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Research article

On certain inclusion relations of functions with bounded rotations associated with Mittag-Leffler functions

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Abstract: Inspired essentially by the excellence of the implementations of the Mittag-Leffler functions in numerous areas of science and engineering, the authors present, in a unified manner, a detailed account of the Mittag-Leffler function and generalized Mittag-Leffler functions and their interesting and useful characteristics. Besides that, we have used generalized Mittag-Leffler functions to define some novel classes associated with bounded boundary and bounded radius rotations. Moreover, several inclusion relations and radius results, along with some integral preserving properties of these newly constructed classes have been investigated. Our derived results are analogous to some of those already present in the literature. The results showed that the proposed findings procedure is dependable and meticulous in presenting the tendencies of subordination, super-ordination and fractional operators techniques.

Keywords: bounded boundary rotations; strongly Janowski type functions; subordination; generalized Mittag-Leffler functions
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1. Introduction

The class of normalized analytic functions in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ denoted by Ω consists of the functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

where $f'(0) - 1 = f(0) = 0$. Let $\ell(z) \in \Omega$ defined by

$$\ell(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{1.2}$$

Then the Hadamard product, also known as the convolution of two function $f(z)$ and $\ell(z)$ denoted by $f * \ell$ is defined as

$$(f * \ell)(z) = f(z) * \ell(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \Delta.$$

Moreover, $f(z) \prec \ell(z)$, if there exist a Schwartz function $\chi(z)$ in A, satisfying the conditions $\chi(0) = 0$ and $|\chi(z)| < 1$, such that $f(z) = \ell(\chi(z))$. The symbol $\prec$ is used to denote subordination.

Let S denote the subclass of Ω of univalent functions in Δ. Let P, C, S^* and K represent the subclasses of S known as the classes of Caratheodory functions, convex funtions, starlike functions, and close-to-convex functions, respectively.

The concept of bounded rotations was introduced by Brannan in [7]. A lot of quality work on the generalization of this concept has already been done. Working in the same manner, we have defined the following new classes.

Definition 1.1. Let

$$\nu(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.3}$$

be analytic in Δ such that $\nu(0) = 1$. Then for $m \geq 2$, $\nu(z) \in P_m(\hbar(z))$, if and only if

$$\nu(z) = (\frac{m}{4} + \frac{1}{2})\nu_1(z) - (\frac{m}{4} - \frac{1}{2})\nu_2(z), \tag{1.4}$$

where $\hbar(z)$ is a convex univalent function in Δ and $\nu_i(z) \prec \hbar(z)$ for $i = 1, 2$.

Particularly, for $m = 2$, we get the class $P(\hbar(z))$.

Definition 1.2. Let $f(z)$ and $\ell(z)$ be two analytic functions as defined in (1.1) and (1.2) such that $(f * \ell)'(z) \neq 0$. Let $\hbar(z)$ be a convex univalent function. Then for $m \geq 2$, $f \in V_m[\hbar(z); \ell(z)]$ if and only if

$$\frac{(z(f * \ell)')'}{(f * \ell)'} \in P_m(\hbar(z)), \quad z \in \Delta. \tag{1.5}$$

Particularly, for $m = 2$, we will get the class $C[\hbar(z); \ell(z)]$. So, a function $f \in C[\hbar(z); \ell(z)]$ if and only if

$$\frac{(z(f * \ell)')'}{(f * \ell)'} \prec \hbar(z), \quad z \in \Delta.$$

Definition 1.3. Let $f(z)$ and $\ell(z)$ be the functions defined in (1.1) and (1.2), then $f(z) \in R_m[\hbar(z); \ell(z)]$ if and only if

$$\frac{z(f * \ell)'}{(f * \ell)} \in P_m(\hbar(z)), \quad z \in \Delta. \tag{1.6}$$

Particularly, for $m = 2$, we get the class $S^\wedge[\hbar(z); \ell(z)]$, i.e., $f \in S^\wedge[\hbar(z); \ell(z)]$ if and only if

$$\frac{z(f * \ell)'}{(f * \ell)} \prec \hbar(z), \quad z \in \Delta.$$

From (1.5) and (1.6) it can be easily noted that $f \in V_m[\hbar(z); \ell(z)]$ if and only if $zf'(z) \in R_m[\hbar(z); \ell(z)]$. For $m = 2$, this relation will hold for the classes $C[\hbar(z); \ell(z)]$ and $S^\wedge[\hbar(z); \ell(z)]$.

