11) *satisfies*

$$\|z(t)\|_{E_\mu}^2 \leqslant \|z(s^+)\|_{E_\mu}^2 e^{-\sigma(t-s)} + c_0 e^{-\sigma t}\int_s^t e^{\sigma s}(\|f(\theta)\|^2 + \|g(\theta)\|^2 + \|\psi(\theta)\|^4)\mathrm{d}\theta, \quad s < t \leqslant T_*. \tag{2.25}$$

Proof. We denote by $z(\cdot) = z(\cdot; s, z(s^+))$ the solution of problems (2.9)–(2.11) corresponding to the initial value $z(s^+)$ at initial time s. Taking the real part of the inner product of (2.9) with $z(\cdot)$ in E_μ gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_{E_\mu}^2 + \mathbf{Re}(\Theta z(t), z(t))_{E_\mu} = \mathbf{Re}(F(z,t), z(t))_{E_\mu}, \quad t > s,\ t \neq t_k,\ k \in \mathbb{Z}. \tag{2.26}$$

Direct computations give

$$\mathbf{Re}(F(z,t),z(t))_{E_\mu} = \mathbf{Re}(-ig(t),\psi(t)) + \mathbf{Re}(f(t),\varphi(t)) + \mathbf{Re}(A|\psi(t)|^2,\varphi(t)), \tag{2.27}$$

$$\mathbf{Re}(-ig(t),\psi(t)) \leqslant \frac{\gamma}{2}\|\psi(t)\|^2 + \frac{1}{2\gamma}\|g(t)\|^2, \;\; \mathbf{Re}(f(t),\varphi(t)) \leqslant \frac{1}{\alpha}\|f(t)\|^2 + \frac{\alpha}{4}\|\varphi(t)\|^2, \tag{2.28}$$

$$\mathbf{Re}(A|\psi(t)|^2,\varphi(t)) \leqslant \frac{\alpha}{4}\|\varphi(t)\|^2 + \frac{16}{\alpha}\|\psi(t)\|^4. \tag{2.29}$$

By [31, Lemma 2.3], we have

$$\mathbf{Re}(\Theta z, z)_{E_\mu} \geqslant \delta(\|u\|_\mu^2 + \|\varphi\|^2) + \frac{\alpha}{2}\|\varphi\|^2 + \gamma\|\psi\|^2. \tag{2.30}$$

Hence, taking (2.26)–(2.30) into account, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_{E_\mu}^2 + \beta\|z(t)\|_{E_\mu}^2 \leqslant c_0(\|f(t)\|^2 + \|g(t)\|^2 + \|\psi(t)\|^4), \; t > s, \; t \neq t_k, \; k \in \mathbb{Z}. \tag{2.31}$$

Now, for the impulsive condition, we have by (H1) that

$$\begin{aligned}
\|z(t_k^+)\|_{E_\mu}^2 =& \|\psi(t_k^+)\|^2 + \|u(t_k)\|_\mu^2 + \|\varphi(t_k^+)\|^2 \\
=& \sum_{j\in\mathbb{Z}} |\psi_j(t_k^+)|^2 + \sum_{j\in\mathbb{Z}} |(Bu(t_k))_j|^2 + \mu\sum_{j\in\mathbb{Z}} |u_j(t_k)|^2 + \sum_{j\in\mathbb{Z}} \left|\dot{u}_j(t_k^+) + \lambda u_j(t_k)\right|^2 \\
=& \sum_{j\in\mathbb{Z}} \left|\psi_j(t_k) + I_{jk}^{\psi}(\psi_j(t_k))\right|^2 + \sum_{j\in\mathbb{Z}} \left|(Bu(t_k))_j\right|^2 + \mu\sum_{j\in\mathbb{Z}} \left|u_j(t_k)\right|^2 \\
&+ \sum_{j\in\mathbb{Z}} \left|\dot{u}_j(t_k) + I_{jk}^{u}(\dot{u}_j(t_k)) + \lambda u_j(t_k)\right|^2 \\
\leqslant& (2+2L^2)\sum_{j\in\mathbb{Z}} |\psi_j(t_k)|^2 + 4\sum_{j\in\mathbb{Z}} |u_j(t_k)|^2 + \mu\sum_{j\in\mathbb{Z}} |u_j(t_k)|^2 \\
&+ (4+4L^2)\sum_{j\in\mathbb{Z}} |\dot{u}_j(t_k)|^2 + 2\lambda^2\sum_{j\in\mathbb{Z}} |u_j(t_k)|^2 \\
\leqslant& (2+2L^2)\|\psi(t_k)\|^2 + (4+\mu+2\lambda^2)\|u(t_k)\|^2 + (4+4L^2)\|\dot{u}(t_k)\|^2 \\
\leqslant& (2+2L^2)\|\psi(t_k)\|^2 + \frac{(4+\mu+2\lambda^2)}{\mu}\|u(t_k)\|_\mu^2 + (4+4L^2)\|\varphi(t_k)\|^2 \\
\leqslant& (4+4L^2 + \frac{(4+\mu+2\lambda^2)}{\mu})\|z(t_k)\|_{E_\mu}^2.
\end{aligned} \tag{2.32}$$

Applying Lemma 2.2 to (2.31) and (2.32) for $y(t) = \|z(t)\|^2$, we obtain

$$\begin{aligned}
\|z(t)\|_{E_\mu}^2 \leqslant& c_0\int_s^t (4+4L^2 + \frac{(4+\mu+2\lambda^2)}{\mu})^{n[\theta,t)} e^{-\beta(t-\theta)}(\|f(\theta)\|^2 + \|g(\theta)\|^2 + \|\psi(\theta)\|^4)\mathrm{d}\theta \\
&+ \|z(s^+)\|_{E_\mu}^2 (4+4L^2 + \frac{(4+\mu+2\lambda^2)}{\mu})^{n(s,t)} e^{-\beta(t-s)}, \;\; \forall\, t > s.
\end{aligned} \tag{2.33}$$

Using (2.13) and (2.23), we get

$$(4+4L^2 + \frac{(4+\mu+2\lambda^2)}{\mu})^{n(s,t)} e^{-\beta(t-s)} \leqslant e^{-\sigma(t-\theta)}. \tag{2.34}$$

Inserting (2.34) into (2.33) gives (2.25). This completes the proof. □

Combining Lemma 2.4 and the extension theorem (see [3, Theorem 2.6]), we assert that the local solution obtained in Lemma 2.1 exists globally on $[s, +\infty)$.

Theorem 2.1. *Let h, γ, α, μ be positive constants satisfying* (2.15) *and assumptions* (H1) and (H2) *hold. Then, for every given $s \in \mathbb{R}$ and $z(s^+) \in E_\mu$, there corresponds a unique solution $z \in PC([s, +\infty); E_\mu) \cap PC^1((s, +\infty); E_\mu)$ to problems* (2.9)–(2.11) *satisfying*

$$\|z(t)\|^2_{E_\mu} \leqslant \|z(s^+)\|^2_{E_\mu} e^{-\sigma(t-s)} + c_0 e^{-\sigma t} \int_s^t e^{\sigma\theta}(\|f(\theta)\|^2 + \|g(\theta)\|^2 + \|\psi(\theta)\|^4)\mathrm{d}\theta, \quad \forall\, t > s. \tag{2.35}$$

Next, we prove that the solution of problems (2.9)–(2.11) depends continuously on its initial value. For brevity, from now on we will employ the notation $a \lesssim b$ (also $a \gtrsim b$) to mean that $a \leqslant cb$ (also $a \geqslant cb$) for a universal constant $c > 0$ that only depends on the parameters coming from the problem.

Theorem 2.2. *Let h, γ, α, μ be positive constants satisfying* (2.15) *and assumptions* (H1) and (H2) *hold. Denote by $z^{(m)}(\cdot) = z^{(m)}(\cdot; s, z^{(m)}(s^+))$ $(m = 1, 2)$, the solution of problems* (2.9)–(2.11) *corresponding to the initial value $z^{(m)}_{s^+}$ $(m = 1, 2)$. Then*

$$\|z^{(1)}(t) - z^{(2)}(t)\|^2_{E_\mu} \lesssim \|z^{(1)}(s^+) - z^{(2)}(s^+)\|^2_{E_\mu} e^{-(\sigma+2c_1-\gamma)(t-s)}, \quad \forall\, t > s, \tag{2.36}$$

where c_1 is a positive constant depending essentially on $z^{(1)}(s^+)$ and $z^{(2)}(s^+)$.

Proof. Let $z^{(m)}(\cdot) = z^{(m)}(\cdot; s, z^{(m)}(s^+))$ $(m = 1, 2)$ be two solutions of problems (2.9)–(2.11) corresponding to the initial values $z^{(m)}(s^+)$ $(m = 1, 2)$, and set $\widetilde{z} = z^{(1)}(\cdot) - z^{(2)}(\cdot)$. Then $\widetilde{z}$ satisfies

$$\frac{\mathrm{d}\widetilde{z}(t)}{\mathrm{d}t} + \Theta\widetilde{z}(t) = F(z^{(1)}, t) - F(z^{(2)}, t), \quad t > s,\ t \neq t_k,\ k \in \mathbb{Z}, \tag{2.37}$$

$$\widetilde{z}(t_k^+) - \widetilde{z}(t_k) = \begin{pmatrix} I_k^\psi(\psi^{(1)}(t_k)) - I_k^\psi(\psi^{(2)}(t_k)) \\ 0 \\ I_k^u(\dot{u}^{(1)}(t_k)) - I_k^u(\dot{u}^{(2)}(t_k)) \end{pmatrix}, \quad k \in \mathbb{Z}, \tag{2.38}$$

$$\widetilde{z}(s^+) = z^{(1)}(s^+) - z^{(2)}(s^+), \quad s \in \mathbb{R}. \tag{2.39}$$

Taking the real part of the inner product of (2.37) with $\tilde{z}$ in E_μ yields that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{z}\|^2_{E_\mu} + \mathbf{Re}(\Theta\widetilde{z}, \widetilde{z})_{E_\mu} = \mathbf{Re}(F(z^{(1)}, t) - F(z^{(2)}, t), \widetilde{z})_{E_\mu}, \quad t > s,\ t \neq t_k,\ k \in \mathbb{Z}. \tag{2.40}$$

