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Research article

Copulas generated by mixtures of weighted distributions

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Abstract: In this paper, we characterize several partial dependencies in a general mixture model of weighted distributions with a parametric weight function that encompasses many well-known frailty models. There are well-known frailty models in survival analysis satisfying the proposed mixture model which are used to examine the results. The mixture-based copula functions associated with the mixture model are characterized. Examples are given to draw the copula functions out from respected mixture models.

Keywords: weighted distribution; mixture model; copula function; dependence structure; semiparametric distributions; stochastic order
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1. Introduction

Weighted distributions are used when sampling procedures occur with unequal sampling probabilities proportional to a given weight function. The introduction of weighted distributions dates back to [12]. The formal definition and several statistical applications can be found in [41]. In 1977 and 1978, Patil and Rao developed further applications of weighted distributions in [39] and [40] respectively. In recent decades, stochastic comparisons of weighted distributions have been studied in the literature in the context of reliability and life testing. To name a few, in 2006 Bartoszewicz and Skolimowska in [4] and in 2008 Misra et al. in [35] obtained results for the preservation of some stochastic orders and aging classes under weighted distributions. Subsequently, the same study using different approaches to prove the preservation results has been done in [15, 18, 19]. Recently, a typical family of univariate weighted distributions with a parameter-indexed weight function imposing on a base distribution that does not depend on the parameter (on which the weight function depends) has been considered by Albabtain et al. in [2]. They have obtained some conservation properties of stochastic orderings under the typical class of weighted distributions. They have also developed their results under mixture weighted distributions. The study of size-biased distribution, as an important weighted distribution, and its mixture has been conducted in [26]. The parametric family of distributions considered in [2] includes many well-known statistical survival models and the extended mixture model also comprises some reputable frailty models where the parameter is an unobservable factor [14]. In particular, a typical family of semiparametric distributions can be viewed as a weighted distribution, which may have further advantages and applications [2].

The present study aims to analyse the concept of dependence in the mixture weighted distribution through various partial dependence structures and also introduce methods for deriving the copula function. Since in a general mixture model the overall population random variable is usually related to the mixing random variable thus the study of the dependence structure between these random variables in the model is worth considering. In the context of some frailty models, some researchers have investigated this kind of dependence problem. For example, in the context of the mixture proportional hazards model, it was demonstrated in [47] that the output random variable and the mixed (frailty) random variable exhibit a negative dependency. The proportional odds model, as found by [32], induces a positive dependence between the overall population random variable and the mixing random variable (random tipping parameter). However, there is a lack of further general and comprehensive methods in the literature to present a a commonly used technique to get the copula functions arisen from mixture models.

The rest of the paper is organized as follows. In Section 2, we present some preliminary concepts including a parametric weighted distribution and its mixture, a semiparametric model and its mixture, several stochastic orders and also a number of partial dependence structures. In Section 3, in the first part we give some necessary and sufficient conditions for the partial dependence structures in the mixture weighted distribution with many examples which are moved to Appendix to enhance the readability. In the second part in Section 3, the copula function associated with the mixture semiparametric model is extracted with some example of well-known frailty models which are transmitted to Appendix. In Section 4, the paper concludes with a brief summary of the results and future studies of the current work.

2. Preliminaries

In this section, some useful notions in distribution theory that will be used throughout the paper which are well-known in the previous literature are presented.

2.1. A parametric weighted distribution

Let X be an absolutely continuous random variable with cumulative distribution function (cdf) F_X and probability density function (pdf) f_X. Denote by $\overline{F}_X = 1 - F_X$ the survival function (sf) of X. For some $\theta \in \chi$, let $w : x \to w(x;\theta)$ be a non-negative function such that $\eta(\theta) = E(w(X;\theta)) < \infty$. The random variable X_w, which is the weighted version of X with weight function $w(\cdot;\theta)$, has cdf

$$F(x;\theta) = \frac{1}{\eta(\theta)} \int_0^x w(x';\theta) dF_X(x'), \tag{2.1}$$

and the pdf associated with (2.1) is

$$f(x;\theta) = \frac{w(x;\theta) f_X(x)}{\eta(\theta)}. \tag{2.2}$$

To develop the mixture model of (2.1) when θ is a realization of a mixing random variable Θ, we consider random variable X^* with cdf

$$F^*(x) = \int_{\theta \in \chi} F(x;\theta) d\Lambda(\theta) = E[v(X) \mid X \le x] F_X(x), \tag{2.3}$$

where $v(x) = E\left[\frac{w(x,\Theta)}{\eta(\Theta)}\right]$, Λ is the cdf of Θ with corresponding pdf λ. The pdf of X^* is obtained as

$$f^*(x) = \int_{\theta \in \chi} f(x,\theta) d\Lambda(\theta) = f_X(x) v(x). \tag{2.4}$$

The mixture model (2.3) is pervasive to be used in different situations since it encompasses many other models depending on variety of choices of the weight function $w(x,\theta)$.

2.2. A semiparametric model: a special weighted distribution

Here, we consider a semiparametric class of distributions that is characterised by having a parameter that is itself a distribution function. If the underlying distribution is F_X then a semiparametric family is said to provide a way to add a new parameter θ, extending the family from which F_X originates. There are many families of distributions that can be assumed to come from the standard distributions over semiparametric families that add a second parameter. Therefore, the study of semiparametric families is useful for two purposes: it provides a new understanding of standard distribution families and it shows methods for extending families to add flexibility in fitting data. A typical family of semiparametric distributions that encompasses several well-known models in reliability and survival analysis is considered. Suppose that X is a random variable with distribution function F_X and that θ is a parameter with values in $\chi \subseteq \mathbb{R}$. The semiparametric family with underlying distribution F_X is defined as

$$F_\theta(x) = d(F_X(x), \theta), \quad x \ge 0, \ \theta \in \chi, \tag{2.5}$$

in which

$$\begin{aligned} d : [0,1] &\to [0,1] \\ u &\to d(u,\theta) \end{aligned}$$

is a non-negative function requiring the following conditions:

(i) $0 \le d(u,\theta) \le 1$, for all $u \in [0,1]$ and $\theta \in \chi$.
(ii) $d(0,\theta) = 0$, for all $\theta \in \chi$.
(iii) $d(1,\theta) = 1$, for all $\theta \in \chi$.
(iv) d is non-decreasing and right continuous for all $\theta \in \chi$.

The function d may be referred to as the generator of the underlying semiparametric family of distribution. Provided that (i)–(iv) hold, $F(\cdot|\theta)$ in (2.1) is a distribution function for any $\theta \in \chi$. By considering

$$\xi : [0,1] \to [0,\infty)$$

$$x \quad \to \xi(x,\theta)$$

a general family of functions d in (2.1) is constructed as follows:

$$d(u,\theta) = \frac{\int_0^u \xi(x,\theta)dx}{\int_0^1 \xi(x,\theta)dx}. \tag{2.6}$$

Specific choices of the function ξ in (2.6) by which several reputable models are built includes:

(i) Proportional hazard rates model. $\xi(x,\theta) = 1-(1-x)^\theta$, where $\theta > 0$.
(ii) Proportional reversed hazard rates model. $\xi(x,\theta) = x^\theta$, where $\theta > 0$.
(iii) Proportional odds model. $\xi(x,\theta) = \frac{1}{(1-\bar{\theta}(1-x))^2}$, where $\theta > 0$.
(iv) Upper tail distribution $\xi(x,\theta) = I(x \geq \theta)$ with $\theta \in [0,1)$.
(v) Lower tail distribution $\xi(x,\theta) = I(x < \theta)$ with $\theta \in (0,1]$.

