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Research article

On the exponential Diophantine equation $\left(\frac{q^{2l}-p^{2k}}{2}n\right)^x + (p^k q^l n)^y = \left(\frac{q^{2l}+p^{2k}}{2}n\right)^z$

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Abstract: Let k, l, m_1, m_2 be positive integers and let both p and q be odd primes such that $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}$ where a is odd prime with $a \equiv 5 \pmod 8$ and $a \not\equiv 1 \pmod 5$. In this paper, using only the elementary methods of factorization, congruence methods and the quadratic reciprocity law, we show that the exponential Diophantine equation $\left(\frac{q^{2l}-p^{2k}}{2}n\right)^x + (p^k q^l n)^y = \left(\frac{q^{2l}+p^{2k}}{2}n\right)^z$ has only the positive integer solution $(x, y, z) = (2, 2, 2)$.

Keywords: exponential Diophantine equations; positive integer solution; quadratic residue
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1. Introduction

Let a, b, c be fixed positive integers. Consider the exponential Diophantine equation

$$a^x + b^y = c^z. \tag{1.1}$$

The following problem proposed by Jeśmanowicz [5] has been actively studied in the field of Eq (1.1).

Conjecture 1.1. Assume that $a^2 + b^2 = c^2$. Then the Eq (1.1) has no positive integer solution (x, y, z) other than $x = y = z = 2$.

The pioneering work related to Conjecture 1.1 was given by Sierpiński [11], he showed that $(2, 2, 2)$ is the unique positive solution of the equation $3^x + 4^y = 5^z$. In the same journal, Jeśmanowicz [5] obtained the same conclusion for the following cases:

$$(a, b, c) = (5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61);$$

and furthermore he proposed Conjecture 1.1. After these works, Conjecture 1.1 has been proved to be true for various particular cases. For recent results, we only refer to the papers of Deng et al. [3], Hu and Le [4], Miyazaki [8,10], Miyazaki et al. [9], Terai [12], Yuan and Han [16] and the references given there. Most of the existing works on Conjecture 1.1 concern the coprimality case, that is, $\gcd(a, b) = 1$. Indeed, all of the above mentioned results treat the coprimality case, and such a case is essential in the study of the Eq (1.1). Actually, the non-coprimalty case, i.e. $\gcd(a, b) > 1$, is a degenerate one in the sense that Eq (1.1) can be often solved only by local arguments using the prime factors of $\gcd(a, b)$. Recently, several authors actively studied the non-coprimality case about Conjecture 1.1. For any primitive Pythagorean triple (a, b, c), we can write

$$A = aN, B = bN, C = cN,$$

where N is a positive integer. Without loss of generality, we may assume b is even. Then, (1.1) becomes

$$(aN)^x + (bN)^y = (cN)^z. \tag{1.2}$$

This equation has been solved for some special triples (a, b, c), without assuming any conditions on N. For some results in this direction, we refer to the papers of Deng and Cohen [2], Yang and Tang [14, 15], Deng [1], Ma and Chen [7], and the references given there. In particular, Tang and Weng [13] very recently solved Eq (1.2) for the case where (a, b, c) is expressed as

$$a = 2^{2^r} - 1, b = 2^{2^{r-1}+1}, c = 2^{2^r} + 1,$$

where r is any positive integer. Note that this is the first result dealing with (1.2) for infinitely many triples (a, b, c). Miyazaki [10] extend this result as follows: If b is a power of 2, then Conjecture 1.1 is true. It is well known that any primitive Pythagorean triple (a, b, c) is parameterized as follows:

$$a = u^2 - v^2, b = 2uv, c = u^2 + v^2,$$

where u, v are co-prime positive integers of different parities with $u > v$. In this notation, the mentioned result of Tang and Weng corresponds to $(u, v) = (2^{2^{r-1}}, 1)$ with $r \geq 1$, and the result of Miyazaki [10] corresponds to $(u, v) = (2^r, 1)$ with $r \geq 1$. In this paper we consider the exponential Diophantine equation

$$\left(\frac{q^{2l} - p^{2k}}{2}n\right)^x + (p^k q^l n)^y = \left(\frac{q^{2l} + p^{2k}}{2}n\right)^z, \tag{1.3}$$

where k, l, n are positive integers and both p and q are odd primes such that $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}$, where a is odd prime with $a \equiv 5 \pmod 8$ and $a \not\equiv 1 \pmod 5$, m_1 and m_2 are positive integers. We obtain the following:

Theorem 1.1. *Let k, l, m_1, m_2 be positive integers and let both p and q be odd primes such that $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}$, where a is odd prime with $a \equiv 5 \pmod 8$ and $a \not\equiv 1 \pmod 5$. Then the Eq (1.3) has only the positive integer solution $(x, y, z) = (2, 2, 2)$.*

This paper is organized as follows. First of all, in Section 2, we show some preliminary lemmas which are needed in the proof of Theorems 1.1. Then in Section 3, we give the proof of Theorem 1.1. Finally in Section 4, we give some examples of applications of Theorems 1.1.

