Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 8250–8262.
DOI: 10.3934/math.2022460
Received: 13 October 2021
Revised: 22 January 2022
Accepted: 21 February 2022
Published: 25 February 2022

Research article

Extension-closed subcategories in extriangulated categories

Lingling Tan and Tiwei Zhao*

School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China

* **Correspondence:** Email: tiweizhao@qfnu.edu.cn.

Abstract: In this paper, we mainly focus on extension-closed subcategories of extriangulated categories. Let $\mathcal{X}$ be an extension-closed subcategory. We show that if C is $\mathcal{X}$-projective and there is a minimal right almost split deflation in $\mathcal{X}$ ending by C, then there is an $\mathfrak{s}$-triangle ending by C which is very similar to an Auslander-Reiten triangle in $\mathcal{X}$. We also show that if the extriangulated category admits a negative first extension $\mathbb{E}^{-1}$, and $\mathcal{X}$ is self-orthogonal with respect to $\mathbb{E}^{-1}$, then $\mathcal{X}$ has an exact structure.

Keywords: extriangulated categories; extension-closed subcategories; almost split morphism; negative first extension; exact categories
Mathematics Subject Classification: 18G25, 18G80

1. Introduction

Exact and triangulated categories are two important structures in category theory. Recently, Nakaoka and Palu [14] introduced the notion of extriangulated categories as a simultaneous generalization of exact categories and extension-closed subcategories of triangulated categories. After that, the study of extriangulated categories has become an active topic, and up to now, many results on exact categories and triangulated categories can be unified in the same framework.

Extension-closed subcategories form an important class of subcategories in representation theory of algebra. The existence of Auslander-Reiten sequences (resp., triangles) in extension-closed subcategories of abelian (resp., triangulated) categories is a very important topic in representation theory, for example, see [4, 12] for the abelian case and [10] for the triangulated case. Jørgensen studied the Auslander-Reiten triangles in subcategories ending at non-Ext-projective objects in [11, Theorem 3.1], and Iyama, Nakaoka and Palu studied the extriangulated version in [8, Proposition 5.15]. In [6], Fedele considered minimal right almost split morphisms ending at Ext-projective objects in an extension-closed category of a triangulated category, and found something quite similar to Auslander-Reiten triangles in a suitable extension-closed subcategory. Extension-closed subcategories admit sometimes nice categorical structures, for example, extension-closed subcategories of abelian categories are exact categories. Extension-closed subcategories of triangulated categories are not necessarily triangulated, but they are extriangulated. In [11], Jørgensen showed that an extension-closed subcategory of a triangulated category admits an exact structure under some assumption. In [17], Zhou considered n-extension closed subcategories of $(n+2)$-angulated categories. Therefore, it is natural to ask whether there is a framework in the setting of extriangulated categories for these results. The aim of this paper is to prove the analogues of the triangulated results [6, Theorem A] and [11, Proposition 2.5(i)] into the extriangulated setup.

In Section 2, we recall some terminologies. In Section 3, we show that given an $\mathfrak{s}$-triangle $Y \xrightarrow{\alpha} X' \xrightarrow{\beta} X \overset{\delta}{\dashrightarrow}$ in $\mathcal{K}$, if β is a minimal right almost split morphism in $\mathcal{X}$, and X is $\mathcal{X}$-projective, then: (1) $Y \notin \mathcal{X}$, (2) α is an $\mathcal{X}$-envelope of Y, (3) Y is indecomposable. This generalizes Fedele's result [6, Theorem A]. In Section 4, let $\mathcal{K}$ be an extriangulated category admitting a negative first extension $\mathbb{E}^{-1}$. In general, extension-closed subcategories of extriangulated categories are still extriangulated, but not necessarily exact. We show that if $\mathcal{X}$ is an extension-closed subcategory of $\mathcal{K}$ satisfying $\mathbb{E}^{-1}(\mathcal{X},\mathcal{X}) = 0$, then $\mathcal{X}$ is an exact category, which generalizes Jørgensen's result [11, Proposition 2.5(i)].

2. Preliminaries

We first recall some notions from [14].

In this section, $\mathcal{K}$ is an additive category and $\mathbb{E} : \mathcal{K}^{\rm op} \times \mathcal{K} \to \mathfrak{Ab}$ is a biadditive functor, where $\mathfrak{Ab}$ is the category of abelian groups.

Let $A, C \in \mathcal{K}$. An element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension*. Two sequences of morphisms

$$A \xrightarrow{x} B \xrightarrow{y} C \ \text{ and } \ A \xrightarrow{x'} B' \xrightarrow{y'} C$$

in $\mathcal{K}$ are said to be *equivalent* if there exists an isomorphism $b \in \operatorname{Hom}_{\mathcal{K}}(B, B')$ such that $x' = bx$ and $y = y'b$. We denote by $[\, A \xrightarrow{x} B \xrightarrow{y} C \,]$ the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$. In particular, we write $0 := [\, A \xrightarrow{\binom{\mathrm{id}_A}{0}} A \oplus C \xrightarrow{(0\ \mathrm{id}_C)} C \,]$.

For an $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$, we briefly write

$$a_\star\delta := \mathbb{E}(C, a)(\delta) \text{ and } c^\star\delta := \mathbb{E}(c, A)(\delta).$$

For two $\mathbb{E}$-extensions $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$, a *morphism* from δ to δ' is a pair (a, c) of morphisms with $a \in \operatorname{Hom}_{\mathcal{K}}(A, A')$ and $c \in \operatorname{Hom}_{\mathcal{K}}(C, C')$ such that $a_\star\delta = c^\star\delta'$.

Definition 2.1. ([14, Definition 2.9]) Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [\, A \xrightarrow{x} B \xrightarrow{y} C \,]$ to each $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. Such $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ provided that it satisfies the following condition.

(R) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions with

$$\mathfrak{s}(\delta) = [\, A \xrightarrow{x} B \xrightarrow{y} C \,] \text{ and } \mathfrak{s}(\delta') = [\, A' \xrightarrow{x'} B' \xrightarrow{y'} C' \,].$$

Then for any morphism $(a, c) : \delta \to \delta'$, there exists $b \in \mathrm{Hom}_{\mathcal{K}}(B, B')$ such that the following diagram

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[d, "c"] \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \end{tikzcd}$$

commutes.

