Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 7888–7897.
DOI: 10.3934/math.2022441
Received: 31 October 2021
Revised: 31 January 2022
Accepted: 07 February 2022
Published: 17 February 2022

Research article

Liftings of metallic structures to tangent bundles of order r

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Abstract: It is well known that the prolongation of an almost complex structure from a manifold M to the tangent bundle of order r on M is also an almost complex structure if it is integrable. The general quadratic structure $F^2 = \alpha F + \beta I$ is a generalization of an almost complex structure where $\alpha = 0$, $\beta = -1$. The purpose of this paper is to characterize a metallic structure defined by the general quadratic structure $F^2 = \alpha F + \beta I$, $\alpha, \beta \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. We show that the r-lift of the metallic structure F in the tangent bundle of order r is also a metallic structure. Furthermore, we deduce a theorem on the projection tensor in the tangent bundle of order r. Moreover, prolongations of G-structures immersed in the metallic structure to the tangent bundle of order r and 2 are discussed. Finally, we construct examples of metallic structures that admit an almost para contact structure on the tangent bundle of order 3 and 4.

Keywords: lifts; metallic structure; integrability; projection tensor; prolongation
Mathematics Subject Classification: 53C15, 58A30

1. Introduction

Let us consider the general quadratic equation $x^2 - \alpha x - \beta = 0$, $\alpha, \beta \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. In [18], De Spinadel presented the fascinating concept of metallic means family (MMF) (or, metallic proportions) as the set of positive solutions $\sigma_{\alpha,\beta} = \frac{1}{2}[\alpha + \sqrt{\alpha^2 + 4\beta}]$ of the equation $x^2 - \alpha x - \beta = 0$, $\alpha, \beta \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.

The name MMF was clearly explained by De Spinadel in [19]. In [19], the author declared that "besides carrying the name of a metals, they have common mathematical properties that attach a fundamental importance to them in modern investigations about the search of universal roads to chaos". These metallic numbers have seen several interesting modern applications in researches that "analyze the behavior of non linear dynamical systems when they proceed from a periodic regime to a chaotic one" [17, 20].

Goldberg, Yano and Petridis [5, 6] defined and studied polynomial structures on the differentiable manifold. The general quadratic structure satisfying $J^2 = \alpha J + \beta I$, $\alpha, \beta \in \mathbb{N}$, where J is a tensor field of type (1,1) and $\mathbb{N}$ is the set of natural numbers, is named metallic structure on the differentiable manifold. Crasmareanu and Hretcanu [7] discussed metallic structures in the Riemannian manifolds. Azmi introduced metallic structures on the tangent bundle of P-Sasakian manifolds and investigated the integrability and parallelity of such metallic structures [1]. Recently, the geometry of metallic structures have been studied in [2, 3, 8, 10].

On the other hand, liftings of tensor fields and connections to tangent bundle were defined and studied by Yano and Davis [22], Ledger and Yano [12] and Yano [21]. Yano and Ishiharo [24] developed the theory of prolongations of these geometric objects to the tangent bundle of order 2 and investigated integrability conditions. Morimoto [13] has studied prolongations of tensor fields, connections and G-structures to the tangent bundle of higher order. The first author has studied the prolongation of G-structure immersed in generalized almost r-contact structure on M to its tangent bundle TM of order 2 [11].

The prolongation of some classical G-structure tensor fields and connections immersed in an almost complex structure, f-structure, generalized almost r-contact structure, etc. to the tangent bundles of order r have been investigated [11, 16, 23]. Inspired by the above mentioned studies, we define and study liftings of metallic structures to tangent bundle of order r.

The main contributions of the paper can be listed as follows:

- The r-lift is applied to the metallic structure F and we show that it is also a metallic structure in $T_r(M)$.
- We show that the metallic structure $F^{(r)}$ in $T_r(M)$ is integrable if and only if F is an integrable metallic structure.
- The projection tensors are defined for the metallic structure and a theorem is proved about them.
- Some classical G-structures defined by tensor fields immersed in metallic structures in $T_r(M)$ and $T_2(M)$ have been investigated.
- Examples on metallic structures that admits an almost para contact structure in $T_3(M)$ and $T_4(M)$ are constructed.

