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*Research article*

# Analysis of the generalized fractional differential system

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**Abstract:** In this paper, we study the existence, uniqueness, and stability of the solution of the fractional differential system with the generalized fractional derivative. First, the solution of the generalized fractional differential system is obtained by the transformation method. Based on the fixed point theorems, we establish the existing and unique theories of the solution. Furthermore, the sufficient criteria of local stabilities of one-dimensional, two-dimensional, and *n*-dimensional linear generalized fractional differential systems are dealt with. In addition, the linearization and stability theorems of the nonlinear generalized fractional differential systems are discussed. Finally, we take the generalized fractional Chen system as an example to illustrate the correctness of the theoretical analysis.

**Keywords:** Erdélyi-Kober fractional derivative; generalized fractional derivative; existence; uniqueness; stability
**Mathematics Subject Classification:** 26A33, 34A08, 34A12, 34D20, 34D30

## 1. Introduction

It has been proved that the fractional calculus is the appropriate tool for characterizing some realistic problems in physics and engineering, such as viscoelastic, anomalous diffusion, and control [1–6]. In particular, Erdélyi-Kober fractional integral frequently appears in the description of diffusive processes governed by the generalized grey Brownian motion [7]. The Erdélyi-Kober fractional integral with order $\alpha\,(\alpha > 0)$ is defined as [8, 9]

$$\begin{aligned} {}_{EK}\mathrm{D}^{-\alpha}_{a,t;\,\sigma,\eta} f(t) &= \frac{t^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^t (t^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma\eta} f(\tau)\,\mathrm{d}\tau^\sigma \\ &= \left[ t^{-(\alpha+\eta)} {}_{RL}\mathrm{D}^{-\alpha}_{a^\sigma,t}\, t^\eta f\left(t^{\frac{1}{\sigma}}\right)\right]\Big|_{t\to t^\sigma},\ t > a \geq 0,\ \sigma > 0,\ \eta \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where $f(t)$ is an adequately smooth function. Obviously, the Erdélyi-Kober fractional integral operator ${}_{EK}\mathrm{D}^{-\alpha}_{a,t;\,\sigma,\eta}\cdot$ is an extension of the Riemann-Liouville fractional integral operator ${}_{RL}\mathrm{D}^{-\alpha}_{a,t}\cdot$. Moreover, the operator ${}_{EK}\mathrm{D}^{-\alpha}_{a,t;\,\sigma,\eta}\cdot$ can be degenerated to Kober-Erdélyi operator ${}_{EK}\mathrm{D}^{-\alpha}_{a,t;\,1,\eta}\cdot$ when $\sigma = 1$ and Erdélyi-Kober operator ${}_{EK}\mathrm{D}^{-\alpha}_{a,t;\,2,\eta}\cdot$ if $\sigma = 2$ [10–13]. Correspondingly, the Erdélyi-Kober fractional derivative of order $\alpha\,(n-1 < \alpha < n \in \mathbb{Z}^+)$ is resoundingly constructed as [8, 9, 14–16]

$$\begin{aligned} {}_{EK}\mathrm{D}^{\alpha}_{a,t;\,\sigma,\eta}\,f(t) &= t^{-\sigma\eta}\left(\frac{1}{\sigma t^{\sigma-1}}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n} t^{\sigma(\eta+n)}{}_{EK}\mathrm{D}^{-(n-\alpha)}_{a,t;\,\sigma,\alpha+\eta}\,f(t) \\ &= \left[t^{-\eta}{}_{RL}\mathrm{D}^{\alpha}_{a^{\sigma},t}\,t^{\alpha+\eta}f\left(t^{\frac{1}{\sigma}}\right)\right]\Big|_{t\to t^{\sigma}},\ t > a \geq 0,\ \sigma > 0,\ \eta \in \mathbb{R}. \end{aligned} \tag{1.2}$$

The Erdélyi-Kober fractional derivative and its fractional differential equation are widely used in Lie symmetry analysis of the time fractional generalized fifth-order KdV equation and the space-time fractional variant Boussinesq system [17, 18], which is of great theoretical significance for studying the fractional nonlinear evolution equations. Therefore, it is necessary to consider the fractional differential equations with Erdélyi-Kober fractional derivative.

For convenience in application, Kiryakova [8] modified the Erdélyi-Kober fractional integral and derivative, and which are known as the generalized fractional integral and derivative. Their expressions are as follows

$$\begin{aligned} {}_{EK}\mathfrak{D}^{-\alpha}_{a,t;\,\sigma,\eta}\,f(t) &= t^{\sigma\alpha}{}_{EK}\mathrm{D}^{-\alpha}_{a,t;\,\sigma,\eta}\,f(t) \\ &= \frac{t^{-\sigma\eta}}{\Gamma(\alpha)}\int_a^t (t^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma\eta}f(\tau)\,\mathrm{d}\tau^{\sigma} \\ &= \left[t^{-\eta}{}_{RL}\mathrm{D}^{-\alpha}_{a^{\sigma},t}\,t^{\eta}f\left(t^{\frac{1}{\sigma}}\right)\right]\Big|_{t\to t^{\sigma}},\ \alpha > 0, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\,\sigma,\eta}\,f(t) &= t^{-\sigma\alpha}{}_{EK}\mathrm{D}^{\alpha}_{a,t;\,\sigma,\eta-\alpha}\,f(t) \\ &= t^{-\sigma\eta}\left(\frac{1}{\sigma t^{\sigma-1}}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n} t^{\sigma\eta}{}_{EK}\mathfrak{D}^{-(n-\alpha)}_{a,t;\,\sigma,\eta}\,f(t) \\ &= \left[t^{-\eta}{}_{RL}\mathrm{D}^{\alpha}_{a^{\sigma},t}\,t^{\eta}f\left(t^{\frac{1}{\sigma}}\right)\right]\Big|_{t\to t^{\sigma}},\ n-1 < \alpha < n \in \mathbb{Z}^+, \end{aligned} \tag{1.4}$$

in which $t > a \geq 0$, $\sigma > 0$ and $\eta \in \mathbb{R}$. At present, there have been some studies on the integral and differential equations involving the generalized fractional integral and derivative. In [19], the authors obtained the explicit solutions to the generalized fractional integral and differential equations below

$$x(t) - \lambda\,{}_{EK}\mathfrak{D}^{-\alpha}_{0,t;\,\sigma,\eta}x(t) = f(t),\ \alpha > 0,\ t > 0,\ \sigma > 0,\ \eta \in \mathbb{R}, \tag{1.5}$$

and

$${}_{EK}\mathfrak{D}^{\alpha}_{0,t;\,\sigma,\alpha+\eta}x(t) - \lambda x(t) = f(t),\ n-1 < \alpha < n \in \mathbb{Z}^+,\ t > 0,\ \sigma > 0,\ \eta \in \mathbb{R}, \tag{1.6}$$

by using the transformation method. With the help of a generalized weakly singular integral inequality, Ma and Pečarić [20] investigated the explicit bound of the solution to the following integral equation with the generalized fractional integral

$$x^{p}(t) = f(t) + \frac{\lambda t^{-\sigma\eta}}{\Gamma(\alpha)}\int_0^t (t^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma\eta}x^{q}(\tau)\,\mathrm{d}\tau^{\sigma},\ t > 0.$$

On this basis, in [21], the authors used a directly computational method and Schauder fixed point theorem to present the existing and unique results of the solution to the following nonlinear integral equation

$$x(t) = b_1(t) + \frac{b_2(t)}{\Gamma(\alpha)} \int_0^t (t^\sigma - \tau^\sigma)^{\alpha-1} \tau^\eta f(\tau, x) \mathrm{d}\tau, \ t \in J = [0, T], \ T > 0,$$

where $\alpha$, $\sigma$ and $\eta$ are positive parameters, $b_i(t)\,(i = 1, 2) : J \to \mathbb{R}$ and $f(t, x) : J \times \mathbb{R} \to \mathbb{R}$. And, the local stability of the solution was discussed.

The existence and uniqueness of the solution play an essential role in the study of fractional differential equation [22–25]. Using Schauder and Tychonov fixed point theorems, Hadid [26] obtained the local and global existing results of the solution of the differential equation involving Riemann-Liouville fractional derivative. Li and Sarwar [27] presented the local, global existence, and continuation theorems for Caputo-type fractional differential equations. Furthermore, some works about the existing and unique studies of differential equations involving other fractional derivatives, such as $\psi$-Caputo fractional derivative, Caputo-Hadamard fractional derivative, and multi-order Erdélyi-Kober fractional derivative, have been found in [28–30]. On the basis of these studies, we give the existence and uniqueness analyses of the generalized fractional differential equation.

Stability analysis is one of the main interests for the research of dynamic systems. There are inevitably inestimable small disturbances in the process of establishing the fractional differential model, which can essentially change the stability of the solution of the fractional differential equation. Therefore, the discussion of stability has important theoretical significance and application value [31–33]. In [34], the author analyzed the stability of the linear fractional differential equations with the Caputo derivative. By using the Laplace transform, Deng et al. studied the stability of $n$-dimensional linear fractional differential equation with time delays [35]. Qian et al. established stability theorems of the zero solutions for the linear, perturbed, and time-delayed systems containing the Riemann-Liouville fractional derivative [36]. In order to determine the stability of hyperbolic equilibrium of the nonlinear system, the linearization theory was proposed in [37]. Recently, Li and Li [38, 39] took into account the stability and decay rate of linear and nonlinear fractional differential systems based on four different fractional derivatives. Besides, effective integral transformations were also provided. In the paper, we discuss the stability of the zero solutions to the linear and nonlinear generalized fractional differential systems.

The structure of the paper is as follows. In Section 2, we recall some basic definitions and properties. In Section 3, the existence and uniqueness of the solution to the generalized fractional differential equation with initial value are considered. In Section 4, we analyze the stability of the linear and nonlinear generalized fractional differential equations. In Section 5, an example explaining the theoretical result is given. The conclusion is showed in Section 6.

## 2. Preliminaries

In this section, we introduce some basic definitions and results which are needed throughout this paper.

**Lemma 1.** [2, 3] *Suppose that* $\tilde{x}\left(t^{\frac{1}{\sigma}}\right)$ *and* $\tilde{f}\left(t^{\frac{1}{\sigma}},\, \tilde{x}\right)$ *are continuous. The initial value problem with Riemann-Liouville derivative*

$$\begin{cases} {}_{RL}\mathrm{D}^{\alpha}_{a^{\sigma},t}\,\tilde{x}\left(t^{\frac{1}{\sigma}}\right) = \tilde{f}\left(t^{\frac{1}{\sigma}},\, \tilde{x}\right),\ n-1 < \alpha < n \in \mathbb{Z}^{+},\ t > a^{\sigma} \geq 0,\ \sigma > 0, \\ \left[{}_{RL}\mathrm{D}^{\alpha-j}_{a^{\sigma},t}\,\tilde{x}\left(t^{\frac{1}{\sigma}}\right)\right]\Big|_{t=a^{\sigma}} = \tilde{x}^{(j)}_{a^{\sigma}},\ j = 1, 2, \ldots, n, \end{cases} \tag{2.1}$$

*is equivalent to the following Volterra integral equation*

$$\tilde{x}(t) = \sum_{j=1}^{n} \frac{\tilde{x}^{(j)}_{a^{\sigma}}}{\Gamma(\alpha+1-j)}(t-a^{\sigma})^{\alpha-j} + \frac{1}{\Gamma(\alpha)}\int_{a^{\sigma}}^{t}(t-\tau)^{\alpha-1}\tilde{f}\left(\tau^{\frac{1}{\sigma}},\, \tilde{x}\right)\mathrm{d}\tau. \tag{2.2}$$

**Lemma 2.** *Let* $f(t) \in C[a, \infty)$*. Then*

$${}_{EK}\mathfrak{D}^{\alpha}_{a,t;\,\sigma,\eta}\,{}_{EK}\mathfrak{D}^{-\alpha}_{a,t;\,\sigma,\eta}\,f(t) = f(t),\ t > a \geq 0, \tag{2.3}$$

*where* $n-1 < \alpha < n \in \mathbb{Z}^{+}$, $\sigma > 0$ *and* $\eta \in \mathbb{R}$.

**Lemma 3.** [4] *Let* $0 < \alpha < 2$, $\beta \in \mathbb{C}$, *and* $\mu \in \mathbb{R}$ *such that* $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$*. Then, for any integer* $p \geq 1$*, the following asymptotic expansions hold, if* $|\arg(z)| \leq \mu$*, then*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}z^{\frac{1-\beta}{\alpha}}\exp(z^{\frac{1}{\alpha}}) - \sum_{k=1}^{p}\frac{z^{-k}}{\Gamma(\beta-\alpha k)} + \mathrm{O}(|z|^{-1-p}),\ |z| \to \infty; \tag{2.4}$$

*and if* $\mu \leq |\arg(z)| \leq \pi$*, then*

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p}\frac{z^{-k}}{\Gamma(\beta-\alpha k)} + \mathrm{O}(|z|^{-1-p}),\ |z| \to \infty, \tag{2.5}$$

*where* $E_{\alpha,\beta}(z)$ *is the Mittag-Leffler function.*

**Definition 1.** *The point* $x_{eq} \in \mathbb{R}^{n}$ *is known as an equilibrium of the generalized fractional differential system*

$${}_{EK}\mathfrak{D}^{\alpha}_{a,t;\,\sigma,\eta}\,x(t) = f(t,x),\ 0 < \alpha < 1,\ \sigma > 0,\ \eta \in \mathbb{R},\ x(t) \in \mathbb{R}^{n}, \tag{2.6}$$

*if* $f(t, x_{eq}) \equiv 0$ *for all* $t > a \geq 0$.

**Definition 2.** *The zero solution to system* (2.6) *with order* $\alpha$ $(0 < \alpha < 1)$ *is said to be:*
(i) *Stable, if for any initial value* $\left[t^{\sigma\eta}\,{}_{EK}\mathfrak{D}^{\alpha-1}_{a,t;\,\sigma,\eta}\,x(t)\right]\Big|_{t=a} = x_a$*, there exist* $\varepsilon > 0$ *and* $\tilde{a}$ *such that* $\|x(t)\| < \varepsilon$ *for all* $t \geq \tilde{a} > a \geq 0$*;*
(ii) *Asymptotically stable, if* $\lim_{t\to+\infty}\|x(t)\| = 0$.

For the autonomous nonlinear generalized fractional differential system

$${}_{EK}\mathfrak{D}^{\alpha}_{a,t;\,\sigma,\eta}\,x(t) = f(x),\ 0 < \alpha < 1,\ t > a \geq 0,\ \sigma > 0,\ \eta \in \mathbb{R},\ x(t) \in \mathbb{R}^{n} \tag{2.7}$$

with the initial value $\left[t^{\sigma\eta}\,{}_{EK}\mathfrak{D}^{\alpha-1}_{a,t;\,\sigma,\eta}\,x(t)\right]\Big|_{t=a} = x_a$. we can define its zero solution, i.e. equilibrium.