By (2.16) and (2.30), we see that there exists a positive constant c_1 depending essentially on $z^{(1)}(s^+)$ and $z^{(2)}(s^+)$ such that

$$\mathbf{Re}(-\Theta\widetilde{z}(t) + (F(z^{(1)}, t) - F(z^{(2)}, t)), \widetilde{z}(t))_{E_\mu} \leqslant c_1\|\widetilde{z}(t)\|^2_{E_\mu}, \quad \forall\, t > s. \tag{2.41}$$

Inserting (2.41) into (2.40) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{z}(t)\|^2_{E_\mu} \leqslant 2c_1\|\widetilde{z}(t)\|^2_{E_\mu}, \quad t > s,\ t \neq t_k,\ k \in \mathbb{Z}. \tag{2.42}$$

For the impulsive condition of $\|\widetilde{z}(\cdot)\|^2_{E_\mu}$, we have by using (H1) that

$$\|\widetilde{z}(t_k^+)\|^2_{E_\mu} = \sum_{j\in\mathbb{Z}} |\widetilde{z}_j(t_k^+)|^2_{E_\mu}$$

$$
\begin{aligned}
&\leqslant 2\sum_{j\in\mathbb{Z}}\left|\widetilde{z}_j(t_k)\right|^2_{E_\mu} + 2\sum_{j\in\mathbb{Z}}\left|I^{\psi}_{jk}(\psi^{(1)}_j(t_k)) - I^{\psi}_{jk}(\psi^{(2)}_j(t_k))\right|^2 \\
&\quad + 2\sum_{j\in\mathbb{Z}}\left|I^{u}_{jk}(\dot{u}^{(1)}_j(t_k)) - I^{u}_{jk}(\dot{u}^{(2)}_j(t_k))\right|^2 \\
&\leqslant 2\sum_{j\in\mathbb{Z}}\left|\widetilde{z}_j(t_k)\right|^2_{E_\mu} + 2L^2\sum_{j\in\mathbb{Z}}\left|\psi^{(1)}_j(t_k) - \psi^{(2)}_j(t_k)\right|^2 + 2L^2\sum_{j\in\mathbb{Z}}|\dot{u}^{(1)}_j(t_k) - \dot{u}^{(2)}_j(t_k)|^2 \\
&\leqslant 2\|\widetilde{z}(t_k)\|^2_{E_\mu} + 2L^2\|\widetilde{\psi}(t_k)\|^2 + 2L^2\|\widetilde{u}(t_k)\|^2 \leqslant (2+2L^2)\|\widetilde{z}(t_k)\|^2_{E_\mu}.
\end{aligned}
\tag{2.43}
$$

Applying Lemma 2.2 to (2.42) and (2.43) for $y(t) = \|v(t)\|^2$ yields

$$
\|\widetilde{z}(t)\|^2_{E_\mu} \leqslant \|\widetilde{z}(s^+)\|^2_{E_\mu}(2+2L^2)^{n(s,t)}e^{-2(c_1+c_2)(t-s)}, \quad \forall\, t > s. \tag{2.44}
$$

Now, from (2.13) and (2.24) we see that

$$
(2+2L^2)^{n(s,t)}e^{-2c_1(t-s)} \leqslant e^{-(\sigma+2c_1-\gamma)(t-s)}. \tag{2.45}
$$

Inserting (2.45) into (2.44) gives (2.36). The proof of Theorem 2.2 is complete. □

3. Existence of the pullback attractor

In this section, we first show that the solution operators of problems (2.9)–(2.11) generate a continuous process $\{S(t,s)\}_{t\geqslant s}$ in E_μ and $\{S(t,s)\}_{t\geqslant s}$ that possesses a bounded pullback absorbing set in E_μ. Then we establish that $\{S(t,s)\}_{t\geqslant s}$ has pullback asymptotically nullness and a pullback attractor.

By Theorem 2.1 we see that the solution operators

$$
S(t,s): z(s^+) \in E_\mu \longmapsto S(t,s)z(s^+) = z(t;s,z(s^+)), \quad \forall\, t \geqslant s, \tag{3.1}
$$

of problems (2.9)–(2.11) generate a process on E_μ, hereinafter $z(\cdot;s,z(s^+))$ denotes the solution of problems (2.9)–(2.11) corresponding to the initial value $z(s^+)$ at initial time s. Moreover, from Theorem 2.2, we see that the process $\{S(t,s)\}_{t\geqslant s}$ is continuous on E_μ, that is, for every given t and s with $s \leqslant t$, the map $S(t,s): E_\mu \longmapsto E_\mu$ is continuous.

In the sequel, we denote by $\mathcal{O}(E_\mu)$ the family of all subsets of E_μ and consider the families of nonempty sets $\widehat{D}_0 = \{D_0(\theta): \theta \in \mathbb{R}\} \subseteq \mathcal{O}(E_\mu)$ parameterized by time t. Let $\mathcal{D}_\sigma$ be the class of families $\widehat{D} = \{D(\theta): \theta \in \mathbb{R}\} \subseteq \mathcal{O}(E_\mu)$ which satisfies

$$
\lim_{s\to-\infty}(e^{\frac{\sigma s}{2}}\sup_{z\in D(s)}\|z\|^2_{E_\mu}) = 0. \tag{3.2}
$$

Definition 3.1.

(1) *A family of sets $\widehat{D}_0 = \{D_0(\theta): \theta \in \mathbb{R}\} \subseteq \mathcal{O}(E_\mu)$, with $D_0(\theta) \subset E_\mu$ bounded for every $\theta \in \mathbb{R}$, is called a bounded pullback $\mathcal{D}_\sigma$-absorbing set for the process $\{S(t,s)\}_{t\geqslant s}$ in E_μ, if for each $t \in \mathbb{R}$ and any $\widehat{D} = \{D(\theta): \theta \in \mathbb{R}\} \in \mathcal{D}_\sigma$, there exists a $s_0(t,\widehat{D}) \leqslant t$ such that $S(t,s)D(s) \subseteq D_0(t)$ for all $s \leqslant s_0(t,\widehat{D})$.*

(2) *The process* $\{S(t,s)\}_{t\geqslant s}$ *is said to have pullback* $\mathcal{D}_\sigma$*-asymptotically nullness if for any given* $t\in\mathbb{R}$*, any* $\epsilon>0$ *and* $\widehat{D}=\{D(\theta):\theta\in\mathbb{R}\}\in\mathcal{D}_\sigma$*, there exist* $M_0=M_0(t,\epsilon,\widehat{D})\in\mathbb{Z}_+$ *and* $s_0=s_0(t,\epsilon,\widehat{D})\leqslant t$ *such that*

$$\sup_{z(s^+)\in D(s)}\sum_{|j|\geqslant M_0}|(S(t,s)z(s^+))_j|^2_{E_\mu}\leqslant\epsilon^2,\quad\forall\, s\leqslant s_0. \tag{3.3}$$

(3) *A family of sets* $\hat{\mathcal{A}}_{\mathcal{D}_\sigma}=\{\mathcal{A}_{\mathcal{D}_\sigma}(t):t\in\mathbb{R}\}\subseteq\mathcal{O}(E_\mu)$ *is said to be a pullback* $\mathcal{D}_\sigma$*-attractor for the process* $\{S(t,s)\}_{t\geqslant s}$ *in* E_μ *if the following properties hold:*
   (a) *Compactness: for any* $t\in\mathbb{R}$*,* $\mathcal{A}_{\mathcal{D}_\sigma}(t)$ *is a nonempty compact subset of* E_μ*;*
   (b) *Invariance:* $S(t,s)\mathcal{A}_{\mathcal{D}_\sigma}(s)=\mathcal{A}_{\mathcal{D}_\sigma}(t)$*,* $\forall\, s\leqslant t$*;*
   (c) *Pullback attraction:* $\hat{\mathcal{A}}_{\mathcal{D}_\sigma}$ *is pullback* $\mathcal{D}_\sigma$*-attracting in the following sense*

$$\lim_{s\to-\infty}\operatorname{dist}_{E_\mu}(S(t,s)D(s),\mathcal{A}_{\mathcal{D}_\sigma}(t))=0,\ \forall\,\widehat{D}=\{D(\theta):\theta\in\mathbb{R}\}\in\mathcal{D}_\sigma,t\in\mathbb{R},$$

where $\operatorname{dist}_{E_\mu}(\cdot,\cdot)$ *denotes the Hausdorff semidistance in* E_μ*.*

For the existence of the bounded pullback $\mathcal{D}_\sigma$-absorbing set for $\{S(t,s)\}_{t\geqslant s}$ in E_μ, we have

Lemma 3.1. *Let h,* γ*,* α*,* μ *be positive constants satisfying* (2.15) *and assumptions* (H1) and (H2) *hold. Then the process* $\{S(t,s)\}_{t\geqslant s}$ *defined by* (3.1) *possesses a bounded pullback* $\mathcal{D}_\sigma$*-absorbing set in* E_μ*.*

Proof. We denote by $R_\sigma(t)>0$ such that

$$\begin{aligned}R^2_\sigma(t)=&1+c_0e^{-\sigma t}\int_{-\infty}^t e^{\sigma\theta}(\|f(\theta)\|^2+\|g(\theta)\|^2+\|\psi(\theta)\|^4)\mathrm{d}\theta\\ \leqslant&1+c_0e^{-\sigma t}\int_{-\infty}^t e^{\sigma\theta}(\|f(\theta)\|^2+\|g(\theta)\|^2)\mathrm{d}\theta\\ &+c_0e^{-\sigma t}\int_{-\infty}^t e^{-\sigma\theta}(\int_{-\infty}^\theta e^{\gamma\eta}\|g(\eta)\|^2\mathrm{d}\eta)^2\mathrm{d}\theta,\quad t\in\mathbb{R}.\end{aligned} \tag{3.4}$$