In the cases where the underlying distribution function is absolutely continuous, the density function associated with (2.5) is given in terms of (2.6) by

$$f_\theta(x) = \frac{\xi(F_X(x),\theta)}{\eta(\theta)} f_X(x), \tag{2.7}$$

with $\eta(\theta) = E[\xi(F_X(X),\theta)] = \int_0^1 \xi(x,\theta)dx$, being the normalizing constant. The mixture model of (2.5) with cdf

$$F^*(x) = \int_{\theta\in\chi} d(F_X(x),\theta)d\Lambda(\theta), \tag{2.8}$$

and pdf

$$f^*(x) = f_X(x)\int_{\theta\in\chi} \frac{\xi(F_X(x),\theta)}{\eta(\theta)} d\Lambda(\theta), \tag{2.9}$$

will also be considered. It is assumed that X^* has pdf and cdf f^* and F^*, respectively, in this case. It can be seen that the family (2.7) is indeed a weighted distribution of F with the corresponding weight function $w(x,\theta) = \xi(F(x),\theta)$, which depends on the underlying distribution function. In spite of that, the problem of studying the proposed family of semiparametric distributions in (2.5) lies in the framework of weighted distributions with a parameter-indexed weight function.

2.3. Stochastic orders

The theory of stochastic orders has been developed to make stochastic comparisons and also study of structural properties of complex stochastic systems in several fields. A number of stochastic orders has been routinely utilized in many applications in economics, finance, insurance, management science, operations research, statistics, and various other contexts [6, 37, 42].

Let X and Y be two non-negative random variables with cdfs F and G, pdfs (whenever they exist) f and g, respectively. Then, the hazard rates of X and Y are defined as $h_X(t) = \frac{f(t)}{\bar{F}(t)}$ for all $t : \bar{F}(t) > 0$ and $h_Y(t) = \frac{g(t)}{\bar{G}(t)}$ for all $t : \bar{G}(t) > 0$, where $\bar{F}$ and $\bar{G}$ are, respectively, the sfs of X and Y. The reversed hazard rates of X and Y are given by $r_X(t) = \frac{f(t)}{F(t)}$ for all $t : F(t) > 0$ and $r_Y(t) = \frac{g(t)}{G(t)}$ for all $t : G(t) > 0$. Four well-known stochastic orders to compare the magnitude of the random variables X and Y are defined as follows.

Definition 1. *The random variable X is said to be less (resp. greater) or equal than the random variable Y in the (see, e.g., [42])*

(i) likelihood ratio order (denoted as $X \leq_{lr}$ (resp. $\geq_{lr}$)Y) whenever $\frac{g(t)}{f(t)}$ is increasing (resp. decreasing) in $t \geq 0$.
(ii) hazard rate order (denoted as $X \leq_{hr}$ (resp. $\geq_{hr}$)Y) whenever $h_X(t) \geq$ (resp. $\leq$)$h_Y(t)$ for all $t \geq 0$ or equivalently if $\frac{\bar{G}(t)}{\bar{F}(t)}$ is increasing (resp. decreasing) in $t \geq 0$.
(iii) reversed hazard rate order (denoted as $X \leq_{rh}$ (resp. $\geq_{rh}$)Y) whenever $r_X(t) \leq$ (resp. $\geq$)$r_Y(t)$ for all $t \geq 0$ or equivalently if $\frac{G(t)}{F(t)}$ is increasing (resp. decreasing) in $t \geq 0$.
(iv) usual stochastic order (denoted as $X \leq_{st}$ (resp. $\geq_{st}$)Y) whenever $\bar{F}(t) \leq$ (resp. $\geq$)$\bar{G}(t)$ for all $t \geq 0$.

The stochastic orders in Definition 1 are connected as indicated in the following chain:

$$\begin{array}{ccc} X \leq_{lr} (\geq_{lr})Y & \longrightarrow & X \leq_{hr} (\geq_{hr})Y \\ \Downarrow & & \Downarrow \\ X \leq_{rh} (\geq_{rh})Y & \longrightarrow & X \leq_{st} (\geq_{st})Y. \end{array}$$

2.4. Dependencies and copulas

The notion of dependence is quantified by some inequalities that represent some well-known concepts of dependence from weakest to strongest in the bivariate case [8, 11, 43]. The following definition introduces totally positive (reverse regular) of order two functions.

Definition 2. *A non-negative real valued bivariate function h is said to be totally positive of order 2 (TP_2) in $(x, y) \in \chi \times \Upsilon$, whenever*

$$h(x_1, y_1)h(x_2, y_2) \geq h(x_1, y_2)h(x_2, y_1), \tag{2.10}$$

for all $x_1 \leq x_2 \in \chi$ and $y_1 \leq y_2 \in \Upsilon$, where χ and γ are two arbitrary subsets of the real line R. If the orientation of the inequality in (2.10) is reversed, then h is called the reverse regular of order 2 (RR_2) in $(x, y) \in \chi \times \Upsilon$

Next, a number of partial dependencies from [38] are presented.

Definition 3. *Suppose that (X, Y) is a random pair with the joint cdf F, the joint sf $\bar{F}$ and the joint pdf f. We write F_X and F_Y for marginal cdf of X and marginal cdf of Y, respectively.*

(i) The random variables X and Y are said to be positive (negative) likelihood ratio dependent, PLRD (NLRD), if $f(x, y)$ is TP_2 (RR_2) in $(x, y) \in \{(x, y) \mid f(x, y) > 0\}$.
(ii) The random variable Y is stochastically increasing (decreasing) in X, SI($Y \mid X$) (SD($Y \mid X$)) if $P(Y > y \mid X = x)$ is increasing (decreasing) in x, for all y.
(iii) The random variables X and Y are left corner set decreasing (increasing), LCSD(X, Y) [LCSI(X, Y)] if $F(x, y)$ is TP_2 (RR_2) in $(x, y) \in \{(x, y) : F(x, y) > 0\}$.
(iv) The random variables X and Y are right corner set increasing (decreasing), RCSI(X, Y) (RCSD(X, Y)) if $\bar{F}(x, y)$ is TP_2 (RR_2) in $(x, y) \in \{(x, y) : \bar{F}(x, y) > 0\}$.

(v) The random variables X and Y are positive (negative) quadrant dependent, PQD(X, Y) [NQD(X, Y)) whenever $F(x, y) \geq (\leq) F_X(x)F_Y(y)$ *for all* $(x, y) \in \mathbb{R}^2$.

The dependence structures in Definition 3 are connected as:

$$SI(Y|X) \iff PLRD(X, Y) \implies LCSD(X, Y), RCSI(X, Y) \implies NQD(X, Y).$$

$$SD(X|Y) \iff NLRD(X, Y) \implies LCSI(X, Y), RCSD(X, Y) \implies PQD(X, Y).$$

The PLRD (NLRD), the RCSI (RCSD), the LCSD (LCSI) and the SI (SD) as well as the PQD (NQD) structures for the mixing random variable Θ and the overall population random variable X^* which follows the cdf (2.3) are characterized in Subsection 3.1 by the stochastic orders given in Subsection 2.3.