Let $\mathfrak{s}$ be a realization of $\mathbb{E}$. If $\mathfrak{s}(\delta) = [\, A \xrightarrow{x} B \xrightarrow{y} C \,]$ for some $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$, then one says that the sequence $A \xrightarrow{x} B \xrightarrow{y} C$ *realizes* δ; and in the condition (R), the triple (a, b, c) *realizes* the morphism (a, c).

For any two equivalence classes $[\, A \xrightarrow{x} B \xrightarrow{y} C \,]$ and $[\, A' \xrightarrow{x'} B' \xrightarrow{y'} C' \,]$, we define

$$[\, A \xrightarrow{x} B \xrightarrow{y} C \,] \oplus [\, A' \xrightarrow{x'} B' \xrightarrow{y'} C' \,] := [\, A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C' \,].$$

Definition 2.2. ([14, Definition 2.10]) A realization $\mathfrak{s}$ of $\mathbb{E}$ is called *additive* if it satisfies the following conditions.

(1) For any $A, C \in \mathcal{K}$, the split $\mathbb{E}$-extension $0 \in \mathbb{E}(C, A)$ satisfies $\mathfrak{s}(0) = 0$.

(2) For any pair of $\mathbb{E}$-extensions $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$, we have $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$.

Definition 2.3. ([14, Definition 2.12]) The triple $(\mathcal{K}, \mathbb{E}, \mathfrak{s})$ is called an *externally triangulated* (or *extriangulated* for short) category if it satisfies the following conditions.

(ET1) $\mathbb{E} : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathfrak{Ab}$ is a biadditive functor.

(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.

(ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions with

$$\mathfrak{s}(\delta) = [\, A \xrightarrow{x} B \xrightarrow{y} C \,] \text{ and } \mathfrak{s}(\delta') = [\, A' \xrightarrow{x'} B' \xrightarrow{y'} C' \,].$$

For any commutative diagram

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \end{tikzcd}$$

in $\mathcal{K}$, there exists a morphism $(a, c) : \delta \to \delta'$ which is realized by the triple (a, b, c).

(ET3)$^{\mathrm{op}}$ Dual of (ET3).

(ET4) Let $\delta \in \mathbb{E}(C, A)$ and $\rho \in \mathbb{E}(F, B)$ be any pair of $\mathbb{E}$-extensions with

$$\mathfrak{s}(\delta) = [\, A \xrightarrow{x} B \xrightarrow{y} C \,] \text{ and } \mathfrak{s}(\rho) = [\, B \xrightarrow{u} D \xrightarrow{v} F \,].$$

Then there exist an object $E \in \mathcal{K}$, an $\mathbb{E}$-extension ξ with $\mathfrak{s}(\xi) = [\, A \xrightarrow{z} D \xrightarrow{w} E \,]$, and a commutative diagram

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, equal] & B \arrow[r, "y"] \arrow[d, "u"] & C \arrow[d, "s"] \\ A \arrow[r, "z"] & D \arrow[r, "w"] \arrow[d, "v"] & E \arrow[d, "t"] \\ & F \arrow[r, equal] & F \end{tikzcd}$$

in $\mathcal{K}$, which satisfy the following compatibilities.

(i) $\mathfrak{s}(y_\star\rho) = [\, C \xrightarrow{s} E \xrightarrow{t} F \,]$.

(ii) $s^\star\xi = \delta$.

(iii) $x_\star\xi = t^\star\rho$.

(ET4)$^{\mathrm{op}}$ Dual of (ET4).

Definition 2.4. ([14, Definitions 2.15, 2.17 and 2.19]) Let $\mathcal{K}$ be an extriangulated category.

(1) A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ in $\mathcal{K}$ is called a *conflation* if it realizes some $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. In this case, x is called an *inflation* and y is called a *deflation*.

(2) If a conflation $A \xrightarrow{x} B \xrightarrow{y} C$ in $\mathcal{K}$ realizes $\delta \in \mathbb{E}(C, A)$, the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ is called an $\mathbb{E}$-*triangle* (or $\mathfrak{s}$-*triangle*, *extriangle*), and write it in the following way:

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow} .$$

We usually do not write this "δ" if it is not used in the argument.

(3) Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be any pair of $\mathbb{E}$-triangles. If a triplet (a, b, c) realizes $(a, c) : \delta \to \delta'$, then we write it as

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] & B \arrow[r, "y"] \arrow[d, "b"'] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"'] & {} \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \arrow[r, dashed, "\delta'"] & {} \end{tikzcd}$$

and call (a, b, c) a *morphism* of $\mathbb{E}$-triangles.

If a, b, c above are isomorphisms, then $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ are said to be *isomorphic*.

(4) Let $\mathcal{X}$ be a full additive subcategory of $\mathcal{K}$, closed under isomorphisms. The subcategory $\mathcal{X}$ is said to be *extension-closed* (or equivalently, *closed under extensions*) if, for any conflation $A \to B \to C$ which satisfies $A, C \in \mathcal{X}$, then $B \in \mathcal{X}$.

Example 2.5. Both exact categories and triangulated categories are extriangulated categories (see [14, Proposition 3.22]) and extension closed subcategories of extriangulated categories are again extriangulated (see [14, Remark 2.18]). Moreover, there exist extriangulated categories which are neither exact categories nor triangulated categories, see [14, Proposition 3.30]. For more examples, see also [7, Theorem 1.2], [16, Example 2.8], [18, Corollary 4.12 and Remark 4.13].

Condition (WIC):

(1) Let $f \in \mathrm{Hom}_{\mathcal{K}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{K}}(B, C)$ be any composable pair of morphisms. If gf is an inflation, then so is f.
(2) Let $f \in \mathrm{Hom}_{\mathcal{K}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{K}}(B, C)$ be any composable pair of morphisms. If gf is a deflation, then so is g.