Then from (2.19), (2.35) and (3.2) we see that there exists a time $s_0=s_0(t,\widehat{D})<t$ such that

$$\|S(t,s)z(s^+)\|^2_{E_\mu}\leqslant R^2_\sigma(t),\ \forall\, s<s_0, \tag{3.5}$$

that is, the family of closed balls $\hat{\mathcal{B}}(t)=\{\mathcal{B}(t)=\mathcal{B}(0,R_\sigma(t)):t\in\mathbb{R}\}$ is the desired bounded pullback $\mathcal{D}_\sigma$-absorbing set for $\{S(t,s)\}_{t\geqslant s}$, where $\mathcal{B}(0,R_\sigma(t))$ denotes the closed ball in E_μ with center zero and radius $R_\sigma(t)$. This completes the proof of Lemma 3.1. □

In order to investigate the pullback-D_σ asymptotic nullness of the process $S(t,s)_{t\geqslant s}$ in E_μ, we shall estimate the truncations of the solutions. Choose a smooth function $\chi(\cdot)\in C^1(\mathbb{R}_+;\mathbb{R}_+)$ satisfying

$$\begin{cases}\chi(x)=0, & 0\leqslant x\leqslant1,\\ 0\leqslant\chi(x)\leqslant1, & 1\leqslant x\leqslant2,\\ \chi(x)=1, & x\geqslant2,\\ |\chi'(x)|\leqslant\chi_0(\text{ positive constant}), & x\geqslant0.\end{cases} \tag{3.6}$$

Consider any given $\widehat{D} = \{D(\theta) : \theta \in \mathbb{R}\} \in \mathcal{D}_\sigma$. We have denoted by $z(t) = z(t; s, z(s^+)) = S(t, s)z(s^+) = (\psi(t), u(t), \varphi(t))^T = (\psi_j(t), u_j(t), \varphi_j(t))^T_{j\in\mathbb{Z}} \in E_\mu$ the solution of problems (2.9)–(2.11) with initial value $z(s^+) \in D(s)$ at the initial time $s \in \mathbb{R}$. Let $M \in \mathbb{Z}_+$ and set

$$\begin{aligned}&\xi_j(t) = \chi(\frac{|j|}{M})\psi_j(t),\;\; v_j(t) = \chi(\frac{|j|}{M})u_j(t),\;\; w_j(t) = \chi(\frac{|j|}{M})\varphi_j(t),\\&\varrho(t) = (\varrho_j(t))_{j\in\mathbb{Z}},\;\; \varrho_j(t) = (\xi_j(t), v_j(t), w_j(t))^T,\;\; j \in \mathbb{Z}.\end{aligned}$$

Taking the real part of the inner product of (2.9) with $\varrho(t) = (\varrho_j(t))_{j\in\mathbb{Z}}$ in E_μ, we get

$$\mathbf{Re}(\dot{z}(t), \varrho(t))_{E_\mu} + \mathbf{Re}(\Theta z(t), \varrho(t))_{E_\mu} = \mathbf{Re}(F(z, t), \varrho(t))_{E_\mu},\;\; \forall\, t > s,\; t \neq t_k,\; k \in \mathbb{Z}. \tag{3.7}$$

We next estimate the three terms in (3.7). Firstly, By some computations, we have

$$\begin{aligned}\mathbf{Re}(\dot{z}(t), \varrho(t))_{E_\mu} =&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})[\mu u_j^2 + \varphi_j^2 + |\psi_j|^2] + \sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})(B\dot{u})_j(Bu)_j\\&+ \sum_{j\in\mathbb{Z}}[\chi(\frac{|j+1|}{M}) - \chi(\frac{|j|}{M})](\dot{u}_{j+1} - \dot{u}_j)u_{j+1}.\end{aligned} \tag{3.8}$$

Using Lemma 3.1 and the Mean Value Theorem, we obtain

$$\mathbf{Re}(\dot{z}(t), \varrho(t))_{E_\mu} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})|z_j|^2_{E_\mu} \geqslant -\frac{R^2_\sigma(t)}{M},\;\; \forall\, t > s_0 \geqslant s,\; t \neq t_k, \tag{3.9}$$

hereinafter $R^2_\sigma(t)$ is the radius of the bounded pullback-$\mathcal{D}_\sigma$ absorbing set and $s_0 = s_0(t, \hat{D})$ is the pullback absorbing time in Lemma 3.1.

Lemma 3.2. *The term* $\boldsymbol{Re}(\Theta z(t), \varrho(t))_{E_\mu}$ *in* (3.7) *satisfies*

$$\begin{aligned}&\boldsymbol{Re}(\Theta z(t), \varrho(t))_{E_\mu} - \sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})[\delta((Bu)_j^2 + \mu u_j^2 + \varphi_j^2) + \frac{\alpha}{2}\varphi_j^2 + \gamma|\psi_j|^2]\\&- \frac{h^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})(Au)_j^2 - \frac{\lambda h^2}{2}\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})(Au)_j^2 \geqslant \frac{R^2_\sigma(t)}{M},\;\; \forall\, t > s_0 \geqslant s,\; \forall\, t \neq t_k.\end{aligned} \tag{3.10}$$

Proof. By direct computations, we get

$$\begin{aligned}\mathbf{Re}(\Theta z(t), \varrho(t))_{E_\mu} =&\gamma(\psi, \xi) + \mathbf{Im}(A\psi, \xi) - \mathbf{Im}(h^2D\psi, \xi) + \lambda(Bu, Bv) + \mu\lambda(u, v) - (B\varphi, Bv)\\&+ \lambda(\lambda - \alpha)(u, w) + (Bu, Bw) + h^2(Du, w) + (\alpha - \lambda)(\varphi, w)\\=&\gamma(\psi, \xi) + \mu\lambda(u, v) + \lambda(\lambda - \alpha)(u, w) + (\alpha - \lambda)(\varphi, w) + \lambda(Bu, Bv)\\&+ (Bu, Bw) - (B\varphi, Bv) + h^2(Du, w) + \mathbf{Im}(A\psi, \xi) - \mathbf{Im}(h^2D\psi, \xi),\end{aligned} \tag{3.11}$$

$$\begin{aligned}&(\psi, \xi) = \sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})|\psi_j|^2, \qquad (u, v) = \sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})|u_j|^2,\\&(u, w) = \sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})u_j\varphi_j, \qquad (\varphi, w) = \sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})\varphi_j^2.\end{aligned} \tag{3.12}$$

Using the Mean Value Theorem and (3.6), we can obtain

$$(Bu, Bw) - (B\varphi, Bv) \gtrsim -\frac{R_\sigma^2(t)}{M}, \quad \forall\, t > s_0 \geqslant s, \tag{3.13}$$

$$(Bu, Bv) \gtrsim \sum_{j\in\mathbb{Z}} \chi(\frac{|j|}{M})(Bu)_j^2 - \frac{2\chi_0 R_\sigma^2(t)}{\mu M}, \quad \forall\, t > s_0 \geqslant s. \tag{3.14}$$

For the terms $\mathbf{Im}(A\psi, \xi)$ and $-\mathbf{Im}(D\psi, \xi)$ in (3.11), we also have, using the Mean Value Theorem, (3.6) and Lemma 3.1,

$$\mathbf{Im}(A\psi, \xi) = -\mathbf{Im}(B\psi, B\xi) \gtrsim -\frac{R_\sigma^2(t)}{M}, \quad \forall\, t > s_0 \geqslant s, \tag{3.15}$$

$$-\mathbf{Im}(D\psi, \xi) = -\mathbf{Im}(A\psi, A\xi) \gtrsim -\frac{R_\sigma^2(t)}{M}, \quad \forall\, t > s_0 \geqslant s. \tag{3.16}$$

For the term $h^2(Du, w)$ in (3.11), we have

$$(Du, \dot{v}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j\in\mathbb{Z}} \chi(\frac{|j|}{M})(Au)_j^2 + \sum_{j\in\mathbb{Z}}(Au)_j[(A\dot{v})_j - \chi(\frac{|j|}{M})(A\dot{u})_j], \quad \forall\, t > s_0 \geqslant s,\ \forall\, t \neq t_k, \tag{3.17}$$

and

$$\sum_{j\in\mathbb{Z}}(Au)_j[(A\dot{v})_j - \chi(\frac{|j|}{M})(A\dot{u})_j] \gtrsim -\frac{R_\sigma^2(t)}{M}, \quad \forall\, t > s_0 \geqslant s,\ \forall\, t \neq t_k. \tag{3.18}$$

Similarly,

$$(Du, v) = (Au, Av) \gtrsim \frac{1}{2}\sum_{j\in\mathbb{Z}} \chi(\frac{|j|}{M})(Au)_j^2 - \frac{R_\sigma^2(t)}{M}, \quad \forall\, t > s_0 \geqslant s. \tag{3.19}$$

Combining (3.17)–(3.19), we obtain

$$\begin{aligned}&(Du, w) = (Du, \dot{v}) + \lambda(Du, v)\\ \gtrsim&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j\in\mathbb{Z}} \chi(\frac{|j|}{M})(Au)_j^2 + \frac{\lambda}{2}\sum_{j\in\mathbb{Z}} \chi(\frac{|j|}{M})(Au)_j^2 - \frac{R_\sigma^2(t)}{M}, \quad \forall\, t > s_0 \geqslant s,\ \forall\, t \neq t_k.\end{aligned} \tag{3.20}$$

Taking (3.11)–(3.20) into account, we have

$$\begin{aligned}&\mathbf{Re}(\Theta z(t), \varrho(t))_{E_\mu} - \frac{h^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j\in\mathbb{Z}} \chi(\frac{|j|}{M})(Au)_j^2 - \frac{\lambda h^2}{2}\sum_{j\in\mathbb{Z}} \chi(\frac{|j|}{M})(Au)_j^2\\ &- \sum_{j\in\mathbb{Z}} \chi(\frac{|j|}{M})[\delta((Bu)_j^2 + \mu u_j^2 + \varphi_j^2) + \frac{\alpha}{2}\varphi_j^2 + \gamma|\psi_j|^2]\\ \gtrsim&\sum_{j\in\mathbb{Z}} \chi(\frac{|j|}{M})[(\lambda-\delta)((Bu)_j^2 + \mu u_j^2) + (\frac{\alpha}{2} - \lambda - \delta)\varphi_j^2 + \lambda(\lambda-\alpha)u_j\varphi_j] - \frac{R_\sigma^2(t)}{M}\\ \geqslant& -\frac{R_\sigma^2(t)}{M}, \quad \forall\, t > s_0 \leqslant s,\ \forall\, t \neq t_k,\end{aligned} \tag{3.21}$$

where we have used the fact that

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})[(\lambda-\delta)((Bu)_j^2+\mu u_j^2)+(\frac{\alpha}{2}-\lambda-\delta)\varphi_j^2+\lambda(\lambda-\alpha)u_j\varphi_j]\\ \geqq&\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})[\mu(\lambda-\delta)u_j^2+(\frac{\alpha}{2}-\lambda-\delta)\varphi_j^2-\lambda\alpha|u_j||\varphi_j|]\\ =&\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})(\sqrt{\mu(\lambda-\delta)}|u_j|-\sqrt{\frac{\alpha}{2}-\lambda-\delta}|\varphi_j|)^2\geqslant 0.\end{aligned}$$