2. Preliminaries

In this section, we present some lemmas that will be used later.

Lemma 2.1. ([6]) *If (x, y, z) is a solution of (1.2) with $(x, y, z) \neq (2, 2, 2)$, then one of the following conditions is satisfied*
(i) $max\{x, y\} > min\{x, y\} > z$;
(ii) $x > z > y$;
(iii) $y > z > x$.

Lemma 2.2. ([2, 10]) *Assume that $n > 1$, then (1.2) has no solution (x, y, z) with $max\{x, y\} > min\{x, y\} > z$.*

Lemma 2.3. *Let k, l, m_1, m_2 be positive integers and let both p and q be odd primes such that $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}$, where a is odd prime with $a \equiv 5 \pmod 8$. Then $m_2 \equiv 1 \pmod 2$.*

Proof. On the contrary suppose that m_2 is even. If m_1 is also even, then we get from the condition

$$p^k = 2^{m_1} - a^{m_2} = (2^{\frac{m_1}{2}} + a^{\frac{m_2}{2}})(2^{\frac{m_1}{2}} - a^{\frac{m_2}{2}})$$

that

$$2^{\frac{m_1}{2}} + a^{\frac{m_2}{2}} = p^{k_1}, \quad 2^{\frac{m_1}{2}} - a^{\frac{m_2}{2}} = p^{k_2}, k_1 > k_2 \geq 0.$$

So

$$2^{\frac{m_1}{2}+1} = p^{k_2}(p^{k_1-k_2} + 1),$$

thus would give $k_2 = 0$ and $2^{\frac{m_1}{2}} - a^{\frac{m_2}{2}} = 1$. If $m_1 > 2$, then taking the equation $2^{\frac{m_1}{2}} - a^{\frac{m_2}{2}} = 1$ modulo 4 yields $-1 \equiv 1 \pmod 4$, which leads to a contradiction. Hence $m_1 = 2, m_2 = 0$, which contradicts the condition that m_2 is a positive integer. If m_1 is odd, then we get from the condition

$$q^l \equiv 2^{m_1} + a^{m_2} \equiv 2 + 1 \equiv 0 \pmod 3$$

that $q = 3$ since q is prime. Taking modulo 4 for the equation $3^l = 2^{m_1} + a^{m_2}$ would give $3^l \equiv 1 \pmod 4$. It follows that l is even and

$$1 = \left(\frac{3^l}{a}\right) = \left(\frac{2}{a}\right) = -1,$$

which leads to a contradiction. This completes the proof. □

Lemma 2.4. *Assume that $n = 1$. If (x, y, z) is a solution of the Eq (1.3) with $x \equiv y \equiv z \equiv 0 \pmod 2$, then $(x, y, z) = (2, 2, 2)$.*

Proof. It is easy to find that $m_1 \geq 3$ by the condition $p^k = 2^{m_1} - a^{m_2}$. We may write $x = 2x_1, y = 2y_1, z = 2z_1$ by the assumption $x \equiv y \equiv z \equiv 0 \pmod 2$. It follows from (1.3) that

$$\left((2^{2m_1} + a^{2m_2})^{z_1} + (2^{2m_1} - a^{2m_2})^{y_1}\right)\left((2^{2m_1} + a^{2m_2})^{z_1} - (2^{2m_1} - a^{2m_2})^{y_1}\right) = a^{2m_1 x_1} 2^{2(m_1+1)x_1}.$$

Since

$$\gcd((2^{2m_1} + a^{2m_2})^{z_1} + (2^{2m_1} - a^{2m_2})^{y_1}, (2^{2m_1} + a^{2m_2})^{z_1} - (2^{2m_1} - a^{2m_2})^{y_1}) = 2,$$

then

$$
\begin{aligned}
(2^{2m_1}+a^{2m_2})^{z_1}+(2^{2m_1}-a^{2m_2})^{y_1} &= 2^{2(m_1+1)x_1-1},\\
(2^{2m_1}+a^{2m_2})^{z_1}-(2^{2m_1}-a^{2m_2})^{y_1} &= 2\cdot a^{2m_2x_1};
\end{aligned} \tag{2.1}
$$

or

$$
\begin{aligned}
(2^{2m_1}+a^{2m_2})^{z_1}+(2^{2m_1}-a^{2m_2})^{y_1} &= 2^{2(m_1+1)x_1-1}\cdot a^{2m_2x_1},\\
(2^{2m_1}+a^{2m_2})^{z_1}-(2^{2m_1}-a^{2m_2})^{y_1} &= 2;
\end{aligned} \tag{2.2}
$$

or

$$
\begin{aligned}
(2^{2m_1}+a^{2m_2})^{z_1}+(2^{2m_1}-a^{2m_2})^{y_1} &= 2\cdot a^{2m_2x_1},\\
(2^{2m_1}+a^{2m_2})^{z_1}-(2^{2m_1}-a^{2m_2})^{y_1} &= 2^{2(m_1+1)x_1-1}.
\end{aligned} \tag{2.3}
$$