2. Preliminaries

2.1. The tangent bundle of order r and lifts of tensor fields

In an n-dimensional differentiable manifold, let $r \geq 1$ be a fixed integer and $\mathbb{R}$ be the real line. Consider the following equivalence relation $\sim$ among all differentiable mappings: "If the mappings $F : \mathbb{R} \to M$ and $G : \mathbb{R} \to M$ meet the following criteria

$$F^h(0) = G^h(0),\ \frac{dF^h(0)}{dt} = \frac{dG^h(0)}{dt}, ..., \frac{dF^r(0)}{dt} = \frac{dG^r(0)}{dt}, \tag{2.1}$$

where F and G are characterized respectively by $x^h = F^h(t)$ and $x^h = G^h(t)$, t is an element of $\mathbb{R}$) with respect to local coordinates (x^h) in a coordinate neighborhood of $\{U, x^h\}$ containing the point

$P = F(0) = G(0)$, then we state that $F \sim G$. Each equivalence relation is called r-jet of M and denoted by $j^r_P(F)$, if this class contains a mapping $F : \mathbb{R} \to M$ such that $F(0) = P$. The point P is called the target of the r-jet $j^r_P(F)$. The set of all r-jets of M is called the tangent bundle of order r and denoted by $T_r(M)$" [16].

Let π_r be the bundle projection such that $\pi_r = T_r(M) \to M$, i.e. $j^r_P(F) = P$. We define $\pi_{sr} : T_r(M) \to T_s(M)$ for $r > s$ by $\pi_{sr}(j^r_P(F)) = j^s_P(F)$. Then we have $\pi_r = \pi_s \circ \pi_{sr}$.

Consider an r-jet $j^r_P(F)$ belonging to $\pi^{-1}(U)$ and the set

$$y^{(\nu)h} = \frac{1}{\nu!}\frac{d^\mu F(0)}{dt^\nu}, \quad \nu = 0, 1, ..., r, \tag{2.2}$$

where $x^h = F^h(t)$, t is an element of $\mathbb{R}$) of F in U and $P = F(0)$. The r-jet $j^r_P(F)$ is represented by the set $(y^{(\nu)h},\ \nu = 0, 1, ..., r)$, where $(y^{(0)h}) = (x^h)$ are coordinates of P in U. Therefore, the system of coordinates $(y^{(\nu)h};\ \nu = 0, 1, ..., r)$ is established in the open set $\pi^{-1}(U)$ of $T_r(M)$ and called induced coordinates in $\pi^{-1}(U)$ [15, 23].

The following notations will be used throughout the paper: Let $\Im^0_0(M),\ \Im^1_0(M),\ \Im^0_1(M),\ \Im^1_1(M)$ be the set of functions, vector fields, 1-forms and tensor fields of type (1,1) in M, respectively.

Let f be a function in M. The λ-lift $f^{(\lambda)}$ of a function f to $T_r(M)$ is defined in [13] as

$$f^{(\lambda)}(j^r_P(F)) = \frac{1}{\lambda!}\frac{d^\lambda(f \circ F)}{dt^\lambda}, \quad \lambda = 0, 1, ..., r. \tag{2.3}$$

By virtue of (2.3), we have

$$(fg)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)} g^{(\lambda-\mu)} \tag{2.4}$$

for all $f, g \in \Im^0_0(M),\ \mu = 1, 2, ..., r$.

Remark 2.1. *In $T_r(M)$ [4]:*

For $r = 1$, $T_1(M) = T(M)$ (tangent bundle) $f^V = f^{(0)},\ f^C = f^{(1)}$.

For $r = 2$, $T_2(M)$, i.e. tangent bundle of order 2, $f^0 = f^{(0)},\ f^I = f^{(1)},\ f^{II} = f^{(2)}$ for any $f \in \Im^0_0(M)$.

We shall first state the following propositions ([23], p. 379, 383, 384).

Proposition 2.1. *For any vector field $X, Y \in \Im^1_0(M),\ F \in \Im^1_1(M),\ f \in \Im^0_0(M),\ \omega \in \Im^0_1(M)$, the λ-lift X^λ of X to $T_r(M)$ is a known result:*

$$[X^{(\lambda)}, Y^{(\mu)}] = [X, Y]^{(\lambda+\mu-r)},\ \ X^{(\lambda)} f^{(\mu)} = (Xf)^{(\lambda+\mu-r)}, \tag{2.5}$$

$$(fX)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)} X^{(\lambda-\mu)},\ \ (f\omega)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)} \omega^{(\lambda-\mu)}, \tag{2.6}$$

$$\omega^{(\lambda)}(X^{(\mu)}) = (\omega(X))^{(\lambda+\mu-r)},\ \ F^{(\lambda)} X^{(\mu)} = (FX)^{(\lambda+\mu-r)}, \tag{2.7}$$

where $\lambda, \mu = 0, 1, ..., r$.