This ends the proof of Lemma 3.2. □

Lemma 3.3. *The term* $\boldsymbol{Re}(F(z,t),y(t))_{E_\mu}$ *in* (3.7) *satisfies*

$$\begin{aligned}&\boldsymbol{Re}(F(z,t),\varrho(t))_{E_\mu}\\ \lesssim&\frac{\gamma}{2}\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|\psi_j|^2+\sum_{|j|\geqslant M}|g_j(t)|^2+\frac{\alpha}{2}\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})\varphi_j^2\\ &+\sum_{|j|\geqslant M}|f_j(t)|^2+R_\sigma^2(t)\|\psi(s^+)\|^2e^{-\sigma(t-s)}+\frac{R_\sigma^2(t)}{M}\int_s^t e^{-\sigma(t-\theta)}R_\sigma^2(\theta)\mathrm{d}\theta\\ &+R_\sigma^2(t)[\int_s^t(\sum_{|j|\geqslant M}(|g_{j+1}(\theta)|^2+|g_j(\theta)|^2+|g_{j-1}(\theta)|^2))e^{-\sigma(t-\theta)}\mathrm{d}\theta],\quad\forall\, t>s_0\geqslant s,\ \forall\, t\neq t_k.\end{aligned}\tag{3.22}$$

Proof. Direct computations give

$$\mathbf{Re}(F(z,t),\varrho(t))_{E_\mu}=\mathbf{Im}(g(t),\xi)+(f(t),w(t))+(A|\psi|^2,w(t)).\tag{3.23}$$

Using Cauchy's inequality and Lemma 3.1, we have

$$\mathbf{Im}(g(t),\xi)\lesssim\frac{\gamma}{2}\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|\psi_j|^2+\sum_{|j|\geqslant M}|g_j(t)|^2,\tag{3.24}$$

$$(f(t),w(t))\lesssim\frac{\alpha}{4}\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})\varphi_j^2+\sum_{|j|\geqslant M}|f_j(t)|^2,\tag{3.25}$$

and

$$\begin{aligned}(A|\psi|^2,w(t))=&\sum_{j\in\mathbb{Z}}(|\psi_{j+1}|^2-2|\psi_j|^2+|\psi_{j-1}|^2)\chi(\frac{|j|}{M})\varphi_j\\ \lesssim&\frac{\alpha}{4}\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})\varphi_j^2+R_\sigma^2(t)(\mathbf{I}_1+\mathbf{I}_2+\mathbf{I}_3),\quad\forall\, t>s_0\geqslant s,\end{aligned}\tag{3.26}$$

where

$$\mathbf{I}_1=\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|\psi_{j+1}|^2,\quad\mathbf{I}_2=\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|\psi_j|^2,\quad\mathbf{I}_3=\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|\psi_{j-1}|^2.$$

Next we will estimate the terms $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_3$ in (3.26). Set

$$\zeta = (\zeta_{j+1})_{j\in\mathbb{Z}} = (\chi(\frac{|j|}{M})\psi_{j+1})_{j\in\mathbb{Z}}.$$

Taking the imaginary part of the inner product $(\cdot,\cdot)$ of Eq (2.5) with ζ and then using Cauchy's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|\psi_{j+1}|^2+\gamma\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|\psi_{j+1}|^2\\
=&\mathbf{Im}\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})\overline{\psi}_{j+1}g_{j+1}(t)-\mathbf{Im}(A\psi,\zeta)+h^2\mathbf{Im}(D\psi,\zeta)\\
\lesssim&\frac{\gamma}{2}\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|\psi_{j+1}|^2+\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|g_{j+1}(t)|^2-\mathbf{Im}(A\psi,\zeta)+\mathbf{Im}(D\psi,\zeta),\ \ \forall t>s_0\geqslant s, t\neq t_k,
\end{aligned}$$

which, together with (3.15) and (3.16), gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|\psi_{j+1}|^2+\gamma\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|\psi_{j+1}|^2\lesssim\sum_{|j|\geqslant M}|g_{j+1}(t)|^2+\frac{R_\sigma^2(t)}{M},\ \ \forall t>s_0\geqslant s, t\neq t_k. \tag{3.27}$$

For the impulsive condition of $\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|\psi_{j+1}(t_k^+)|^2$, we have

$$\begin{aligned}
&\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|\psi_{j+1}(t_k^+)|^2=\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})\left|\psi_{j+1}(t_k)+I^{\psi}_{j+1,k}(\psi_{j+1}(t_k))\right|^2\\
\leqslant&2\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|\psi_{j+1}(t_k)|^2+2\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})\left|I^{\psi}_{j+1,k}(\psi_{j+1}(t_k))\right|^2\\
\leqslant&(2+2L^2)\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|\psi_{j+1}(t_k)|^2.
\end{aligned} \tag{3.28}$$

Note that

$$y(s^+)=\sum_{|j|\geqslant M}\chi(\frac{|j|}{M})|\psi_{j+1}(s^+)|^2\leqslant\|\psi(s^+)\|^2. \tag{3.29}$$

From (2.24), (3.27)–(3.29) and Lemma 2.2, we obtain

$$\begin{aligned}
\mathbf{I}_1\lesssim&\|\psi(s^+)\|^2(2+2L^2)^{n(s,t)}e^{-\gamma(t-s)}\\
&+\int_s^t(2+2L^2)^{n[\theta,t)}e^{-\gamma(t-\theta)}(\sum_{|j|\geqslant M}|g_{j+1}(\theta)|^2+\frac{R_\sigma^2(\theta)}{M})\mathrm{d}\theta\\
\lesssim&\|\psi(s^+)\|^2e^{-\sigma(t-s)}\\
&+\int_s^te^{-\sigma(t-\theta)}\sum_{|j|\geqslant M}|g_{j+1}(\theta)|^2\mathrm{d}\theta+\frac{1}{M}\int_s^te^{-\sigma(t-\theta)}R_\sigma^2(\theta)\mathrm{d}\theta,\ \ \forall t>s_0\geqslant s.
\end{aligned} \tag{3.30}$$

Similarly, we have

$$\mathbf{I}_2 \lesssim \|\psi(s^+)\|^2 e^{-\sigma(t-s)} + \int_s^t e^{-\sigma(t-\theta)} \sum_{|j|\geqslant M} (|g_j(\theta)|^2 + \frac{R_\sigma^2(\theta)}{M}) \mathrm{d}\theta, \ \ \forall t > s_0 \geqslant s, \tag{3.31}$$

$$\mathbf{I}_3 \lesssim \|\psi(s^+)\|^2 e^{-\sigma(t-s)} + \int_s^t e^{-\sigma(t-\theta)} \sum_{|j|\geqslant M} (|g_{j-1}(\theta)|^2 + \frac{R_\sigma^2(\theta)}{M}) \mathrm{d}\theta, \ \ \forall t > s_0 \geqslant s. \tag{3.32}$$

Taking (3.23)–(3.26) and (3.30)–(3.32) into account, we obtain (3.22) and end the proof. □

Now we begin to establish the pullback-D_σ asymptotic nullness of the process $\{S(t,s)\}_{t\geqslant s}$ in E_μ.

Lemma 3.4. *Let h, γ, α, μ be positive constants satisfying* (2.15) *and assumptions* (H1) and (H2) *hold. Then for any given $t \in \mathbb{R}$, any $\epsilon > 0$ and $\widehat{D} = \{D(\theta) : \theta \in \mathbb{R}\} \in \mathcal{D}_\sigma$, there exist $M_0 = M_0(t, \epsilon, \widehat{D}) \in \mathbb{Z}_+$ and $s_0 = s_0(t, \epsilon, \widehat{D}) \leqslant t$ such that*

$$\sup_{z(s^+)\in D(s)} \sum_{|j|\geqslant M_0} |(S(t,s)z(s))_j|_{E_\mu}^2 \leqslant \epsilon^2, \ \ \forall\, s \leqslant s_0. \tag{3.33}$$

Proof. Combining (3.8), (3.10) and (3.22), we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \sum_{j\in\mathbb{Z}} \chi(\frac{|j|}{M})[|z_j|_{E_\mu}^2 + h^2(Au)_j^2] + \beta \sum_{j\in\mathbb{Z}} \chi(\frac{|j|}{M})[|z_j|_{E_\mu}^2 + h^2(Au)_j^2] \\
\lesssim & R_\sigma^2(t)\|\psi(s^+)\|^2 e^{-\sigma(t-s)} + \sum_{|j|\geqslant M} |g_j(t)|^2 + \sum_{|j|\geqslant M} |f_j(t)|^2 + \frac{R_\sigma^2(t)}{M} \\
&+ \frac{R_\sigma^2(t)}{M} \int_s^t e^{-\sigma(t-\theta)} R_\sigma^2(\theta) \mathrm{d}\theta \\
&+ R_\sigma^2(t)[\int_s^t (\sum_{|j|\geqslant M} (|g_{j+1}(\theta)|^2 + |g_j(\theta)|^2 + |g_{j-1}(\theta)|^2)) e^{-\sigma(t-\theta)} \mathrm{d}\theta], \ \ \forall s \leqslant s_0 \leqslant t.
\end{aligned} \tag{3.34}$$