The copula function provides a unique concept of dependence that is completely detached from all other characteristics of the underlying marginal distributions. The copula function enables the dependence structure to be separated from the marginal distributions (see, e.g., [9, 20, 21, 30, 38]. The copula is a measure playing a central role in modelling the dependence between the components of a random pair. More precisely, let X and Y be continuous random variables with joint distribution function H and marginal distribution functions F and G, respectively. The copula C associated with the random pair (X, Y) is the joint distribution function of the uniform pair (U, V) such that $U = F(X)$ and $V = G(Y)$. According to the well-known Sklar's theorem, the copula can express in terms of the joint and marginal distribution functions as follows:

$$C(u, v) = H(F^{-1}(u), G^{-1}(v)), \quad \text{for all } (u, v) \in [0, 1]^2,$$

where F^{-1} and G^{-1} denote the inverse functions of F and G. Notice that the previous formula is equivalent to

$$H(x, y) = C(F(x), G(y)), \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

Let $\bar{H}$, $\bar{F}$ and $\bar{G}$ denote the survival functions of (X, Y), X and Y, respectively. The previous formula leads to

$$\bar{H}(x, y) = \hat{C}(\bar{F}(x), \bar{G}(y)), \quad \text{for all} (x, y) \in \mathbb{R}^2.$$

The function $\hat{C}$ is called the survival copula. It is linked to C via the next formula

$$\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v), \quad \text{for all } (u, v) \in [0, 1]^2.$$

Remark that C and $\hat{C}$ coincide with independent copula when X and Y are independent, that is,

$$C(u, v) = \hat{C}(u, v) = \Pi(u, v) = uv, \quad \text{for all } (u, v) \in [0, 1]^2.$$

It is well-known that for any copula C and for all $u, v \in [0, 1]$,

$$\max(u + v - 1, 0) \leq C(u, v) \leq \min(u, v).$$

previous bivariate functions $W(u, v) = \max(u+v-1, 0)$ and $M(u, v) = \min(u, v)$ are themselves copulas called Fréchet-Hoeffding bounds. For more on the construction of these bounds, see [13] and for more details on the notion of copula, we refer the interested reader to [38]

3. Main results

This section presents the novelties of the paper including characterizations of partial dependencies enumerated in Definition 3 among two random variables contributed to the mixture model (2.3) using stochastic orders given in Subsection 2.3. The copula function which identifies completely (not regularly or partially), the dependency phenomenon in the mixture semiparametric model (2.8) is derived.

3.1. Characterizations of dependencies in the mixture weighted distribution

Based on the mixture model (2.3), the magnitude the random variable X^* has is affected by that of the random variable Θ. The model (2.3) implies that $F^*(x) = E[F(x;\Theta)]$ where $F(\cdot;\theta)$ is the conditional cdf of X^* given $\Theta = \theta$. Thus the probability of X^* being smaller than x depends on the average of probabilities of X^* given Θ is less or equal than x where the average is taken over all possible values of Θ. For instance, F^* may be the income distribution in a population where Θ is the age of a randomly drawn individual. F^* may be, as another example, the lifetime distribution of an electrical used device so that Θ is the duration during which it has been inactive since the last time it has been in use. In such circumstances, characterizations of the dependencies between X^* and Θ and also a need for the associated copula function are considered to be significant. The association (or the dependence structure) X^* and Θ has can be assessed through the bivariate density function

$$f^*(x,\theta) = f^*(x|\theta)d\Lambda(\theta) = \frac{w(x;\theta)f_X(x)}{\eta(\theta)}\lambda(\theta), \tag{3.1}$$

in which the statement inserted after the second identity is valid when Θ has an absolutely continuous cdf Λ with pdf λ. In (3.1), $f^*(x|\theta) = \frac{w(x,\theta)}{\eta(\theta)}f_X(x)$ is the conditional density of X^* given that Θ equals θ.

Since the joint pdf (3.1) is a bivariate weighted distribution with original distribution $g(x,\theta) = f_X(x)\lambda(\theta)$ representing the case when X and Θ are independent, thus the strongest dependence concept between X^* and Θ follows from Theorem 1 in [16]. Formally,

$$\text{if } w(x,\theta) \text{ is } TP_2\ (RR_2) \text{ in } (x,\theta), \text{ then } X^* \text{ and } \Theta \text{ are PLRD (NLRD).}$$

Generally, in 2016, Izadkhah et al. [16] proved that under some sufficient conditions the partial dependencies given in Definition 3 are preserved under the transformation $(X,Y) \mapsto (X_w,Y_w)$ in which (X,Y) has a joint pdf $f(x,y)$ associated with the original distribution and (X_w,Y_w) has a joint pdf $f_w(x,y) = \frac{w(x,y)}{E[w(X,Y)]}f(x,y)$. In this paper, (X,Θ) is considered as the random pair with original distribution in which X and Θ have been assumed to be independent. However, our goal is not here to obtain sufficient condition(s) for preservation of a dependence structure under $(X,\Theta) \mapsto (X^*,\Theta)$. The first aim of the current study is to present some necessary and sufficient conditions for (X^*,Θ) to follow the dependence structures (i)–(v) in Definition 3. The equivalent conditions we will present to this end are particularly useful to obtain the dependency induced by the model (2.8) between X^* and Θ.

Suppose that F_X^{-1} is the right continuous inverse function of F_X given by $F_X^{-1}(u) = \inf\{\tau | F_X(\tau) \geq u\}$ for $u \in [0,1]$. Suppose that U_θ is a random variable contained in $[0,1]$ having cdf

$$G_\theta(u) = \frac{\int_0^u w(F_X^{-1}(y);\theta)dy}{\int_0^1 w(F_X^{-1}(y);\theta)dy}. \tag{3.2}$$

Therefore, the cdf of the mixture model (2.4) when $F(x;\theta) = \int_{-\infty}^{x} f(t;\theta)d\theta$ where $f(t;\theta)$ is as given in (2.2), is represented as

$$F^*(x) = \int_{-\infty}^{+\infty} F(x;\theta)d\Lambda(\theta)$$

$$= \int_{-\infty}^{+\infty} \frac{\int_{-\infty}^{x} w(t;\theta)f_X(t)dt}{\int_{-\infty}^{+\infty} w(t;\theta)f_X(t)dt}\, d\Lambda(\theta)$$

$$= \int_{-\infty}^{+\infty} \frac{\int_{0}^{F_X(x)} w\left(F_X^{-1}(y);\theta\right)dy}{\int_{0}^{1} w(F_X^{-1}(y);\theta)dy}\, d\Lambda(\theta)$$

$$= \int_{-\infty}^{+\infty} G_\theta(F_X(x))\, d\Lambda(\theta). \tag{3.3}$$

The density function of X^* is also

$$f^*(x) = \int_{-\infty}^{+\infty} f_X(x)g_\theta(F_X(x))\, d\Lambda(\theta), \tag{3.4}$$

where $g_\theta(\cdot)$ is the derivative (density) of $G_\theta(\cdot)$. Let $w(\cdot;\theta)$ be the weight function in the family of the weighted distributions (2.3), i.e. $w(x;\theta) = \xi(F_X(x),\theta)$ which is prevalently arisen with many statistical models. In such a case the cdf G_θ in (3.2) does not depend on F_X and plays the role of generator of the cdf of X^* as revealed in (3.3). In fact, in the cases where a semiparametric family as considered in Subsection 2.2 is given, we will have $G_\theta(u) = d(u,\theta)$.

There are studies conducted in the literature where the dependencies in the pair (X, Y) in Definition 3 are characterized by some stochastic orders, namely, likelihood ratio order ($\leq_{lr}$), hazard rate order ($\leq_{hr}$), reversed hazard rate order ($\leq_{rh}$) and usual stochastic order ($\leq_{st}$) of conditional random variables $[Y|X = x]$, $[Y|X > x]$ and $[Y|X \leq x]$ (see, for instance, Lemma 1 in [16]).

The next result identifies necessary and sufficient conditions for the dependency structures in Definition 3 in the random pair (X^*, Θ) based on the generator (3.2). The class of weighted distributions corresponding to the semiparametric class of distributions can be also considered as a typical application. The distinctive dependencies given in Definition 3 are characterized with stochastic orders of U_θ with respect to θ as follows:

Theorem 4. *The following assertions hold:*

(i) X^* *and* Θ *are PLRD (NLRD) if, and only if,* $U_{\theta_1} \leq_{lr} (\geq_{lr})\, U_{\theta_2}$ *for all* $\theta_1 \leq \theta_2 \in \chi$.
(ii) X^* *and* Θ *are RCSI (RCSD) whenever* $U_{\theta_1} \leq_{hr} (\geq_{hr})\, U_{\theta_2}$ *for all* $\theta_1 \leq \theta_2 \in \chi$.
(iii) X^* *and* Θ *are LCSD (LCSI) whenever* $U_{\theta_1} \leq_{rh} (\geq_{rh})\, U_{\theta_2}$ *for all* $\theta_1 \leq \theta_2 \in \chi$.
(iv) X^* *is SI (SD) in* Θ *if, and only if,* $U_{\theta_1} \leq_{st} (\geq_{st})\, U_{\theta_2}$ *for all* $\theta_1 \leq \theta_2 \in \chi$.