Proposition 2.2. *If for $\tilde{S}$ and $\tilde{T} \in \Im^0_s(T_r(M))$,*

$$\tilde{S}(X^{(r)}_s, ..., X^{(r)}_1) = \tilde{T}(X^{(r)}_s, ..., X^{(r)}_1),$$

for $X_1, ..., X_s \in \Im^1_0(M)$, then $\tilde{S} = \tilde{T}$.

2.2. Almost para contact structure

Let M be an n-dimensional differentiable manifold of class C^∞. Suppose that there is given a tensor field F of type (1,1) satisfying

$$F^2 = I - U \otimes \omega, \quad \omega(U) = 1, \quad FU = 0, \quad \omega \circ F = 0, \tag{2.8}$$

where $U \in \Im_0^1(M),\ \omega \in \Im_1^0(M)$.

The structure (F, U, ω) of such fields F, U, ω is said to be an almost para contact structure ([23], p. 66).

3. Lifts of metallic structures

Let M be an n-dimensional differentiable manifold and $F \in \Im_0^1(M)$. Then F is called metallic structure on M satisfying [8]

$$F^2 - \alpha F - \beta I = 0,\ \alpha, \beta \in \mathbb{N}, \tag{3.1}$$

where I is the unit vector field.

Let F and G be tensor fields of type (1,1) in M. Then ([23], p. 393)

$$G^{(r)}F^{(r)} = (GF)^{(r)}.$$

Let $P(t)$ be a polynomial of t and $F^{(r)}$ its r-lift in $T_r(M)$. Then ([23], p. 393)

$$P(F^{(r)}) = (P(F))^r, \forall F \in \Im_0^1(M). \tag{3.2}$$

Theorem 3.1. *Let $F \in \Im_0^1(M)$. Then $F^{(r)}$ is a metallic structure in $T_r(M)$ if and only if F is a metallic structure in M.*

Proof. By operating the r-lift of Eq (3.1) and by using Eq (3.2), we obtain

$$\begin{aligned} (F^2 - \alpha F - \beta I)^{(r)} &= 0, \\ (F^2)^{(r)} - \alpha F^{(r)} - \beta I^{(r)} &= 0, \quad I^{(r)} = I, \\ (F^{(r)})^2 - \alpha F^{(r)} - \beta I) &= 0. \end{aligned}$$

Hence, $F^{(r)}$ is a metallic structure in $T_r(M)$.

Theorem 3.2. *Let F and $F^{(r)}$ be metallic structures in M and $T_r(M)$, respectively. Then $F^{(r)}$ is integrable in $T_r(M)$ if and only if F is integrable in M.*

Proof. Let N_F and $N_{F^{(r)}}$ denote the Nijenhuis tensors of F and $F^{(r)}$ respectively. Then ([23], p. 393)

$$N_{F^{(r)}} = (N_F)^{(r)}, \tag{3.3}$$

since F is integrable if and only if $N_F = 0$ [9]. So, from (3.3), we obtained $N_{F^{(r)}} = 0$. Hence, the proof is completed.

Let l and m be the projection tensors defined in [8]:

$$l = \frac{F^2 - \alpha F}{\beta}, \tag{3.4}$$

and

$$m = I - \left(\frac{F^2 - \alpha F}{\beta}\right), \tag{3.5}$$

where I denotes the identity operator in M.

Theorem 3.3. *Let l and m be the projection tensors. Then*

$$\begin{aligned} & l + m = 0, \\ & l^2 = l, \quad m^2 = m, \quad lm = ml = 0, \\ & Fl = lF = F, \quad Fm = mF = 0. \end{aligned} \tag{3.6}$$