If (2.1) holds, then taking modulo 4 for the former equation we get that

$$1+(-1)^{y_1}\equiv 0 \pmod 4.$$

It follows that y_1 is odd. On the other hand, subtracting the right equation from the left one yields

$$(2^{(m_1+1)x_1-1}+a^{m_2x_1})(2^{(m_1+1)x_1-1}-a^{m_2x_1})=(2^{m_1}+a^{m_2})^{y_1}(2^{m_1}-a^{m_2})^{y_1}.$$

As

$$\gcd(2^{(m_1+1)x_1-1}+a^{m_2x_1},(2^{(m_1+1)x_1-1}-a^{m_2x_1}))=1$$

and

$$2^{m_1}+a^{m_2}=q^l, 2^{m_1}-a^{m_2}=p^k,$$

we get

$$
\begin{aligned}
2^{(m_1+1)x_1-1}+a^{m_2x_1} &= (2^{m_1}+a^{m_2})^{y_1},\\
2^{(m_1+1)x_1-1}-a^{m_2x_1} &= (2^{m_1}-a^{m_2})^{y_1}.
\end{aligned}
$$

Adding the two equations gives

$$2^{(m_1+1)x_1}=(2^{m_1}+a^{m_2})^{y_1}+(2^{m_1}-a^{m_2})^{y_1}. \tag{2.4}$$

We claim that $y_1=1$. On the contrary suppose $y_1>1$. Note that y_1 is odd, Eq (2.4) would give that

$$2^{(m_1+1)(x_1-1)}=\sum_{r=0}^{(y_1-1)/2}\binom{y_1}{2r}2^{m_1(y_1-2r-1)}a^{2rm_2}.$$

Thus $y_1a^{m_2(y_1-1)}\equiv 0 \pmod 2$, which is a contradiction. Therefore $y_1=1$ and $2^{(m_1+1)x_1}=2^{m_1+1}$ yields that $x_1=1$. Substituting these values $x=y=2$ and $n=1$ into Eq (1.3) gives $z=2$.

If (2.2) holds, then taking modulo 4 for the former equation we get that

$$1+(-1)^{y_1}\equiv 0 \pmod 4.$$

It follows that y_1 is odd. We then get taking modulo a for the former equation that

$$(-1)^{m_1 z_1} \equiv \left(2^{\frac{a-1}{2}}\right)^{m_1 z_1} \equiv -\left(2^{\frac{a-1}{2}}\right)^{m_1 y_1} \equiv -(-1)^{m_1} \pmod{a}.$$

It follows that z_1 is even. Finally adding the two equations and then dividing it by 2 gives that

$$(2^{2m_1} + a^{2m_2})^{z_1} = 2^{2(m_1+1)x_1-2} \cdot a^{2m_2 x_1} + 1, \tag{2.5}$$

which is impossible. Hence the Eq (2.2) is not true.

The Eq (2.3) is obviously not true since

$$2^{2(m_1+1)x_1-1} \geq 2 \cdot 2^{2m_1 x_1} > 2 \cdot a^{2m_2 x_1}.$$

This completes the proof. □

Lemma 2.5. *Assume that $n = 1$. Then the Eq (1.3) has only the positive integer solution $(x, y, z) = (2, 2, 2)$.*

Proof. It is easy to find that is enough to prove that $x \equiv y \equiv z \equiv 0 \pmod{2}$ by Lemma 2.4. Substituting the conditions $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}$ into the Eq (1.3) gives

$$a^{m_2 x} 2^{(m_1+1)x} + (2^{2m_1} - a^{2m_2})^y = (2^{2m_1} + a^{2m_2})^z. \tag{2.6}$$

Taking modulo 4 for the above equation we get that

$$(-1)^y \equiv 1 \pmod{4}.$$

It follows that $y \equiv 0 \pmod{2}$. We now prove that z is even. Taking modulo a for the Eq (2.6) gives

$$1 \equiv (-1)^{m_1 y} \equiv \left(2^{\frac{a-1}{2}}\right)^{m_1 y} \equiv \left(2^{\frac{a-1}{2}}\right)^{m_1 z} \equiv (-1)^{m_1 z} \pmod{a}.$$