Now for given $t \in \mathbb{R}$, $\widehat{D} = \{D(\theta) : \theta \in \mathbb{R}\} \in \mathcal{D}_\sigma$ and any $\epsilon > 0$, there exist $s_1 = s_1(t, \epsilon, \widehat{D})$ and $M_1 = M_1(t, \epsilon)$ such that

$$\frac{R_\sigma^2(t)}{M} \leqslant \frac{\sigma\epsilon^2}{12}, \ \ \forall\, M \geqslant M_1, \tag{3.35}$$

$$R_\sigma^2(t)\|\psi(s^+)\|^2 e^{-\sigma(t-s)} \leqslant R_\sigma^2(t)\|z(s^+)\|_{E_\mu}^2 e^{-\sigma(t-s)} \leqslant \frac{\sigma\epsilon^2}{12}, \ \ \forall\, s \leqslant s_1 \leqslant s_0 \leqslant t. \tag{3.36}$$

From (2.14) and (3.4) we see that for above t and ϵ exists $M_2 = M_2(t, \epsilon, \widehat{D})$ such that

$$\frac{R_\sigma^2(t)}{M} \int_s^t e^{-\sigma(t-\theta)} R_\sigma^2(\theta) \mathrm{d}\theta \leqslant \frac{\sigma\epsilon^2}{12}, \ \ \forall\, M \geqslant M_2. \tag{3.37}$$

Also, from (2.14), (3.2) and (3.4) we see that there exists $M_3 = M_3(t, \epsilon, \widehat{D}) \in \mathbb{N}$ such that

$$R_\sigma^2(t) \int_s^t (\sum_{|j|\geqslant M} (|g_j(\theta)|^2 + |g_{j+1}(\theta)|^2 + |g_{j-1}(\theta)|^2)) e^{-\sigma(t-\theta)} \mathrm{d}\theta \leqslant \frac{\sigma\epsilon^2}{36}, \ \ \forall\, M \geqslant M_3. \tag{3.38}$$

Picking $M_4 = \max\{M_1, M_2, M_3\}$ and then inserting the estimations (3.35)–(3.38) into (3.34), we get

$$\begin{aligned}&\frac{d}{dt}\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})[|z_j(t)|^2_{E_\mu}+h^2(A(u(t)))^2_j]+\beta\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})[|z_j(t)|^2_{E_\mu}+h^2(A(u(t)))^2_j]\\&\lesssim\sum_{|j|\geqslant M}|g_j(t)|^2+\sum_{|j|\geqslant M}|f_j(t)|^2+\frac{\sigma\epsilon^2}{3},\ \ t>s_0>s_1\geqslant s,\ M>M_4,\ t\neq t_k,\ \ k\in\mathbb{Z}.\end{aligned}\tag{3.39}$$

We next investigate the impulsive condition of $y(t)=\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})(|z_j(t)|^2_{E_\mu}+h^2(A(u(t)))^2_j)$ with $M\geqslant M_4$. Note that $u_j(t_k^+)-u_j(t_k)=0$ for each $j\in\mathbb{Z}$. Hence, for the impulsive condition of this $y(t)$, we have

$$\begin{aligned}y(t_k^+)=&\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})|z_j(t_k^+)|^2_{E_\mu}\\=&\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})(\left|\psi_j(t_k)+I^\psi_{jk}(\psi(t_k))\right|^2+|(Bu(t_k))_j|^2+\mu|u_j(t_k)|^2)\\&+\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})\left|\dot u_j(t_k)+\lambda u_j(t_k)+I^u_{jk}(\dot u(t_k))\right|^2\\\leqslant&(4+4L^2+\frac{\mu+4+4\lambda^2}{\mu})\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})|z_j(t_k)|^2_{E_\mu}.\end{aligned}\tag{3.40}$$

Note that

$$y(s^+)=\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})|z_j(s^+)|^2_{E_\mu}\leqslant\|z(s^+)\|^2_{E_\mu}.\tag{3.41}$$

From Lemma 2.2, (3.39)–(3.41), we obtain

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\chi(\frac{|j|}{M})|z_j(t)|^2_{E_\mu}\\\lesssim&\|z(s^+)\|^2_{E_\mu}(2+2L^2+\frac{4+\mu+4\lambda^2}{\mu})^{n(s,t)}e^{-\beta(t-s)}\\&+\int_s^t(2+2L^2+\frac{4+\mu+4\lambda^2}{\mu})^{n[\theta,t)}e^{-\beta(t-\theta)}(\sum_{|j|\geqslant M}|g_j(t)|^2+\sum_{|j|\geqslant M}|f_j(t)|^2+\frac{\sigma\epsilon^2}{3})d\theta\\\lesssim&\|z(s^+)\|^2_{E_\mu}e^{-\sigma(t-s)}+\int_s^te^{-\sigma(t-\theta)}(\sum_{|j|\geqslant M}|g_j(\theta)|^2+\sum_{|j|\geqslant M}|f_j(\theta)|^2)d\theta+\frac{\epsilon^2}{3}.\end{aligned}\tag{3.42}$$

Now by assumption (H2), we see that for the above $\epsilon>0$, there exists $M_5=M_5(t,\epsilon)\in\mathbb{N}$ such that

$$\begin{aligned}&e^{-\sigma t}\int_s^te^{\sigma\theta}\sum_{|j|\geqslant M}|g_j(\theta)|^2d\theta+e^{-\sigma t}\int_s^t\sum_{|j|\geqslant M}e^{\sigma\theta}|f_j(\theta)|^2d\theta\\\leqslant&e^{-\sigma t}\sum_{|j|\geqslant M}\int_{-\infty}^te^{\sigma\theta}|g_j(\theta)|^2d\theta+e^{-\sigma t}\sum_{|j|\geqslant M}\int_{-\infty}^te^{\sigma\theta}|f_j(\theta)|^2d\theta\leqslant\frac{\epsilon^2}{3},\ \ \forall M\geqslant M_5.\end{aligned}\tag{3.43}$$

At the same time, for the above $\epsilon > 0$, there exists a time $s_2 = s_2(t, \epsilon, \hat{\mathcal{D}})$ such that

$$e^{-\sigma t} \cdot e^{\sigma s} \sup_{z_{s^+} \in \mathcal{D}(s)} \|z(s^+)\|_{E_\mu}^2 \leqslant \frac{\epsilon^2}{3}, \quad \forall\, s \leqslant s_2. \tag{3.44}$$

Pick $M^* = \max\{M_4, M_5\}$ and $s^* = \min\{s_1,\ s_2\}$. It then follows from (3.42)–(3.44) that

$$\sup_{z(s^+)\in\mathcal{D}(s)} \sum_{|j|\geqslant 2M^*} |(S(t,s)z(s^+))_j|_{E_\mu}^2 = \sup_{z(s^+)\in\mathcal{D}(s)} \sum_{|j|\geqslant 2M^*} |z_j(t; s, z_m(s^+))|_{E_\mu}^2 \leqslant \epsilon^2, \quad \forall\, s \leqslant s^*. \tag{3.45}$$

The proof of Lemma 3.4 is complete. □

The main result of this section reads as follows.

Theorem 3.1. *Let h, γ, α, μ be positive constants satisfying* (2.15) *and assumptions* (H1) and (H2) *hold. Then the process $\{S(t,s)\}_{t\geqslant s}$ defined by* (3.1) *has a pullback $\mathcal{D}_\sigma$-attractor (denoted by) $\hat{\mathcal{A}}(t) = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ satisfying Definition* (3.1) (3).

Proof. Since the process $\{S(t,s)\}_{t\geqslant s}$ is continuous on E_μ, the result of Theorem 3.1 is obtained directly by using Lemma 3.1, Lemma 3.2 and [20, Theorem 2.1]. □

4. Statistical solution and piecewise Liouville type theorem

The goal of this section is to establish that there is a family of invariant Borel probability measures contained in the pullback attractors, and that this family of measures satisfies piecewise the Liouville type theorem and is a statistical solution of the impulsive discrete Zakharov equations.

In the sequel, $\int_{E_\mu} \Psi(z)\mathrm{d}\rho_t(z)$ denotes the Bochner integral, where ρ_t is a Borel probability measure on E_μ and $\Psi \in C(E_\mu)$ (the collection of continuous and real-valued function on E_μ).

Lemma 4.1. *Let h, γ, α, μ be positive constants satisfying* (2.15)*, and assumptions* (H1) and (H2) *hold. Then, for every given $t_* \in \mathbb{R}$ and $z_* \in E_\mu$, the E_μ-valued function $s \mapsto S(t_*, s)z_*$ is bounded on* $(-\infty, t_*]$.

Proof. It is a direct consequence of Theorem 2.1, Lemma 3.1 and assumption (H2). In fact, for any given $t_* \in \mathbb{R}$ and $z_* \in E_\mu$, we have that

$$\|z(t_*; s, z_*)\|_{E_\mu}^2 \leqslant \|z_*\|_{E_\mu}^2 + c_0 e^{-\sigma t_*} \int_{-\infty}^{t_*} e^{\sigma\theta}(\|f(\theta)\|^2 + \|g(\theta)\|^2 + \|\psi(\theta)\|^4)\mathrm{d}\theta, \quad \forall\, t_* > s. \tag{4.1}$$

The right-hand side of (4.1) is bounded by a quantity independent of s. The proof is complete. □

We next establish two auxiliary lemmas about some type of continuity of $S(t,s)z_*$ with respect to the parameters t and s.

Lemma 4.2. *Let h, γ, α, μ be positive constants satisfying* (2.15) *and assumptions* (H1) and (H2) *hold. Also let $s_* \in \mathbb{R}$ and $z_* \in E_\mu$ be given. Then, for any $\nu > 0$, there exists a positive $\kappa = \kappa(\nu, s_*, z_*)$ small enough, such that there holds*

$$\|S(t,s)z_* - z_*\|_{E_\mu} < \nu, \quad \forall\, s \in (s_*, s_* + \kappa), \quad \forall\, t \in (s, s_* + \kappa). \tag{4.2}$$

Proof. Let $s_* \in \mathbb{R}$ and $z_* \in E_\mu$ be given. Without loss of generality, we assume $s_* \in (t_{k_0}, t_{k_0+1}]$ for some $k_0 \in \mathbb{Z}$. We next split the proof into two cases.
Case 1. $s_* \in (t_{k_0}, t_{k_0+1})$. In this case, we set $2d = \min\{s_* - t_{k_0}, t_{k_0+1} - s_*\} > 0$ and consider $s_* < s \leqslant t \leqslant s_* + d$.