Proof. In the view of Eqs (3.4) and (3.5), then

$$\begin{aligned} l + m &= \frac{F^2 - \alpha F}{\beta} + I - \left(\frac{F^2 - \alpha F}{\beta}\right) = I, \\ l^2 &= \left(\frac{F^2 - \alpha F}{\beta}\right)\left(\frac{F^2 - \alpha F}{\beta}\right) \\ &= \left(\frac{F^2 - \alpha F}{\beta}\right)\left(\frac{\beta I}{\beta}\right), \ \ as \ \ F^2 - \alpha F = \beta I, \\ &= l, \\ m^2 &= \left[I - \left(\frac{F^2 - \alpha F}{\beta}\right)\right]^2 = I - 2\left(\frac{F^2 - \alpha F}{\beta}\right) + \left(\frac{F^2 - \alpha F}{\beta}\right)^2 \\ &= I - 2\left(\frac{F^2 - \alpha F}{\beta}\right) + \left(\frac{F^2 - \alpha F}{\beta}\right)\left(\frac{\beta I}{\beta}\right) \\ &= I - 2\left(\frac{F^2 - \alpha F}{\beta}\right) + \frac{F^2 - \alpha F}{\beta} \\ &= I - \left(\frac{F^2 - \alpha F}{\beta}\right) \\ &= m, \\ lm &= \left(\frac{F^2 - \alpha F}{\beta}\right)\left[I - \left(\frac{F^2 - \alpha F}{\beta}\right)\right], \\ &= 0 = ml. \end{aligned}$$

Similarly, other identities can be easily proved.

Let D_l and D_m be the complementary distributions corresponding to l and m, respectively in M. Let $\mathrm{rank}(F) = s$, therefore dimension of D_l is s and the dimension of D_m is $(n - s)$, where the dimension of M is n.

Theorem 3.4. *Let m be a projection tensor in M. Then the r-lift $m^{(r)}$ of m is a projection tensor in $T_r(M)$ and the distribution $\tilde{D}$ determined by $m^{(r)}$ in $T_r(M)$ is integrable if and only if the distribution D determined by m is so in M.*

Proof. Let D be a distribution in M. D is determined by a projection tensor m, i.e. m is an element of $\Im_1^1(M)$ such that $m^2 = m$. From Eq (3.2) and $m^2 = m$, we have

$$(m^{(r)})^2 = m^{(r)},$$

that is the r-lift $m^{(r)}$ of m is a projection tensor in $T_r(M)$. The distribution D is integrable if and only if

$$l[mX, mY] = 0, \tag{3.7}$$

where $l = I - m$ denotes a projection tensor complementary to m and $[,]$ is the Lie bracket.

By applying r-lift on (3.7), we obtain

$$l^{(r)}[m^{(r)}X^{(r)}, m^{(r)}Y^{(r)}] = 0, \tag{3.8}$$

where $l^{(r)} = (I - m)^{(r)} = I - m^{(r)}$.

Thus, conditions (3.7) and (3.8) are equivalent to each other.

This finishes the proof.

4. Prolongation of G-structures immersed in metallic structures to the tangent bundle of order r

In this section, we study the tangent bundle of order r on some classical G-structures, which are defined by tensor fields immersed in metallic structures.

Let $P(M, \pi^*, G)$ be a G-structure over a manifold M, where G is a Lie subgroup of $GL(n, \mathbb{R})$. "A G-structure on M is a G-subbundle $P(M, \pi^*, G)$ of the frame bundle FM over M" [14]. Let $u = U$ be an open covering of M such that in each U there exists an n-frame $\{X_i\}$ which is adapted to the G-structure $P(M, \pi^*, G)$. The structure group $T_r(GL(n, \mathbb{R}))$ of $T(T_r(M))$ is reducible to the tangent group $T_r(G)$ of order r, that is, the tangent bundle $T_r(M)$ of order r admits a $T_r(G)$-structure $\tilde{P}$, which is called the prolongation of G-structure P in M to $T(T_r(M))$ [23].

Let $\overset{o}{F}$ be a tensor field of type (1,1) in $\mathbb{R}^n$, which is invariant by G. We consider that M admits a G-structure P. Consider a coordinate neighborhood $\left\{U, X^h\right\}$ of M and an n-frame $\{X_{(i)}\}$ in U. Thus, if we set

$$\overset{o}{F} = \overset{o}{F}{}_i^{(h)} \; X_{(h)}\theta^{(i)} \tag{4.1}$$

in $\left\{U, \theta^{(i)}\right\}$ being the n-coframe dual to $\{X_{(i)}\}$ in U and $\overset{o}{F}{}_i^{(h)}$ are the components of $\overset{o}{F}$ in R^n. The local tensor field F is defined by Eq (4.1) in M. Hence F defines a global tensor field, which is called the tensor field induced in M from $(\overset{o}{F}, P)$ [23].

Now, we state the following proposition for later use ([23], p. 406).