3. Envelopes and almost split morphisms in extension-closed subcategories

Assume that k is a field, and $\mathcal{K}$ a skeletally small k-linear Hom-finite Krull-Schmidt extriangulated category satisfying Condition (WIC). All subcategories are full, and closed under isomorphisms and direct summands.

Assume that $\mathcal{X}$ is an additive subcategory of $\mathcal{K}$ which is closed under extensions. We will investigate minimal right almost split deflations in $\mathcal{X}$, and give a relation between minimal right almost split deflations of $\mathcal{X}$-projective objects and $\mathcal{X}$-envelopes in an $\mathfrak{s}$-triangle which generalizes [6, Theorem A]. The results follow closely part of Sections 2 and 3 from [6] and the proofs also follow by very similar arguments.

Definition 3.1. Let $A, B \in \mathcal{X}$.

(1) A morphism $\alpha : A \to B$ is called *right almost split* in $\mathcal{X}$ if α is not a split epimorphism, and for any $C \in \mathcal{X}$, any morphism $\beta : C \to B$ which is not a split epimorphism factors through α. Dually, the notion of *left almost split morphisms* is given.
(2) A morphism $\alpha : A \to B$ is called *minimal right almost split* in $\mathcal{X}$ if it is both right minimal and right almost split in $\mathcal{X}$. A minimal left almost split morphism in $\mathcal{X}$ is defined dually.

Lemma 3.2. *Let $\beta : B \to C$ be a right almost split morphism in $\mathcal{X}$. Then C is indecomposable. Moreover, if β is right minimal and $\beta' : B' \to C$ is a minimal right almost split morphism in $\mathcal{X}$, then there is an isomorphism $\varphi : B \to B'$ such that $\beta = \beta'\varphi$.*

Proof. The proof follows from the same argument of [6, Proposition 2.7]. □

Definition 3.3. An object $X \in \mathcal{X}$ is called *$\mathcal{X}$-injective* if $\mathbb{E}(\mathcal{X}, X) = 0$. An object $X \in \mathcal{X}$ is called *$\mathcal{X}$-projective* if $\mathbb{E}(X, \mathcal{X}) = 0$.

Lemma 3.4. *([6, Lemma 3.4]) Assume that $f = (f_1, f_2) : A_1 \oplus A_2 \to B$ is right minimal in $\mathcal{K}$. Then f_1 and f_2 are right minimal.*

Lemma 3.5. *Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \overset{\delta}{\dashrightarrow}$ be an $\mathfrak{s}$-triangle in $\mathcal{K}$. Then*

(1) $\beta \in \mathrm{rad}_{\mathcal{K}}(B, C)$ if and only if α is left minimal.
(2) $\alpha \in \mathrm{rad}_{\mathcal{K}}(A, B)$ if and only if β is right minimal.

Proof. (1) By [14, Proposition 3.3], there is an exact sequence

$$\mathrm{Hom}_{\mathcal{K}}(C, B) \xrightarrow{\mathrm{Hom}_{\mathcal{K}}(\beta, B)} \mathrm{Hom}_{\mathcal{K}}(B, B) \xrightarrow{\mathrm{Hom}_{\mathcal{K}}(\alpha, B)} \mathrm{Hom}_{\mathcal{K}}(A, B).$$

Then the result holds by the dual of [9, Lemma 1.1].

(2) is similar. □

Lemma 3.6. *Let $X \in \mathcal{X}$ be indecomposable $\mathcal{X}$-projective, and $Y \overset{\alpha}{\longrightarrow} X' \overset{\beta}{\longrightarrow} X \overset{\delta}{\dashrightarrow}$ be a non-split $\mathfrak{s}$-triangle with $X' \in \mathcal{X}$. Then α is an $\mathcal{X}$-envelope of Y.*

Moreover, if β is right minimal, then Y is indecomposable.

Proof. Applying $\mathrm{Hom}_{\mathcal{K}}(-, \mathcal{X})$ to the given $\mathfrak{s}$-triangle, we have an exact sequence

$$\mathrm{Hom}_{\mathcal{K}}(X', \mathcal{X}) \xrightarrow{\mathrm{Hom}_{\mathcal{K}}(\alpha, \mathcal{X})} \mathrm{Hom}_{\mathcal{K}}(Y, \mathcal{X}) \longrightarrow \mathbb{E}(X, \mathcal{X})\ .$$

Since X is $\mathcal{X}$-projective, $\mathbb{E}(X, \mathcal{X}) = 0$, and hence $\mathrm{Hom}_{\mathcal{K}}(\alpha, \mathcal{X})$ is epic. This shows that α is an $\mathcal{X}$-preenvelope of Y.

Since $\mathcal{K}$ is Krull-Schmidt, we can write $X' = X'_1 \oplus \cdots \oplus X'_t$, where $X'_1, \cdots, X'_t$ are indecomposable, and write $\beta = (\beta_1, \cdots, \beta_t)$, where $\beta_i : X'_i \to X$, $i = 1, \cdots, t$. Since $\delta \neq 0$, β is not a split epimorphism, and hence each β_i is not a split epimorphism. By [3, Appendix, Proposition 3.5], each $\beta_i \in \mathrm{rad}_{\mathcal{K}}(X'_i, X)$. Thus $\beta \in \mathrm{rad}_{\mathcal{K}}(X', X)$ by [3, Appendix, Lemma 3.4]. By Lemma 3.5(1), α is left minimal, and hence α is an $\mathcal{X}$-envelope of Y.