Proof.

(i) By virtue of (3.2), U_θ has the pdf

$$g_\theta(u) = \frac{w(F_X^{-1}(y);\theta)}{\int_0^1 w(F_X^{-1}(y);\theta)dy}.$$

It is known that X^* given $\Theta = \theta$ has pdf

$$\begin{aligned} f^*(x|\theta) &= \frac{w(x;\theta)f_X(x)}{\eta(\theta)} \\ &= \frac{w(x;\theta)f_X(x)}{\int_0^1 w(F_X^{-1}(y);\theta)dy} \\ &= g_\theta(F_X(x))f_X(x). \end{aligned}$$

Notice that X^* and Θ are PLRD (NLRD) whenever $f^*(x \mid \theta)$ is TP_2 (RR_2) in (x,θ). On the other hand, it is apparently evident that $U_{\theta_1} \leq_{lr} (\geq_{lr}) U_{\theta_2}$ for all $\theta_1 \leq \theta_2$ if, and only if, $g(u \mid \theta)$ is TP_2 (RR_2) in $(u,\theta) \in [0,1] \times \chi$. By the consequences B.2. and B.3. of Definition B.1. in [34], the last property holds if, and only if, $g(F_X(x)|\theta)f_X(x)$ is TP_2 (RR_2) in $(x,\theta) \in \mathbb{R} \times \chi$. The proof of (i) is complete.

(ii) Let us write for the joint cdf of X^* and Θ

$$\begin{aligned} \bar{F}^*(x,\theta) &= \int_\theta^{+\infty}\int_x^\infty \frac{w(x';\theta')f_X(x')}{\eta(\theta')}\lambda(\theta')\,dx'd\theta' \\ &= \int_\theta^{+\infty}\int_{F_X(x)}^1 \frac{w(F_X^{-1}(y);\theta')}{\eta(\theta')}\lambda(\theta')\,dyd\theta' \\ &= \int_\theta^\infty (1 - G(F_X(x)|\theta'))\lambda(\theta')\,d\theta' \\ &= \int_{\mathbb{R}} \bar{G}(F_X(x)|\theta')\lambda(\theta')I[\theta > \theta']\,d\theta'. \end{aligned}$$

Note that $U_{\theta_1} \leq_{hr} (\geq_{hr}) U_{\theta_2}$ for all $\theta_1 \leq \theta_2 \in \chi$ if, and only if, $\bar{G}_{\theta'}(u)$ is TP_2 (RR_2) in $(u,\theta') \in [0,1] \times \chi$ which further implies that $\bar{G}_{\theta'}(F_X(x))$ is TP_2 (RR_2) in $(x,\theta') \in \mathbb{R} \times \chi$. We know that $\lambda(\theta')I[\theta > \theta']$ is TP_2 in (θ,θ'). Therefore, by the well-known general composition theorem of [22], $\bar{F}^*(x,\theta)$ is TP_2 (RR_2) in $(x,\theta) \in \mathbb{R} \times \chi$. This completes the proof of (ii).

(iii) The proof is similar to (ii).

(iv) It is plain to see that $U_{\theta_1} \leq_{st} (\geq_{st}) U_{\theta_2}$ for all $\theta_1 \leq \theta_2 \in \chi$ if, and only if, $G_{\theta_1}(u) \geq (\leq) G_{\theta_2}(u)$, for all $\theta_1 \leq \theta_2 \in \chi$ and for all $u \in [0,1]$, or equivalently, $G_{\theta_1}(F_X(x)) \geq (\leq) G_{\theta_2}(F_X(x))$, for all $\theta_1 \leq \theta_2 \in \chi$ and for all $x \in \mathbb{R}$. To establish the SI (SD) property, we first observe that

$$\begin{aligned} \bar{F}^*(x|\theta) &= \int_x^\infty \frac{w(x';\theta)f_X(x')}{\eta(\theta)}dx' \\ &= \int_{F_X(x)}^1 \frac{w(F_X^{-1}(y);\theta)}{\eta(\theta)}dy \\ &= 1 - \int_0^{F_X(x)} \frac{w(F_X^{-1}(y);\theta)}{\eta(\theta)}dy \\ &= 1 - G_\theta(F_X(x)). \end{aligned}$$

The proof of (iv) is complete as a direct consequence of Definition 3.2 (ii). □

Remark 5. *To establish the PQD (NQD) property, using the statement given for the joint survival function of X^* and Θ in the proof of Theorem 4(ii), we conclude that X^* and Θ are PQD (NQD) if, and only if, $E[\bar{G}_\Theta(u) \mid \Theta > \theta] \geq (\leq)\ E[\bar{G}_\Theta(u)]$, holds for all $u \in [0, 1]$ and for all $\theta \in \mathbb{R}$.*

Examples of reputable statistical mixture (or frailty) models is taken into consideration in Appendix to construct the distribution function (3.2) and examine the correctness of the results reported in Theorem 4 (see Examples A.1–A.10 in Appendix).

Remark 6. *It is generally evident that the generator distribution function G_θ in (3.2) does not depend on the baseline distribution function F_X when considering the mixture model of a classical semiparametric family of distributions presented in Section 2. In this case, the distribution function G_θ in (3.2) and the generator of the underlying semiparametric distribution in (2.2) actually coincide.*

3.2. Derivation of copula functions in the mixture semiparametric model

The study of copulas in a variety of mixture models and also analysis of joint frailty-copula models have been recently considered (see, e.g., [3, 5, 10, 28, 36, 46]). For evaluating the influence the mixing random variable Θ has on variation of the resultant random variable X^* in the weighted mixture model (2.3), identification of the partial dependence structures enumerated in Definition 3 between Θ and X^* has its limitations. This is because there may be situations where these dependencies are not fulfilled. Drawing the copula function out is, therefore, important as the copula is a principal reference to provide a comprehensive analysis of dependency in the pair (X^*, Θ). Since X^* follows a mixture model, thus the implied copula function is supposed to be mixture-based.

To obtain the mixture-based copula function associated with the pair (X^*, Θ) specified in the mixture weighted model (2.4), we recall the notion of copula function.

The joint distribution function of (X^*, Θ) is needed. From (1.4), the joint distribution function is

$$\begin{aligned} F^*(x, \theta) &= \int_{-\infty}^{\theta} \int_{-\infty}^{x} \frac{w(x', \theta') f_X(x')}{\eta(\theta')} \lambda(\theta')\, dx' d\theta' \\ &= \int_{-\infty}^{\theta} \frac{\lambda(\theta')}{\eta(\theta')} \left(\int_{-\infty}^{x} w(x', \theta') f_X(x') dx' \right) d\theta' \\ &= \int_{-\infty}^{\theta} \frac{\lambda(\theta')}{\eta(\theta')} \left(\int_{0}^{F_X(x)} w(F_X^{-1}(y), \theta') dy \right) d\theta'. \end{aligned}$$

Applying Sklar's theorem for distribution functions, one obtains the copula function of the joint distribution of X^* and Θ, as a result

$$\begin{aligned} C(u, v) &= F^*(F_{X^*}^{-1}(u), \Lambda^{-1}(v)) \\ &= \int_{-\infty}^{\Lambda^{-1}(v)} \frac{\lambda(\theta')}{\eta(\theta')} \int_{0}^{F_X(F_{X^*}^{-1}(u))} w(F_X^{-1}(y); \theta') dy d\theta' \end{aligned}$$

$$= \int_{-\infty}^{\Lambda^{-1}(v)} \lambda(\theta')G_{\theta'}(F_X(F_{X^*}^{-1}(u)))d\theta'$$

$$= vE[G_\Theta(F_X(F_{X^*}^{-1}(u)))|\Theta \leq \Lambda^{-1}(v)], for\ \ all\ \ u,v \in [0,1]. \quad (3.5)$$