Firstly, set $c_* = \|z_*\|^2_{E_\mu} + \int_{s_*-d}^{s_*+d} (\|f(\theta)\|^2 + \|g(\theta)\|^2 + \|\psi(\theta)\|^4)\mathrm{d}\theta$ and we prove

$$\int_s^t \|\frac{\mathrm{d}S(\theta, s)z_*}{\mathrm{d}\theta}\|^2_{E_\mu}\mathrm{d}\theta \lesssim c_*. \tag{4.3}$$

Indeed, from (2.9) we see that

$$\|\frac{\mathrm{d}S(\theta, s)z_*}{\mathrm{d}\theta}\|^2 \lesssim \|\Theta(S(\theta, s)z_*)\|^2_{E_\mu} + \|F(S(\theta, s)z_*, t)\|^2_{E_\mu}, \quad s \leqslant \theta \leqslant t. \tag{4.4}$$

Direct computations and estimations give

$$\begin{aligned}\|\Theta z(\theta, s)\|^2_{E_\mu} =&\|\gamma\psi - iA\psi + ih^2D\psi\|^2 + \mu\|\lambda u - \varphi\|^2\\ &+ \|\lambda(\lambda - \alpha)u + \mu u - Au + h^2Du + (\alpha - \lambda)\varphi\|^2\\ \lesssim&\|\psi\|^2 + \|D\psi\|^2 + \|A\psi\|^2 + \|u\|^2_\mu + \|\varphi\|^2 + \|Au\|^2 + \|u\|^2 + \|Du\|^2 + \|\varphi\|^2\\ \lesssim&\|z(\theta, s)\|^2_{E_\mu},\end{aligned} \tag{4.5}$$

that is,

$$\|\Theta S(\theta, s)z_*\|^2_{E_\mu} \lesssim \|S(\theta, s)z_*\|^2_{E_\mu}. \tag{4.6}$$

Similarly,

$$\begin{aligned}\|F(z(\theta, s), \theta)\|^2_{E_\mu} =&\| - i\psi u - ig(\theta)\|^2 + \|f(\theta) + A|\psi|^2\|^2\\ \lesssim&\|f(\theta)\|^2 + \|g(\theta)\|^2 + \|\psi(\theta)\|^4 + R_*,\end{aligned} \tag{4.7}$$

where

$$R_* = \max_{t\in[t_*-d,t_*+d]} R^4_\sigma(t) = \max_{t\in[t_*-d,t_*+d]} [1 + c_0e^{-\sigma t}\int_{-\infty}^t e^{\sigma\vartheta}(\|f(\vartheta)\|^2 + \|g(\vartheta)\|^2 + \|\psi(\vartheta)\|^4)\mathrm{d}\vartheta]^2.$$

Inserting (4.6) and (4.7) into (4.4) and then integrating the resulting inequality with respect to θ over $[s, t]$, we obtain (4.3).

Secondly, we observe that

$$\|S(t, s)z_* - z_*\|^2_{E_\mu} = \int_s^t \frac{\mathrm{d}\|S(\theta, s)z_*\|^2_{E_\mu}}{\mathrm{d}\theta}\mathrm{d}\theta - 2\mathbf{Re}\,(S(t, s)z_* - z_*, z_*). \tag{4.8}$$

By (2.19) and (2.31) we have

$$\int_s^t \frac{\mathrm{d}\|S(\theta, s)z_*\|^2_{E_\mu}}{\mathrm{d}\theta}\mathrm{d}\theta \lesssim \int_s^t (\|f(\theta)\|^2 + \|g(\theta)\|^2)\mathrm{d}\theta + \int_s^t (\|z_*\|^2_{E_\mu}e^{-\sigma(\theta-s)} + e^{-\sigma\theta}\int_s^\theta e^{\sigma\eta}\|g(\eta)\|^2)^2\mathrm{d}\eta. \tag{4.9}$$

By (H2), we can pick $\kappa' = \kappa'(\nu, s_*, f, g) \in (0, d)$ such that

$$\int_s^t \frac{\mathrm{d}\|S(\theta, s)z_*\|_{E_\mu}^2}{\mathrm{d}\theta}\mathrm{d}\theta \lesssim \frac{\nu^2}{2}, \quad s_* < s \leqslant t \leqslant s_* + \kappa'. \tag{4.10}$$

Using Cauchy's inequality and (4.3) gives

$$\begin{aligned}\Big|(S(\theta, s)z_* - z_*, z_*)\Big| =& \Big|\Big(\int_s^t \frac{\mathrm{d}S(\theta, s)z_*}{\mathrm{d}\theta}\mathrm{d}\theta, z_*\Big)\Big| \leqslant \|z_*\|_{E_\mu}\int_s^t \|\frac{\mathrm{d}S(\theta, s)z_*}{\mathrm{d}\theta}\|_{E_\mu}\mathrm{d}\theta \\ \leqslant& \|z_*\|_{E_\mu}\Big(\int_s^t \|\frac{\mathrm{d}S(\theta, s)z_*}{\mathrm{d}\theta}\|_{E_\mu}^2\mathrm{d}\theta\Big)^{1/2}(t-s)^{1/2} \leqslant c_*^{1/2}\|z_*\|_{E_\mu}(t-s)^{1/2},\end{aligned} \tag{4.11}$$

which implies that there exists some $\kappa'' = \kappa''(\nu, s_*, z_*) \in (0, d)$ such that

$$\Big|(S(\theta, s)z_* - z_*, z_*)\Big| < \frac{\nu^2}{4}, \quad s_* < s \leqslant t \leqslant s_* + \kappa''. \tag{4.12}$$

Set $\kappa_* = \min\{\kappa', \kappa''\}$. We obtain (4.2) from (4.8)–(4.12). The case $s_* \in (t_{k_0}, t_{k_0+1})$ is proved.
Case 2. $s_* = t_{k_0+1}$. In this case we denote $2d = t_{k_0+2} - t_{k_0+1}$ and consider $s_* < s \leqslant t \leqslant s_* + d$. The main difference in the proof to *case 1* is that the constants c_* and R_* are replaced with

$$\begin{aligned}c_*' =& \|z_*\|_{E_\mu}^2 + \int_{s_*}^{s_*+d} (\|f(\theta)\|^2 + \|g(\theta)\|^2 + \|\psi(\theta)\|^4)\mathrm{d}\theta, \\ R_*' =& \max_{t\in[t_*, t_*+d]} R_\sigma^4(t) = \max_{t\in[t_*, t_*+d]} [1 + c_0 e^{-\sigma t}\int_{-\infty}^t e^{\sigma\vartheta}(\|f(\vartheta)\|^2 + \|g(\vartheta)\|^2 + \|\psi(\vartheta)\|^4)\mathrm{d}\vartheta]^2,\end{aligned}$$

respectively. Here c_*' and R_*' are also constants independent of s and t. We omit the details here. □

Similarly to Lemma 4.2, we have

Lemma 4.3. *Let h, γ, α, μ be positive constants satisfying* (2.15) *and assumptions* (H1) and (H2) *hold. Also let $s_* \in \mathbb{R}$ and $z_* \in E_\mu$ be given. Then, for any $\nu > 0$, there exists a positive $\kappa = \kappa(\nu, s_*, z_*)$ such that*

$$\|S(t, s)z_* - z_*\|_{E_\mu} < \nu, \quad \forall\, s \in (s_* - \kappa, s_*],\ \forall\, t \in [s, s_*]. \tag{4.13}$$

Now, we begin to investigate some kind of continuity of the E_μ-valued function $s \mapsto S(t_*, s)z_*$ for every given $t_* \in \mathbb{R}$ and $z_* \in E_\mu$.

Lemma 4.4. *Let h, γ, α, μ be positive constants satisfying* (2.15) *and assumptions* (H1) and (H2) *hold. Then, for every given $t_* \in \mathbb{R}$ and $z_* \in E_\mu$, the E_μ-valued function $s \mapsto S(t_*, s)z_*$ belongs to $PC((-\infty, t_*]; E_\mu)$, that is,*

1. *$S(t_*, s)z_*$ is left continuous for $s \in (-\infty, t_*]$;*
2. *$S(t_*, s)z_*$ is continuous for $s \in (-\infty, t_*]$ with $s \neq t_k \in (-\infty, t_*]$, $k \in \mathbb{Z}$;*
3. *$S(t_*, s)z_*$ has right limit at the impulsive points $t_k \in (-\infty, t_*]$, $k \in \mathbb{Z}$.*

Proof. Firstly, we prove item (1). Consider any given $s_* \in (-\infty, t_*]$. We shall prove that $S(t_*, s)z_*$ is left-continuous at $s = s_*$. Indeed, without loss of generality, we assume that $t_* \in (t_{k_0}, t_{k_0+1}]$ for some $k_0 \in \mathbb{Z}$. Then for any $s \in (t_{k_0}, s_*]$ we have that, by the invariance property of the process,

$$\|S(t_*, s)z_* - S(t_*, s_*)z_*\|_{E_\mu} = \|S(t_*, s_*)S(s_*, s)z_* - S(t_*, s_*)z_*\|_{E_\mu}. \tag{4.14}$$

Since t_* and s_* are fixed, $S(t_*, s_*) : E_\mu \mapsto E_\mu$ is continuous. The left-continuity of $S(t_*, s)z_*$ at $s = s_*$ follows from (4.2) and (4.14).

Secondly, we prove item (2). Without loss of generality, we just prove, in view of the result of item (1), that $S(t_*, s)z_*$ is right-continuous on $(t_{k_0}, t_{k_0+1}) \cap (-\infty, t_*]$ for some $k_0 \in \mathbb{Z}$. Let $s_* \in (t_{k_0}, t_{k_0+1}) \cap (-\infty, t_*]$ be given and $s_* < s < t_{k_0+1} \leqslant t_*$. Using the invariance property of the process and (2.36), we have

$$\begin{aligned}\|S(t_*, s_*)z_* - S(t_*, s)z_*\|_{E_\mu} =& \|S(t_*, s)S(s, s_*)z_* - S(t_*, s)z_*\|_{E_\mu}\\ \leqslant& \|S(s, s_*)z_* - z_*\|_{E_\mu} e^{-(\sigma+2c_1-\gamma)(t_*-s)}.\end{aligned} \tag{4.15}$$

The right-continuity of $S(t_*, s)z_*$ at $s = s_*$ follows from (4.15) and the fact that $U(\cdot, s_*)z_* = z(\cdot) \in C((t_{k_0}, t_{k_0+1}); E_\mu)$.