**Definition 3.** *The origin is an equilibrium of system* (2.7) *iff* $f(0) \equiv 0$ *for all* $t > a \geq 0$.

**Definition 4.** *Assume that the origin is an equilibrium of system* (2.7)*, and all the eigenvalues* $\lambda(f'(0))$ *of the linearized matrix* $f'(0)$ *satisfy* $|\lambda(f'(0))| \neq 0$ *and* $|\arg(\lambda(f'(0)))| \neq \frac{\pi\alpha}{2}$*, then the origin is a hyperbolic equilibrium of system* (2.7).

Let $\mathcal{V}$, $\mathcal{W} \subseteq \mathbb{R}^n$. If $f(x)$ and $g(y)$ are continuous vector fields defined on $\mathcal{V}$ and $\mathcal{W}$, respectively, and they generate flows $\phi_{t,f} : \mathcal{V} \to \mathcal{V}$ and $\phi_{t,g} : \mathcal{W} \to \mathcal{W}$. Then the definition of topological equivalence can be given as follows.

**Definition 5.** *If there exists a homeomorphism* $h : \mathcal{V} \to \mathcal{W}$ *such that* $h \circ \phi_{t,f}(x) = \phi_{t,g} \circ h(x)$ *for* $x \in \delta(x_a) \subset \mathcal{V}, \forall x_a \in \mathcal{V}$*, then* $f(x)$ *and* $g(y)$ *are locally topologically equivalent.*

## 3. Existence and uniqueness theorems

In this section, we give the solution of the generalized fractional differential equation by using the transformation method proposed in [19]. Further, the local existence and uniqueness of solutions to generalized fractional differential equations are carried out.

Motivated by [19], suppose that the linear transmutation operator $T = t^{-\eta}\,(\eta \in \mathbb{R})$. Then the following transformation relations hold

$$(1)\, T\,{}_{RL}\mathrm{D}^{-\alpha}_{a^\sigma,t}\cdot = {}_{EK}\mathfrak{D}^{-\alpha}_{a^\sigma,t;1,\eta}T\cdot; \tag{3.1}$$

$$(2)\, \Omega^{-1}\,{}_{EK}\mathfrak{D}^{\alpha}_{a^\sigma,t;1,\eta}\cdot = {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\,\Omega^{-1}\cdot, \tag{3.2}$$

where the operator $\Omega^{-1} : f(t) \to f(t^\sigma)$, $\sigma > 0$.

In order to get the solution to the generalized fractional differential equation, we introduce a lemma.

**Lemma 4.** *In* $C[a^\sigma, \infty)\,(a^\sigma > 0)$*, the following relation between two fractional derivative operators* ${}_{RL}\mathrm{D}^{\alpha}_{a^\sigma,t}\cdot$ *and* ${}_{EK}\mathfrak{D}^{\alpha}_{a^\sigma,t;1,\eta}\cdot$ *holds*

$$\left(T\,{}_{RL}\mathrm{D}^{\alpha}_{a^\sigma,t}\right)\tilde{x}\left(t^{\frac{1}{\sigma}}\right) = \left({}_{EK}\mathfrak{D}^{\alpha}_{a^\sigma,t;1,\eta}T\right)\tilde{x}\left(t^{\frac{1}{\sigma}}\right) - \sum_{j=1}^{n}\frac{\tilde{x}^{(j)}_{a^\sigma}}{\Gamma(1-j)}t^{-\eta}(t-a^\sigma)^{-j},\ j = 1,2,\ldots,n, \tag{3.3}$$

*for* $n-1 < \alpha < n \in \mathbb{Z}^+$, $t > a^\sigma \geq 0$, $\sigma > 0$ *and* $\eta \in \mathbb{R}$.

*Proof.* Using Lemma 2 and Eq (3.1), one can get that

$$\begin{aligned}
&\left({}_{EK}\mathfrak{D}^{\alpha}_{a^\sigma,t;1,\eta}T\,{}_{RL}\mathrm{D}^{-\alpha}_{a^\sigma,t}\,{}_{RL}\mathrm{D}^{\alpha}_{a^\sigma,t}\right)\tilde{x}\left(t^{\frac{1}{\sigma}}\right)\\
&= \left({}_{EK}\mathfrak{D}^{\alpha}_{a^\sigma,t;1,\eta}\,{}_{EK}\mathfrak{D}^{-\alpha}_{a^\sigma,t;1,\eta}T\,{}_{RL}\mathrm{D}^{\alpha}_{a^\sigma,t}\right)\tilde{x}\left(t^{\frac{1}{\sigma}}\right)\\
&= \left(T\,{}_{RL}\mathrm{D}^{\alpha}_{a^\sigma,t}\right)\tilde{x}\left(t^{\frac{1}{\sigma}}\right).
\end{aligned}$$

From [2], it's true that

$$\left({}_{RL}\mathrm{D}^{-\alpha}_{a^\sigma,t}\,{}_{RL}\mathrm{D}^{\alpha}_{a^\sigma,t}\right)\tilde{x}\left(t^{\frac{1}{\sigma}}\right) = \tilde{x}\left(t^{\frac{1}{\sigma}}\right) - \sum_{j=1}^{n}\frac{\tilde{x}^{(j)}_{a^\sigma}}{\Gamma(\alpha+1-j)}(t-a^\sigma)^{\alpha-j}.$$

Hence,

$$\begin{aligned}
&\left(T\ {}_{RL}\mathrm{D}^{\alpha}_{a^{\sigma},t}\right)\ \tilde{x}\left(t^{\frac{1}{\sigma}}\right)\\
&=\left({}_{EK}\mathfrak{D}^{\alpha}_{a^{\sigma},t;1,\eta}T\right)\ \tilde{x}\left(t^{\frac{1}{\sigma}}\right)-\sum_{j=1}^{n}\frac{\tilde{x}^{(j)}_{a^{\sigma}}}{\Gamma(\alpha+1-j)}\left({}_{EK}\mathfrak{D}^{\alpha}_{a^{\sigma},t;1,\eta}T\right)(t-a^{\sigma})^{\alpha-j}\\
&=\left({}_{EK}\mathfrak{D}^{\alpha}_{a^{\sigma},t;1,\eta}T\right)\ \tilde{x}\left(t^{\frac{1}{\sigma}}\right)-\sum_{j=1}^{n}\frac{\tilde{x}^{(j)}_{a^{\sigma}}}{\Gamma(\alpha+1-j)}t^{-\eta}{}_{RL}\mathrm{D}^{\alpha}_{a^{\sigma},t}(t-a^{\sigma})^{\alpha-j}\\
&=\left({}_{EK}\mathfrak{D}^{\alpha}_{a^{\sigma},t;1,\eta}T\right)\ \tilde{x}\left(t^{\frac{1}{\sigma}}\right)-\sum_{j=1}^{n}\frac{\tilde{x}^{(j)}_{a^{\sigma}}}{\Gamma(1-j)}t^{-\eta}(t-a^{\sigma})^{-j}.
\end{aligned}$$

This completes the proof. □

With the help of Lemmas 1 and 4, the following theorem can be given.

**Theorem 1.** *Suppose that $x(t)$ and $f(t,\ x)$ are continuous, then the initial value problem with the generalized fractional derivative*

$$\begin{cases}{}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\,x(t)=f(t,\ x),\ n-1<\alpha<n\in\mathbb{Z}^{+},\ t>a\geq 0,\ \sigma>0,\ \eta\in\mathbb{R},\\ \left[t^{\sigma\eta}\,{}_{EK}\mathfrak{D}^{\alpha-j}_{a,t;\sigma,\eta}\,x(t)\right]\Big|_{t=a}=x^{(j)}_{a},\ \ j=1,2,\ldots,n,\end{cases}\tag{3.4}$$

*is equivalent to the nonlinear Volterra integral equation*

$$x(t)=\sum_{j=1}^{n}\frac{x^{(j)}_{a}}{\Gamma(\alpha-j+1)}t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{\alpha-j}+\frac{t^{-\sigma\eta}}{\Gamma(\alpha)}\int_{a}^{t}(t^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma\eta}f(\tau,\ x)\,\mathrm{d}\tau^{\sigma}.\tag{3.5}$$

*Proof.* First, consider the solution of the following system

$$\begin{cases}{}_{EK}\mathfrak{D}^{\alpha}_{a^{\sigma},t;1,\eta}\,\hat{x}\left(t^{\frac{1}{\sigma}}\right)=\hat{f}\left(t^{\frac{1}{\sigma}},\ \hat{x}\right),\ n-1<\alpha<n\in\mathbb{Z}^{+},\ t>a^{\sigma}\geq 0,\ \sigma>0,\ \eta\in\mathbb{R},\\ \left[t^{\eta}\,{}_{EK}\mathfrak{D}^{\alpha-j}_{a^{\sigma},t;1,\eta}\,\hat{x}\left(t^{\frac{1}{\sigma}}\right)\right]\Big|_{t=a^{\sigma}}=\hat{x}^{(j)}_{a^{\sigma}},\ \ j=1,2,\ldots,n.\end{cases}\tag{3.6}$$

Applying the operator $T$ on both sides of ${}_{RL}\mathrm{D}^{\alpha}_{a^{\sigma},t}\,\tilde{x}\left(t^{\frac{1}{\sigma}}\right)=\tilde{f}\left(t^{\frac{1}{\sigma}},\ \tilde{x}\right)$, we have

$${}_{EK}\mathfrak{D}^{\alpha}_{a^{\sigma},t;1,\eta}\,\hat{x}\left(t^{\frac{1}{\sigma}}\right)=T\tilde{f}\left(t^{\frac{1}{\sigma}},\ \tilde{x}\right)+\sum_{j=1}^{n}\frac{\tilde{x}^{(j)}_{a^{\sigma}}}{\Gamma(1-j)}t^{-\eta}(t-a^{\sigma})^{-j},$$

where $\hat{x}\left(t^{\frac{1}{\sigma}}\right)=T\tilde{x}\left(t^{\frac{1}{\sigma}}\right)$. It's clear that

$$\begin{aligned}
\tilde{f}\left(t^{\frac{1}{\sigma}},\ \tilde{x}\right)&=T^{-1}\left\{\hat{f}\left(t^{\frac{1}{\sigma}},\ \hat{x}\right)-\sum_{j=1}^{n}\frac{\tilde{x}^{(j)}_{a^{\sigma}}}{\Gamma(1-j)}t^{-\eta}(t-a^{\sigma})^{-j}\right\}\\
&=t^{\eta}\hat{f}\left(t^{\frac{1}{\sigma}},\ \hat{x}\right)-\sum_{j=1}^{n}\frac{\tilde{x}^{(j)}_{a^{\sigma}}}{\Gamma(1-j)}(t-a^{\sigma})^{-j}.
\end{aligned}$$

Because $t > a^\sigma$, the function $\sum_{j=1}^{n} \frac{\tilde{x}_{a^\sigma}^{(j)}}{\Gamma(1-j)}(t-a^\sigma)^{-j}$ is equal to zero, then

$$\tilde{f}\left(t^{\frac{1}{\sigma}},\, \tilde{x}\right) = t^\eta \hat{f}\left(t^{\frac{1}{\sigma}},\, \hat{x}\right).$$

From $\hat{x}\left(t^{\frac{1}{\sigma}}\right) = T\tilde{x}\left(t^{\frac{1}{\sigma}}\right)$ and Eq (2.2), the solution of system (3.6) is derived as

$$\hat{x}\left(t^{\frac{1}{\sigma}}\right) = \sum_{j=1}^{n} \frac{\hat{x}_{a^\sigma}^{(j)}}{\Gamma(\alpha-j+1)} t^{-\eta}(t-a^\sigma)^{\alpha-j} + \frac{t^{-\eta}}{\Gamma(\alpha)} \int_{a^\sigma}^{t} (t-\tau)^{\alpha-1}\tau^\eta \hat{f}\left(\tau^{\frac{1}{\sigma}},\, \hat{x}\right) \mathrm{d}\tau.$$

Introducing the operator $\Omega^{-1}$, we can get the explicit solution of system (3.4) via using Eq (3.2)

$$x(t) = \sum_{j=1}^{n} \frac{x_a^{(j)}}{\Gamma(\alpha-j+1)} t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-j} + \frac{t^{-\sigma\eta}}{\Gamma(\alpha)} \int_{a}^{t} (t^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma\eta} f(\tau,\, x)\, \mathrm{d}\tau^\sigma.$$

On the other hand, use Eq (3.5) to derive Eq (3.4). Since $f(t, x)$ is continuous, then $x(t)$ is a differentiable function with regard to $t$. Utilizing the operator ${}_{EK}\mathfrak{D}_{a,t;\sigma,\eta}^{\alpha}\cdot$ to both sides of Eq (3.5), one gets

$$\begin{aligned}
&{}_{EK}\mathfrak{D}_{a,t;\sigma,\eta}^{\alpha}\left[\sum_{j=1}^{n} \frac{x_a^{(j)}}{\Gamma(\alpha-j+1)} t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-j}\right] \\
&= \sum_{j=1}^{n} \frac{x_a^{(j)}}{\Gamma(\alpha-j+1)} {}_{EK}\mathfrak{D}_{a,t;\sigma,\eta}^{\alpha} t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-j} \\
&= \sum_{j=1}^{n} \frac{x_a^{(j)}}{\Gamma(n-\alpha)\Gamma(\alpha-j+1)} t^{-\sigma\eta}\left(\frac{1}{\sigma t^{\sigma-1}}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t^\sigma-\tau^\sigma)^{n-\alpha-1}(\tau^\sigma-a^\sigma)^{\alpha-j}\, \mathrm{d}\tau^\sigma \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t\to a} t^{\sigma\eta}{}_{EK}\mathfrak{D}_{a,t;\sigma,\eta}^{\alpha-j}\left[\sum_{k=1}^{n} \frac{x_a^{(k)}}{\Gamma(\alpha-k+1)} t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-k}\right] \\
&= \lim_{t\to a} \sum_{k=1}^{n} \frac{x_a^{(k)}}{\Gamma(\alpha-k+1)} t^{\sigma\eta}{}_{EK}\mathfrak{D}_{a^+;\sigma,\eta}^{\alpha-j} t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-k} \\
&= \lim_{t\to a} \sum_{k=1}^{n} \frac{x_a^{(k)}}{\Gamma(j-k+1)} (t^\sigma - a^\sigma)^{j-k} \\
&= \lim_{t\to a} \sum_{k=1}^{j-1} \frac{x_a^{(k)}}{\Gamma(j-k+1)} (t^\sigma - a^\sigma)^{j-k} + x_a^{(j)} \\
&\quad + \lim_{t\to a} \sum_{k=j+1}^{n} \frac{x_a^{(k)}}{\Gamma(j-k+1)} (t^\sigma - a^\sigma)^{j-k} \\
&= x_a^{(j)}, \;\; j = 1, 2, \ldots, n.
\end{aligned}$$

Combining the above discussion and Lemma 2, the proof of this theorem is completed. □

**Remark 1.** *The initial condition of the initial value problem with the generalized fractional derivative is not unique. Since* $\lim_{t\to a} t^{\sigma\eta}\,{}_{EK}\mathfrak{D}^{\alpha-j}_{a,t;\sigma,\eta}\,x(t) = \lim_{t\to a}\Gamma(1-j+\alpha)\,t^{\sigma\eta}(t^\sigma - a^\sigma)^{j-\alpha}x(t)\,(j = 1,2,\ldots,n)$. *Then the initial conditions* $\left[t^{\sigma\eta}\,{}_{EK}\mathfrak{D}^{\alpha-j}_{a,t;\sigma,\eta}\,x(t)\right]\Big|_{t=a} = x_a^{(j)}$ *and* $\left[\Gamma(1-j+\alpha)\,t^{\sigma\eta}(t^\sigma - a^\sigma)^{j-\alpha}x(t)\right]\Big|_{t=a} = x_a^{(j)}$ *can transform each other.*

Before considering the existence and uniqueness, we first make the following hypothesis.