Definition 1.4. Let $f(z)$ and $\ell(z)$ be analytic function as defined in (1.1) and (1.2) and $m \geq 2$. Let $\hbar(z)$ be the convex univalent function. Then, $f \in T_m[\hbar(z); \ell(z)]$ if and only if there exists a function $\psi(z) \in S^\wedge[\hbar(z); \ell(z)]$ such that

$$\frac{z(f * \ell)'}{\psi * \ell} \in P_m(\hbar(z)), \quad z \in \Delta. \tag{1.7}$$

It is interesting to note that the particular cases of our newly defined classes will give us some well-known classes already discussed in the literature. Some of these special cases have been elaborated below.

Special Cases: Let $\ell(z)$ be the identity function defined as $\frac{z}{1-z}$ denoted by I i.e., $f * \ell = f * I = f$. Then
(1) For $\hbar(z) = \frac{1+z}{1-z}$ we have $P_m(\hbar(z)) = P_m, R_m[\hbar(z); \ell(z)] = R_m$ introduced by Pinchuk [23] and the class $V_m[\hbar(z); \ell(z)] = V_m$ defined by Paatero [21]. For $m = 2$, we will get the well-known classes of convex functions C and the starlike functions $S^\wedge$.
(2) Taking $\hbar(z) = \frac{1+(1-2\delta)z}{1-z}$, we get the classes $P_m(\delta), R_m(\delta)$ and $V_m(\delta)$ presented in [22]. For $m = 2$, we will get the classes $C(\delta)$ and $S^\wedge(\delta)$.
(3) Letting $\hbar(z) = \frac{1+Az}{1+Bz}$, with $-1 \leq B < A \leq 1$ introduced by Janowski in [12], the classes $P_m[A, B], R_m[A, B]$ and $V_m[A, B]$ defined by Noor [16,17] can be obtained. Moreover, the classes $C[A, B]$ and $S^\wedge[A, B]$ introduced by [12] can be derived by choosing $m = 2$.

A significant work has already been done by considering $\ell(z)$ to be different linear operators and $\hbar(z)$ to be any convex univalent function. For the details see ([4,9,18,19,24]).

The importance of Mittag-Leffler functions have tremendously been increased in the last four decades due to its vast applications in the field of science and technology. A number of geometric properties of Mittag-Leffler function have been discussed by many researchers working in the field of Geometric function theory. For some recent and detailed study on the Geometric properties of Mittag-Leffler functions see ([2,3,31]).

Special function theory has a vital role in both pure and applied mathematics. Mittag-Leffler functions have massive contribution in the theory of special functions, they are used to investigate certain generalization problems. For details see [11, 26]

There are numerous applications of Mittag-Leffler functions in the analysis of the fractional generalization of the kinetic equation, fluid flow problems, electric networks, probability, and statistical distribution theory. The use of Mittag-Leffler functions in the fractional order integral equations and differential equations attracted many researchers. Due to its connection and applications in fractional calculus, the significance of Mittag-Leffler functions has been amplified. To get a look into the applications of Mittag-Leffler functions in the field of fractional calculus, (see [5,27–30]).

Here, in this article we will use the operator $H^{\gamma,\kappa}_{\lambda,\eta} : \Omega \to \Omega$, introduced by Attiya [1], defined as

$$H^{\gamma,\kappa}_{\lambda,\eta}(f) = \mu^{\gamma,\kappa}_{\lambda,\eta} * f(z), \quad z \in \Delta, \tag{1.8}$$

where $\eta, \gamma \in C$, $\Re(\lambda) > max\{0, \Re(k) - 1\}$ and $\Re(k) > 0$. Also, $\Re(\lambda) = 0$ when $\Re(k) = 1; \eta \neq 0$. Here, $\mu^{\gamma,\kappa}_{\lambda,\eta}$ is the generalized Mittag-Leffler function, defined in [25]. The generalized Mittag-Leffler function has the following representation.

$$\mu^{\gamma,\kappa}_{\lambda,\eta} = z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma + n\kappa)\Gamma(\lambda + \eta)}{\Gamma(\gamma + \kappa)\Gamma(\eta + \lambda n)n!} z^n.$$