Thirdly, the fact that $S(t_*, s)z_*$ has right-hand limit at the impulsive points $t_k \in (-\infty, t_*]$ can be obtained directly from Lemma 4.2 and the invariance property of the process, by using Cauchy's criterion for convergence. The proof of Lemma 4.4 is therefore complete. □

Combining Lemma 4.1 and Lemma 4.4, we have the following lemma.

Lemma 4.5. *The process $\{S(t, s)\}_{t\geqslant s}$ possesses the so-called* ***PC-s-continuity*** *in the sense that, for every given $t_* \in \mathbb{R}$ and $z_* \in E_\mu$, the E_μ-valued function $s \mapsto S(t_*, s)z_* \in PC((-\infty, t_*]; E_\mu)$ and is bounded on $(-\infty, t_*]$.*

In order to state the results concerning the existence of invariant Borel probability measure, we will use the definition of generalized Banach limit and its property.

Definition 4.1. ([6, 13]) *A generalized Banach limit is any linear functional, denoted by $\mathrm{LIM}_{t\to+\infty}$, defined on the space of all bounded real-valued functions on $[0, +\infty)$ and satisfying*

(1) $\mathrm{LIM}_{t\to+\infty}\zeta(t) \geqslant 0$ *for nonnegative functions $\zeta(\cdot)$ on $[0, +\infty)$;*
(2) $\mathrm{LIM}_{t\to+\infty}\zeta(t) = \lim_{t\to+\infty} \zeta(t)$ *if the usual limit $\lim_{t\to+\infty} \zeta(t)$ exists.*

Let B_+ be the collection of all bounded real-valued functions on $[0, +\infty)$. For any generalized Banach limit $\mathrm{LIM}_{t\to+\infty}$, the following useful property

$$|\mathrm{LIM}_{t\to+\infty}\zeta(t)| \leqslant \limsup_{t\to+\infty} |\zeta(t)|, \quad \forall\, \zeta(\cdot) \in B_+, \tag{4.16}$$

is presented in [6, (1.38)] and in [5, (2.3)].

Remark 4.1. *Since we consider the "pullback" asymptotic behavior, we will thus use the generalized Banach limits as $s \to -\infty$. For a given real-valued function ζ defined on $(-\infty, 0]$ and a given generalized Banach limit $\mathrm{LIM}_{t\to+\infty}$, we define*

$$\mathrm{LIM}_{t\to-\infty}\zeta(t) = \mathrm{LIM}_{t\to+\infty}\zeta(-t). \tag{4.17}$$

We next refine the result of [13, Theorem 3.1] to construct a family of invariant Borel probability measures for the process $\{S(t,s)\}_{t\geqslant s}$ on E_μ, via the generalized Banach limit and the pullback attractor $\hat{\mathcal{A}}(t)$ obtained in Theorem 3.1.

Theorem 4.1. *Let h, γ, α, μ be positive constants satisfying* (2.15) *and assumptions* (H1) and (H2) *hold. Also let $v(\cdot):\mathbb{R}\mapsto E_\mu$ be a continuous map such that $v(\cdot)\in\mathcal{D}_\sigma$. Then for a given generalized Banach limit* $\mathrm{LIM}_{t\to+\infty}$ *there exists a unique family of Borel probability measures $\{m_t\}_{t\in\mathbb{R}}$ on E_μ such that the support of the measure m_t is contained in $\mathcal{A}(t)$ and*

$$\begin{aligned}
&\mathrm{LIM}_{s\to-\infty}\frac{1}{t-s}\int_s^t\Psi(S(t,\theta)v(\theta))\mathrm{d}\theta\\
=&\int_{\mathcal{A}(t)}\Psi(z)\mathrm{d}m_t(z)=\int_{E_\mu}\Psi(z)\mathrm{d}m_t(z) \qquad (4.18)\\
=&\mathrm{LIM}_{s\to-\infty}\frac{1}{t-s}\int_s^t\int_{E_\mu}\Psi(S(t,\theta)z)\mathrm{d}m_\theta(z)\mathrm{d}\theta, \qquad (4.19)
\end{aligned}$$

for any $\Psi\in C(E_\mu)$. Moreover, m_t is invariant in the sense that

$$\int_{\mathcal{A}(t)}\Psi(z)\mathrm{d}m_t(z)=\int_{\mathcal{A}(s)}\Psi(S(t,s)z)\mathrm{d}m_s(z),\quad t\geqslant s. \qquad (4.20)$$

Proof. The proof is similar to that of [28, Theorem 4.1]. For the convenience of the reader, we present the main steps of the proof.

Fix $\Psi(\cdot)\in C(E_\mu)$ and a continuous map $v(\cdot):\mathbb{R}\mapsto E_\mu$ such that $v(\cdot)\in\mathcal{D}_\sigma$. For given $t\in\mathbb{R}$, we claim that, for every compact interval $[t_0,t]\subset\mathbb{R}$, the function $s\mapsto\Psi(S(t,s)v(s))$ is bounded on $[t_0,t]$ with each $t_0<t$. In fact, on the one hand, from (1.5) we see that the impulsive points $\{t_k\}_{k\in\mathbb{Z}}$ belonging to the interval $[t_0,t]$ are only a finite number. We denote these impulsive points by t_{k_0+1}, $t_{k_0+2},\cdots,t_{k_0+N}$ for some $N\in\mathbb{Z}_+$. Then by Lemma 4.4, the function $s\mapsto\Psi(S(t,s)v(s))$ is continuous on $[t_0,t]\backslash\{t_{k_0+1},t_{k_0+1},\cdots,t_{k_0+N}\}$, and is left continuous on $(t_0,t]$ and has right-hand limit at the points t_0, $t_{k_0+1},\cdots,t_{k_0+N}$. Therefore, $\Psi(S(t,s)v(s))$ is bounded on the compact interval $[t_0,t]$. On the other hand, from Lemma 3.1 we see $\Psi(S(t,s)v(s))$ is also bounded on the interval $(-\infty,t_0+1]$ for t_0 sufficiently large and negative, since $v(\cdot)\in\mathcal{D}_\sigma$ and $\{S(t,s)\}_{t\geqslant s}$ has pullback $\mathcal{D}_\sigma$-attracting property. Hence, we have proved that the function $\Psi(S(t,s)v(s))$ is bounded on $(-\infty,t]$ and the function

$$s\longmapsto\frac{1}{t-s}\int_s^t\Psi(S(t,\theta)v(\theta))\mathrm{d}\theta$$

is bounded on the interval $(-\infty,t]$. In view of this fact, we define

$$L(\Psi)=\mathrm{LIM}_{s\to-\infty}\frac{1}{t-s}\int_s^t\Psi(S(t,\theta)v(\theta))\mathrm{d}\theta.$$

The rest of the proof follows closely the that of [13, Theorem 3.1] and we omit the details here. □

We now investigate the statistical solution for Eq (2.9). First we introduce the class $\mathcal{T}$ of test functions associated to the statistical solution. Write Eq (2.9) as

$$\frac{\mathrm{d}z}{\mathrm{d}t}=G(z,t)=F(z(t),t)-\Theta z,\quad t\neq t_k,\ k\in\mathbb{Z}. \qquad (4.21)$$

Then $G(z,t) : E_\mu \times \mathbb{R} \longmapsto E_\mu$. We expect that the test function $\Phi \in \mathcal{T}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(z(t)) = (\Phi'(z), G(z,t)), \quad t \neq t_k,\ k \in \mathbb{Z}, \tag{4.22}$$

for every solution $z(t)$ of Eqs (2.9)–(2.11).

Definition 4.2. (cf. [27, Definition 4.2] *We define the class $\mathcal{T}$ of test functions to be the set of real-valued functionals $\Phi = \Phi(\cdot)$ on E_μ that are bounded on bounded subset of E_μ and satisfy*

(a) *for any $z \in E_\mu$, the Fréchet derivative $\Phi'(z)$ exists: for each $z \in E_\mu$ there exists an element $\Phi'(z)$ such that*
$$\frac{|\Phi(z+w) - \Phi(z) - (\Phi'(z), w)|}{\|w\|} \longrightarrow 0 \ \text{as}\ \|w\| \to 0, \ \ w \in E_\mu;$$
(b) *$\Phi'(z) \in E_\mu$ for all $z \in E_\mu$, and the mapping $z \longmapsto \Phi'(z)$ is continuous and bounded as a function from E_μ to E_μ;*
(c) *for every global solution $z(t)$ of Eq* (4.21)*,* (4.22) *holds true.*

For example, we can consider the cylindrical test functions (cf. [6, page 178]) defined on E_μ. Let $\phi \in E_\mu$ and ϱ be a continuously differentiable real-valved function on $\mathbb{R}$ with compact support. For each $z \in E_\mu$, define $\Phi(z)$ via
$$\Phi(z) = \varrho((\phi, z)).$$
Then the function $\Phi(\cdot)$ is obviously continuous from E_μ to $\mathbb{R}$ and, is Fréchet differentiable on E_μ, with Fréchet derivative $\Phi'(\cdot)$ at $z \in E_\mu$ given by

$$\Phi'(z) = \varrho'((\phi, u))\phi \in E_\mu. \tag{4.23}$$

The cylindrical test functions of the above form satisfy Definition 4.2.

We now state the definition of statistical solution for Eq (4.21) and prove the existence.