Hypothesis [H]: If $f(t,\,x) : [a,\,\infty)\times\Omega \to \mathbb{R}$ is a continuous function, then $f(t,\,x)$ is continuous bounded map defined on $[a,\,a+h^*]\times\Omega_0$, where $\Omega_0$ is a bounded subset of $\Omega\subset\mathbb{R}$. For convenience, let $X_a(t) = \sum_{j=1}^{n}\frac{x_a^{(j)}}{\Gamma(\alpha-j+1)}t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-j},\ j = 1,2,\ldots,n.$

Here, we only consider the case of $\eta > 0$, other cases can be obtained similarly.

**Theorem 2.** *Postulate that the hypothesis* [H] *holds. Then there exits at least one solution* $x(t)\in\Omega_0$ *to Eq* (3.4)*. The constant h can be determined as follows*

$$h := \begin{cases} h^*, & if\ M^* = 0,\\ \min\left\{h^*,\ (\Gamma(1+\alpha)K/M^*)^{\frac{1}{\alpha\sigma}} - a\right\}, & if\ M^* \neq 0, \end{cases} \tag{3.7}$$

*in which the positive constants* $M^*$ *and* $K$ *satisfy* $M^* := \sup_{t\in[a,\,a+h^*]}|f(t,\,x)|$ *and* $\|x - X_a\|_{C[a,\,a+h^*]} \leq K$*, respectively.*

*Proof.* If $M^* = 0$, evidently $x(t) = X_a(t)$ is the solution of Eq (3.4). Hence there is a solution in this case.

For $M^*\neq 0$, we first define a set $U$ as

$$U := \{x\in C[a, a+h] : \|x - X_a\|_{C[a,\,a+h^*]} \leq K\}. \tag{3.8}$$

It is clear that $U$ is a nonempty, bounded, closed, and convex subset. From Theorem 1, on the set $U$, the operator $B$ can be expressed as

$$(Bx)(t) = X_a(t) + \frac{t^{-\sigma\eta}}{\Gamma(\alpha)}\int_a^t (t^\sigma - \tau^\sigma)^{\alpha-1}\tau^{\sigma\eta}f(\tau,\,x)\,\mathrm{d}\tau^\sigma. \tag{3.9}$$

In the following, we prove that $B$ has a fixed point on $U$, the proof is divided into three steps.

**Step 1**: $BU\subset U$.

For every $x\in C[a,\,a+h]$, one has

$$|(Bx)(t) - X_a(t)| \leq \frac{M^*}{\Gamma(\alpha)}\int_a^x (x^\sigma - t^\sigma)^{\alpha-1}\,\mathrm{d}t^\sigma \leq \frac{M^*}{\Gamma(1+\alpha)}(a+h)^{\sigma\alpha},$$

which implies that

$$\|(Bx)(t) - X_a(t)\|_{C[a,\,a+h]} \leq \frac{M^*}{\Gamma(1+\alpha)}(a+h)^{\sigma\alpha} \leq K.$$

Therefore, we have $Bx\in U$ for every $x\in U$.

**Step 2**: $B$ is continuous.

Since $f(t, x)$ is continuous, it is uniformly continuous on compact set $[a, a+K] \times U$. For any $\varepsilon > 0$, there exists $\delta_0$ ($\delta_0 > 0$), if $\|x_m - x\|_{C[a,a+h]} < \delta_0$ with $m \to \infty$ such that the following result holds,

$$\|f(t, x_m) - f(t, x)\|_{C[a,a+h]} < \frac{\varepsilon}{(a+h)^{\sigma\alpha}} \Gamma(\alpha+1). \tag{3.10}$$

Further, we can get that

$$\begin{aligned}
&|(Bx_m)(t) - (Bx)(t)| \\
&= \left| \frac{t^{-\sigma\eta}}{\Gamma(\alpha)} \int_a^t (t^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma\eta} \left[ f(\tau, x_m) - f(\tau, x) \right] \mathrm{d}\tau^\sigma \right| \\
&\le \frac{t^{-\sigma\eta}}{\Gamma(\alpha)} \int_a^t (t^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma\eta} \left| f(\tau, x_m) - f(\tau, x) \right| \mathrm{d}\tau^\sigma \\
&\le \frac{(a+h)^{\sigma\alpha}}{\Gamma(1+\alpha)} \|f(\tau, x_m) - f(\tau, x)\|_{C[a,a+h]} \\
&< \varepsilon,
\end{aligned} \tag{3.11}$$

which completes the proof of $B \in C(U)$.

**Step 3**: $BU$ is equicontinuous.

Let $x_m$ ($m \in \mathbb{N}$) be a sequence on $U$, it gives that

$$\begin{aligned}
|(Bx_m)(t)| &\le \|X_a(t)\|_{C[a,a+h]} + \frac{t^{-\sigma\eta}}{\Gamma(\alpha)} \int_a^t (t^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma\eta} g(\tau, x)\, \mathrm{d}\tau^\sigma \\
&\le M + K.
\end{aligned} \tag{3.12}$$

So, $BU$ is uniformly bounded.

Next, we complete the proof that $BU$ is equicontinuous. If there are $t_1$ and $t_2$ such that $a \le a + t_1 \le a + t_2 \le a + h$, then

$$\begin{aligned}
&|(Bx)(a+t_2) - (Bx)(a+t_1)| \\
&\le |X_a(a+t_2) - X_a(a+t_1)| \\
&\quad + \left| \frac{(a+t_2)^{-\sigma\eta}}{\Gamma(\alpha)} \int_a^{a+t_2} \left[ (a+t_2)^\sigma - \tau^\sigma \right]^{\alpha-1} \tau^{\sigma\eta} f(\tau, x_n)\, \mathrm{d}\tau^\sigma \right| \\
&\quad - \left| \frac{(a+t_1)^{-\sigma\eta}}{\Gamma(\alpha)} \int_a^{a+t_1} \left[ (a+t_1)^\sigma - \tau^\sigma \right]^{\alpha-1} \tau^{\sigma\eta} f(\tau, x_n)\, \mathrm{d}\tau^\sigma \right| \\
&\le |X_a(a+t_2) - X_a(a+t_1)| \\
&\quad + \left| \frac{(a+t_2)^{-\sigma\eta} - (a+t_1)^{-\sigma\eta}}{\Gamma(\alpha)} \int_a^{a+t_1} \left[ (a+t_2)^\sigma - \tau^\sigma \right]^{\alpha-1} \tau^{\sigma\eta} f(\tau, x_n)\, \mathrm{d}\tau^\sigma \right| \\
&\quad + \left| \frac{(a+t_1)^{-\sigma\eta}}{\Gamma(\alpha)} \int_a^{a+t_1} \left[ ((a+t_2)^\sigma - \tau^\sigma)^{\alpha-1} - ((a+t_1)^\sigma - \tau^\sigma)^{\alpha-1} \right] \tau^{\sigma\eta} f(\tau, x_n)\, \mathrm{d}\tau^\sigma \right| \\
&\quad + \left| \frac{(a+t_2)^{-\sigma\eta}}{\Gamma(\alpha)} \int_{a+t_1}^{a+t_2} \left[ (a+t_2)^\sigma - \tau^\sigma \right]^{\alpha-1} \tau^{\sigma\eta} f(\tau, x_n)\, \mathrm{d}\tau^\sigma \right| \\
&:= |X_a(a+t_2) - X_a(a+t_1)| + I_1 + I_2 + I_3.
\end{aligned} \tag{3.13}$$

First, consider $I_1$. It is clear that

$$\begin{aligned} I_1 &\le \left|\frac{(a+t_2)^{-\sigma\eta}-(a+t_1)^{-\sigma\eta}}{\Gamma(\alpha)}\right|\left|\int_a^{a+t_1}\left[(a+t_2)^\sigma-\tau^\sigma\right]^{\alpha-1}\tau^{\sigma\eta}f(\tau,x)\,\mathrm{d}\tau^\sigma\right| \\ &\le \frac{M^*(a+h)^{\alpha\sigma}}{\Gamma(\alpha+1)}\left|(a+t_2)^{-\sigma\eta}-(a+t_1)^{-\sigma\eta}\right|. \end{aligned} \tag{3.14}$$

As far as $I_2$, three cases need to be considered, i.e., $\alpha < 1$, $\alpha = 1$, $\alpha > 1$. For $\alpha = 1$, the value of $I_2$ is zero. In the case of $\alpha < 1$, one has $[(a+t_1)^\sigma-\tau^\sigma]^{\alpha-1} \ge [(a+t_2)^\sigma-\tau^\sigma]^{\alpha-1}$. Thus

$$\begin{aligned} I_2 &\le \left|\frac{(a+t_1)^{-\sigma\eta}}{\Gamma(\alpha)}\right|\left|\int_a^{a+t_1}\left[((a+t_1)^\sigma-\tau^\sigma)^{\alpha-1}-((a+t_2)^\sigma-\tau^\sigma)^{\alpha-1}\right]\tau^{\sigma\eta}f(\tau,x)\,\mathrm{d}\tau^\sigma\right| \\ &\le \frac{M^*}{\Gamma(\alpha+1)}\left\{[(a+t_2)^\sigma-(a+t_1)^\sigma]^\alpha+[(a+t_1)^\sigma-a^\sigma]^\alpha-[(a+t_2)^\sigma-a^\sigma]^\alpha\right\} \\ &\le \frac{M^*}{\Gamma(\alpha+1)}\left[(a+t_2)^\sigma-(a+t_1)^\sigma\right]^\alpha. \end{aligned} \tag{3.15}$$

If $\alpha > 1$, it is valid that $[(a+t_2)^\sigma-\tau^\sigma]^{\alpha-1} \ge [(a+t_1)^\sigma-\tau^\sigma]^{\alpha-1}$. Then

$$\begin{aligned} I_2 &\le \left|\frac{(a+t_1)^{-\sigma\eta}}{\Gamma(\alpha)}\right|\left|\int_a^{a+t_1}\left[((a+t_2)^\sigma-\tau^\sigma)^{\alpha-1}-((a+t_1)^\sigma-\tau^\sigma)^{\alpha-1}\right]\tau^{\sigma\eta}f(\tau,x)\,\mathrm{d}\tau^\sigma\right| \\ &\le \frac{M^*}{\Gamma(\alpha+1)}\left\{[(a+t_2)^\sigma-a^\sigma]^\alpha-[(a+t_1)^\sigma-a^\sigma]^\alpha-[(a+t_2)^\sigma-(a+t_1)^\sigma]^\alpha\right\} \\ &\le \frac{M^*}{\Gamma(\alpha+1)}\left[((a+t_2)^\sigma-a^\sigma)^\alpha-((a+t_1)^\sigma-a^\sigma)^\alpha\right]. \end{aligned} \tag{3.16}$$

Finally, we discuss $I_3$. It can be got that

$$\begin{aligned} I_3 &\le \frac{(a+t_2)^{-\sigma\eta}}{\Gamma(\alpha)}\int_{a+t_1}^{a+t_2}\left|[(a+t_2)^\sigma-\tau^\sigma]^{\alpha-1}\tau^{\sigma\eta}f(\tau,x)\right|\mathrm{d}\tau^\sigma \\ &\le \frac{M^*}{\Gamma(\alpha+1)}\left[(a+t_2)^\sigma-(a+t_1)^\sigma\right]^\alpha. \end{aligned} \tag{3.17}$$

Thus, in the case of $\alpha \le 1$, it can be deduced as

$$\begin{aligned} &|(Bx)(a+t_2)-(Bx)(a+t_1)| \\ &\le |X_a(a+t_2)-X_a(a+t_1)|+\frac{M^*(a+h)^{\alpha\sigma}}{\Gamma(\alpha+1)}\left|(a+t_2)^{-\sigma\eta}-(a+t_1)^{-\sigma\eta}\right| \\ &\quad+\frac{2M^*}{\Gamma(\alpha+1)}\left[(a+t_2)^\sigma-(a+t_1)^\sigma\right]^\alpha \\ &\le |X_a(a+t_2)-X_a(a+t_1)|+\frac{\sigma\eta\, M^*a^{-\sigma\eta-1}(a+h)^{\alpha\sigma}}{\Gamma(\alpha+1)}(t_2-t_1) \\ &\quad+\frac{2\sigma^\alpha M^*b^{\alpha\sigma}}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha. \end{aligned}$$

Since $X_a(t)$ is continuous and suppose that $|t_2-t_1| < \delta_0$, we have

$$|(Bx)(a+t_2)-(Bx)(a+t_1)| \le M_1\delta_0+\frac{2\sigma^\alpha M^*b^{\alpha\sigma}}{\Gamma(\alpha+1)}\delta_0^\alpha, \tag{3.18}$$

in which $M_1$ is a positive constant and independent of $x$, $t_1$, $t_2$, and the right-hand side of inequality (3.18) has no relevance to $x$. Hence, $BU$ is equicontinuous. Similarly, in the case of $\alpha > 1$, the conclusion still holds. In accordance with the Arzelà-Ascoli theorem [29], $BU$ is precompact. Therefore, $B$ is complete. From the Schauder Fixed Point theorem [29], it can come to the conclusion that $B$ has at least a fixed point. Then the fixed point is the required solution of Eq (3.4). Thereby, the theorem is proved. □

**Theorem 3.** *Suppose that the hypothesis* [H] *is satisfied. The function* $f : [a, \infty)\times\Omega \to \mathbb{R}$ *is continuous and fulfills the Lipschitz condition with respect to the second variable, i.e.,*

$$|f(t, x_2) - f(t, x_1)| \leq L|x_2 - x_1|, \tag{3.19}$$

*where the constant* $L > 0$ *is uncorrelated with* $t$, $x_1$, *and* $x_2$. *Then, there exists a unique solution* $x(t) \in C[a, a+h]$ *for the initial value problem with the generalized fractional derivative Eq* (3.4).