It follows that $m_1 z \equiv 0 \pmod{2}$. We claim that $z \equiv 0 \pmod{2}$. On the contrary suppose that z is odd, then we must have that m_1 is an even. Hence m_2 is odd by Lemmas 2.3. On the other hand, since

$$\left(\frac{2}{2^{m_1} - a^{m_2}}\right) = -1, \quad \left(\frac{a}{2^{m_1} - a^{m_2}}\right) = \left(\frac{2^{m_1}}{a}\right) = 1, \quad \left(\frac{2^{2m_1} + a^{2m_2}}{2^{m_1} - a^{m_2}}\right) = \left(\frac{2 \cdot a^{2m_2}}{2^{m_1} - a^{m_2}}\right) = -1,$$

then $(-1)^x = (-1)^z$ would give x is odd. Again taking the Eq (2.6) modulo 3 will lead to $a^{m_2 x} \equiv 1 \pmod{3}$. Since m_2 and x are both odd, this means that $a \equiv 1 \pmod{3}$. Therefore, $q^l \equiv 0 \pmod{3}$, so that $q = 3$. This gives

$$1 = \left(\frac{a}{3}\right) = \left(\frac{3}{a}\right) = \left(\frac{2}{a}\right)^{m_1} = (-1)^{m_1} = -1,$$

a contradiction. Therefore z is even. Finally we prove x is also even. The congruence modulo $2^{m_1} - a^{m_2}$ of the Eq (2.6) gives

$$a^{m_2 x} 2^{(m_1+1)x} \equiv (2^{2m_1} + a^{2m_2})^z \pmod{2^{m_1} - a^{m_2}}.$$

Notice that

$$\left(\frac{a}{2^{m_1} - a^{m_2}}\right) = \left(\frac{2^{m_1}}{a}\right) = (-1)^{m_1},$$

we have that

$$(-1)^x = (-1)^{(2m_1+1)x} = \left(\frac{a}{2^{m_1} - a^{m_2}}\right)^{m_2 x} \left(\frac{2}{2^{m_1} - a^{m_2}}\right)^{(m_1+1)x} = \left(\frac{2}{2^{m_1} - a^{m_2}}\right)^z = (-1)^z.$$

This means that x is also even. This completes the proof. □

3. Proof of Theorem 1.1

Assume that (x, y, z) is a positive integer solution with $(x, y, z) \neq (2, 2, 2)$. Then we have by Lemmas 2.1, 2.2 and 2.5 that $n > 1$ and either $x > z > y$ or $y > z > x$. We shall discuss separately two cases.

The case $x > z > y$. Then dividing Eq (1.3) by n^y yields

$$(p^k q^l)^y = n^{z-y}\left(\left(\frac{q^{2l}+p^{2k}}{2}\right)^z - \left(\frac{q^{2l}-p^{2k}}{2}\right)^x n^{x-z}\right). \tag{3.1}$$

Since $\gcd\left((p^k q^l)^y, \left(\frac{q^{2l}+p^{2k}}{2}\right)^z\right) = 1$, we can observe that the two factors on the right-hand side are co-prime. Hence then Eq (3.1) yields $n = p^u$ for some positive integer u and

$$q^{ly} = \left(\frac{q^{2l}+p^{2k}}{2}\right)^z - \left(\frac{q^{2l}-p^{2k}}{2}\right)^x p^{u(x-z)}, \tag{3.2}$$

or $n = q^v$ for some positive integer v and

$$p^{ky} = \left(\frac{q^{2l}+p^{2k}}{2}\right)^z - \left(\frac{q^{2l}-p^{2k}}{2}\right)^x q^{v(x-z)}, \tag{3.3}$$

or $n = p^u q^v$ for some positive integers u and v and

$$1 = \left(\frac{q^{2l}+p^{2k}}{2}\right)^z - \left(\frac{q^{2l}-p^{2k}}{2}\right)^x p^{u(x-z)} q^{v(x-z)}. \tag{3.4}$$

If (3.2) holds, then substituting the conditions $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}$ into Eq (3.2) would give

$$2^{(m_1+1)x} a^{m_2 x} p^{u(x-z)} = (2^{2m_1} + a^{2m_2})^z - (2^{m_1} + a^{m_2})^y. \tag{3.5}$$

Taking modulo 8 for Eq (3.5) leads to $a^{m_2 y} \equiv 1 \pmod 8$. It follows that y is even since m_2 is odd by Lemma 2.3. Taking modulo 8 for equation $p^k = 2^{m_1} - a^{m_2}$ leads to $p \equiv -a \equiv 3 \pmod 8$. Again taking modulo p for Eq (3.5) leads to

$$(2 \cdot a^{2m_2})^z \equiv (2^{m_1} + a^{m_2})^y \pmod p.$$

It follows that

$$(-1)^z = \left(\frac{2}{p}\right)^z = \left(\frac{2^{m_1} + a^{m_2}}{p}\right)^y = 1.$$