Proposition 4.1. *"The prolongation $\tilde{P}$ of a G-structure P given in M is integrable in the tangent bundle $T(M)$ if and only if the G-structure P is integrable in M".*

Theorem 4.1. *Let M denote a manifold that admits a metallic structure P (as a G-structure) defined by a tensor field F of type (1,1) such that $F^2 = \alpha F + \beta I$. Let the tangent bundle $T_r(M)$ of order r be $\tilde{P}$ of P is the metallic structure defined by the r-lift $F^{(r)}$ of F to $T_r(M)$. Then the metallic structure P is integrable in M if and only if $\tilde{P}$ of P to $T_r(M)$ is integrable.*

Proof. Let $\overset{o}{F}$ be a tensor of type (1,1) in $\mathbb{R}^{2n}$ such that $\overset{o}{F}^2 = \alpha \overset{o}{F} + \beta I$ and denote by $GL(n, \mathbb{C})$ the group of all elements of $G = GL(2n, \mathbb{C})$ which leave $\overset{o}{F}$ invariant. Then the r-lift $\overset{o}{F}^{(r)}$ of $\overset{o}{F}$ to $T_r(R^{2n})$ is a tensor of type (1,1) satisfying $(\overset{o}{F}^{(r)})^2 = \alpha \overset{o}{F}^{(r)} + \beta I$ and the tangent group $T_r(G)$ leaves $\overset{o}{F}^{(r)}$ invariant. Thus, we obtain $T_r(G) = GL(3n, \mathbb{C})$. By Proposition 4.1, the metallic structure P is integrable in M if and only if $\tilde{P}$ of P to $T_r(M)$ is integrable.

This finishes the proof.

Theorem 4.2. *If a manifold M of $2n$-dimension admits almost para contact structure P (as a G-structure) determined by (F, U, ω) given in Eq (2.8), then, on the tangent bundle $T_2(M)$, $\tilde{P}$ of P is the metallic structure determined by the tensor field*

$$\overset{o}{J} = \frac{\alpha}{2} - \left(\frac{2\sigma_{\alpha,\beta} - \alpha}{2}\right)\left(\overset{o}{F}^{II} + \overset{o}{\eta}^{0} \otimes \overset{o}{\xi}^{0} + \overset{o}{\eta}^{II} \otimes \overset{o}{\xi}^{II}\right), \quad \overset{o}{U} = \overset{o}{\xi}^{I}, \quad \overset{o}{\omega} = \overset{o}{\eta}^{I}.$$

Proof. $G = GL(n, \mathbb{C}) \times I$. Let the rank of $\overset{o}{F}$ be $2s$. Let $\overset{o}{\xi}$ denote a contravariant vector field and $\overset{o}{\omega}$ a 1-form in $\mathbb{R}^{2n}$ such that

$$\overset{o}{F}^2 = I - \overset{o}{\eta} \otimes \overset{o}{\xi}, \tag{4.2}$$

where

$$\begin{aligned} &\text{(i)} \quad \overset{o}{F} \circ \overset{o}{\xi} = 0, \\ &\text{(ii)} \quad \overset{o}{\eta}(\overset{o}{F}) = 0, \\ &\text{(iii)} \quad \overset{o}{\eta}(\overset{o}{\xi}) = 1. \end{aligned} \tag{4.3}$$

If we denote by G, the group of all the elements of $GL(2n, \mathbb{C})$, which leave $\overset{o}{F}, \overset{o}{\xi}, \overset{o}{\eta}$ invariant, then it is obvious that

$$G = GL(n, \mathbb{C}) \times I \subset GL(2n, \mathbb{R}),$$

where I denotes the trivial group.

We set

$$\overset{o}{J} = \frac{\alpha}{2} - \left(\frac{2\sigma_{\alpha,\beta} - \alpha}{2}\right)\left(\overset{o}{F}^{II} + \overset{o}{\eta}^{0} \otimes \overset{o}{\xi}^{0} + \overset{o}{\eta}^{II} \otimes \overset{o}{\xi}^{II}\right), \quad \overset{o}{U} = \overset{o}{\xi}^{I}, \quad \overset{o}{\omega} = \overset{o}{\eta}^{I}. \tag{4.4}$$

By operating 2-lift of both sides of (4.2) and (4.3), we get

$$(F^{II})^2 = (F^2)^{II} = I - \eta^{II} \otimes \xi^0 - \eta^0 \otimes \xi^{II}, \tag{4.5}$$

$$\eta^0(\xi^{II}) = \eta^{II}(\xi^0) = 1, \ \ \eta^0(\xi^0) = \eta^{II}(\xi^{II}) = 0, \tag{4.6}$$

$$F^{II}(\xi^0) = F^{II}(\xi^{II}) = 0, \ \ \eta^0 \circ F^{II} = \eta^{II} \circ F^0 = 0. \tag{4.7}$$