*Proof.* Inspired by [23], we first complete the proof that $B$ has a unique fixed point. For $x_1, x_2 \in U$, one has

$$\| B^m(x_2) - B^m(x_1)\|_{C[a,a+t]} \leq \frac{L^m[(a+t)^\sigma - a^\sigma]^{m\alpha}}{\Gamma(m\alpha+1)}\|x_2 - x_1\|_{C[a,a+t]}, \tag{3.20}$$

where $m \in \mathbb{N}$, $a+t \in [a, a+h]$. This can be seen by induction. When $m = 0$, the result is true. Assume that Eq (3.20) holds for $m-1$. Then, it can be arrived at

$$\begin{aligned}
&\|B^m(x_2) - B^m(x_1)\|_{C[a,a+t]}\\
&= \|B[B^{m-1}(x_2)] - B[B^{m-1}(x_1)]\|_{C[a,a+t]}\\
&\leq \frac{L}{\Gamma(\alpha)} \sup_{a\leq\omega\leq a+t}\left|\omega^{-\sigma\eta}\int_a^\omega (\omega^\sigma - \tau^\sigma)^{\alpha-1}\tau^{\sigma\eta}\left|B^{m-1}x_2(\tau) - B^{m-1}x_1(\tau)\right| \mathrm{d}\tau^\sigma\right|\\
&\leq \frac{L}{\Gamma(\alpha)}(a+t)^{-\sigma\eta}\int_a^{a+t} [(a+t)^\sigma - \tau^\sigma])^{\alpha-1}\tau^{\sigma\eta}\sup_{a\leq\omega\leq\tau}\left|B^{m-1}x_2(\omega) - B^{m-1}x_1(\omega)\right| \mathrm{d}\tau^\sigma\\
&\leq \frac{L^m\|x_2 - x_1\|_{C[a,a+t]}}{\Gamma(\alpha)\Gamma(m\alpha+1)}\int_a^{a+t}[(a+t)^\sigma - \tau^\sigma]^{\alpha-1}(\tau^\sigma - a^\sigma)^{m\alpha}\,\mathrm{d}\tau^\sigma\\
&= \frac{L^m[(a+t)^\sigma - a^\sigma]^{m\alpha}}{\Gamma(m\alpha+1)}\|x_2 - x_1\|_{C[a,a+t]}.
\end{aligned}$$

Since

$$\sum_{k=0}^{\infty}\frac{L^{k+1}[(a+h)^\sigma - a^\sigma]^{(k+1)\alpha}}{\Gamma((k+1)\alpha+1)} = E_\alpha\left(L[(a+h)^\sigma - a^\sigma]^\alpha\right),$$

in accordance with the Banach Fixed Point Theorem [29], the proof is accomplished. □

**Remark 2.** *Theorems 1–3 only deal with the one-dimensional generalized fractional differential equation, one could extend such results to the n-dimensional case* ($n > 1$)*, which can be verified similarly.*

## 4. Stability analysis

By applying Lyapunov's stability criterion and linearization theory, the present section provides the stability analysis of the generalized fractional differential systems.

### 4.1. The linear fractional differential system

#### 4.1.1. One-dimensional case

Consider the one-dimensional generalized fractional differential system as follows

$$\begin{cases} {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\, x(t) = \lambda x(t),\ 0<\alpha<1,\ t>a\geq 0,\ \lambda\in\mathbb{R},\ \sigma>0,\ \eta\in\mathbb{R}, \\ \left[t^{\sigma\eta}{}_{EK}\mathfrak{D}^{\alpha-1}_{a,t;\sigma,\eta}\, x(t)\right]\Big|_{t=a} = x_a. \end{cases} \tag{4.1}$$

In order to obtain its solution, we give the lemma below.

**Lemma 5.** [4] *The initial value problem of the one-dimensional fractional differential system with Riemann-Liouville derivative*

$$\begin{cases} {}_{RL}\mathrm{D}^{\alpha}_{a^{\sigma},t}\, \tilde{x}\left(t^{\frac{1}{\sigma}}\right) = \lambda\tilde{x}\left(t^{\frac{1}{\sigma}}\right),\ 0<\alpha<1,\ t>a^{\sigma}\geq 0,\ \lambda\in\mathbb{R}, \\ \left[{}_{RL}\mathrm{D}^{\alpha-1}_{a^{\sigma},t}\, \tilde{x}\left(t^{\frac{1}{\sigma}}\right)\right]\Big|_{t=a^{\sigma}} = \tilde{x}_{a^{\sigma}}, \end{cases} \tag{4.2}$$

*is equivalent to*

$$\tilde{x}\left(t^{\frac{1}{\sigma}}\right) = \tilde{x}_{a^{\sigma}}\,(t-a^{\sigma})^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-a^{\sigma})^{\alpha}\right). \tag{4.3}$$

**Theorem 4.** *The solution of system* (4.1) *is*

$$x(t) = x_a\, t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t^{\sigma}-a^{\sigma})^{\alpha}\right). \tag{4.4}$$

*Proof.* Introduce operator $T$ in Lemma 5. The following system can be derived

$$\begin{cases} {}_{EK}\mathfrak{D}^{\alpha}_{a^{\sigma},t;1,\eta}\, \hat{x}\left(t^{\frac{1}{\sigma}}\right) = \lambda\hat{x}\left(t^{\frac{1}{\sigma}}\right),\ 0<\alpha<1,\ t>a^{\sigma}\geq 0,\ \sigma>0,\ \eta\in\mathbb{R}, \\ \left[t^{\eta}{}_{EK}\mathfrak{D}^{\alpha-1}_{a^{\sigma},t;1,\eta}\, \hat{x}\left(t^{\frac{1}{\sigma}}\right)\right]\Big|_{t=a^{\sigma}} = \hat{x}_{a^{\sigma}}. \end{cases} \tag{4.5}$$

From Theorem 1, the solution of system (4.5) is expressed as

$$x(t) = x_a\, t^{-\eta}(t-a^{\sigma})^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-a^{\sigma})^{\alpha}\right). \tag{4.6}$$

With the help of Eq (3.2), we obtain the solution of system (4.1). Therefore, the proof is completed. □

**Theorem 5.** *Suppose that* $0<\alpha<1$, $t>a\geq 0$ *and* $\sigma>0$. *The following statements hold:*
(1) *Let* $\lambda<0$. *Under the condition of* $\eta>-\alpha-1$, *the zero solution of system* (4.1) *is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{-\alpha-1}\right)$. *Under the condition of* $\eta=-\alpha-1$, *the zero solution of system* (4.1) *is stable but not asymptotically stable. In residual conditions, the zero solution of system* (4.1) *is unstable.*
(2) *Let* $\lambda=0$. *Under the condition of* $\eta>\alpha-1$, *the zero solution of system* (4.1) *is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{\alpha-1}\right)$. *Under the condition of* $\eta=\alpha-1$, *the zero solution of system* (4.1) *is stable but not asymptotically stable. In residual conditions, the zero solution of system* (4.1) *is unstable.*
(3) *If* $\lambda>0$, *then the zero solution of system* (4.1) *is unstable.*

*Proof.* (1) If $\lambda < 0$, one has

$$\begin{aligned} E_{\alpha,\alpha}\left(\lambda(t^{\sigma}-a^{\sigma})^{\alpha}\right) &= -\sum_{k=2}^{p}\frac{1}{\Gamma(\alpha-\alpha k)}\left[\lambda(t^{\sigma}-a^{\sigma})^{\alpha}\right]^{-k} + O\left(\left(\lambda(t^{\sigma}-a^{\sigma})^{\alpha}\right)^{-1-p}\right) \\ &= -\frac{1}{\lambda^{2}\Gamma(-\alpha)}(t^{\sigma}-a^{\sigma})^{-2\alpha} + O\left(\left(\lambda(t^{\sigma}-a^{\sigma})^{\alpha}\right)^{-3}\right). \end{aligned}$$

It is clear that

$$x(t) = -\frac{x_a}{\lambda^{2}\Gamma(-\alpha)}\,t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{-\alpha-1} + x_a O\left(\lambda^{-3}t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{-2\alpha-1}\right).$$

Hence,

$$\lim_{t\to+\infty}|x(t)| = \begin{cases} 0, & \eta > -\alpha-1, \\ \dfrac{x_a}{\lambda^{2}\Gamma(-\alpha)}, & \eta = -\alpha-1, \\ \infty, & \eta < -\alpha-1, \end{cases}$$

according to Definition 2, the expected results are obtained.

(2) When $\lambda = 0$, there holds

$$\lim_{t\to+\infty}|x(t)| = \lim_{t\to+\infty}\left|\frac{x_a}{\Gamma(\alpha)}t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{\alpha-1}\right| = \begin{cases} 0, & \eta > \alpha-1, \\ \dfrac{x_a}{\Gamma(\alpha)}, & \eta = \alpha-1, \\ \infty, & \eta < \alpha-1. \end{cases}$$

Thus, when $\eta > \alpha - 1$, the zero solution of system (4.1) is asymptotically stable, and the decay rate is $O\left(t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{\alpha-1}\right)$. When $\eta = \alpha - 1$, the zero solution is stable but not asymptotically stable. When $\eta < \alpha - 1$, the zero solution is unstable.

(3) For $\lambda > 0$, we can show that

$$\begin{aligned} E_{\alpha,\alpha}\left(\lambda(t^{\sigma}-a^{\sigma})^{\alpha}\right) &= \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}(t^{\sigma}-a^{\sigma})^{1-\alpha}\exp\left[\lambda^{\frac{1}{\alpha}}(t^{\sigma}-a^{\sigma})\right] \\ &\quad - \sum_{k=2}^{p}\frac{\left[\lambda(t^{\sigma}-a^{\sigma})^{\alpha}\right]^{-k}}{\Gamma(\alpha-\alpha k)} + \mathrm{O}((\lambda(t^{\sigma}-a^{\sigma})^{\alpha})^{-1-p}) \\ &= \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}(t^{\sigma}-a^{\sigma})^{1-\alpha}\exp\left[\lambda^{\frac{1}{\alpha}}(t^{\sigma}-a^{\sigma})\right] \\ &\quad - \frac{1}{\lambda^{2}\Gamma(-\alpha)}(t^{\sigma}-a^{\sigma})^{-2\alpha} + \mathrm{O}((\lambda(t^{\sigma}-a^{\sigma})^{\alpha})^{-3}). \end{aligned}$$

Then,

$$\lim_{t\to+\infty}|x(t)| = \lim_{t\to+\infty}\left|x_a\,t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t^{\sigma}-a^{\sigma})^{\alpha}\right)\right| = \infty,$$

which yields that the zero solution of system (4.1) is unstable.

Thus, the proof is completed. □

4.1.2. Two-dimensional case

Consider the following two-dimensional generalized fractional differential system

$$\begin{cases} {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\, x(t) = Ax(t),\ 0<\alpha<1,\ t>a\geq 0,\ A\in\mathbb{R}^{2\times 2},\ \sigma>0,\ \eta\in\mathbb{R}, \\ \left[t^{\sigma\eta}\,{}_{EK}\mathfrak{D}^{\alpha-1}_{a,t;\sigma,\eta}\, x(t)\right]\Big|_{t=a} = x_a, \end{cases} \tag{4.7}$$

where $x(t)=(x_1(t),\ x_2(t))^T$ and $x_a=(x_{a1},\ x_{a2})^T$.

**Case 1:** If the matrix A is diagonalizable, then there exists an invertible matrix $T$ satisfying $T^{-1}AT = \begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}$, in which $\lambda_i\,(i = 1,\ 2)$ are the eigenvalues of matrix $A$. Because nonsingular transformation does not change stability, then we can directly write $A = \begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}$. Following Theorem 4, we get the solution of system (4.7)

$$\begin{cases} x_1(t) = x_{a1}\, t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_1(t^{\sigma}-a^{\sigma})^{\alpha}\right), \\ x_2(t) = x_{a2}\, t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_2(t^{\sigma}-a^{\sigma})^{\alpha}\right). \end{cases} \tag{4.8}$$

Next, we give the stability theorems of the zero solution to system (4.7).

**Theorem 6.** *Let* $0<\alpha<1,\ t>a\geq 0,\ \sigma>0$ *and* $\lambda_1,\ \lambda_2\in\mathbb{R}$. *Then*
(1) *If* $\lambda_j<0\,(j=1,\ 2)$*, then, in the case of* $\eta>-\alpha-1$*, the zero solution of system* (4.7) *is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{-\alpha-1}\right)$*. In the case of* $\eta=-\alpha-1$*, the zero solution is stable but not asymptotically stable. Otherwise, the zero solution is unstable.*
(2) *If at least one of* $\lambda_j\,(j=1,\ 2)$ *is equal to* 0*, and the rest is less than* 0*, then, in the case of* $\eta>\alpha-1$*, the zero solution of system* (4.7) *is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{\alpha-1}\right)$*. In the case of* $\eta=\alpha-1$*, the zero solution is stable but not asymptotically stable. Otherwise, the zero solution is unstable.*
(3) *If at least one of* $\lambda_j\,(j=1,\ 2)$ *is greater than* 0*, then the zero solution of system* (4.7) *is unstable.*

**Theorem 7.** *Let* $0<\alpha<1,\ t>a\geq 0,\ \sigma>0$ *and* $\lambda_1=\overline{\lambda_2}\in\mathbb{C}$. *Then one gets*
(1) *If* $|\arg\lambda_1|>\frac{\pi\alpha}{2}$*, then, under the circumstance of* $\eta>-\alpha-1$*, the zero solution is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{-\alpha-1}\right)$*. Under the circumstance of* $\eta=-\alpha-1$*, the zero solution is stable but not asymptotically stable. In other circumstances, the zero solution is unstable.*
(2) *If* $|\arg\lambda_1|=\frac{\pi\alpha}{2}$*, then, under the circumstance of* $\eta>0$*, the zero solution is asymptotically stable, and the decay rate is* $O\,(t^{-\sigma\eta})$*. Under the circumstance of* $\eta=0$*, the zero solution is stable but not asymptotically stable. In other circumstances, the zero solution is unstable.*
(3) *If* $|\arg\lambda_1|<\frac{\pi\alpha}{2}$*, then the zero solution is unstable.*

*Proof.* (1) For $|\arg\lambda_1|>\frac{\pi\alpha}{2}$, the proof of the result can be given in the same way as the result (1) of Theorem 5.

(2) When $|\arg\lambda_1| = \frac{\pi\alpha}{2}$, assume that $\lambda_1 = r\exp\left(\pm i\frac{\pi\alpha}{2}\right)$ $(r > 0)$. One has

$$\begin{aligned}
&(t^\sigma - a^\sigma)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t^\sigma - a^\sigma)^\alpha\right)\\
&= \frac{1}{\alpha}r^{\frac{1-\alpha}{\alpha}}\exp\left[\pm i\left(\frac{1-\alpha}{2}\pi + r^{\frac{1}{\alpha}}(t^\sigma - a^\sigma)\right)\right]\\
&\quad - \sum_{k=2}^{p}\frac{r^{-k}(t^\sigma - a^\sigma)^{\alpha(1-k)-1}}{\Gamma(\alpha - \alpha k)}\exp\left(\pm i\frac{\alpha k\pi}{2}\right)\\
&\quad + \mathrm{O}\left(r^{-1-p}(t^\sigma - a^\sigma)^{-1-\alpha p}\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\lim_{t\to+\infty}|x(t)| &= \lim_{t\to+\infty}\left|x_{a1}\,t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t^\sigma - a^\sigma)^\alpha\right)\right|\\
&= \begin{cases} 0, & \eta > 0,\\ \frac{1}{\alpha}r^{\frac{1-\alpha}{\alpha}}x_{a1}, & \eta = 0,\\ \infty, & \eta < 0,\end{cases}
\end{aligned}$$

which shows that, for $\eta > 0$, the zero solution is asymptotically stable, and the decay rate is $O\left(t^{-\sigma\eta}\right)$. For $\eta = 0$, the zero solution is stable but not asymptotically stable. While the zero solution is not stable for $\eta < 0$.