Therefore z is even. Then we get from Eq (3.5) that

$$2^{(m_1+1)x} a^{m_2 x} p^{u(x-z)} = ((2^{2m_1} + a^{2m_2})^{z/2} + (2^{m_1} + a^{m_2})^{y/2})((2^{2m_1} + a^{2m_2})^{z/2} - (2^{m_1} + a^{m_2})^{y/2}).$$

Since

$$(2^{2m_1} + a^{2m_2})^{z/2} + (2^{m_1} + a^{m_2})^{y/2} \equiv 2 \pmod 4,$$

so it follows that

$$2^{(m_1+1)x-1} | (2^{2m_1} + a^{2m_2})^{z/2} - (2^{m_1} + a^{m_2})^{y/2},$$

however, this is impossible since

$$2^{(m_1+1)x-1} \geq 2^{(m_1+1)z} = (4 \cdot 2^{2m_1})^{z/2} > (2^{2m_1} + a^{2m_2})^{z/2} - (2^{m_1} + a^{m_2})^{y/2}.$$

If (3.3) holds, then substituting the conditions $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}$ into Eq (3.3) would give

$$2^{(m_1+1)x} a^{m_2 x} q^{v(x-z)} = (2^{2m_1} + a^{2m_2})^z - (2^{m_1} - a^{m_2})^y. \tag{3.6}$$

Then taking modulo 4 for Eq (3.6) leads to $(-1)^y \equiv 1 \pmod 4$. It follows that y is even. Taking modulo 8 for equation $q^k = 2^{m_1} + a^{m_2}$ leads to $q \equiv a \equiv 5 \pmod 8$. Again taking modulo q for Eq (3.6) leads to

$$(2 \cdot a^{2m_2})^z \equiv (2^{m_1} - a^{m_2})^y \pmod q.$$

It follows that

$$(-1)^z = \left(\frac{2}{q}\right)^z = \left(\frac{2^{m_1} - a^{m_2}}{q}\right)^y = 1.$$

Therefore z is even. Then similarly we get from Eq (3.6) that

$$2^{(m_1+1)x-1} | (2^{2m_1} + a^{2m_2})^{z/2} - (2^{m_1} + a^{m_2})^{y/2},$$

which is impossible by the above result that has been proved.

If (3.4) holds, then substituting the conditions $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}$ into Eq (3.4) would give

$$2^{(m_1+1)x} a^{m_2 x} p^{u(x-z)} q^{v(x-z)} = (2^{2m_1} + a^{2m_2})^z - 1. \tag{3.7}$$

Taking modulo 8 for equation $q^k = 2^{m_1} + a^{m_2}$ leads to $q \equiv a \equiv 5 \pmod 8$. Again taking modulo q for Eq (3.6) leads to

$$(2 \cdot a^{2m_2})^z \equiv 1 \pmod q.$$

It follows that $(-1)^z = \left(\frac{2}{q}\right)^z = \left(\frac{1}{q}\right) = 1$. Therefore z is even. Then similarly we get from Eq (3.7) that

$$2^{(m_1+1)x-1} | (2^{2m_1} + a^{2m_2})^{z/2} - 1,$$

which is impossible by the above result that has been proved.

The case $y > z > x$. Then dividing Eq (1.3) by n^x yields

$$a^{m_2 x} 2^{(m_1+1)x} = n^{z-x}((2^{2m_1} + a^{2m_2})^z - (2^{2m_1} - a^{2m_2})^y n^{y-z}). \tag{3.8}$$

It is easy to see that the two factors on the right-hand side are co-prime. Thus, Eq (3.8) yields $n = a^s$ for some positive integer s and

$$2^{(m_1+1)x} = (2^{2m_1} + a^{2m_2})^z - (2^{2m_1} - a^{2m_2})^y a^{s(y-z)}, \tag{3.9}$$

or $n = 2^r$ for some positive integer r with $(m_1 + 1)x = r(z - x)$ and

$$a^{m_2 x} = (2^{2m_1} + a^{2m_2})^z - (2^{2m_1} - a^{2m_2})^y 2^{r(y-z)}, \tag{3.10}$$

or $n = 2^r a^s$ for some positive integers r and s and

$$1 = (2^{2m_1} + a^{2m_2})^z - (2^{2m_1} - a^{2m_2})^y 2^{r(y-z)} a^{s(y-z)}. \tag{3.11}$$