Then

$$\overset{o}{J}(\overset{o}{\xi}^{0}) = \frac{\alpha}{2} \overset{o}{\xi}^{0} - \left(\frac{2\sigma_{\alpha,\beta} - \alpha}{2}\right)(\overset{o}{\xi}^{II}), \tag{4.8}$$

$$\overset{o}{J}(\overset{o}{\xi}^{II}) = \frac{\alpha}{2} \overset{o}{\xi}^{II} - \left(\frac{2\sigma_{\alpha,\beta} - \alpha}{2}\right)(\overset{o}{\xi}^{0}), \tag{4.9}$$

$$\overset{o}{J}(F^{II}\tilde{X}) = \frac{\alpha}{2}F^{II}\tilde{X} - (\frac{2\sigma_{\alpha,\beta} - \alpha}{2})(\tilde{X} - \eta^{0}(\tilde{X})\xi^{0} - \eta^{II}(\tilde{X})\xi^{II}) \tag{4.10}$$

and

$$\overset{o}{J}^{2}(\tilde{X}) = \alpha\, \overset{o}{J}(\tilde{X}) + \beta(\tilde{X}).$$

So, $(\overset{o}{J}, \overset{o}{\eta}, \overset{o}{\xi})$ is a metallic structure in $T_2(r)$. Hence, $T_2(R^{2n})$ leaves $(\overset{o}{J}, \overset{o}{\eta}, \overset{o}{\xi})$ invariant. Thus, we obtain

$$T(G) \subset GL(3n, C) \times I \subset GL(6n, R).$$

This finishes the proof.

5. Examples of metallic structures admitting an almost para contact structure

Let M be a $2n$-dimensional differential manifold and $T_3(M)$ and $T_4(M)$ be its tangent bundles of order 3 and 4. We construct the following examples on metallic structures that admits an almost para contact structure.

Example 5.1. If a manifold M of $2n$-dimension admits an almost para contact structure P (as a G-structure) determined by (F, U, ω) given in Eq (2.8). Then, on the tangent bundle $T_3(M)$, $\tilde{P}$ of P is the metallic structure is determined by the tensor field

$$\tilde{J} = \frac{\alpha}{2}I - (\frac{2\sigma_{\alpha,\beta} - \alpha}{2})(F^{II} + \eta^{0} \otimes \xi^{0} + \eta^{I} \otimes \xi^{I} - \eta^{II} \otimes \xi^{II} - \eta^{III} \otimes \xi^{III}).$$

Example 5.2. If a manifold M of $2n$-dimension admits an almost para contact structure P (as a G-structure) determined by (F, U, ω) given in Eq (2.8), then, on the tangent bundle $T_4(M)$, $\tilde{P}$ of P is the metallic structure is determined by the tensor field

$$\begin{aligned} \tilde{J} &= \frac{\alpha}{2}I - (\frac{2\sigma_{\alpha,\beta} - \alpha}{2})(F^{(4)} + \eta^{(0)} \otimes \xi^{(0)} + \eta^{(1)} \otimes \xi^{(1)} - \eta^{(3)} \otimes \xi^{(3)}) - \eta^{(4)} \otimes \xi^{(4)}), \\ \tilde{U} &= \xi^{(2)}, \quad \tilde{\omega} = \eta^{(2)}. \end{aligned}$$

6. Conclusions

In this work, we have characterized a metallic structure given by the general quadratic structure $F^2 = \alpha F + \beta I$, $\alpha, \beta \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. We have proved that the r-lift of the metallic structure F in the tangent bundle of order r is also a metallic structure. Furthermore, the projection tensor in the tangent bundle of order r is studied. Moreover, we have discussed prolongations of G-structures immersed in the metallic structure to the tangent bundle of order r and 2. Finally, the examples are given to validate obtained results. Future studies could fruitfully explore this issue further by considering the polynomial structure $Q(F) = F^n + a_n F^{n-1} + \ldots.. + a_2 F + a_1 I$, where F is the tensor field of type (1,1) on the differentiable manifold M.

Acknowledgments

The researchers would like to thank the Deanship of Scientific Research, Qassim University for funding the publication of this project.

Conflict of interest

The authors declare no conflicts of interest in this paper.

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