Now let β be right minimal, and let $Y = Y_1 \oplus \cdots \oplus Y_r$, where each Y_i is indecomposable. Let $\mathrm{inc} : Y_i \to Y$ be the inclusion. For any $X'' \in \mathcal{X}$, and any $g : Y_i \to X''$, consider the diagram

$$\begin{tikzcd}
Y_i \arrow[rr, "\alpha\circ\mathrm{inc}"] \arrow[dr, "\mathrm{inc}"] \arrow[ddr, "g"'] & & X' \arrow[ddl, dotted, "h"] \\
& Y \arrow[ur, "\alpha"'] \arrow[d, dotted, "\overline{g}"] & \\
& X''. &
\end{tikzcd}$$

Then there exists $\overline{g} : Y \to X''$ such that $g = \overline{g} \circ \mathrm{inc}$. Since α is an $\mathcal{X}$-envelope of Y, there is $h : X' \to X''$ with $\overline{g} = h\alpha$. Thus $g = \overline{g} \circ \mathrm{inc} = h \circ \alpha \circ \mathrm{inc}$, which shows that $\alpha \circ \mathrm{inc} : Y_i \to X'$ is an $\mathcal{X}$-preenvelope of Y_i. Since $\mathcal{K}$ is Krull-Schmidt, there exists an $\mathcal{X}$-envelope $\alpha_i : Y_i \to X_i$. Thus we get an $\mathcal{X}$-envelope $\oplus_{i=1}^r \alpha_i : \oplus_{i=1}^r Y_i \to \oplus_{i=1}^r X_i$. But $\mathcal{X}$-envelopes are unique up to isomorphisms, so we may assume that $\alpha = \oplus_{i=1}^r \alpha_i$ and $X' = \oplus_{i=1}^r X_i$. Note that since each $\alpha_i : Y_i \to X_i$ is an $\mathcal{X}$-envelope, we have a commutative diagram

$$\begin{tikzcd}
Y_i \arrow[r, "\alpha_i"] \arrow[d, "\mathrm{inc}"] & X_i \arrow[d, dotted] \\
Y \arrow[r, "\alpha"] & X'.
\end{tikzcd}$$

Then $\alpha \circ \mathrm{inc}$ is an inflation implies that α_i is an inflation by Condition (WIC). Thus there exists an $\mathfrak{s}$-triangle $Y_i \overset{\alpha_i}{\longrightarrow} X_i \overset{\gamma_i}{\longrightarrow} Z_i \overset{\delta_i}{\dashrightarrow}$. Consider the following diagram

$$\begin{tikzcd}
Y_i \arrow[r, "\alpha_i"] \arrow[d, "\mathrm{inc}"] & X_i \arrow[r, "\gamma_i"] \arrow[d, "\mathrm{inc}"] & Z_i \arrow[r, dashed, "\delta_i"] & {} \\
Y \arrow[r, "\alpha"] & X' \arrow[r, "\beta"] & X \arrow[r, dashed, "\delta"] & {}.
\end{tikzcd}$$

where the left square is commutative. By (ET3), there exists $w_i : Z_i \to X$ such that the following diagram

$$\begin{tikzcd} Y_i \arrow[r, "\alpha_i"] \arrow[d, "\mathrm{inc}"] & X_i \arrow[r, "\gamma_i"] \arrow[d, "\mathrm{inc}"] & Z_i \arrow[r, dashed, "\delta_i"] \arrow[d, "w_i"] & {} \\ Y \arrow[r, "\alpha"] & X' \arrow[r, "\beta"] & X \arrow[r, dashed, "\delta"] & {} \end{tikzcd}$$

commutates. Therefore, we get a commutative diagram

$$\begin{tikzcd} \oplus_{i=1}^t Y_i \arrow[r, "\oplus_{i=1}^t \alpha_i"] \arrow[d, equal] & \oplus_{i=1}^t X_i \arrow[r, "\oplus_{i=1}^t \gamma_i"] \arrow[d, equal] & \oplus_{i=1}^t Z_i \arrow[r, dashed, "\oplus_{i=1}^t \delta_i"] \arrow[d, "\oplus_{i=1}^t w_i"] & {} \\ Y \arrow[r, "\alpha"] & X' \arrow[r, "\beta"] & X \arrow[r, dashed, "\delta"] & {} \end{tikzcd}.$$

By [14, Corollary 3.6], $X \cong \oplus_{i=1}^t Z_i$. But X is indecomposable, so we may assume that $X \cong Z_1$, and $Z_i = 0$ for $i \neq 1$. This shows that $w_i = \gamma_i = 0$ and α_i is an isomorphism for $i \neq 1$.

Now suppose that $\beta = (\beta_1, \cdots, \beta_r)$ is right minimal, by Lemma 3.4, each β_i is right minimal. On the other hand, for $i \neq 1, \beta_i = \beta \circ \mathrm{inc} = w_i \gamma_i = 0$, thus

$$\begin{aligned} \beta \circ 0 = 0 = \beta = \beta \circ 1_{B_i},\ \forall i \neq 1 &\Rightarrow 0 = 1_{B_i},\ \forall i \neq 1 \\ &\Rightarrow X_i = 0,\ \forall i \neq 1 \\ &\Rightarrow Y_i = 0,\ \forall i \neq 1 \\ &\Rightarrow Y = Y_1 \text{ is indecomposable.} \end{aligned}$$

□

Theorem 3.7. *Let* $Y \xrightarrow{\alpha} X' \xrightarrow{\beta} X \overset{\delta}{\dashrightarrow}$ *be an* $\mathfrak{s}$*-triangle in* $\mathcal{K}$*. Assume that* β *is a minimal right almost split morphism in* $\mathcal{X}$*, and* X *is* $\mathcal{X}$*-projective. Then*

(i) (1) $Y \notin \mathcal{X}$*,*

(2) α *is an* $\mathcal{X}$*-envelope of* Y*,*

(3) Y *is indecomposable.*

(ii) If there is an $\mathfrak{s}$*-triangle* $\widetilde{Y} \xrightarrow{\widetilde{\alpha}} \widetilde{X'} \xrightarrow{\widetilde{\beta}} \widetilde{X} \overset{\widetilde{\delta}}{\dashrightarrow}$ *such that* $\widetilde{\beta}$ *is a minimal right almost split morphism in* $\mathcal{X}$ *and* $\widetilde{X}$ *is* $\mathcal{X}$*-projective, then* $X \cong \widetilde{X}$ *if and only if* $Y \cong \widetilde{Y}$*.*

Proof. (i) Since β is a right almost split morphism in $\mathcal{X}$, X is indecomposable by Lemma 3.2. Now assume $Y \in \mathcal{X}$. Since X is $\mathcal{X}$-projective, we have $\mathbb{E}(X, Y) = 0$, and hence $\delta = 0$. Thus β is a split epimorphism, which is a contradiction. Hence $Y \notin \mathcal{X}$. By Lemma 3.6, α is an $\mathcal{X}$-envelope of Y. Moreover, since β is right minimal, Y is indecomposable by Lemma 3.6.