(3) For $|\arg\lambda_1| < \frac{\pi\alpha}{2}$, let $\lambda_1 = r\exp(i\theta)$ $(r > 0,\ |\theta| < \frac{\alpha}{2})$. One has

$$\begin{aligned}
&(t^\sigma - a^\sigma)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right)\\
&= \frac{1}{\alpha}r^{\frac{1-\alpha}{\alpha}}\exp\left[r^{\frac{1}{\alpha}}(t^\sigma - a^\sigma)\cos\left(\frac{\theta}{\alpha}\right)\right]\exp\left[i\left(\frac{1-\alpha}{\alpha}\theta + r^{\frac{1}{\alpha}}(t^\sigma - a^\sigma)\sin\left(\frac{\theta}{\alpha}\right)\right)\right]\\
&\quad - \sum_{k=2}^{p}\frac{r^{-k}(t^\sigma - a^\sigma)^{\alpha(1-k)-1}}{\Gamma(\alpha - \alpha k)}\exp(-ik\theta) + \mathrm{O}\left(r^{-1-p}(t^\sigma - a^\sigma)^{-1-\alpha p}\right).
\end{aligned}$$

Then,

$$\lim_{t\to+\infty}|x(t)| = \lim_{t\to+\infty}\left|x_{a1}\,t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t^\sigma - a^\sigma)^\alpha\right)\right| = \infty,$$

in accordance with Definition 2, the zero solution is unstable.

Consequently, this theorem is finished. □

**Case 2:** Suppose the matrix $A$ is similar to a Jordan canonical form, i.e. $T^{-1}AT = \begin{pmatrix}\lambda & 1\\ & \lambda\end{pmatrix}$, where $\lambda$ and $T$ are a real number and an invertible matrix, respectively. Without affecting the stability, we can also consider $A = \begin{pmatrix}\lambda & 1\\ & \lambda\end{pmatrix}$. Correspondingly, the solution of the system (4.7) has the following form

$$\begin{cases} x_1(t) = x_{a1}\,t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t^\sigma - a^\sigma)^\alpha\right) + \frac{x_{a2}}{\alpha}\,t^{-\sigma\eta}(t^\sigma - a^\sigma)^{2\alpha-1}\\ \qquad\qquad \times\left[E_{\alpha,2\alpha-1}\left(\lambda(t^\sigma - a^\sigma)^\alpha\right) - (\alpha-1)E_{\alpha,2\alpha}\left(\lambda(t^\sigma - a^\sigma)^\alpha\right)\right],\\ x_2(t) = x_{a2}\,t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t\sigma - a^\sigma)^\alpha\right).\end{cases} \tag{4.9}$$

**Theorem 8.** *Let* $0 < \alpha < 1$, $t > a \geq 0$, $\sigma > 0$ *and* $\lambda \in \mathbb{R}$. *Then*
(1) *If* $\lambda < 0$, *then, when* $\eta > -\alpha - 1$, *the zero solution is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^\sigma - a^\sigma)^{-\alpha-1}\right)$. *When* $\eta = -\alpha - 1$, *the zero solution is stable but not asymptotically stable. When* $\eta < -\alpha - 1$, *the zero solution is unstable.*
(2) *If* $\lambda = 0$, *then, when* $\eta > 2\alpha - 1$, *the zero solution is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^\sigma - a^\sigma)^{2\alpha-1}\right)$. *When* $\eta = 2\alpha - 1$, *the zero solution is stable but not asymptotically stable. When* $\eta < -\alpha - 1$, *the zero solution is unstable.*
(3) *If* $\lambda > 0$, *then the zero solution is unstable.*

*Proof.* (1) If $\lambda < 0$, it can be deduced that

$$\begin{aligned}&\frac{1}{\alpha}(t^\sigma - a^\sigma)^{2\alpha-1}\left[E_{\alpha,2\alpha-1}(\lambda(t^\sigma - a^\sigma)^\alpha) - (\alpha-1)E_{\alpha,2\alpha}(\lambda(t^\sigma - a^\sigma)^\alpha)\right]\\ =\ &\frac{2}{\lambda^3\Gamma(-\alpha)}(t^\sigma - a^\sigma)^{-\alpha-1} + O\left(|\lambda|^{-4}(t^\sigma - a^\sigma)^{-2\alpha-1}\right).\end{aligned}$$

Combining the result (1) of Theorem 5, we have

$$\lim_{t\to+\infty}|x_1(t)| = \begin{cases}0, & \eta > -\alpha - 1,\\ \dfrac{x_{a1}}{\lambda^2\Gamma(-\alpha)} + \dfrac{2\,x_{a2}}{\lambda^3\Gamma(-\alpha)}, & \eta = -\alpha - 1,\\ \infty, & \eta < -\alpha - 1.\end{cases}$$

By Definition 2, a direct calculation can be seen that, as far as $\eta > -\alpha - 1$, the zero solution is asymptotically stable, and the decay rate is $O\left(t^{-\sigma\eta}(t^\sigma - a^\sigma)^{-\alpha-1}\right)$. As far as $\eta = -\alpha - 1$, the zero solution is stable but not asymptotically stable. However, the zero solution is not stable for $\eta < -\alpha - 1$.

(2) When $\lambda = 0$, the solution of system (4.7) can be rewritten as

$$\begin{cases}x_1(t) = \dfrac{x_{a1}}{\Gamma(\alpha)}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} + \dfrac{x_{a2}}{\Gamma(2\alpha)}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{2\alpha-1},\\ x_2(t) = \dfrac{x_{a2}}{\Gamma(\alpha)}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1},\end{cases}$$

which means that, if $\eta > 2\alpha - 1$ meets, then the zero solution is asymptotically stable, and the decay rate is $O\left(x^{-\sigma\eta}(t^\sigma - a^\sigma)^{2\alpha-1}\right)$. If $\eta = 2\alpha - 1$ meets, then the zero solution is stable but not asymptotically stable. If $\eta < 2\alpha - 1$ meets, the zero solution is not stable.

(3) For $\lambda > 0$, from Eq (4.9) and the result (3) of Theorem 5, it yields that the solution is unstable.

The proof of the theorem is now fulfilled. □

4.1.3. *n*-dimensional case ($n \geq 3$)

In accordance with the stability analyses of one-dimensional and two-dimensional cases, we extend the content to the $n$-dimensional case ($n \geq 3$). Here, we focus on the linear fractional differential system with the generalized fractional derivative

$$\begin{cases}{}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\, x(t) = Ax(t),\ 0 < \alpha < 1,\ t > a \geq 0,\ A \in \mathbb{R}^{n\times n},\ \sigma > 0,\ \eta \in \mathbb{R},\\ \left[t^{\sigma\eta}\,{}_{EK}\mathfrak{D}^{\alpha-1}_{a,t;\sigma,\eta}\, x(t)\right]\Big|_{t=a} = x_a,\end{cases} \tag{4.10}$$

where $x(t) = (x_1(t),\ x_2(t), \cdots, x_n(t))^T$ and $x_a = (x_{a1},\ x_{a2}, \cdots, x_{an})^T$.

**Case 1:** If the matrix $A$ is diagonalizable, then we can find an invertible matrix $P$ fulfilling $P^{-1}AP = J$, where

$$J = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}. \tag{4.11}$$

Without affecting the stability, we consider the matrix $A = J$ in order to simplify the calculation. Theorem 4 implies that the solution of system (4.10) in the case is

$$\begin{cases} x_1(t) = x_{a1}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right), \\ x_2(t) = x_{a2}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_2(t^\sigma - a^\sigma)^\alpha\right), \\ \qquad \vdots \\ x_n(t) = x_{an}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_n(t^\sigma - a^\sigma)^\alpha\right). \end{cases} \tag{4.12}$$

**Theorem 9.** *Let* $0 < \alpha < 1$, $t > a \geq 0$ *and* $\sigma > 0$. *Suppose that* $\lambda_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$. *Then there hold*

(1) *If* $\lambda_j < 0\,(j = 1, 2, \ldots, n)$*, then, under the condition of* $\eta > -\alpha - 1$*, the zero solution is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^\sigma - a^\sigma)^{-\alpha-1}\right)$. *Under the condition of* $\eta = -\alpha - 1$*, the zero solution is stable but not asymptotically stable. In residual condition, the zero solution is unstable.*

(2) *If at least one of* $\lambda_j\,(j = 1, 2, \ldots, n)$ *is equal to* $0$*, and the rest are less than* $0$*, then, under the condition of* $\eta > \alpha - 1$*, the zero solution is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1}\right)$. *Under the condition of* $\eta > \alpha - 1$*, the zero solution is stable but not asymptotically stable. In residual condition, the zero solution is unstable.*

(3) *If at least one of* $\lambda_j\,(j = 1, 2, \ldots, n)$ *is greater than* $0$*, then the zero solution is unstable.*

When the eigenvalues $\lambda_j'$s are complex numbers, we assume that $\lambda_l \in \mathbb{C}$, $l = 1, 2, \ldots, 2k$ and $\lambda_m \in \mathbb{R}$, $m = 2k + 1, \ldots, n$, where $k$ is from 1 to $\left[\frac{n}{2}\right]$ at most and $\lambda_1 = \overline{\lambda}_2, \ldots, \lambda_{2k-1} = \overline{\lambda}_{2k}$.

**Theorem 10.** *Let* $0 < \alpha < 1$, $t > a \geq 0$ *and* $\sigma > 0$. *If* $\lambda_l \in \mathbb{C}$, $\lambda_m \in \mathbb{R}$ *and* $\lambda_1 = \overline{\lambda}_2, \ldots, \lambda_{2k-1} = \overline{\lambda}_{2k}$*, then*

(1) *If* $|\arg \lambda_l| > \frac{\pi\alpha}{2}\,(l = 1, 2, \ldots, 2k)$ *and* $\lambda_m < 0\,(m = 2k + 1, \ldots, n)$*, then, in the case of* $\eta > -\alpha - 1$*, the zero solution is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^\sigma - a^\sigma)^{-\alpha-1}\right)$. *In the case of* $\eta = -\alpha - 1$*, the zero solution is stable but not asymptotically stable. Otherwise, the zero solution is unstable.*

(2) *If at least one of* $|\arg \lambda_l|\,(l = 1, 2, \ldots, 2k)$ *is equivalent to* $\frac{\pi\alpha}{2}$ *and others are smaller than* $\frac{\alpha\pi}{2}$*, and the rest eigenvalues satisfy* $\lambda_m \leq 0\,(m = 2k + 1, \ldots, n)$*, then, in the case of* $\eta > 0$*, the zero solution is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}\right)$. *In the case of* $\eta = 0$*, the zero solution is stable but not asymptotically stable. Otherwise, the zero solution is unstable.*

(3) *If at least one of* $\lambda_m\,(m = 2k + 1, \ldots, n)$ *is equivalent to* $0$ *and others are smaller than* $0$*, and the rest eigenvalues satisfy* $|\arg \lambda_l| > \frac{\pi\alpha}{2}\,(l = 1, 2, \ldots, 2k)$*, then, in the case of* $\eta > \alpha - 1$*, the zero solution is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1}\right)$. *In the case of* $\eta = \alpha - 1$*, the zero*

*solution is stable but not asymptotically stable. Otherwise, the zero solution is unstable.*
(4) *If at least one of* $\lambda_m < 0\,(m = 2k+1, \ldots, n)$ *or* $|\arg \lambda_l| < \frac{\pi\alpha}{2}\,(l = 1, 2, \ldots, 2k)$*, then the zero solution is unstable.*

**Case 2:** Let the matrix $A$ be similar to a Jordan canonical form, i.e., there exists an invertible matrix $T$ such that

$$T^{-1}AT = J = \operatorname{diag}(J_1, J_2, \ldots, J_v), \tag{4.13}$$

here $J_i\,(i = 1, 2, \ldots, v)$ have the following form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}_{n_i \times n_i}, \ \lambda_i \in \mathbb{C},\ n_i \geq 2, \tag{4.14}$$

and $\sum_{i=1}^{v} n_i = n$.

The following theorem considers the stability of the $n$-dimensional linear system (4.10).

**Theorem 11.** *Let* $0 < \alpha < 1$, $t > a \geq 0$, $\sigma > 0$ *and* $\lambda_i \in \mathbb{C}\,(i = 1, 2, \ldots, v)$*. Then*
(1) *If* $|\arg \lambda_i| > \frac{\pi\alpha}{2}\,(i = 1, 2, \ldots, v)$*, then, under the circumstance of* $\eta > -\alpha - 1$*, the zero solution is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^\sigma - a^\sigma)^{-\alpha-1}\right)$*. Under the circumstance of* $\eta = -\alpha - 1$*, the zero solution is stable but not asymptotically stable. In the remaining circumstance, the zero solution is unstable.*
(2) *If at least one of* $|\arg \lambda_i|\,(i = 1, 2, \ldots, v)$ *is equivalent to* $\frac{\pi\alpha}{2}$ *and others are smaller than* $\frac{\alpha\pi}{2}$*, then, under the circumstance of* $\eta > n_i - 1$*, the zero solution is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^\sigma - a^\sigma)^{n_i-1}\right)$*. Under the circumstance of* $\eta = n_i - 1$*, the zero solution is stable but not asymptotically stable. In the remaining circumstance, the zero solution is unstable.*
(3) *If there are some* $\lambda_j = 0$*, and all other eigenvalues have* $|\arg \lambda_i| > \frac{\pi\alpha}{2}\,(i = 1, 2, \ldots, j-1, j+1, \ldots, v)$*, then, under the circumstance of* $\eta > n_j\alpha - 1$*, the zero solution is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^\sigma - a^\sigma)^{n_j\alpha-1}\right)$*. under the circumstance of* $\eta = n_j\alpha - 1$*, the zero solution is stable but not asymptotically stable. In the remaining circumstance, the zero solution is unstable.*
(4) *If at least one of* $|\arg \lambda_i|\,(i = 1, 2, \ldots, v)$ *is less than* $\frac{\pi\alpha}{2}$*, then the zero solution is unstable.*

*Proof.* Without loss of generality, we suppose that $A = J_1$. Then the solution has the following representation

$$\begin{cases} x_1(t) = x_{a1}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right) \\ \qquad + x_{a2}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} \dfrac{\partial}{\partial \lambda_1} E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right) \\ \qquad + \dfrac{x_{a3}}{2!}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} \dfrac{\partial^2}{\partial \lambda_1^2} E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right) + \cdots \\ \qquad + \dfrac{x_{an_1}}{(n_1-1)!}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} \dfrac{\partial^{n_1-1}}{\partial \lambda_1^{n_1-1}} E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right), \end{cases}$$

$$
\begin{cases}
x_2(t) = x_{a2}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right) \\
\qquad + x_{a3}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} \dfrac{\partial}{\partial\lambda_1} E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right) + \cdots \\
\qquad + \dfrac{x_{an_1}}{(n_1-2)!}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} \dfrac{\partial^{n_1-2}}{\partial\lambda_1^{n_1-2}} E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right), \\
\qquad \vdots \\
x_{n_1-1}(t) = x_{a(n_1-1)}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right) \\
\qquad + x_{an_1}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} \dfrac{\partial}{\partial\lambda_1} E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right), \\
x_{n_1}(t) = x_{an_1}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right).
\end{cases}
\tag{4.15}
$$

(1) Suppose that $|\arg\lambda_1| > \frac{\pi\alpha}{2}$ and $m = 0, 1, \ldots, n_1 - 1$. We have

$$
\begin{aligned}
&\frac{1}{m!}(t^\sigma - a^\sigma)^{\alpha-1} \frac{\partial^m}{\partial\lambda_1^m} E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right) \\
&= \frac{(-1)^{m+1}(m+1)}{\lambda_1^{2+m}\Gamma(-\alpha)}(t^\sigma - a^\sigma)^{-\alpha-1} + \mathrm{O}\left(|\lambda_1|^{-3-m}(t^\sigma - a^\sigma)^{-2\alpha-1}\right),
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\lim_{t\to+\infty}\left|\frac{1}{m!} t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} \frac{\partial^m}{\partial\lambda_1^m} E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right)\right| \\
&= \begin{cases}
0, & \eta > -\alpha - 1, \\
\dfrac{(-1)^{m+1}(m+1)}{\lambda_1^{2+m}\Gamma(-\alpha)}, & \eta = -\alpha - 1, \\
\infty, & \eta < -\alpha - 1.
\end{cases}
\end{aligned}
$$

Thus, when $\eta > -\alpha - 1$, the zero solution is asymptotically stable, and the decay rate is $O\left(t^{-\sigma\eta}(t^\sigma - a^\sigma)^{-\alpha-1}\right)$. When $\eta = -\alpha - 1$, the zero solution is stable but not asymptotically stable. When $\eta < -\alpha - 1$, the zero solution is not stable.