On the other hand, the joint survival function of (X^*, Θ) as given in the proof of Theorem 4 (ii) is

$$\bar{F}^*(x,\theta) = \int_\theta^\infty \frac{\lambda(\theta')}{\eta(\theta')}\left(\int_{F_X(x)}^1 w(F_X^{-1}(y),\theta')dy\right)d\theta'.$$

Applying Sklar's theorem for survival functions, the survival copula of X^* and Θ is derived as

$$\widehat{C}(u,v) = \bar{F}^*(\bar{F}_{X^*}^{-1}(u), \bar{\Lambda}^{-1}(v))$$

$$= \bar{F}^*(F_{X^*}^{-1}(1-u), \Lambda^{-1}(1-v))$$

$$= \int_{\Lambda^{-1}(1-v)}^{+\infty} \frac{\lambda(\theta')}{\eta(\theta')} \int_{F_X(F_{X^*}^{-1}(1-u))}^1 w(F_X^{-1}(y);\theta')dyd\theta'$$

$$= \int_{\Lambda^{-1}(1-v)}^{+\infty} \lambda(\theta')(1 - G_{\theta'}(F_X(F_{X^*}^{-1}(1-u))))d\theta'$$

$$= v(1 - E[G_\Theta(F_X(F_{X^*}^{-1}(1-u)))|\Theta > \Lambda^{-1}(1-v)]), for\ \ all\ \ u,v \in [0,1]. \quad (3.6)$$

However, the copula function C in (3.5) and the survival copula $\widehat{C}$ in (3.6) are connected to each other as $\widehat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$ [38]. In mixture model of the underlying semiparametric family of distributions which introduced and described in Section 2, the derivation of the copula function (3.5) and/or the survival copula function (3.6) is more abstract. To this end, for $w(x,\theta) = \xi(F_X(x),\theta)$ one has

$$\begin{aligned} F_{X^*}(x) &= E[d(F_X(x),\Theta)] \\ &= \int_{-\infty}^{+\infty} d(F_X(x),\theta)d\Lambda(\theta), \end{aligned}$$

from which we can define

$$\begin{aligned} K(u) &= F_{X^*}(F_X^{-1}(u)) \\ &= \int_{-\infty}^{+\infty} d(u,\theta)d\Lambda(\theta) \\ &= \int_0^1 d(u,\Lambda^{-1}(y))dy. \end{aligned}$$

To derive an explicit and attainable expression for the composition function $F_X o F_{X^*}^{-1}$ in (3.5) and (3.6), we use the fact that the function K^{-1} and the function $F_X o F_{X^*}^{-1}$ are identical. Therefore, in view of (3.5),

$$\begin{aligned} C(u,v) &= \int_{-\infty}^{\Lambda^{-1}(v)} \lambda(\theta') G_{\theta'}(F_X(F_{X^*}^{-1}(u)))\, d\theta' \\ &= \int_{-\infty}^{\Lambda^{-1}(v)} \lambda(\theta') d(K^{-1}(u), \theta')\, d\theta' \\ &= \int_0^v d(K^{-1}(u), \Lambda^{-1}(y))\, dy, \quad \forall\, u, v \in [0,1]. \end{aligned} \tag{3.7}$$

The survival copula is obtained by (3.6) as

$$\begin{aligned} \widehat{C}(u,v) &= \int_{\Lambda^{-1}(1-v)}^{+\infty} \lambda(\theta')(1 - G_{\theta'}(F_X(F_{X^*}^{-1}(1-u))))d\theta' \\ &= \int_{\Lambda^{-1}(1-v)}^{+\infty} \lambda(\theta') d(K^{-1}(1-u), \theta')\, d\theta' \\ &= \int_0^v d(K^{-1}(1-u), \Lambda^{-1}(1-y))\, dy, \quad \forall\, u, v \in [0,1]. \end{aligned} \tag{3.8}$$

Remark 7. *To extract the copula function associated with mixture models in Subsections 2.1 and 2.2, as by the Sklar's theorem the copula function does not depend on the marginal distributions of X^* and Θ, thus it suffices to move forward along an arbitrary continuous distribution for Θ. The marginal distribution function of X^* is then obtained in the spirit of (3.3) as $F_{X^*}(x) = E[G_\Theta(F_X(x))] = \int_{-\infty}^{+\infty} G_\theta(F_X(x))\, d\Lambda(\theta)$ where F_X is a given (baseline) cdf where the expectation is taking with respect to the cdf of Θ.*

Remark 8. *That the implied copula function (3.5) and also the copula function (3.7) have a closed expression depends on whether the function $K(u)$ has a closed form. It also depends on whether the integrations (3.5) and (3.7) are algebraically solvable. In general, the equation $K(u) = y$ used to determine the inverse function K^{-1} may have an exact solution or be inherently a transcendental equation. In the latter case, the desired copula function has no closed nice expression.*

Remark 9. *The approach proposed in this subsection to derive the copula function assumed that X^* and Θ in both of the mixture models (2.3) and (2.8) are continuous random variables which is a basic assumption for uniqueness of the copula function. In spite of that, there may be situations where Θ is a discrete random variable where the uniqueness of copula is a brittle assumption. To describe such a situation, consider the random sequence $X_1, X_2, \ldots$ of independent and identical random variables, then the extreme order statistic $X_{1:N} = \min\{X_1, \ldots, X_N\}$ follows the proportional hazard rates model and the extreme order statistic $X_{N:N} = \max\{X_1, \ldots, X_N\}$ follows the proportional reversed hazard rates model where N is the mixing component that needs to be counted. Methods for finding copulas in such cases have been developed in the literature recently (see, e.g., [1, 44]).*

Examples have been presented in Appendix to derive the copula function in some well-known frailty models (see Examples A.11–A.14 in Appendix).

3.3. Further application in modeling bivariate variable dependency

In general, the model (2.4) is applicable in a broader setting when knowing that there are always some weight functions that relates two densities. Let us assume that the random pair (X, Y) has a joint density $f(x, y)$ (whenever it exists) where X and Y are continues univariate random variables following marginal densities $f_X(x)$ and $f_Y(y)$, respectively. Denote by $f(x|y)$ and $g(y|x)$ the conditional densities of of X given $Y = y$ and that of Y given $X = x$, respectively. It is well-known that

$$f_X(x) = \int_{-\infty}^{\infty} f(x|y) f_Y(y)\, dy \tag{3.9}$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} g(y|x) f_X(x)\, dx \tag{3.10}$$

To exploit (3.9) and (3.10) to an advantage in terms of the model (2.4), one rewrites them as follows:

$$f_X(x) = f_Z(x) \int_{-\infty}^{\infty} \frac{w(x;y)}{\eta(y)} f_Y(y)\, dy = f_Z(x) \int_{-\infty}^{\infty} w(x;y) f_Y(y)\, dy, \tag{3.11}$$

where $w(x;y) = f(x|y)/f_Z(x)$ and $\eta(y) = \int_{-\infty}^{\infty} w(x;y) f_Z(x)\, dx = 1$ where f_Z is the pdf of Z which is assumed to be a random variable with an absolutely continuous distribution function. In parallel, we analogously have

$$f_Y(y) = f_W(y) \int_{-\infty}^{\infty} \frac{w^*(y;x)}{\eta^*(y)} f_X(x)\, dx = f_W(y) \int_{-\infty}^{\infty} w^*(y;x) f_X(x)\, dx, \tag{3.12}$$