(ii) Using (i), we have that $\widetilde{Y} \notin \mathcal{X}$, $\widetilde{Y}$ is indecomposable, and $\widetilde{\alpha}$ is an $\mathcal{X}$-envelope of $\widetilde{Y}$.

The "only if" part. Assume that $X \cong \widetilde{X}$, that is, there is an isomorphism $\varphi : X \to \widetilde{X}$. Then $\widetilde{\beta} : \widetilde{X'} \to \widetilde{X}$ and $\varphi\beta : X' \to \widetilde{X}$ are minimal right almost split morphisms. By Lemma 3.2, there exists an isomorphism $\psi : X' \to \widetilde{X'}$ such that $\varphi\beta = \widetilde{\beta}\psi$. Then we get the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, dotted, "\phi"] & X' \arrow[r, "\beta"] \arrow[d, "\psi"] & X \arrow[r, dashed, "\delta"] \arrow[d, "\varphi"] & {} \\ \widetilde{Y} \arrow[r, "\widetilde{\alpha}"] & \widetilde{X'} \arrow[r, "\widetilde{\beta}"] & \widetilde{X} \arrow[r, dashed, "\widetilde{\delta}"] & {} \end{tikzcd}.$$

By [14, Corollary 3.6], ϕ is an isomorphism, and hence $Y \cong \widetilde{Y}$.

The "if" part. Assume $Y \cong \widetilde{Y}$, that is, there is an isomorphism $\phi : Y \to \widetilde{Y}$. Then α and $\widetilde{\alpha}\phi$ are $\mathcal{X}$-envelopes of Y. Then $X' \cong \widetilde{X'}$ since envelopes are unique up to isomorphisms. That is, there exists an isomorphism $\psi : X' \to \widetilde{X'}$ such that $\widetilde{\alpha}\phi = \psi\alpha$. Then we have the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, "\phi"] & X' \arrow[r, "\beta"] \arrow[d, "\psi"] & X \arrow[r, dashed, "\delta"] \arrow[d, dotted, "\varphi"] & {} \\ \widetilde{Y} \arrow[r, "\widetilde{\alpha}"] & \widetilde{X'} \arrow[r, "\widetilde{\beta}"] & \widetilde{X} \arrow[r, dashed, "\widetilde{\delta}"] & {} . \end{tikzcd}$$

By [14, Corollary 3.6], φ is an isomorphism, and hence $X \cong \widetilde{X}$. □

Example 3.8. Let $\Lambda = kQ$ be a finite-dimensional hereditary path algebra over an algebraically closed field k, where Q is the quiver

$$1 \leftarrow 2 \leftarrow 3.$$

Then the Auslander-Reiten quiver of Λ is as follows ([3, Chapter IV, Example 4.10]):

$$\begin{tikzcd}[column sep=small] {[S_1]} \arrow[rr, dotted, no head] \arrow[dr] & & {[S_2]} \arrow[rr, dotted, no head] \arrow[dr] & & {[S_3]} \\ & {[P_2]} \arrow[ur] \arrow[rr, dotted, no head] \arrow[dr] & & {[I_2]} \arrow[ur] & \\ & & {[P_3],} \arrow[ur] & & \end{tikzcd}$$

where the symbol $[M]$ denotes the isomorphism class of a module M, and

(1) $S_1 = (k \leftarrow 0 \leftarrow 0)$, $S_2 = (0 \leftarrow k \leftarrow 0)$ and $S_3 = (0 \leftarrow 0 \leftarrow k)$ are all simple modules;
(2) $P_1 = S_1$, $P_2 = (k \leftarrow k \leftarrow 0)$ and $P_3 = (k \leftarrow k \leftarrow k)$ are all indecomposable projective modules;
(3) $I_1 = P_3$, $I_2 = (0 \leftarrow k \leftarrow k)$ and $I_3 = S_3$ are all indecomposable injective modules.

Let $\mathcal{K} = \mathrm{mod}\Lambda$ be the category of finitely generated left Λ-modules. Then $\mathrm{mod}\Lambda$ is an extriangulated category with $\mathbb{E} = \mathrm{Ext}^1_\Lambda$. Set $\mathcal{X} = \mathrm{add}(I_2 \oplus S_3)$. Then $\mathcal{X}$ is closed under extensions. Clearly, there is an Auslander-Reiten sequence $0 \to S_2 \xrightarrow{\alpha} I_2 \xrightarrow{\beta} I_3 \to 0$ in $\mathrm{mod}\Lambda$. In particular, β is a minimal right almost split morphism. Moreover, since $\mathrm{Ext}^1_\Lambda(S_3, I_2 \oplus S_3) \cong D\mathrm{Hom}_\Lambda(I_2 \oplus S_3, \tau S_3) = D\mathrm{Hom}_\Lambda(I_2 \oplus S_3, S_2) = 0$, we have that S_3 is $\mathcal{X}$-projective. By Theorem 3.7, α is an $\mathcal{X}$-envelope.

Remark 3.9. (1) An $\mathfrak{s}$-triangle in $\mathcal{K}$ of the form $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \overset{\delta}{\dashrightarrow}$ with $A, B, C \in \mathcal{X}$ is said to be an *Auslander-Reiten $\mathfrak{s}$-triangle* in $\mathcal{X}$ if

(i) $\delta \neq 0$,
(ii) α is left almost split in $\mathcal{X}$,
(iii) β is right almost split in $\mathcal{X}$.

Clearly, the $\mathfrak{s}$-triangle given in Theorem 3.7 is not Auslander-Reiten $\mathfrak{s}$-triangle in $\mathcal{X}$ since the starting object is not in $\mathcal{X}$ at least.