If (3.9) holds, taking modulo 4 for Eq (3.9) leads to $(-1)^y \equiv 1 \pmod 4$. It follows that y is even. Taking modulo a for Eq (3.9) leads to

$$2^{(m_1+1)x} \equiv 2^{2m_1 z} \pmod a.$$

It follows that

$$(-1)^{(m_1+1)x} = \left(\frac{2}{a}\right)^{(m_1+1)x} = \left(\frac{2}{a}\right)^{2m_1 z} = 1.$$

It follows that $(m_1+1)x$ is even. Then we get from Eq (3.9) that

$$(2^{m_1} + a^{m_2})^y | ((2^{2m_1} + a^{2m_2})^{z/2} + 2^{(m_1+1)x/2})((2^{2m_1} + a^{2m_2})^{z/2} - 2^{(m_1+1)x/2}).$$

It follows either

$$(2^{m_1} + a^{m_2})^y | (2^{2m_1} + a^{2m_2})^{z/2} + 2^{(m_1+1)x/2},$$

or

$$(2^{m_1} + a^{m_2})^y | (2^{2m_1} + a^{2m_2})^{z/2} - 2^{(m_1+1)x/2}.$$

Hence

$$(2^{m_1} + a^{m_2})^y \le (2^{2m_1} + a^{2m_2})^{z/2} + 2^{(m_1+1)x/2},$$

which is impossible since

$$(2^{m_1} + a^{m_2})^y > (2^{2m_1} + a^{2m_2} + 2^{m_1+1} \cdot a^{m_2})^{z/2} > (2^{2m_1} + a^{2m_2})^{z/2} + 2^{(m_1+1)x/2}.$$

If (3.10) holds, we first prove that $r(y-z) = 2$. In fact, taking modulo 4 for Eq (3.10) yields $2^{r(y-z)} \equiv 0 \pmod 4$. Therefore $r(y-z) \ge 2$. On the other hand, if $r(y-z) \ge 3$, then taking modulo 8 for Eq (3.10) leads to $a^{m_2 x} \equiv 1 \pmod 8$. It follows that $m_2 x$ is even. Taking modulo 3 for Eq (3.10) leads to $1 \equiv a^{m_2 x} \equiv 2^z \equiv (-1)^z \pmod 3$, which implies that z is even. Then we get from Eq (3.10) that

$$(2^{m_1} + a^{m_2})^y | ((2^{2m_1} + a^{2m_2})^{z/2} + a^{m_2 x/2})((2^{2m_1} + a^{2m_2})^{z/2} - a^{m_2 x/2}).$$

It follows either

$$(2^{m_1} + a^{m_2})^y | (2^{2m_1} + a^{2m_2})^{z/2} + a^{m_2 x/2},$$

or

$$(2^{m_1} + a^{m_2})^y | (2^{2m_1} + a^{2m_2})^{z/2} - a^{m_2 x/2}.$$

Hence

$$(2^{m_1} + a^{m_2})^y \le (2^{2m_1} + a^{2m_2})^{z/2} + a^{m_2 x/2},$$

which is impossible since

$$(2^{m_1} + a^{m_2})^y > (2^{2m_1} + a^{2m_2} + 2^{m_1+1} \cdot a^{m_2})^{z/2} > (2^{2m_1} + a^{2m_2})^{z/2} + a^{m_2 x/2}.$$

So $r(y-z) \le 2$ and $r(y-z) = 2$. We now prove that

$$m_1 \equiv m_2 \equiv x \equiv z \equiv 1 \pmod 2, \quad y \equiv 0 \pmod 2.$$

First taking modulo 8 for Eq (3.10) we get that

$$a^{m_2 x} \equiv 1 - 4 \equiv 5 \pmod 8.$$

It follows that $m_2 x$ is odd. Again taking modulo $2^{m_1} - a^{m_2}$ for Eq (3.10) leads to

$$a^{m_2 x} \equiv (2 \cdot a^{2m_2})^z \pmod{2^{m_1} - a^{m_2}}.$$

It follows that

$$(-1)^{m_1 m_2 x} = \left(\frac{a}{2^{m_1} - a^{m_2}}\right)^{m_2 x} = \left(\frac{2}{2^{m_1} - a^{m_2}}\right)^{z} = (-1)^z$$

since

$$\left(\frac{a}{2^{m_1} - a^{m_2}}\right) = \left(\frac{2}{a}\right)^{m_1} = (-1)^{m_1}.$$

Thus $m_1 m_2 x \equiv z \pmod 2$. If z is even, then we get that m_1 is also even. Finally taking modulo a for Eq (3.10) leads to $2^{2m_1 z} \equiv 2^{2m_1 y+2} \pmod a$. It follows that

$$1 \equiv (-1)^{m_1 z} \equiv (2^{\frac{a-1}{2}})^{m_1 z} \equiv (2^{\frac{a-1}{2}})^{m_1 y+1} \equiv (-1)^{m_1 y+1} = -1 \pmod a,$$

which leads to a contradiction. Therefore we must have

$$m_1 \equiv m_2 \equiv x \equiv z \equiv 1 \pmod 2, \quad y \equiv 0 \pmod 2.$$

Notice that $(m_1 + 1)x = r(z - x)$ and $r(y - z) = 2$, we must have that

$$z = \frac{(m_1+3)x}{2}, \;\; y = \frac{(m_1+3)x}{2} + 1, \;\; m_1 \equiv 3 \pmod 4, \quad m_2 \equiv x \equiv 1 \pmod 2. \tag{3.12}$$