Definition 4.3. *We say that a family of Borel probability measures $\{\rho_t\}_{t\in\mathbb{R}}$ on E_μ is a statistical solution of Eq* (4.21) *if the following conditions are satisfied:*

(a) *for every $\Psi \in C(E_\mu)$, the function $t \mapsto \int_{E_\mu} \Psi(z)\mathrm{d}\rho_t(z) \in PC(\mathbb{R};\mathbb{R})$;*
(b) *for almost $t \in \mathbb{R}$, the function $z \mapsto (w, G(z,t))$ is ρ_t-integrable for every $w \in E_\mu$. Moreover, the map $t \mapsto \int_{E_\mu} (w, G(z,t))\mathrm{d}\rho_t(z)$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$ for every $w \in E_\mu$;*
(c) *for any test function $\Phi \in \mathcal{T}$, there holds that*
$$\int_{E_\mu} \Phi(z)\mathrm{d}\rho_t(z) - \int_{E_\mu} \Phi(z)\mathrm{d}\rho_s(z) = \int_s^t \int_{E_\mu} (\Phi'(z), G(z,\theta))\mathrm{d}\rho_\theta(z)\mathrm{d}\theta,$$
for $t, s \in (t_k, t_{k+1})$ with $t \geqslant s$ and $k \in \mathbb{Z}$.

Theorem 4.2. *Let h, γ, α, μ be positive constants satisfying* (2.15) *and assumptions* (H1) and (H2) *hold. Then the family of Borel probability measures $\{\mathsf{m}_t\}_{t\in\mathbb{R}}$ obtained in Theorem 4.1 is a statistical solution of Eq* (4.21).

Proof. We prove that the family of Borel probability measures $\{m_t\}_{t\in\mathbb{R}}$ obtained in Theorem 4.1 satisfies items (a)–(c) of Definition 4.3.

Firstly, we prove item (a). Consider any $k \in \mathbb{Z}$ and any given $t_* \in (t_k, t_{k+1}]$.

In case $t_* \in (t_k, t_{k+1})$, we establish that for every $\Psi \in C(E_\mu)$ there holds

$$\lim_{t\to t_*} \int_{E_\mu} \Psi(z)\mathrm{d}m_t(z) = \int_{E_\mu} \Psi(z)\mathrm{d}m_{t_*}(z). \tag{4.24}$$

In fact, from (4.18) and (4.20) we can see that for $t > t_*$,

$$\int_{E_\mu} \Psi(z)\mathrm{d}m_t(z) - \int_{E_\mu} \Psi(z)\mathrm{d}m_{t_*}(z) = \int_{\mathcal{A}(t_*)} (\Psi(S(t,t_*)z) - \Psi(z))\mathrm{d}m_{t_*}(z). \tag{4.25}$$

Since $S(t,t_*)z \longrightarrow z$ strongly in E_μ as $t \to t_*^+$, $\Psi \in C(E_\mu)$ and $\mathcal{A}(t_*)$ is compact in E_μ, (4.25) implies

$$\lim_{t\to t_*^+} \int_{E_\mu} \Psi(z)\mathrm{d}m_t(z) = \int_{E_\mu} \Psi(z)\mathrm{d}m_{t_*}(z).$$

Similarly, for $t < t_*$,

$$\int_{E_\mu} \Psi(z)\mathrm{d}m_{t_*}(z) - \int_{E_\mu} \Psi(z)\mathrm{d}m_t(z) = \int_{\mathcal{A}(t)} (\Psi(S(t_*,t)z) - \Psi(z))\mathrm{d}m_t(z). \tag{4.26}$$

Since $s(t_*,t)z \longrightarrow z$ strongly in E_μ as $t \to t_*^-$ (see Lemma 4.3), and $\Psi \in C(E_\mu)$, $m_t(\mathcal{A}(t)) \leqslant 1$ for every $t \in \mathbb{R}$, (4.26) implies

$$\lim_{t\to t_*^-} \int_{E_\mu} \Psi(z)\mathrm{d}m_t(z) = \int_{E_\mu} \psi(z)\mathrm{d}m_{t_*}(z). \tag{4.27}$$

In case $t_* = t_{k+1}$, we use the same proof as (4.27) to obtain the left-continuity of $\int_{E_\mu} \Psi(z)\mathrm{d}m_t(z)$ at $t = t_*$. To establish the existence of $\lim_{t\to t_*^+} \int_{E_\mu} \Psi(z)\mathrm{d}m_t(z)$, we consider $t_* < t' \leqslant t'' < t_{k+2}$ and have

$$\int_{E_\mu} \Psi(z)\mathrm{d}m_{t''}(z) - \int_{E_\mu} \Psi(z)\mathrm{d}m_{t'}(z) = \int_{\mathcal{A}(t')} (\Psi(S(t'',t')z) - \Psi(z))\mathrm{d}m_{t'}(z). \tag{4.28}$$

Since $\Psi \in C(E_\mu)$, $m_{t'}(\mathcal{A}(t')) \leqslant 1$, (4.28) and Lemma 4.2 imply the existence of $\lim_{t\to t_*^+} \int_{E_\mu} \Psi(z)\mathrm{d}m_t(z)$. Thus item (a) is proved.

Secondly, we establish item (b). For every $t \in \mathbb{R}$ we have proved that m_t is carried by $\mathcal{A}(t) \subset E_\mu$. Now, for every $z, w \in E_\mu$, we define

$$\Psi(z) = (w, G(z,\cdot)). \tag{4.29}$$

Then $\Psi(\cdot) : E_\mu \longmapsto \mathbb{R}$. We next establish $\Psi(\cdot) \in C(E_\mu)$ and thus $\Psi(\cdot)$ satisfies (4.24). Let $z_* \in E_\mu$ be fixed and consider $z \in E_\mu$ with $\|z_* - z\| \leqslant 1$. Then by (2.10), (2.41) and (4.21) we obtain

$$|\Psi(z_*) - \Psi(z)| = |(w, G(z_*,\cdot) - G(z,\cdot))|$$

$$\lesssim |(w, F(z_*, t) - F(z, t))| + |(w, \Theta z_* - \Theta z)| \lesssim c_1 \|w\| \|z_* - z\|_{E_\mu}, \tag{4.30}$$

where c_1 is same as in Eq (2.41). (4.30) implies that the real-valued function $\Psi(\cdot)$ defined by (4.29) is continuous on E_μ. From (4.18) and (4.29) we conclude that the function $z \mapsto (w, G(z, \cdot)) = \Psi(z)$ is m_t-integrable for every $w \in E_\mu$. At the same time, we have proved in the previous item that the function

$$t \mapsto \int_{E_\mu} (w, G(z, t)) \mathrm{d}m_t(z) = \int_{E_\mu} \Psi(z) \mathrm{d}m_t(z)$$

is piecewise continuous on $\mathbb{R}$ and its discontinuities $\{t_k\}_{k \in \mathbb{Z}}$ are of the first-kind. Hence it belongs to $L^1_{\text{loc}}(\mathbb{R})$ for every $w \in E_\mu$.

Thirdly, we prove item (c). For any $\Phi \in \mathcal{T}$, we deduce from (4.22) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \Phi(z(t)) = (\Phi'(z), G(z, t)), \quad t \neq t_k, \ k \in \mathbb{Z}.$$

Hence, for all t, $s \in (t_k, t_{k+1})$ with $t \geqslant s$ and $k \in \mathbb{Z}$, there holds

$$\Phi(z(t)) - \Phi(z(s)) = \int_s^t (\Phi'(z(\theta)), G(z(\theta), \theta)) \mathrm{d}\theta. \tag{4.31}$$

Now for any $\zeta \leqslant s$, let $z_* \in E_\mu$ and $z(\theta) = S(\theta, \zeta) z_*$ for $\theta \in [s, t]$. We use (4.31) to derive

$$\Phi(S(t, \zeta) z_*) - \Phi(S(s, \zeta) z_*) = \int_s^t (\Phi'(S(\theta, \zeta) z_*), G(S(\theta, \zeta) z_*, \theta)) \mathrm{d}\theta. \tag{4.32}$$

By (4.18), (4.19) and (4.32), we obtain after some calculations that

$$\begin{aligned} & \int_{E_\mu} \Phi(z) \mathrm{d}m_t(z) - \int_{E_\mu} \Phi(z) \mathrm{d}m_s(z) \\ =& \mathrm{LIM}_{M \to -\infty} \frac{1}{s - M} \int_M^s \int_s^t \int_{E_\mu} (\Phi'(S(\theta, \zeta) z_*), G(S(\theta, \zeta) z_*, \theta)) \mathrm{d}m_\zeta(u_*) \mathrm{d}\theta \mathrm{d}\zeta, \quad t_k < s \leqslant t < t_{k+1}, \end{aligned}$$

where we have used Fubini's Theorem to change the order of integration. Now, by (4.20) and the invariance property of the process $S(\theta, \zeta) = S(\theta, s) S(s, \zeta)$, we have

$$\int_{E_\mu} (\Phi'(S(\theta, \zeta) z_*), G(S(\theta, \zeta) z_*, \theta)) \mathrm{d}m_\zeta(z_*) = \int_{E_\mu} (\Phi'(S(\theta, s) z_*), G(S(\theta, s) z_*, \theta)) \mathrm{d}m_s(z_*), \tag{4.33}$$

where the right-hand side of (4.33) is independent of ζ. Therefore,

$$\begin{aligned} & \int_{\mathcal{A}(t)} \Phi(z) \mathrm{d}m_t(z) - \int_{\mathcal{A}(\tau)} \Phi(z) \mathrm{d}m_s(z) = \int_s^t \int_{E_\mu} (\Phi'(S(\theta, s) z_*), G(S(\theta, s) z_*, \theta)) \mathrm{d}m_s(z_*) \mathrm{d}\theta \\ =& \int_s^t \int_{E_\mu} (\Phi'(z), G(z(\zeta), \zeta)) \mathrm{d}m_\zeta(z) \mathrm{d}\zeta, \quad t_k < s \leqslant t < t_{k+1}. \end{aligned} \tag{4.34}$$

The proof of Theorem 4.2 is complete. □

5. Conclusions

In this article, we first prove that the problem of the discrete Zakharov equations with impulsive effect is global well-posed. Then we establish that the process formed by the solution operators possesses a pullback attractor and that there is a family of invariant Borel probability measures contained in the pullback attractor. Further, we verify that this family of measures satisfies the Liouville type theorem piecewise and is a statistical solution of the impulsive discrete Zakharov equations. Our results reveal that the Liouville type equation for impulsive system will not always hold true on the interval containing any impulsive point.

Conflict of interest

Authors have no conflicts of interest.

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