(2) Let $\mathcal{K} = \mathcal{T}$ be a triangulated category, [1] the shift functor, and $\mathbb{E} = \mathrm{Hom}_{\mathcal{T}}(-, -[1])$. Then Theorem 3.7 recovers [6, Theorem A].

(3) Dual to Theorem 3.7, we have the following result: Let $Y \xrightarrow{\alpha} X' \xrightarrow{\beta} X \overset{\delta}{\dashrightarrow}$ be an $\mathfrak{s}$-triangle in $\mathcal{K}$. Assume that α is a minimal left almost split morphism in $\mathcal{X}$, and Y is $\mathcal{X}$-injective. Then

(i) (a) $X \notin \mathcal{X}$,
(b) β is an $\mathcal{X}$-cover of X,
(c) X is indecomposable.

(ii) If there is an $\mathfrak{s}$-triangle $\widetilde{Y} \xrightarrow{\widetilde{\alpha}} \widetilde{X'} \xrightarrow{\widetilde{\beta}} \widetilde{X} \overset{\widetilde{\delta}}{\dashrightarrow}$ such that $\widetilde{\alpha}$ is a minimal left almost split morphism in $\mathcal{X}$ and $\widetilde{Y}$ is $\mathcal{X}$-injective, then $Y \cong \widetilde{Y}$ if and only if $X \cong \widetilde{X}$.

4. An exact structure of extension-closed subcategories

Assume that $\mathcal{K}$ is an extriangulated category, and $\mathcal{X}$ be an extension-closed subcategory of $\mathcal{K}$. By [14, Remark 2.18], $\mathcal{X}$ is also an extriangulated category. In this section, we will show that $\mathcal{X}$ has an exact structure under some assumptions, which generalizes [11, Proposition 2.5]. The results in this section follow closely part of Section 2 from [11] and that the proofs also follow by very similar arguments. An important difference is that in our setup one has to add the assumption that the extriangulated category has a negative first extension in order to obtain the needed exact sequences in our arguments, while in [11] these exact sequences come for free from the triangulated structure.

Definition 4.1. A *homotopy cartesian square* in $\mathcal{K}$ is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "y"] & B \arrow[d, "v"] \\ C \arrow[r, "u"] & D \end{tikzcd}$$

in $\mathcal{K}$ such that $A \xrightarrow{\binom{x}{y}} B \oplus C \xrightarrow{(-v,u)} D$ is a conflation.

Let $\mathcal{X}$ be an additive subcategory of $\mathcal{K}$. We set

$$\mathcal{E}_{\mathcal{X}} := \{\text{conflations } X_1 \to X_2 \to X_3 \text{ in } \mathcal{K} \text{ with } X_1, X_2, X_3 \in \mathcal{X}\}.$$

Definition 4.2. ([1, Definition 2.3]) A *negative first extension structure* on $\mathcal{K}$ consists of the following data:

(NE1) $\mathbb{E}^{-1} : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to Ab$ is an additive bifunctor.

(NE2) For each $\delta \in \mathbb{E}(C, A)$, there exist two natural transformations

$$\delta_\sharp^{-1} : \mathbb{E}^{-1}(-, C) \to \mathrm{Hom}_{\mathcal{K}}(-, A)$$
$$\delta_{-1}^\sharp : \mathbb{E}^{-1}(A, -) \to \mathrm{Hom}_{\mathcal{K}}(C, -)$$

such that for each $\mathbb{E}$-triangle $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$ and each $W \in \mathcal{K}$, two sequences

$$\mathbb{E}^{-1}(W, A) \xrightarrow{\mathbb{E}^{-1}(W,f)} \mathbb{E}^{-1}(W, B) \xrightarrow{\mathbb{E}^{-1}(W,g)} \mathbb{E}^{-1}(W, C) \xrightarrow{(\delta_\sharp^{-1})_W} \mathrm{Hom}_{\mathcal{K}}(W, A) \xrightarrow{\mathrm{Hom}_{\mathcal{K}}(W,f)} \mathrm{Hom}_{\mathcal{K}}(W, B)$$

$$\mathbb{E}^{-1}(C, W) \xrightarrow{\mathbb{E}^{-1}(g,W)} \mathbb{E}^{-1}(B, W) \xrightarrow{\mathbb{E}^{-1}(f,W)} \mathbb{E}^{-1}(A, W) \xrightarrow{(\delta_{-1}^\sharp)_W} \mathrm{Hom}_{\mathcal{K}}(C, W) \xrightarrow{\mathrm{Hom}_{\mathcal{K}}(g,W)} \mathrm{Hom}_{\mathcal{K}}(B, W)$$

are exact.

In this case, we call $(\mathcal{K}, \mathbb{E}, \mathfrak{s}, \mathbb{E}^{-1})$ an *extriangulated category with a negative first extension*.

In what follows, we assume that $\mathcal{K}$ is an extriangulated category with a negative first extension $\mathbb{E}^{-1}$, and $\mathcal{X}$ is an additive subcategory of $\mathcal{K}$ which is closed under extensions and $\mathbb{E}^{-1}(\mathcal{X}, \mathcal{X}) = 0$.

Lemma 4.3. *Each conflation* $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$ *in* $\mathcal{E}_{\mathcal{X}}$ *is a kernel-cokernel pair in* $\mathcal{X}$.