Taking modulo 3 for Eq (3.10) leads to $a \equiv a^{m_2 x} \equiv 2^z \equiv 2 \pmod 3$. Taking modulo 3 for equation $p^k = 2^{m_1} - a^{m_2}$ leads to $p^k \equiv 0 \pmod 3$. It follows that $p = 3$ and taking modulo 4 for equation $3^k = 2^{m_1} - a^{m_2}$ we get that k is odd. If $k \equiv 3 \pmod 4$, we then get taking modulo 5 for equation $3^k = 2^{m_1} - a^{m_2}$ that $a \equiv 1 \pmod 5$, which contradicts to the condition $a \not\equiv 1 \pmod 5$. If $k \equiv 1 \pmod 4$, then taking modulo 5 for equation $3^k = 2^{m_1} - a^{m_2}$ leads to

$$a^{m_2} \equiv 0 \pmod 5.$$

It follows that $a = 5$ since a is prime. Substituting the Eq (3.12) and $a = 5$ into the Eq (3.10), we get that

$$5^{m_2 x} = (2^{2m_1} + 5^{2m_2})^{\frac{(m_1+3)x}{2}} - 4 \cdot (2^{2m_1} - 5^{2m_2})^{\frac{(m_1+3)x}{2}+1}. \tag{3.13}$$

If $m_1 \equiv m_2 \pmod 3$, say $m_1 \equiv m_2 \equiv \lambda \pmod 3$. Then applying Fermat's little theorem to Eq (3.13) yields

$$2^{2m_1} - 5^{2m_2} \equiv 2^{2\lambda} - (-2)^{2\lambda} \equiv 0 \pmod 7,$$

which implies that $5^{m_2 x} \equiv 2^{(2\lambda+1)z} \pmod 7$. This leads to $-1 = \left(\frac{5}{7}\right) = \left(\frac{2}{7}\right) = 1$, a contradiction. So in the following discussion we will assume that $m_1 \not\equiv m_2 \pmod 3$. We now distinguish three cases.

Case 1: $m_1 \equiv 0 \pmod 3$. Then $m_1 \equiv 3 \pmod{12}$.

Subcase 1.1: $m_2 \equiv 1 \pmod 3$. Then $m_2 \equiv 1 \pmod 6$. Applying Fermat's little theorem to Eq (3.13), we get that $2^x \equiv 3 \pmod 7$. It follows that

$$1 = \left(\frac{2}{7}\right) = \left(\frac{3}{7}\right) = -\left(\frac{7}{3}\right) = -1,$$

which is a contradiction.

Subcase 1.2: $m_2 \equiv 2 \pmod 3$. Then $m_2 \equiv 5 \pmod 6$. Applying Fermat's little theorem to Eq (3.13), we get that $4^x \equiv 5 \pmod 7$. It follows that

$$1 = \left(\frac{4^x}{7}\right) = \left(\frac{5}{7}\right) = \left(\frac{2}{5}\right) = -1,$$

which is a contradiction.

Case 2: $m_1 \equiv 1 \pmod 3$. Then $m_1 \equiv 7 \pmod{12}$.

Subcase 2.1: $m_2 \equiv 0 \pmod 3$. Then $m_2 \equiv 3 \pmod 6$. Applying Fermat's little theorem to Eq (3.13), we get that $2^{x+1} + 2^{2x} \equiv 1 \pmod 7$. If $x \equiv 0 \pmod 3$, then $x \equiv 3 \pmod 6$, which yields to $3 \equiv 1 \pmod 7$, a contradiction. If $x \equiv 2 \pmod 3$, then $x \equiv 5 \pmod 6$, which yields to $3 \equiv 1 \pmod 7$, a contradiction again. Therefore $x \equiv 1 \pmod 3$, then $x \equiv 1 \pmod 6$. Again applying Euler's theorem to Eq (3.13), we have that $-1 \equiv 5^5 \equiv -7 \pmod 9$, which is also impossible.

Subcase 2.2: $m_2 \equiv 2 \pmod 3$. Then $m_2 \equiv 5 \pmod 6$. Applying Fermat's little theorem to Eq (3.13), we get that $-1 \equiv 0 \pmod 7$, which is a contradiction.