(2) For $\left|\arg(\lambda_1)\right| = \frac{\pi\alpha}{2}$, it can be deduced as

$$
\begin{aligned}
&(t^\sigma - a^\sigma)^{\alpha-1} \frac{1}{m!} \frac{\partial^m}{\partial\lambda_1^m} E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right) \\
&= \frac{1}{m!} \exp\left(\lambda_1^{\frac{1}{\alpha}}(t^\sigma - a^\sigma)\right) \left\{ \frac{(1-\alpha)(1-2\alpha)\cdots(1-m\alpha)}{\alpha^{m+1}} \lambda_i^{\frac{1-(m+1)\alpha}{\alpha}} + \cdots \right. \\
&\quad \left. + \frac{C_{m+1}^{m-1}(1-\alpha)}{\alpha^{m+1}} \lambda_1^{\frac{m-(m+1)\alpha}{\alpha}} (t^\sigma - a^\sigma)^{m-1} + \frac{1}{\alpha^{m+1}} \lambda_1^{\frac{(m+1)(1-\alpha)}{\alpha}} (t^\sigma - a^\sigma)^m \right\} \\
&\quad - \sum_{k=2}^{p} \frac{(-1)^m (k+m-1)!}{m!(k-1)!\Gamma(\alpha - \alpha k)} \lambda_i^{-k-m+1} (t^\sigma - a^\sigma)^{-\alpha k} \\
&\quad + \mathrm{O}\left(|\lambda_1|^{-1-p-m} |(t^\sigma - a^\sigma)^\alpha|^{-1-p}\right)
\end{aligned}
$$

$$= \frac{1}{m!}\exp\left(\lambda_1^{\frac{1}{\alpha}}(t^\sigma - a^\sigma)\right)\left\{\frac{(1-\alpha)(1-2\alpha)\cdots(1-m\alpha)}{\alpha^{m+1}}\lambda_i^{\frac{1-(m+1)\alpha}{\alpha}} + \cdots\right.$$
$$\left. + \frac{C_{m+1}^{m-1}(1-\alpha)}{\alpha^{m+1}}\lambda_1^{\frac{m-(m+1)\alpha}{\alpha}}(t^\sigma - a^\sigma)^{m-1} + \frac{1}{\alpha^{m+1}}\lambda_1^{\frac{(m+1)(1-\alpha)}{\alpha}}(t^\sigma - a^\sigma)^m\right\} \tag{4.16}$$
$$+ \frac{(-1)^{m+1}(m+1)}{\lambda_1^{2+m}\Gamma(-\alpha)}(t^\sigma - a^\sigma)^{-\alpha-1} + \mathrm{O}\left(|\lambda_1|^{-3-m}(t^\sigma - a^\sigma)^{-2\alpha-1}\right),$$

where $m = 0,\ 1, \ldots, n_1 - 1$. Substitute $\lambda_1 = r\exp\left(\pm i\frac{\pi\alpha}{2}\right)$ $(r > 0)$ into Eq (4.16), then there exists

$$\lim_{t\to+\infty}\left|\frac{1}{m!}t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1}\frac{\partial^m}{\partial\lambda_1^m}E_{\alpha,\alpha}\left(r\exp\left(\pm i\frac{\alpha\pi}{2}\right)(t^\sigma - a^\sigma)^\alpha\right)\right|$$
$$= \begin{cases} 0, & \eta > n_1 - 1, \\ \dfrac{1}{\alpha^{n_1}(n_1-1)!}\lambda_1^{\frac{(n_1)(1-\alpha)}{\alpha}}, & \eta = n_1 - 1, \\ \infty, & \eta < n_1 - 1, \end{cases}$$

which means that, for $\eta > n_1 - 1$, the zero solution is asymptotically stable, and the decay rate is $O\left(t^{-\sigma\eta}(t^\sigma - a^\sigma)^{n_1-1}\right)$. For $\eta = n_1 - 1$, the zero solution is stable but not asymptotically stable. For $\eta < n_1 - 1$, the zero solution is unstable.

(3) If $\lambda_1 = 0$, for $m = 0,\ 1, \ldots, n_1 - 1$, we find that

$$(t^\sigma - a^\sigma)^{\alpha-1}\frac{1}{m!}\frac{\partial^m}{\partial\lambda_1^m}E_{\alpha,\alpha}\left(\lambda_1(t^\sigma - a^\sigma)^\alpha\right) = \frac{1}{\Gamma(\alpha+\alpha m)}(t^\sigma - a^\sigma)^{\alpha m+\alpha-1}.$$

The solution of the system (4.10) takes the form

$$\begin{cases} x_1(t) = \dfrac{x_{a1}}{\Gamma(\alpha)}\,t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} + \cdots + \dfrac{x_{an_1}}{\Gamma(\alpha n_1)}\,t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha n_1-1}, \\ x_2(t) = \dfrac{x_{a2}}{\Gamma(\alpha)}\,t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} + \cdots + \dfrac{x_{an_1}}{\Gamma(\alpha(n_1-1))}\,t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha(n_1-1)-1}, \\ \quad\vdots \\ x_{n_1}(t) = \dfrac{x_{an_1}}{\Gamma(\alpha)}\,t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1}. \end{cases}$$

It indicates that, if $\eta > n_1\alpha - 1$, then the zero solution is asymptotically stable, and the decay rate is $O\left(x^{-\sigma\eta}(t^\sigma - a^\sigma)^{n_1\alpha-1}\right)$. if $\eta = n_1\alpha - 1$, the zero solution is stable but not asymptotically stable. Otherwise, the zero solution is unstable.

(4) Using Eqs (4.15) and (4.16), the desired result can be achieved.

Thus, the proof of this theorem is completed. □

From the above discussion, we can get the general theorem.

**Theorem 12.** *Let* $0 < \alpha < 1$ *and* $t > a \geq 0$. *Then*
(1) *If all the eigenvalues* $\lambda(A)$ *of* $A$ *meet* $|\arg\lambda(A)| > \frac{\pi\alpha}{2}$, *then, in the case of* $\eta > -\alpha - 1$, *the zero solution is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^\sigma - a^\sigma)^{-\alpha-1}\right)$. *In the case of* $\eta = -\alpha - 1$, *the zero solution is stable but not asymptotically stable. Otherwise, the zero solution is*

*unstable.*

(2) *If the zero eigenvalues of A have the same algebraic and geometric multiplicities, and the rest eigenvalues meet* $|\arg\lambda(A)| > \frac{\pi\alpha}{2}$*, then, in the case of* $\eta > \alpha - 1$*, the zero solution is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^{\sigma} - a^{\sigma})^{\alpha-1}\right)$*. In the case of* $\eta = \alpha - 1$*, the zero solution is stable but not asymptotically stable. Otherwise, the zero solution is unstable.*

(3) *If at least one of the eigenvalues of A meeting* $|\arg\lambda(A)| = \frac{\pi\alpha}{2}$ *has the same algebraic and geometric multiplicities, and the rest eigenvalues are less than* 0*, then, in the case of* $\eta > 0$*, the zero solution is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}\right)$*. In the case of* $\eta = 0$*, the zero solution is stable but not asymptotically stable. Otherwise, the zero solution is unstable.*

(4) *If the eigenvalues of A subject to* $|\arg\lambda(A)| = \frac{\pi\alpha}{2}$ *have the different algebraic and geometric multiplicities, and others are greater than* $\frac{\pi\alpha}{2}$*, then, in the case of* $\eta > n_k - 1$*, the zero solution is asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^{\sigma} - a^{\sigma})^{n_k-1}\right)$*. In the case of* $\eta = n_k - 1$*, the zero solution is stable but not asymptotically stable. Otherwise, the zero solution is unstable, where* $n_k$ $(2 \le n_k < n,\ n \in \mathbb{Z}^+)$ *is the algebraic multiplicities.*

(5) *If the zero eigenvalues of A have the different algebraic and geometric multiplicities, and other eigenvalues have* $|\arg\lambda_i| > \frac{\pi\alpha}{2}$*, then, in the case of* $\eta > n_k\alpha - 1$*, the zero solution is asymptotically stable, and the decay rate is* $O\left(x^{-\sigma\eta}(t^{\sigma} - a^{\sigma})^{n_k\alpha-1}\right)$*. In the case of* $\eta = n_k\alpha - 1$*, the zero solution is stable but not asymptotically stable. Otherwise, the zero solution is unstable for* $\eta < n_k\alpha - 1$*, where* $n_k \in \mathbb{Z}^+$ $(2 \le n_k < n,\ n \in \mathbb{Z}^+)$ *is the algebraic multiplicities of the zero eigenvalues.*

(6) *If at least one of the eigenvalues of A is less than* $\frac{\pi\alpha}{2}$*, then the zero solution is unstable.*

### *4.2. The autonomous nonlinear fractional differential system*

We restrict our attention to the following $n$-dimensional nonlinear generalized fractional differential system

$$\begin{cases} {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\, x(t) = f(x),\ 0 < \alpha < 1,\ t > a \ge 0,\ \sigma > 0,\ \eta \in \mathbb{R}, \\ \left[t^{\sigma\eta}\, {}_{EK}\mathfrak{D}^{\alpha-1}_{a,t;\sigma,\eta}\, x(t)\right]\Big|_{t=a} = x_a, \end{cases} \tag{4.17}$$

where $x(t) \in \mathbb{R}^n$ and $f(x) \in \mathbb{R}^n$ satisfies the Lipschitz condition. Next, we will discuss the stability of the zero solution to system (4.17) with $f(0) \equiv 0$.

**Lemma 6.** *The function* $f(x)$ *is continuous and the solution* $x(t)$ *is also continuous in system* (4.17)*. Then* $\varphi_t$ *has the following properties*

(1) $\varphi_a = \frac{t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{\alpha-1}}{\Gamma(\alpha)} x_a$.

(2) $\varphi_{t+s} = \varphi_t \circ \theta_t \circ \varphi_s,\ t > a \ge 0,\ s > a \ge 0$*, there exists a linear map* $\theta_t$ *satisfying*

$$\begin{aligned} \theta_t \circ \varphi_s(x_a) = \frac{t^{-\sigma\eta}(t^{\sigma} - a^{\sigma})^{\alpha-1}}{\Gamma(\alpha)} x_a + \frac{(t^{\sigma} + s^{\sigma} - a^{\sigma})^{-\eta}}{\Gamma(\alpha)} \\ \times \int_a^s (t^{\sigma} + s^{\sigma} - \tau^{\sigma} - a^{\sigma})^{\alpha-1} \tau^{\sigma\eta} f\left(\varphi_\tau(x_a)\right)) \,\mathrm{d}\tau^{\sigma}, \end{aligned} \tag{4.18}$$

*and when* $s = a, \theta_t\left(\frac{t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{\alpha-1}}{\Gamma(\alpha)} x_a\right) = \frac{t^{-\sigma\eta}(t^{\sigma}-a^{\sigma})^{\alpha-1}}{\Gamma(\alpha)} x_a$.

(3) $(t,\ x_a) \to \varphi_t\,(x_a)$ *is a continuous map from* $[a,\ +\infty) \times \mathbb{R}$ *onto* $\mathbb{R}$.

*Proof.* From Theorem 1, we know that system (4.17) has a solution

$$x(t) = \frac{t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1}}{\Gamma(\alpha)} x_a + \frac{t^{-\sigma\eta}}{\Gamma(\alpha)} \int_a^t (t^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma\eta} f(x(\tau))\, \mathrm{d}\tau^\sigma. \tag{4.19}$$

Suppose that the operator $\varphi_t$ has the following expression

$$\varphi_t(x_a) = \frac{t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1}}{\Gamma(\alpha)} x_a + \frac{t^{-\sigma\eta}}{\Gamma(\alpha)} \int_a^t (t^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma\eta} f(\varphi_\tau(x_a))\, \mathrm{d}\tau^\sigma. \tag{4.20}$$

Obviously, properties (1) and (3) hold.

Next, we consider the property (2). Assume that $y = \theta_t \circ \varphi_s(x_a) = \frac{t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1}}{\Gamma(\alpha)} x_a + \frac{(t^\sigma + s^\sigma - a^\sigma)^{-\eta}}{\Gamma(\alpha)} \int_a^s (t^\sigma + s^\sigma - \tau^\sigma - a^\sigma)^{\alpha-1} (\tau^\sigma + s^\sigma - a^\sigma)^\eta f(\varphi_\tau(x_a)))\, \mathrm{d}\tau^\sigma$. Then it leads to

$$\begin{aligned}
&\varphi_t \circ \theta_t \circ \varphi_s(x_a) \\
= &\frac{t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1}}{\Gamma(\alpha)} x_a + \frac{t^{-\sigma\eta}}{\Gamma(\alpha)} \int_a^t (t^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma\eta} f(\varphi_\tau(\theta_\tau(\varphi_s(x_a)))) \, \mathrm{d}\tau^\sigma \\
&+ \frac{(t^\sigma + s^\sigma - a^\sigma)^{-\eta}}{\Gamma(\alpha)} \int_a^s (t^\sigma + s^\sigma - \tau^\sigma - a^\sigma)^{\alpha-1} \tau^{\sigma\eta} f(\varphi_\tau(x_a))) \, \mathrm{d}\tau^\sigma \\
= &\frac{t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1}}{\Gamma(\alpha)} x_a \\
&+ \frac{(t^\sigma + s^\sigma - a^\sigma)^{-\eta}}{\Gamma(\alpha)} \int_a^s (t^\sigma + s^\sigma - \tau^\sigma - a^\sigma)^{\alpha-1} \tau^{\sigma\eta} f(\varphi_\tau(x_a))) \, \mathrm{d}\tau^\sigma \\
&+ \frac{(t^\sigma + s^\sigma - a^\sigma)^{-\eta}}{\Gamma(\alpha)} \int_a^{(t^\sigma + s^\sigma - a^\sigma)^{1/\sigma}} (t^\sigma + s^\sigma - \tau^\sigma - a^\sigma)^{\alpha-1} \tau^{\sigma\eta} \\
&\times f\left(\varphi_{(\tau^\sigma + s^\sigma - a^\sigma)^{1/\sigma}}(\theta_{(\tau^\sigma + s^\sigma - a^\sigma)^{1/\sigma}}(\varphi_s(x_a)))\right) \mathrm{d}\tau^\sigma.
\end{aligned}$$

Let

$$\begin{cases} v_\tau(x_a) = \varphi_\tau(x_a), \ \ \tau \le s, \\ v_\tau(x_a) = \varphi_{(\tau^\sigma + s^\sigma - a^\sigma)^{1/\sigma}}(\theta_{(\tau^\sigma + s^\sigma - a^\sigma)^{1/\sigma}}(\varphi_s(x_a))), \ \ \tau > s. \end{cases} \tag{4.21}$$

Since $v_\tau(x_a)$ is continuous with respect to $\tau$, then,

$$\begin{aligned}
v_{t+s}(x_a) = \varphi_t \circ \theta_t \circ \varphi_s(x_a) = &\frac{t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1}}{\Gamma(\alpha)} x_a \\
&+ \frac{(t^\sigma + s^\sigma - a^\sigma)^{-\eta}}{\Gamma(\alpha)} \int_a^{(\tau^\sigma + s^\sigma - a^\sigma)^{1/\sigma}} (t^\sigma + s^\sigma - \tau^\sigma - a^\sigma)^{\alpha-1} \tau^{\sigma\eta} f(v_\tau(x_a))) \, \mathrm{d}\tau^\sigma.
\end{aligned} \tag{4.22}$$

From Theorem 3, we know that the solution is unique, then

$$\varphi_{t+s}(x_a) = v_{t+s}(x_a) = \varphi_t \circ \theta_t \circ \varphi_s(x_a). \tag{4.23}$$

All these yield the Lemma. □

Inspired by [37, 39], we establish the following linearization theorem of the nonlinear generalized fractional differential system.