Proof. For any $X \in \mathcal{X}$, we have exact sequences

$$\mathbb{E}^{-1}(X, X_3) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(X, X_1) \xrightarrow{\mathrm{Hom}_{\mathcal{K}}(X,f)} \mathrm{Hom}_{\mathcal{K}}(X, X_2) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(X, X_3) \tag{4.1}$$

and

$$\mathbb{E}^{-1}(X_1, X) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(X_3, X) \xrightarrow{\mathrm{Hom}_{\mathcal{K}}(g,X)} \mathrm{Hom}_{\mathcal{K}}(X_2, X) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(X_1, X)\ . \tag{4.2}$$

By assumption, $\mathbb{E}^{-1}(X, X_3) = 0 = \mathbb{E}^{-1}(X_1, X)$. Thus the sequence (4.1) implies that f is a kernel of g, and the sequence (4.2) implies that g is a cokernel of f. Thus $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$ is a kernel-cokernel pair in $\mathcal{X}$. □

Lemma 4.4. *Assume that the following diagram*

$$\begin{tikzcd} X_1 \arrow[r, "x"] \arrow[d, "y"] & X_2 \arrow[d, "v"] \\ X_3 \arrow[r, "u"] & X_4 \end{tikzcd}$$

is a homotopy cartesian square with each $X_i \in \mathcal{X}$. *Then it is a pullback and a pushout in* $\mathcal{X}$.

Proof. By definition, there is an $\mathfrak{s}$-triangle $X_1 \xrightarrow{\binom{x}{y}} X_2 \oplus X_3 \xrightarrow{(-v,u)} X_4 \overset{\delta}{\dashrightarrow}$. For any $X \in \mathcal{X}$, there are exact sequences

$$\mathbb{E}^{-1}(X, X_4) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(X, X_1) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(X, X_2 \oplus X_3) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(X, X_4) \tag{4.3}$$

and

$$\mathbb{E}^{-1}(X_1, X) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(X_4, X) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(X_2 \oplus X_3, X) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(X_1, X)\ . \tag{4.4}$$

By assumption, $\mathbb{E}^{-1}(X, X_4) = 0 = \mathbb{E}^{-1}(X_1, X)$. Thus the sequence (4.3) implies that the given diagram is a pullback in $\mathcal{X}$, and the sequence (4.4) implies that the given diagram is a pushout in $\mathcal{X}$. □

We recall the definition of exact categories from [5, Definition 2.1]. As proven by Bühler, it is equivalent to the original definition by Quillen from [15, Section 2]. Let $\mathfrak{C}$ be an additive category. A kernel-cokernel pair (i, p) in $\mathfrak{C}$ is a pair of composable morphisms $A \xrightarrow{i} B \xrightarrow{p} C$ such that i is a kernel of p and p is a cokernel of i. If a class $\mathcal{E}$ of kernel-cokernel pairs on $\mathfrak{C}$ is fixed, a morphism i is called an inflation if there is a morphism p with $(i, p) \in \mathcal{E}$. Deflations are defined dually.

Let $\mathfrak{C}$ be an additive category. An *exact structure* on $\mathfrak{C}$ is a class $\mathcal{E}$ of kernel-cokernel pairs which is closed under isomorphisms and satisfies the following axioms:

[E0] $\forall A \in \mathfrak{C}$, 1_A is an inflation.

[E0$^{\rm op}$] $\forall A \in \mathfrak{C}$, 1_A is a deflation.

[E1] The class of inflations is closed under composition.

[E1$^{\rm op}$] The class of deflations is closed under composition.

[E2] The pushout of an inflation along an arbitrary morphism exists and yields an inflation.

[E2$^{\rm op}$] The pullback of a deflation along an arbitrary morphism exists and yields a deflation.

An *exact category* is a pair $(\mathfrak{C}, \mathcal{E})$ consisting of an additive category $\mathfrak{C}$ and an exact structure $\mathcal{E}$ on $\mathfrak{C}$.

Theorem 4.5. $(\mathcal{X}, \mathcal{E}_{\mathcal{X}})$ *is an exact category.*

Proof. We will check [E0], [E1], [E2] and their dual.

[E0] and [E0$^{\rm op}$]: For each $X \in \mathcal{X}$, there are $\mathfrak{s}$-triangles $X \xrightarrow{1_X} X \longrightarrow 0 \overset{0}{\dashrightarrow}$ and $0 \longrightarrow X \xrightarrow{1_X} X \overset{0}{\dashrightarrow}$. Clearly, $X \xrightarrow{1_X} X \longrightarrow 0$ and $0 \longrightarrow X \xrightarrow{1_X} X$ belong to $\mathcal{E}_{\mathcal{X}}$. Thus 1_A is an inflation and a deflation.

[E1]: Let $x : X_0 \to X_1$ and $y : X_1 \to X_2$ be two inflations in $\mathcal{X}$. Then there exist two conflations $X_0 \xrightarrow{x} X_1 \longrightarrow X_3$ and $X_1 \xrightarrow{y} X_2 \longrightarrow X_4$ in $\mathcal{E}_{\mathcal{X}}$. By (ET4), we have a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "x"] \arrow[d, equal] & X_1 \arrow[r] \arrow[d, "y"] & X_3 \arrow[d] \\
X_0 \arrow[r, "yx"] & X_2 \arrow[r] \arrow[d] & X_5 \arrow[d] \\
& X_4 \arrow[r, equal] & X_4,
\end{tikzcd}$$

where all rows and columns are conflations. Since $X_3, X_4 \in \mathcal{X}$, we have $X_5 \in \mathcal{X}$, and hence $X_0 \xrightarrow{yx} X_2 \longrightarrow X_5 \in \mathcal{E}_{\mathcal{X}}$. Thus yx is an inflation in $\mathcal{X}$. That is, inflations in $\mathcal{X}$ are closed under compositions.

[E1$^{\rm op}$]: Dual to [E1].