where $w^*(y;x) = g(y \mid x)/f_W(y)$ and $\eta^*(y) = \int_{-\infty}^{\infty} w^*(y;x) f_X(y)\, dy = 1$ where f_W is the pdf of W which is assumed to be a random variable with an absolutely continuous distribution function. It is noticeable that (3.11) applies when there exists a random variable Z for which $(X \mid Y = y)$ is equal in distribution with $Z_{w(\cdot;y)}$ which is the weighted version of Z with y-indexed weight function $w(\cdot;y)$. The representation (3.12) also applies when there exists a random variable W for which $(Y \mid X = x)$ is equal in distribution with $W_{w(\cdot;x)}$ which is the weighted version of W with x-indexed weight function $w(\cdot;x)$. For instance, in light of the results of the paper, the bivariate distributions for which there exists a random variable Z with cdf F_Z and a function $\phi(\cdot;y) \geq 0$ such that

$$P(X \leq x|Y = y) = \phi(F_Z(x); y)$$

and the bivariate distributions for which there exists a random variable W with cdf F_W and a function $\psi(\cdot;x) \geq 0$ that fulfills the relation

$$P(Y \leq y|X = x) = \psi(F_W(y); x), \tag{3.13}$$

are applicable and the formula (3.7) gives the copula function of (X, Y) in such cases. There are some data sets whose observations are distribution functions rather than the single numerical point value of classical data [45]. Consider a situation where the regression of $F_Z(Y)$ on $g(X)$ is carried out and the linear form

$$F_W(Y_i) = \alpha + \beta g(X_i) + \varepsilon_i,\ i = 1, 2, \ldots, n$$

is under consideration in which $\varepsilon_i \sim N(0, \sigma^2)$ and α and β are two real-valued parameters and g is a proper function. It is ordinarily assumed that X_i and ε_i are independent. To characterize the copula function of (X, Y) the approach proposed in this paper can be adopted. We have

$$\begin{aligned} P(Y \leq y | X = x) &= P(F_Z^{-1}(\alpha + \beta g(X)) \leq y | X = x) \\ &= P(\varepsilon_i \leq F_W(y) - \alpha - \beta g(X) | X = x) \\ &= \Phi(\sigma(F_W(y) - \alpha - \beta g(x))) \\ &= \psi_{\alpha,\beta,\sigma}(F_W(y); x), \end{aligned}$$

where Φ is the cdf of standard normal distribution and $\psi_{\alpha,\beta,\sigma}$ is the function fulfilling the identity (3.13).

It is noticeable that given the value of y the weight $w(x; y)$ is a weight function on the pdf of X and on the other hand given the value of x the weight $w^*(y; x)$ is a weight function onpdf of Y. Therefore, the realization y from Y and the realization x from X play the role of the parameter θ in the mixture weighted distribution (2.4). The model is, therefore, can be applied in the context of any bivariate distribution. The random pair (X, Y) with baseline Z and the random pair (Y, X) with baseline W can be considered in place of (X^*, Θ) so that the selection of distribution of Z and the selection of distribution of W are at the disposal of the model. The objective of putting (X, Y) in the framework of the mixture model (2.4) is to have a representation of the associated copula function in situations where it does not have a closed form or situations in which the Sklar's theorem does not provide the expression of the copula function.

4. Conclusions

In this paper, we have achieved two goals. Firstly, we have established necessary and sufficient conditions for the partial dependence structures PLRD (NLRD), RCSI (RCSD), LCSD (LCSI), SI (SD) and PQD (NQD) with respect to the stochastic orders $\leq_{lr}$,$\leq_{hr}$, $\leq_{rh}$ and $\leq_{st}$ between two parameter-indexed copies of a random variable from a baseline generator distribution (see, cdf (3.2)). It was shown that the strongest dependence property holds if the parameter-indexed weight function $w(x, \theta)$ has a regular form that is TP_2 for positive dependence (or RR_2 for alternative negative dependence). It is shown that the situation described in this theorem is applicable in the context of proportional hazards (reversed), proportional odds, upper (lower) tail, residual life (inactivity time) models, and scale change model when the model is mixed and the parameter for the individual in the resulting samples is randomly drawn. The weaker dependence structures are also characterized in the mixed model of residual lifetime (inactivity time) and the mixed model of scale change by several known aging notions.

Secondly, since each bivariate distribution function induces a particular copula function, the copula function of the random pair was derived from the mixture (unobservable) and the resultant (overall population). Such an investigation seems to be important and plays an essential role in the literature to focus on further aspects of dependencies using the implicit copula functions. It was found that in some semiparametric distribution families, including the very well-known proportional hazard and proportional reverse hazard models, the copula functions have explicit closed forms.

In the future study of this work, mixture-based copula functions of weighted distributions in high (more than two) dimensions will be considered and also possible extensions of the results of this paper to the multivariate case will be gone over. Further analysis of dependence characteristics using the derived bivariate copula function and the presented multivariate copula function is performed. The authors are quite optimistic about the goal of the current study, which will be pursued by many researchers in this field.

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Conflict of interest

There is no conflict of interest declared by the authors.

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Appendix

Example A.1 (Proportional hazards model with random effects). Suppose that $w(x;\theta) = \bar{F}^{\theta-1}(x), \theta > 0$. It is seen that $(\partial^2/\partial\theta\partial x)\ln(w(x;\theta)) = -h_X(x) < 0$ for all $x \geq 0$ and $\theta > 0$ where h_X is the hazard rate function of X given by $h_X(x) = f(x)/\bar{F}(x)$. Hence, $w(x;\theta)$ is RR_2 in (x,θ) and thus from Theorem 3.1, we conclude that X^* with sf $E[\bar{F}_X^{\Theta}(x)]$ and Θ admit the Nlrd property. In addition, to determine the distribution of U_θ, (3.2) provides that $G_\theta(u) = 1 - (1-u)^\theta$ which is the cdf of $Beta(1,\theta)$. It is trivially apparent that $U_{\theta_1} \geq_{lr} U_{\theta_2}$ for all $0 < \theta_1 < \theta_2$ by which the result of Theorem 4 (i) is validated.

Example A.2 (Proportional reversed hazards model with random effects). Suppose that $w(x;\theta) = F^{\theta-1}(x), \theta > 0$. It is obtained that $(\partial^2/\partial\theta\partial x)\ln(w(x;\theta)) = \widetilde{h}_X(x) > 0$ for all $x \geq 0$ and $\theta > 0$ where $\widetilde{h}_X$ is the reversed hazard rate function of X, defined as $\widetilde{h}_X(x) = f(x)/F(x)$.Hence, $w(x;\theta)$ is TP_2 in (x,θ) which implies that X^* with cdf $E[F_X^{\Theta}(x)]$ and Θ have the PLRD property. Furthermore, to identify the distribution of U_θ, (3.2) gives $G_\theta(u) = u^\theta$ which is the cdf of $Beta(\theta,1)$. It can simply be checked that $U_{\theta_1} \leq_{lr} U_{\theta_2}$ for all $0 < \theta_1 < \theta_2$ fulfilling the result of Theorem 4 (i).

Example A.3 (Proportional odds model with random tilt parameter). Let us assume that $w(x;\theta) = 1/(1-\bar{\theta}\bar{F}(x))^2, \theta > 0$ with $\bar{\theta} = 1-\theta$. It can be seen that $(\partial^2/\partial\theta\partial x)\ln(w(x;\theta)) = 2f_X(x)/(1-\bar{\theta}\bar{F}_X(x))^2 > 0$ for all $x \geq 0$ and for all $\theta > 0$. Therefore, $w(x;\theta)$ is TP_2 in (x,θ) and therefore, X^* with cdf $E[F_X(x)/(1-\bar{\Theta}\bar{F}_X(x))^2]$ and Θ satisfy the PLRD property. To characterize the distribution of U_θ by (3.2), we have $G_\theta(u) = u/(1-\bar{\theta}(1-u))$. The stochastic ordering relation $U_{\theta_1} \leq_{lr} U_{\theta_2}$ for all $0 < \theta_1 < \theta_2$ holds true and Theorem 4 (i) is valid.