**Theorem 13.** *If the origin is a hyperbolic equilibrium of system* (4.17)*, then vector field* $f(x)$ *is locally topologically equivalent with its linearization vector field* $Ax = f'(0)x$ *in the neighborhood* $\delta(0)$ *of the origin.*

*Proof.* Let $\lambda_i\,(i = 1, 2, \ldots, n)$ be the eigenvalues of the linearization matrix $f'(0)$ and satisfy $|\arg(\lambda_i)| > \frac{\pi\alpha}{2}\,(i = 1, 2, \ldots, n_1)$ and $|\arg(\lambda_i)| < \frac{\pi\alpha}{2}\,(i = n_1 + 1, n_1 + 2, \ldots, n)$. To linearize system (4.17), we introduce a nonsingular linear transformation operator $T_0 : \mathbb{R}^n \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ to system (4.17), and $T_0 : x(t) \to y(t) = (y_1(t), y_2(t))\,(y_1(t) \in \mathbb{R}^{n_1}, y_2 \in \mathbb{R}^{n_2}, n_2 = n - n_1)$. Then system (4.17) can be changed into

$$\begin{cases} {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\, y_1(t) = Ay_1(t) + F_1(y_1(t), y_2(t)), \\ {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\, y_2(t) = By_2(t) + F_2(y_1(t), y_2(t)), \end{cases} \tag{4.24}$$

in which $A$ has the eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_{n_1}$, $B$ has the eigenvalues $\lambda_{n_1+1}, \lambda_{n_1+2} \ldots \lambda_n$, $F_i = o(\|y_1(t)\| + \|y_2(t))\|)$ as $y_i(t) \to 0\,(i = 1, 2)$. Excited by Theorems (1) and (4), the solution $\varphi_t(y) = (y_1(t), y_2(t))$ of the system (4.24) takes the form below

$$\begin{aligned} y_1(t) &= y_{a1}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} E_{\alpha,\alpha}\left[A(t^\sigma - a^\sigma)^\alpha\right] \\ &\quad + \frac{t^{-\sigma\eta}}{\Gamma(\alpha)} \int_a^t (t^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma\eta} E_{\alpha,\alpha}\left[A(t^\sigma - a^\sigma)^\alpha\right]\, F_1\,(y_1(\tau), y_2(\tau))\, \mathrm{d}\tau^\sigma \\ &= y_{a1}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} E_{\alpha,\alpha}\left[A(t^\sigma - a^\sigma)^\alpha\right] + P_1(t, y_{a1}, y_{a2}), \end{aligned}$$

and

$$\begin{aligned} y_2(t) &= y_{a2}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} E_{\alpha,\alpha}\left[B(t^\sigma - a^\sigma)^\alpha\right] \\ &\quad + \frac{t^{-\sigma\eta}}{\Gamma(\alpha)} \int_a^t (t^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma\eta} E_{\alpha,\alpha}\left[B(t^\sigma - a^\sigma)^\alpha\right]\, F_2\,(y_1(\tau), y_2(\tau))\, \mathrm{d}\tau^\sigma \\ &= y_{a2}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} E_{\alpha,\alpha}\left[A(t^\sigma - a^\sigma)^\alpha\right] + P_2(t, y_{a1}, y_{a2}), \end{aligned}$$

where $y_{ai} = y_i(a) = {}_{EK}\mathfrak{D}^{\alpha-1}_{a,t;\sigma,\eta}\, y_i(t)\big|_{t=a}$ and $P_i = o(\|y_{a1}\| + |y_{a2}\|)$ as $y_i(t) \to 0\,(i = 1, 2)$. Thus we can find a constant $c > 0$ such that $\|P_i\| < c(\|y_{a1}\| + \|y_{a2}\|)\,(i = 1, 2)$ when $(y_{a1}, y_{a2}) \in \delta(0)$. If $(y_{a1}, y_{a2}) \notin \delta(0)$, there are $P_i \equiv 0$ due to $F_i(y_{a1}, y_{a2}) \equiv 0\,(i = 1, 2)$.

Consider the homogeneous linear system of the system (4.24)

$$\begin{cases} {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\, u_1(t) = Au_1(t), \\ {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\, u_2(t) = Bu_2(t), \end{cases} \tag{4.25}$$

where $u(t) = (u_1(t), u_2(t))$, $u_1(t) \in \mathbb{R}^{n_1}$ and $u_2(t) \in \mathbb{R}^{n_2}$. From Theorem (4), the solution $L_t(u) = (u_1(t), u_2(t))$ of the system (4.25) can be expressed as

$$\begin{cases} u_1(t) = u_{a1}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} E_{\alpha,\alpha}\left(A(t^\sigma - a^\sigma)^\alpha\right), \\ u_2(t) = u_{a2}\, t^{-\sigma\eta}(t^\sigma - a^\sigma)^{\alpha-1} E_{\alpha,\alpha}\left(B(t^\sigma - a^\sigma)^\alpha\right), \end{cases}$$

in which $u_{ai} = u_i(a) = {}_{EK}\mathfrak{D}^{\alpha-1}_{a,t;\sigma,\eta}\, u_i(t)\big|_{t=a}\,(i = 1, 2)$.

By Definition 5, we need to find a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $h \circ \varphi_t = L_t \circ h$. In order to achieve it, the proof shall be divided into three steps.

**Step 1:** Let $t = (a^\sigma + 1)^{1/\sigma}$. There is a continuous map $h_a : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\theta_s \circ L_{(a^\sigma+1)^{1/\sigma}} \circ h_a = h_a \circ \theta_s \circ \varphi_{(a^\sigma+1)^{1/\sigma}},\ s \in (a,\ (a^\sigma + 1)^{1/\sigma}). \tag{4.26}$$

And $h_a$ can be represented by the following coordinate transformations

$$\begin{cases} u_{a1} = U(y_{a1},\ y_{a2}), \\ u_{a2} = V(y_{a1},\ y_{a2}). \end{cases} \tag{4.27}$$

According to Eqs (4.26) and (4.27), there are

$$\begin{cases} \quad \theta_s(a^\sigma + 1)^{-\eta} E_{\alpha,\alpha}(A) U(y_{a1},\ y_{a2}) \\ = U(\theta_s((a^\sigma + 1)^{-\eta} E_{\alpha,\alpha}(A)\, y_{a1} + P_1((a^\sigma + 1)^{1/\sigma},\ y_{a1},\ y_{a2})), \\ \quad \theta_s((a^\sigma + 1)^{-\eta} E_{\alpha,\alpha}(B) y_{a2} + P_2((a^\sigma + 1)^{1/\sigma},\ y_{a1},\ y_{a2}))), \\ \quad \theta_s(a^\sigma + 1)^{-\eta} E_{\alpha,\alpha}(B) V(y_{a1},\ y_{a2}) \\ = V(\theta_s((a^\sigma + 1)^{-\eta} E_{\alpha,\alpha}(A)\, y_{a1} + P_1((a^\sigma + 1)^{1/\sigma},\ y_{a1},\ y_{a2})), \\ \quad \theta_s((a^\sigma + 1)^{-\eta} E_{\alpha,\alpha}(B)\, y_{a2} + P_2((a^\sigma + 1)^{1/\sigma},\ y_{a1},\ y_{a2}))). \end{cases} \tag{4.28}$$

From Eqs (4.28), it is clear that

$$\begin{aligned} & V(y_{a1},\ y_{a2}) \\ = & (E_{\alpha,\alpha}(B))^{-1} \theta_s^{-1}\, V(\theta_s(E_{\alpha,\alpha}(A)\, y_{a1} + (a^\sigma + 1)^\eta P_1((a^\sigma + 1)^{1/\sigma},\ y_{a1},\ y_{a2})), \\ & \theta_s(E_{\alpha,\alpha}(B)\, y_{a2} + (a^\sigma + 1)^\eta P_2((a^\sigma + 1)^{1/\sigma},\ y_{a1},\ y_{a2}))). \end{aligned} \tag{4.29}$$

The solution of Eq (4.29) can be got by using successive approximations. Assume that the following result holds

$$\begin{cases} V_0(y_{a1},\ y_{a2}) = y_{a2}, \\ V_k(y_{a1},\ y_{a2}) = (E_{\alpha,\alpha}(B))^{-1} \theta_s^{-1}\, V_{k-1}(\theta_s(E_{\alpha,\alpha}(A)\, y_{a1} + (a^\sigma + 1)^\eta P_1((a^\sigma + 1)^{1/\sigma},\ y_{a1},\ y_{a2})), \\ \qquad \theta_s(E_{\alpha,\alpha}(B)\, y_{a2} + (a^\sigma + 1)^\eta P_2((a^\sigma + 1)^{1/\sigma},\ y_{a1},\ y_{a2}))), \end{cases} \tag{4.30}$$

where $k = 1,\ 2,\ \ldots$. Then

$$V_1(y_{a1},\ y_{a2}) = y_{a2} + (E_{\alpha,\alpha}(B))^{-1}(a^\sigma + 1)^\eta P_2((a^\sigma + 1)^{1/\sigma},\ y_{a1},\ y_{a2}). \tag{4.31}$$

Presume that $\|E_{\alpha,\alpha}(A)\| = \iota$ and $\|E_{\alpha,\alpha}(B)\|^{-1} = \kappa$. First, we consider the case of $\kappa < \frac{1}{\iota}$. For small enough $\rho\,(\rho > 0)$, we can get

$$r = b\|\theta_s\|^{-1}(2\max\{\iota\|\theta_s\|,\ 2c(a^\sigma + 1)^\eta\|\theta_s\|,\ \kappa^{-1}\|\theta_s\|\})^\rho < 1.$$

Since $P_2 = o(\|y_{a1}\| + |y_{a2}\|)$ as $y_{ai} \to 0\,(i = 1,\ 2)$, then for a constant $M > 0$, one has

$$\|V_1(y_{a1},\ y_{a2}) - V_0(y_{a1},\ y_{a2})\| < Mr(\|y_{a1}\| + \|y_{a2}\|)^\rho.$$

If $\|V_k(y_{a1}, y_{a2}) - V_{k-1}(y_{a1}, y_{a2})\| < Mr^k(\|y_{a1}\| + \|y_{a2}\|)^\rho$, then

$$\begin{aligned}
&\|V_{k+1}(y_{a1}, y_{a2}) - V_k(y_{a1}, y_{a2})\| \\
\leq\ &\|E_{\alpha,\alpha}(B)\|^{-1}\|\theta_s\|^{-1}Mr^k(\|\theta_s(E_{\alpha,\alpha}(A)\,y_{a1} + (a^\sigma + 1)^\eta P_1((a^\sigma + 1)^{1/\sigma}, y_{a1}, y_{a2}))\| \\
&+ \|\theta_s(E_{\alpha,\alpha}(B)\,y_{a2} + (a^\sigma + 1)^\eta P_2((a^\sigma + 1)^{1/\sigma}, y_{a1}, y_{a2}))\|)^\rho \\
\leq\ &M\kappa r^k\|\theta_s\|^{-1}(\|\theta_s\|(\iota\|y_{a1}\| + \kappa^{-1}\|y_{a2}\|) + c(a^\sigma + 1)^\eta(\|y_{a1}\| + \|y_{a2}\|))^\rho \\
\leq\ &M\kappa r^k\|\theta_s\|^{-1}(2\max\{\iota\|\theta_s\|, 2c(a^\sigma + 1)^\eta\|\theta_s\|, \kappa^{-1}\|\theta_s\|\})^\rho(\|y_{a1}\| + \|y_{a2}\|)^\rho \\
\leq\ &Mr^{k+1}(\|y_{a1}\| + \|y_{a2}\|)^\rho,
\end{aligned}$$

where $\kappa < \|\theta_s\| < \frac{1}{\iota}$. Thus, the sequence $V_k(y_{a1}, y_{a2})$ uniformly converges to a continuous function $V(y_{a1}, y_{a2})$ and

$$\begin{aligned}
V(y_{a1}, y_{a2}) &= V_0(y_{a1}, y_{a1}) + \sum_{k=1}^{\infty}[V_k(y_{a1}, y_{a2}) - V_{k-1}(y_{a1}, y_{a2})] \\
&= y_{a2} + o(\|y_{a1}\| + \|y_{a2}\|).
\end{aligned}$$

In the same way, the following result can be achieved at

$$U(y_{a1}, y_{a2}) = y_{a1} + o(\|y_{a1}\| + \|y_{a2}\|).$$

Similar to $\kappa < \frac{1}{\sigma}$, the case $\kappa \geq \frac{1}{\sigma}$ can get the identical conclusion. In accordance with the above discussion, we find the continuous map $h_a$ which satisfies $h_a(0, 0) = (0, 0)$ and $h_a(y_{a1}, y_{a2}) = (y_{a1}, y_{a2})$ when $(y_{a1}, y_{a2}) \notin \delta(0)$. Moreover, the uniqueness is obvious.