[E2]: Given a diagram

$$\begin{tikzcd}
X_1 \arrow[r, "x"] \arrow[d, "y"] & X_2 \\
X_3 &
\end{tikzcd}$$

in $\mathcal{X}$, where x is an inflation. Then there is an $\mathfrak{s}$-triangle $X_1 \xrightarrow{x} X_2 \longrightarrow X_4 \overset{\delta}{\dashrightarrow}$ with $X_4 \in \mathcal{X}$. By [13, Proposition 1.20], there exists a morphism $\varphi : X_2 \to X_5$ such that there is a morphism of $\mathfrak{s}$-triangles

$$\begin{tikzcd}
X_1 \arrow[r, "x"] \arrow[d, "y"] & X_2 \arrow[r] \arrow[d, "\varphi"] & X_4 \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
X_3 \arrow[r, "\phi"] & X_5 \arrow[r] & X_4 \arrow[r, dashed, "y_\star\delta"] & {}
\end{tikzcd}$$

and meanwhile, there is a conflation $X_1 \xrightarrow{\binom{x}{y}} X_2 \oplus X_3 \xrightarrow{(-\varphi,\phi)} X_5$. That is, the diagram

$$\begin{tikzcd} X_1 \arrow[r, "x"] \arrow[d, "y"] & X_2 \arrow[d, "\varphi"] \\ X_3 \arrow[r, "\phi"] & X_5 \end{tikzcd}$$

is a homotopy cartesian diagram. Moreover, since $X_3, X_4 \in \mathcal{X}$, we have $X_5 \in \mathcal{X}$. Thus by Lemma 4.4, this homotopy cartesian diagram is a pushout in $\mathcal{X}$, and ϕ is an inflation in $\mathcal{X}$.

[E2$^{\mathrm{op}}$] Dual to [E2]. □

Remark 4.6. Let $\mathcal{K} = \mathcal{T}$ be a triangulated category, [1] the shift functor, $\mathbb{E} = \mathrm{Hom}_{\mathcal{T}}(-, -[1])$, and $\mathbb{E}^{-1} = \mathrm{Hom}_{\mathcal{T}}(-[1], -)$. Then Theorem 4.5 recovers [11, Proposition 2.5(1)].

5. Conclusions

Let $\mathcal{X}$ be an extension-closed subcategory of an extriangulated category $\mathcal{K}$. We show that if C is $\mathcal{X}$-projective and there is a minimal right almost split deflation in $\mathcal{X}$ ending by C, then there is an $\mathfrak{s}$-triangle ending by C which is very similar to an Auslander-Reiten triangle in $\mathcal{X}$, and it generalizes [6, Theorem A]. We also show that if $\mathcal{K}$ admits a negative first extension $\mathbb{E}^{-1}$ and $\mathbb{E}^{-1}(\mathcal{X}, \mathcal{X}) = 0$, then the subcategory $\mathcal{X}$ has an exact structure which generalizes [11, Proposition 2.5(i)].

Acknowledgments

This work was supported by the NSF of China (11901341), the project ZR2021QA001 supported by Shandong Provincial Natural Science Foundation, the project funded by China Postdoctoral Science Foundation (2020M682141), and Innovation Foundation for Graduate Dissertation of Qufu Normal University (LWCXB202105). The authors would like to thank the anonymous reviewers for their comments and suggestions.

Conflict of interest

The authors declare no conflict of interests.

References

1. T. Adachi, H. Enomoto, M. Tsukamoto, Intervals of s-torsion pairs in extriangulated categories with negative first extensions. Available from: `https://arxiv.org/pdf/2103.09549.pdf`.
2. F. W. Anderson, K. R. Fuller, *Rings and categories of modules*, Springer-Verlag, New York, 1992. https://doi.org/10.1007/978-1-4612-4418-9
3. I. Assem, D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras*, LMSST Vol. 65, Cambridge University Press, Cambridge, 2010. https://doi.org/10.1017/CBO9780511614309

4. M. Auslander, S. O. Smalø, Almost split sequences in subcategories, *J. Algebra*, **69** (1981), 426–454. https://doi.org/10.1016/0021-8693(81)90214-3
5. T. Bühler, Exact categories, *Expo. Math.*, **28** (2010), 1–69. https://doi.org/10.1016/j.exmath.2009.04.004
6. F. Fedele, Almost split morphisms in subcategories of triangulated categories, *J. Algebra Appl.*, 2022. https://doi.org/10.1142/S0219498822502395
7. X. Fu, J. Hu, D. Zhang, H. Zhu, Balanced pairs on triangulated categories, arXiv: 2109.00932. Available from: `https://arxiv.org/pdf/2109.00932.pdf`.
8. O. Iyama, H. Nakaoka, Y. Palu, Auslander-Reiten theory in extriangulated categories, arXiv:1805.03776. Available from: `https://arxiv.org/pdf/1805.03776.pdf`.
9. G. Jasso, S. Kvamme, An introduction to higher Auslander-Reiten theory, *Bull. London Math. Soc.*, **51** (2019), 1–24. https://doi.org/10.1112/blms.12204
10. P. Jørgensen, Auslander-Reiten triangles in subcategories, *J. K-Theory*, **3** (2009), 583–601. https://doi.org/10.1017/is008007021jkt056
11. P. Jørgensen, Abelian subcategories of triangulated categories induced by simple minded systems, arXiv: 2010.11799. Available from: `https://arxiv.org/pdf/2010.11799.pdf`.
12. S. Liu, P. Ng, C. Paquette, Almost split sequences and approximations, *Algebr. Represent. Th.*, **16** (2013), 1809–1827. https://doi.org/10.1007/s10468-012-9383-x
13. Y. Liu, H. Nakaoka, Hearts of twin cotorsion pairs on extriangulated categories, *J. Algebra*, **528** (2019), 96–149. https://doi.org/10.1016/j.jalgebra.2019.03.005
14. H. Nakaoka, Y. Palu, Extriangulated categories, Hovey twin cotorsion pairs and model structures, *Cah. Topol. Géom. Différ. Catég.*, **60** (2019), 117–193.
15. D. Quillen, *Higher algebraic K-theory: I*, Bass H. (eds) Higher K-Theories, Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, 1973. https://doi.org/10.1007/BFb0067053
16. T. Zhao, Z. Huang, Phantom ideals and cotorsion pairs in extriangulated categories, *Taiwan. J. Math.*, **23** (2019), 29–61. https://doi.org/10.11650/tjm/180504
17. P. Zhou, N-extension closed subcategories of $(n+2)$-angulated categories, arXiv: 2109.02068. Available from: `https://arxiv.org/pdf/2109.02068.pdf`.
18. P. Zhou, B. Zhu, Triangulated quotient categories revisited, *J. Algebra*, **502** (2018), 196–232. https://doi.org/10.1016/j.jalgebra.2018.01.031