Example A.4 (Upper tail mixture model). Let us assume that $w(x;\theta) = I[x > F_X^{-1}(\theta)]$ where $\theta \in [0,1]$. For all $x_1 \leq x_2$ and for all $\theta_1 \leq \theta_2 \in [0,1]$, by considering all possible ordering arrangements between x_1 and x_2 together with θ_1 and θ_2 it holds that $I[x_1 > F_X^{-1}(\theta_1)]I[x_2 > F_X^{-1}(\theta_2)] \geq I[x_1 > F_X^{-1}(\theta_2)]I[x_2 > F_X^{-1}(\theta_1)]$. It thus follows that $w(x;\theta) = I[x > F_X^{-1}(\theta)]$ is TP_2 in (x,θ) and thus, X^* with pdf $f_X(x)\int_0^{F_X(x)}(1-\theta)d\Lambda(\theta)$ and Θ fulfill the PLRD structure. The distribution of U_θ in (3.2) is determined as

$$G_\theta(u) = \frac{u-\theta}{1-\theta}I[u > \theta],$$

and thus U_θ has uniform distribution on $[\theta,1]$. The density function of U_θ is $g_\theta(u) = I[\theta \leq u \leq 1]/(1-\theta)$ which for all $u_1 \leq u_2$ and for all $\theta_{\leq}\theta_2$ satisfies

$$\begin{aligned} g_{\theta_1}(u_1)g_{\theta_2}(u_2) &= \frac{I[\theta_1 \leq u_1 \leq 1]I[\theta_2 \leq u_2 \leq 1]}{(1-\theta_1)(1-\theta_2)} \\ &\geq \frac{I[\theta_2 \leq u_1 \leq 1]I[\theta_1 \leq u_2 \leq 1]}{(1-\theta_1)(1-\theta_2)} \\ &= g_{\theta_2}(u_1)g_{\theta_1}(u_2), \end{aligned}$$

thus, $g_\theta(u)$ is TP_2 in $(u,\theta) \in [0,1]^2$. Hence, $U_{\theta_1} \leq_{lr} U_{\theta_2}$ for all $0\theta_1 < \theta_2 \in [0,1]$ which confirms Theorem 4 (i).

Example A.5 (Lower tail mixture model) Assume that $w(x;\theta) = I[x \leq F_X^{-1}(\theta)]$ where $\theta \in [0,1]$. It can be observed as previous case that $w(x;\theta) = I[x \leq F_X^{-1}(\theta)]$ is TP_2 in (x,θ) and thus it follows that that X^* with pdf $f_X(x)\int_{F_X(x)}^{1}(1/\theta)d\Lambda(\theta)$ and Θ are PLRD. In this case, the distribution of U_θ is $G_\theta(u) = \min\{u/\theta, 1\}$ and thus U_θ has uniform distribution on $[0,\theta]$. The associated density function is $g_\theta(u) = I[0 \leq u \leq \theta]/\theta$. It is readily seen again that $g_\theta(u)$ is TP_2 in $(u,\theta) \in [0,1]^2$. Hence, $U_{\theta_1} \leq_{lr} U_{\theta_2}$ for all $0 < \theta_1 < \theta_2$ which fulfils Theorem 4 (i).

In the following mixture models unlike the foregoing cases where the generator distribution G_θ is independent of F_X, they depend on F_X.

Example A.6 (Right truncation mixture model). Assume that $w(x;\theta) = I[x > \theta]$ [28]. For all $x_1 \leq x_2$ and for all $\theta_1 \leq \theta_2$, by considering all possible ordering arrangements between x_1 and x_2 together with θ_1 and θ_2 one has $I[x_1 > \theta_1]I[x_2 > \theta_2] \geq I[x_1 > \theta_2]I[x_2 > \theta_1]$. Hence, $w(x;\theta) = I[x > \theta]$ is TP_2 in (x,θ) and from thus, X^* with pdf $f_X(x)\int_{-\infty}^{x}(1/\bar{F}_X(\theta))d\Lambda(\theta)$ and Θ satisfy the PLRD property. To identify the distribution of U_θ as given in (3.2), we get

$$G_\theta(u) = \frac{u - \min\{u, F_X(\theta)\}}{\bar{F}_X(\theta)},$$

and thus U_θ has uniform distribution on $[F_X(\theta), 1]$ provided that $F_X(\theta) < 1$. The density function of U_θ is $g_\theta(u) = I[F_X(\theta) \leq u \leq 1]/(\bar{F}_X(\theta))$ which for all $u_1 \leq u_2$ and for all $\theta_1 \leq \theta_2$ satisfies

$$\begin{aligned} g_{\theta_1}(u_1)g_{\theta_2}(u_2) &= \frac{I[u_1 > F_X(\theta_1)]I[u_2 > F_X(\theta_2)]}{\bar{F}_X(\theta_1)\bar{F}_X(\theta_2)} \\ &\geq \frac{I[u_1 > F_X(\theta_2)]I[u_2 > F_X(\theta_1)]}{\bar{F}_X(\theta_1)\bar{F}_X(\theta_2)} \\ &= g_{\theta_2}(u_1)g_{\theta_1}(u_2), \end{aligned}$$

which establishes that $g_\theta(u)$ is TP_2 in (u,θ). Hence, the stochastic ordering relation $U_{\theta_1} \leq_{lr} U_{\theta_2}$ for all $0 < \theta_1 < \theta_2$ holds true which is an indication of the correctness of Theorem 4 (i).

Example A.7 (Left truncated mixture model). Let $w(x;\theta) = I[x \leq \theta]$. It is then proved that $w(x;\theta) = I[x \leq \theta]$ is TP_2 in (x,θ) and therefore, X^* with pdf $f_X(x)\int_{x}^{+\infty}(1/F_X(\theta))d\Lambda(\theta)$ and Θ have the PLRD property. To characterize the distribution of U_θ in (3.2), one realizes that $G_\theta(u) = \min\{u, F_X(\theta)\}/F_X(\theta)$ and thus U_θ follows uniform distribution on $[0, F_X(\theta)]$ provided that $F_X(\theta) > 0$. In this case, the density function of U_θ is $g_\theta(u) = I[0 \leq u \leq F_X(\theta)]/(F_X(\theta))$. It can be shown, as in the previous case, that $g_\theta(u)$ is TP_2 in (u,θ). Hence, $U_{\theta_1} \leq_{lr} U_{\theta_2}$ for all $0 < \theta_1 < \theta_2$ which is in agreement with Theorem 3.2 (i).

Example A.8 (The residual life mixture model). Let $w(x;\theta) = f_X(x+\theta)/f_X(x)$ in which θ is a certain survival time [25]. It can be demonstrated that $w(x;\theta) = f_X(x+\theta)/f_X(x)$ is TP_2 (RR_2) in (x,θ) if, and only if, X has a log-convex (log-concave) density function which concludes that X^* and Θ have the PLRD (NLRD) property. The distribution of U_θ in (3.2) for the cdf F_X with $F_X(0) = 0$ and $F_X(\theta) < 1$ is derived as

$$G_\theta(u) = \frac{\int_0^u (f_X(F_X^{-1}(x)+\theta))/(f_X(F_X^{-1}(x)))\,dx}{\int_0^1 (f_X(F_X^{-1}(x)+\theta))/(f_X(F_X^{-1}(x)))\,dx}$$

$$= \frac{F_X(\theta + F_X^{-1}(u)) - F_X(\theta)}{\bar{F}_X(\theta)}$$

$$= F_\theta(F_X^{-1}(u)), \quad u \in [0, 1],$$

where F_θ is the cdf of $X_\theta = (X - \theta \mid X > \theta)$ with $F_X(\theta) < 1$. The random variable X_θ is the residual lifetime, after the time θ, of a lifetime unit whose length of life is X. The density function of U_θ is

$$g_\theta(u) = \frac{f_X(\theta + F_X^{-1}(u))}{f_X(F_X^{-1}(u))\bar{F}_X(\theta)}, \quad u \in [0, F_X(F_X^{-1}(1) - \theta)].$$

It can be seen that $g_\theta(u)$ is TP_2 (RR_2) in (u, θ) whenever X has a log-convex (log-concave) density function which further establishes that $U_{\theta_1} \leq_{lr} U_{\theta_2}$ for all $0 < \theta_1 < \theta_2$. This is a confirmation for the result of Theorem 3.2 (i).