**Step 2:** We prove that $h_a$ is a homeomorphism. From Step 1, we can find a continuous map $h_a^*$ satisfying

$$h_a^* \circ \theta_s \circ L_{(a^\sigma+1)^{1/\sigma}} = \theta_s \circ \varphi_{(a^\sigma+1)^{1/\sigma}} \circ h_a^*,\ s \in (a, (a^\sigma + 1)^{1/\sigma}).$$

Therefore, it can be deduced as

$$h_a \circ h_a^* \circ \theta_s \circ L_{(a^\sigma+1)^{1/\sigma}} = h_a \circ \theta_s \circ \varphi_{(a^\sigma+1)^{1/\sigma}} \circ h_a^* = \theta_s \circ L_{(a^\sigma+1)^{1/\sigma}} \circ h_a \circ h_a^*,\ s \in (a, (a^\sigma + 1)^{1/\sigma}),$$

and

$$\theta_s \circ L_{(a^\sigma+1)^{1/\sigma}} \circ h_a^* \circ h_a = h_a^* \circ \theta_s \circ \varphi_{(a^\sigma+1)^{1/\sigma}} \circ h_a = h_a^* \circ h_a \circ \theta_s \circ \varphi_{(a^\sigma+1)^{1/\sigma}},\ s \in (a, (a^\sigma + 1)^{1/\sigma}).$$

Because $h_a$ and $h_a^*$ are unique, one has

$$h_a \circ h_a^* = Id, h_a^* \circ h_a = Id,$$

which signifies that $h_a^{-1} = h_a^*$, and $h_a^{-1}$ is continuous. Thus, $h_a$ is a homeomorphism.

**Step 3:** Let $h = \int_a^{a+(a^\sigma+1)^{1/\sigma}} L_s \circ h_a \circ \varphi_s^{-1} \mathrm{d}s$. Then

$$
\begin{aligned}
L_t \circ \theta_t \circ h &= \int_{a+t}^{a+(a^\sigma+1)^{1/\sigma}+t} L_t \circ \theta_t \circ L_{s-t} \circ h_a \circ \varphi_{s-t}^{-1} \mathrm{d}s \\
&= \int_{a+t}^{a+(a^\sigma+1)^{1/\sigma}} L_s \circ h_a \circ \varphi_s^{-1} \mathrm{d}s \circ \varphi_t \circ \theta_t \\
&\quad + \int_a^{a+t} L_s \circ \theta_s \circ L_{(a^\sigma+1)^{1/\sigma}} \circ h_a \circ \varphi_{(a^\sigma+1)^{1/\sigma}}^{-1} \circ \theta_s^{-1} \circ \varphi_s^{-1} \mathrm{d}s \circ \varphi_t \circ \theta_t \\
&= \int_a^{a+(a^\sigma+1)^{1/\sigma}} L_s \circ h_a \circ \varphi_s^{-1} \mathrm{d}s \circ \varphi_t \circ \theta_t. \\
&= h \circ \varphi_t \circ \theta_t.
\end{aligned}
$$

By Step 2, it can be got that $h$ is a homeomorphism. Now, we consider

$$L_t \circ \theta_t \circ h(x_a) = h \circ \varphi_t \circ \theta_t(x_a).$$

Since $\theta_t(x_a) = x_a, \theta_t \circ h(x_a) = h(x_a)$, one gets

$$L_t \circ h(x_a) = h \circ \varphi_t(x_a).$$

Hence, the proof is completed. □

With the help of Theorems 12 and 13, we get the following stability theorem about hyperbolic equilibrium represented by zero solution of the nonlinear generalized fractional differential system.

**Theorem 14.** *Let* $0 < \alpha < 1$, $t > a \geq 0$ *and* $\sigma > 0$. *Then*
(1) *If all the eigenvalues* $\lambda(f'(0))$ *of the Jacobian matrix* $f'(0)$ *satisfy* $|\arg(\lambda(f'(0)))| > \frac{\pi\alpha}{2}$, *then, when* $\eta > -\alpha - 1$, *the zero solution of system* (4.17) *is locally asymptotically stable, and the decay rate is* $O\left(t^{-\sigma\eta}(t^\sigma - a^\sigma)^{-\alpha-1}\right)$. *When* $\eta = -\alpha - 1$, *the zero solution of system* (4.17) *is stable but not asymptotically stable. When* $\eta < -\alpha - 1$, *the zero solution of system* (4.17) *is unstable.*
(2) *If at least one of the eigenvalues* $\lambda(f'(0))$ *of the Jacobian matrix* $f'(0)$ *satisfy* $|\arg(\lambda(f'(0)))| < \frac{\pi\alpha}{2}$, *then the zero solution of system* (4.17) *is unstable.*

**Remark 3.** *If the Jacobian matrix* $f'(0)$ *has critical values, i.e.,* $\lambda(f'(0)) = 0$ *and/or* $|\arg(\lambda(f'(0)))| = \frac{\pi\alpha}{2}$, *which indicates that the origin is non-hyperbolic. Then the stability of the zero solution of system* (4.17) *cannot be judged by Theorem 14.*

## 5. Stability of the generalized fractional Chen system

In the section, we deal with the stability of the generalized fractional Chen system, which is described by the following autonomous fractional differential system

$$
\begin{cases}
{}_{EK}\mathfrak{D}_{a,t;\sigma,\eta}^{\alpha}\, x_1(t) = \bar{a}(x_2(t) - x_1(t)), \\
{}_{EK}\mathfrak{D}_{a,t;\sigma,\eta}^{\alpha}\, x_2(t) = (\bar{c} - \bar{a})x_1(t) + \bar{c}x_2(t) - x_1(t)x_3(t), \\
{}_{EK}\mathfrak{D}_{a,t;\sigma,\eta}^{\alpha}\, x_3(t) = x_1(t)x_2(t) - \bar{b}x_3(t), \\
\left[\Gamma(\alpha)\, t^{\sigma\eta}(t^\sigma - a^\sigma)^{1-\alpha} x_i(t)\right]\Big|_{t=a} = x_{ai},\ i = 1,\ 2,\ 3,
\end{cases}
\tag{5.1}
$$

where $0 < \alpha < 1$, $t > a \geq 0$, $\sigma > 0$, $\eta \in \mathbb{R}$ and $\bar{a}$, $\bar{b}$, $\bar{c}$ are positive real numbers.

**Theorem 15.** *System* (5.1) *can be written in the form*

$$\begin{cases} {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\bar{X}(t) = \bar{A}\,\bar{X}(t) + \bar{B}\bar{X}(t),\ 0<\alpha<1,\ t>a\geq 0,\ \sigma>0,\ \eta\in\mathbb{R}, \\ \left[\Gamma(\alpha)\, t^{\sigma\eta}(t^{\sigma}-a^{\sigma})^{1-\alpha}\bar{X}(t)\right]\Big|_{t=a} = \bar{X}_a, \end{cases} \tag{5.2}$$

*in which*

$$\bar{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix},\ \bar{X}_a = \begin{pmatrix} x_{a1} \\ x_{a2} \\ x_{a3} \end{pmatrix},\ \bar{A} = \begin{pmatrix} -\bar{a} & \bar{a} & 0 \\ \bar{c}-\bar{a} & \bar{c} & 0 \\ 0 & 0 & -\bar{b} \end{pmatrix} \text{ and } \bar{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

*Then it possesses the unique solution.*

*Proof.* Let $\Omega_0 := \left\{\bar{X}(t)\in\mathbb{R}^3 :\ \left|\bar{X}-\bar{X}_a\right| \leq\ K\right\}$ and $f\left(\bar{X}(t)\right) = \bar{A}\,\bar{X}(t) + \bar{B}\bar{X}(t)$. Then

$$\left|f\left(\bar{X}(t)\right) - f\left(\bar{Y}(t)\right)\right| \leq L\left|\bar{X}(t)-\bar{Y}(t)\right| \tag{5.3}$$

where $L = \left\|\bar{A}\right\| + \left\|\bar{B}\right\|\left(2\left|\bar{X}_a\right| + K\right)$. It implies that $f\left(\bar{X}(t)\right)$ satisfies Lipschitz condition. In accordance with Theorem 3, this proof is realized. □

**Theorem 16.** *System* (5.1) *with the equilibrium* $x_{eq} = (x_{eq_1},\ x_{eq_2},\ x_{eq_3})$ *can be linearized into*

$$\begin{cases} {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\,\varepsilon_1 = \bar{a}\,(\varepsilon_2-\varepsilon_1), \\ {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\,\varepsilon_2 = (\bar{c}-\bar{a}-x_{eq_3})\varepsilon_1 + \bar{c}\varepsilon_2 - x_{eq_1}\varepsilon_3, \\ {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\,\varepsilon_3 = (x_{eq_1}\varepsilon_2 + x_{eq_2}\varepsilon_1) - \bar{b}\varepsilon_3, \\ \left[\Gamma(\alpha)\, t^{\sigma\eta}(t^{\sigma}-a^{\sigma})^{1-\alpha}\,(x_{eq_i}+\varepsilon_i)\right]\Big|_{t=a} = x_{ai},\ i=1,\ 2,\ 3. \end{cases} \tag{5.4}$$

*Proof.* Let $x_i(t) = x_{eq_i} + \varepsilon_i(t)\,(i=1,\ 2,\ 3)$. One has

$$\begin{cases} {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\,(x_{eq_1}+\varepsilon_1) = \bar{a}\,(x_{eq_2}-x_{eq_1}+\varepsilon_2-\varepsilon_1), \\ {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\,(x_{eq_2}+\varepsilon_2) = (\bar{c}-\bar{a})(x_{eq_1}+\varepsilon_1) + \bar{c}(x_{eq_2}+\varepsilon_2) - (x_{eq_1}+\varepsilon_1)(x_{eq_3}+\varepsilon_3), \\ {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\,(x_{eq_3}+\varepsilon_3) = (x_{eq_1}+\varepsilon_1)(x_{eq_2}+\varepsilon_2) - \bar{b}(x_{eq_3}+\varepsilon_3), \\ \left[\Gamma(\alpha)\, t^{\sigma\eta}(t^{\sigma}-a^{\sigma})^{1-\alpha}\,(x_{eq_i}+\varepsilon_i)\right]\Big|_{t=a} = x_{ai},\ i=1,\ 2,\ 3. \end{cases}$$

Furthermore,

$$\begin{cases} {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\,\varepsilon_1 = \bar{a}\,(\varepsilon_2-\varepsilon_1), \\ {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\,\varepsilon_2 = (\bar{c}-\bar{a}-x_{eq_3})\varepsilon_1 + \bar{c}\varepsilon_2 - x_{eq_1}\varepsilon_3, \\ {}_{EK}\mathfrak{D}^{\alpha}_{a,t;\sigma,\eta}\,\varepsilon_3 = (x_{eq_1}\varepsilon_2 + x_{eq_2}\varepsilon_1) - \bar{b}\varepsilon_3, \\ \left[\Gamma(\alpha)\, t^{\sigma\eta}(t^{\sigma}-a^{\sigma})^{1-\alpha}\,(x_{eq_i}+\varepsilon_i)\right]\Big|_{t=a} = x_{ai},\ i=1,\ 2,\ 3. \end{cases} \tag{5.5}$$

From Theorem 13, system (5.5) is the linearized system of system (5.1), which completes the proof. □

**Theorem 17.** (1) *If* $\bar{a} > 2\bar{c}$ *and* $\eta > -\alpha-1$, *all the eigenvalues* $\lambda_i$ *satisfy* $|\arg(\lambda_i)| > \frac{\pi\alpha}{2}$ $(i=1,\ 2,\ 3)$, *then, the equilibrium* (0, 0, 0) *of system* (5.1) *is locally asymptotically stable.*
(2) *If* $\bar{a} < 2\bar{c}$, *the equilibrium* (0, 0, 0) *of system* (5.1) *is unstable.*

To illustrate the theoretical analysis, we give the phase diagrams applying the fractional Adams-Bashforth-Moulton method. The details of fractional Adams-Bashforth-Moulton method can be found in [9]. Some system parameters are chosen as $(\bar{a}, \bar{b}, \alpha, \sigma, \eta, \tilde{a}, a) = (35, 3, 0.87, 2, 0.2, 2, 0.5)$ and $t \in [\tilde{a}, 25]$. From Theorem 14, we take $\bar{c} = 16$ and the initial value $(x_{a1}, x_{a2}, x_{a3}) = (17.1428, 8.5714, 1.7143)$, the phase diagram is shown in Figure 1. Obviously, the equilibrium (0, 0, 0) of system (5.1) is asymptotically stable. For $\bar{c} = 20$ and the initial value $(x_{a1}, x_{a2}, x_{a3}) = (0.0171, 0.0171, 0.0171)$, Figure 2 signifies the equilibrium (0, 0, 0) of system (5.1) is unstable. In Figure 3, increase the value of $\bar{c}$ to 27, there is a chaotic phenomenon with $(x_{a1}, x_{a2}, x_{a3}) = (1.716, 3.4286, 5.1428)$.

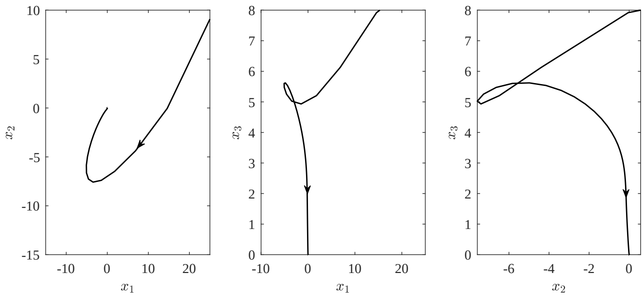


**Figure 1.** The phase diagrams for $\bar{c} = 16$.

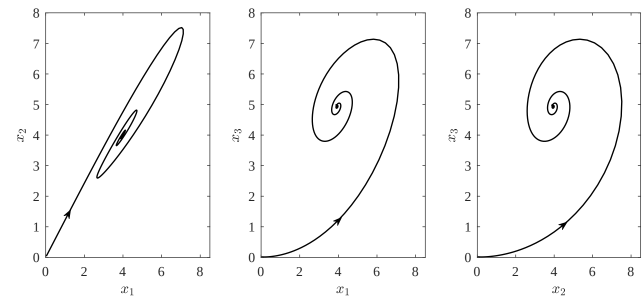


**Figure 2.** The phase diagrams for $\bar{c} = 20$.

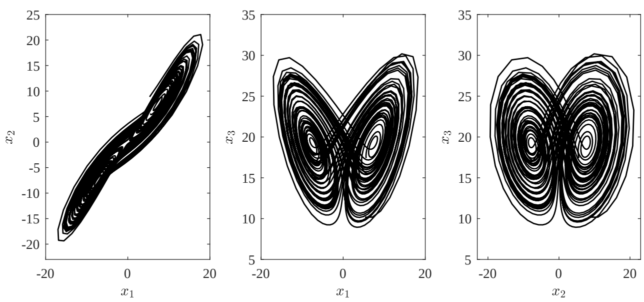


**Figure 3.** The phase diagrams for $\bar{c} = 27$.

## 6. Conclusions

The paper discusses the existence, uniqueness, and stability of solutions for generalized fractional differential equations. Using the transformation method, the solution to the generalized fractional differential equation is obtained, which shows that the initial value problem of the generalized fractional differential equation is equivalent to the nonlinear Volterra integral equation. Furthermore, we explain the solution is existing and unique by the fixed point theorems. In addition, via stability analysis, it can be concluded that the stability condition of generalized fractional differential systems is determined by the argument of eigenvalues and $\eta$. Finally, the generalized fractional Chen system is taken as an example to illustrate the theoretical results.

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## Conflict of interest

All authors declare no conflicts of interest in this paper.

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