Case 3: $m_1 \equiv 2 \pmod 3$. Then $\frac{m_1+3}{2} \equiv 7 \pmod{12}$ or $\frac{m_1+3}{2} \equiv 1 \pmod{12}$.

Subcase 3.1: $m_2 \equiv 0 \pmod 3$. Then $m_2 \equiv 3 \pmod 6$. Applying Fermat's little theorem to Eq (3.13), we get that $3^{x-1} \equiv 1 \pmod 7$. It follows that $x \equiv 1 \pmod 6$, which implies either $x \equiv 1 \pmod{12}$ or $x \equiv 7 \pmod{12}$. If the first case holds, then applying Fermat's little theorem to Eq (3.13), we get either

$$5, 8 \equiv 5^{m_2} \equiv (2^{22} + 5^6)^7 - 4(2^{22} - 5^6)^8 \equiv 12 \pmod{13},$$

or

$$5, 8 \equiv 5^{m_2} \equiv (2^{22} + 5^6) - 4(2^{22} - 5^6)^2 \equiv 6 \pmod{13},$$

which leads to a contradiction. If the last case holds, then using Fermat's little theorem to Eq (3.13), we get either

$$5, 8 \equiv 5^{7m_2} \equiv (2^{22} + 5^6) - 4(2^{22} - 5^6)^2 \equiv 6 \pmod{13},$$

or

$$5, 8 \equiv 5^{7m_2} \equiv (2^{22} + 5^6)^7 - 4(2^{22} - 5^6)^8 \equiv 12 \pmod{13},$$

which leads to a contradiction again.

Subcase 3.2: $m_2 \equiv 1 \pmod 3$. Then $m_2 \equiv 1 \pmod 6$. Using Fermat's little theorem to Eq (3.13), we get that

$$5^x \equiv (2^{22} + 5^2)^{7x} - 4(2^{22} - 5^2)^{7x+1} \equiv -1 - 2^x \equiv -1 + 5^x \pmod 7.$$

This leads to $0 \equiv 1 \pmod 7$, a contradiction.

If (3.11) holds, taking modulo $2^{m_1} - a^{m_2}$ for Eq (3.11) leads to

$$(2 \cdot a^{2m_2})^z \equiv 1 \pmod{2^{m_1} - a^{m_2}}.$$

It follows that

$$(-1)^z = \left(\frac{2}{2^{m_1} - a^{m_2}}\right)^z = \left(\frac{1}{2^{m_1} - a^{m_2}}\right) = 1.$$

Thus z is even. Then we get from Eq (3.11) that

$$(2^{m_1} + a^{m_2})^y | ((2^{2m_1} + a^{2m_2})^{z/2} + 1)((2^{2m_1} + a^{2m_2})^{z/2} - 1).$$

It follows either

$$(2^{m_1} + a^{m_2})^y | (2^{2m_1} + a^{2m_2})^{z/2} + 1$$

or

$$(2^{m_1} + a^{m_2})^y | (2^{2m_1} + a^{2m_2})^{z/2} - 1.$$

Hence

$$(2^{m_1} + a^{m_2})^y \leq (2^{2m_1} + a^{2m_2})^{z/2} + 1,$$

which is impossible since

$$(2^{m_1} + a^{m_2})^y > (2^{2m_1} + a^{2m_2} + 2^{m_1+1} \cdot a^{m_2})^{z/2} > (2^{2m_1} + a^{2m_2})^{z/2} + 1.$$

This completes the proof.

4. Applications

In this section, we give some examples of applications of the result (see Table 1).

Table 1. Some examples of applications of the Theorem1.1.

p^k	q^l	m_1	m_2	a
3	13	3	1	5
3	29	4	1	13
3^3	37	5	1	5
3	61	5	1	29
3^3	101	6	1	37
3^5	269	8	1	13
3	1021	9	1	509
3^5	7949	12	1	3853
3^5	16141	13	1	7949

From the Table 1, one can easily see that the Conjecture 1.1 is true for the following cases:

$$(a, b, c) = (80n, 39n, 89n), (416n, 87n, 425n), (320n, 999n, 1049n), (1856n, 183n, 1865n),$$

$$(4736n, 2727n, 5465n), (6656n, 65367n, 65705n), (521216n, 3063n, 521225n),$$

$$(31563776n, 1931607n, 31622825n), (130236416n, 3922263n, 130295465n).$$

5. Conclusions

Jeśmanowicz' conjecture is true for the following set of Pythagorean numbers:

$$\frac{q^{2l}-p^{2k}}{2}n,\ p^kq^ln,\ \frac{q^{2l}+p^{2k}}{2}n,$$

where p and q are odd primes such that $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}$, a is odd prime with $a \equiv 5 \pmod 8$ and $a \not\equiv 1 \pmod 5$.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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