Example A.9 (The average inactivity time model). Let $w(x; \theta) = f_X(\theta - x)/f_X(x)$ where θ is the time of observation of failure [27]. It is observed in this case that $w(x; \theta) = f_X(\theta - x)/f_X(x)$ is TP_2 (resp. RR_2) in (x, θ) if, and only if, X has a log-concave (resp. log-convex) density function and thus X^* and Θ have the PLRD (NLRD) property. The distribution of U_θ in (3.2) for the cdf F_X with $F_X(0) = 0$ and $F_X(\theta) > 0$ is derived as

$$G_\theta(u) = \frac{\int_0^u (f_X(\theta - F_X^{-1}(x)))/(f_X(F_X^{-1}(x)))\, dx}{\int_0^1 (f_X(\theta - F_X^{-1}(x)))/(f_X(F_X^{-1}(x)))\, dx}$$

$$= \frac{F_X(\theta) - F_X(\theta - F_X^{-1}(u))}{F_X(\theta)}$$

$$= F_{(\theta)}(F_X^{-1}(u)), \quad u \in [0, 1],$$

where $F_{(\theta)}$ is the cdf of $X_{(\theta)} = (\theta - X \mid X \leq \theta)$ with $F_X(\theta) > 0$. The random variable $X_{(\theta)}$ is the inactivity time of a lifetime unit with lifespan X at the time θ (see [17,23,24,29,31]). The density function of U_θ is thus obtained as

$$g_\theta(u) = \frac{f_X(\theta - F_X^{-1}(u))}{f_X(F_X^{-1}(u))F_X(\theta)}, \quad u \in [0, F_X(\theta)].$$

As a function of u and of θ, $g_\theta(u)$ is TP_2 (resp. RR_2) in (u, θ) whenever X has a log-concave (log-convex) density function which holds if, and only if, $U_{\theta_1} \leq_{lr} U_{\theta_2}$ for all $0 < \theta_1 < \theta_2$. The result of Theorem 4 (i) is once again confirmed and validated.

Example A.10 (Scale change mixture model). Let $w(x; \theta) = f_X(\theta x)/f_X(x)$ where $\theta > 0$ [33]. From Definition 2.3(2) in [7], the weight $w(x; \theta) = f_X(\theta x)/f_X(x)$ is TP_2 (RR_2) in (x, θ) if, and only if, X has a decreasing (resp. increasing) proportional likelihood ratio and thus X^* and Θ satisfy the PLRD (NLRD) property. The cdf of U_θ in (3.2) when $F_X(0) = 0$ is obtained as

$$G_\theta(u) = \frac{\int_0^u (f_X(\theta F_X^{-1}(x)))/(f_X(F_X^{-1}(x)))\, dx}{\int_0^1 (f_X(\theta F_X^{-1}(x)))/(f_X(F_X^{-1}(x)))\, dx}$$

$$= \frac{F_X(\theta F_X^{-1}(u))}{F_X(\theta F_X^{-1}(1))}.$$

The associated density function is

$$g_\theta(u) = \frac{f_X(\theta F_X^{-1}(u))}{f_X(F_X^{-1}(u))F_X(\theta F_X^{-1}(1))}, \quad u \in [0,1].$$

We see that $g_\theta(u)$ is TP_2 (RR_2) in (u,θ) whenever X has decreasing (increasing) proportional likelihood ratio which holds if, and only if, $U_{\theta_1} \leq_{lr} U_{\theta_2}$ for all $0 < \theta_1 < \theta_2$ which fulfills the result of Theorem 4 (i) again.

Example A.11 (Proportional hazards model). The generator of the semiparametric distribution in this case as recognized in Section 2 is $d(u,\theta) = 1-(1-u)^\theta$ with $\theta > 0$. Let us assume that Θ has exponential distribution with mean one. It is seen that $K(u) = -\ln(1-u)/(1-\ln(1-u))$ and thus $K^{-1}(u) = 1-\exp\{-u/(1-u)\}$ for all $u \in (0,1)$. From (3.7), since $\Lambda^{-1}(y) = -\ln(1-y)$, thus we can get

$$C(u,v) = \int_0^v (1-(1-K^{-1}(u))^{\Lambda^{-1}(y)})\,dy$$

$$= \int_0^v (1-(1-y)^{\frac{u}{1-u}})\,dy$$

$$= v + (1-u)((1-v)^{\frac{1}{1-u}}-1), \; for\ all\ \ u,v \in [0,1].$$

Example A.12 (Proportional reversed hazards model). The generator of the semiparametric distribution in this case as characterized in Section 2 is $d(u,\theta) = u^\theta$ with $\theta > 0$. As in Example 4.1 we assume again that Θ has exponential distribution with mean one. One has $K(u) = 1/(1-\ln(u))$ and it follows that $K^{-1}(u) = \exp\{(u-1)/u\}$ for all $u \in (0,1)$. From (3.7) one obtains

$$\begin{aligned} C(u,v) &= \int_0^v (K^{-1}(u))^{\Lambda^{-1}(y)})\,dy \\ &= \int_0^v (1-y)^{\frac{1-u}{u}}\,dy \\ &= u(1-(1-v)^{\frac{1}{u}}), \;\; for\ all\ \ u,v \in [0,1]. \end{aligned}$$

Example A.13 (Upper tail mixture model). The generator d in this case is $d(u,\theta) = (u-\theta)I[u > \theta]/(1-\theta)$ with $\theta \in [0,1]$. We assume that Θ has beta distribution with density function $\lambda(\theta) = 2(1-\theta)$. Hence, $\Lambda^{-1}(y) = 1-\sqrt{1-y}$. It is seen that $K(u) = u^2$ and thus $K^{-1}(u) = \sqrt{u}$ for all $u \in (0,1)$. By substituting in (3.7), the copula function is identified as

$$C(u,v) = \int_0^v \frac{\sqrt{u}-(1-\sqrt{1-y})}{1-(1-\sqrt{1-y})} I[\sqrt{u} > (1-\sqrt{1-y})]\,dy$$

$$= \int_0^{\min\{v,1-(1-\sqrt{u})^2\}} \left(1 + \frac{\sqrt{u}-1}{\sqrt{1-y}}\right) dy$$
$$= 2(1-\sqrt{u})(\sqrt{1-\min\{v, 1-(1-\sqrt{u})^2\}} - 1) + \min\{v, 1-(1-\sqrt{u})^2\}.$$

Example A.14 (Lower tail mixture model). In this case we have $d(u,\theta) = \min\{u/\theta, 1\}$ for $\theta \in [0,1]$. Suppose that Θ has beta distribution with density function $\lambda(\theta) = 2\theta$ and thus $\Lambda^{-1}(y) = \sqrt{y}$. We also see that $K(u) = 2u - u^2$ and thus $K^{-1}(u) = 1 - \sqrt{1-u}$ for all $u \in (0,1)$. By using (3.7),

$$C(u,v) = \int_0^v \min\{\frac{K^{-1}(u)}{\Lambda^{-1}(y)}, 1\}\, dy$$

$$= \int_0^v \min\{\frac{1-\sqrt{1-u}}{\sqrt{y}}, 1\}\, dy$$

$$= (1-\sqrt{1-u})(2\sqrt{v} - 1 + \sqrt{1-u})_+ + \min\{v, 1-(1-\sqrt{1-u})^2\},$$

where $a_+ = \max\{0, a\}$.