So, the operator defined in (1.8) can be rewritten as:

$$H^{\gamma,\kappa}_{\lambda,\eta}(f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma + n\kappa)\Gamma(\lambda + \eta)}{\Gamma(\gamma + \kappa)\Gamma(\eta + \lambda n)n!} a_n z^n, \quad z \in \Delta. \tag{1.9}$$

Attiya [1] presented the properties of the aforesaid operator as follows:

$$z\left(H^{\gamma,\kappa}_{\lambda,\eta}(f(z))\right)' = \left(\frac{\gamma + \kappa}{\kappa}\right)\left(H^{\gamma+1,\kappa}_{\lambda,\eta}(f(z))\right) - \left(\frac{\gamma}{\kappa}\right)\left(H^{\gamma,\kappa}_{\lambda,\eta}(f(z))\right), \tag{1.10}$$

and

$$z\left(H^{\gamma,\kappa}_{\lambda,\eta+1}(f(z))\right)' = \left(\frac{\lambda + \eta}{\lambda}\right)\left(H^{\gamma,\kappa}_{\lambda,\eta}(f(z))\right) - \left(\frac{\eta}{\lambda}\right)\left(H^{\gamma,\kappa}_{\lambda,\eta+1}(f(z))\right). \tag{1.11}$$

However, as essential as real-world phenomena are, discovering a solution for the commensurate scheme and acquiring fundamentals with reverence to design variables is challenging and time-consuming. Among the most pragmatically computed classes, we considered the new and novel class which is very useful for efficiently handling complex subordination problems. Here, we propose a suitably modified scheme in order to compute the Janowski type function of the form $\hbar(z) = \left(\frac{1+Az}{1+Bz}\right)^{\beta}$, where $0 < \beta \leq 1$ and $-1 \leq B < A \leq 1$, which is known as the strongly Janowski type function. Moreover, for $\ell(z)$, we will use the function defined in (1.9). So, the classes defined in Definition 1.1–1.4 will give us the following novel classes.

Definition 1.5. A function $\nu(z)$ as defined in Eq (1.3) is said to be in the class $P_{(m,\beta)}[A, B]$ if and only if for $m \geq 2$ there exist two analytic functions $\nu_1(z)$ and $\nu_2(z)$ in Δ, such that

$$\nu(z) = (\frac{m}{4} + \frac{1}{2})\nu_1(z) - (\frac{m}{4} - \frac{1}{2})\nu_2(z),$$

where $\nu_i(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\beta}$ for $i = 1, 2$. For $m = 2$, we get the class of strongly Janowki type functions $P_{\beta}[A, B]$.

Moreover,

$$V_{(m,\beta)}[A, B; \gamma, \eta] = \{f \in \Omega : \frac{\left(z\left(H^{\gamma,\kappa}_{\lambda,\eta} f(z)\right)'\right)'}{\left(H^{\gamma,\kappa}_{\lambda,\eta} f(z)\right)'} \in P_{(m,\beta)}[A, B]\},$$

$$R_{(m,\beta)}[A,B;\gamma,\eta]=\{f\in\Omega:\frac{z\left(H^{\gamma,\kappa}_{\lambda,\eta}f(z)\right)'}{H^{\gamma,\kappa}_{\lambda,\eta}f(z)}\in P_{(m,\beta)}[A,B]\},$$

$$C_{\beta}[A,B,\gamma,\eta]=\{f\in\Omega:\frac{\left(z\left(H^{\gamma,\kappa}_{\lambda,\eta}f(z)\right)'\right)'}{\left(H^{\gamma,\kappa}_{\lambda,\eta}f(z)\right)'}\in P_{\beta}[A,B]\},$$

$$S^{\Lambda}_{\beta}[A,B,\gamma,\eta]=\{f\in\Omega:\frac{z\left(H^{\gamma,\kappa}_{\lambda,\eta}f(z)\right)'}{H^{\gamma,\kappa}_{\lambda,\eta}f(z)}\in P_{\beta}[A,B]\},$$

$$T_{(m,\beta)}[A,B;\gamma,\eta]=\{f\in\Omega:\frac{z\left(H^{\gamma,\kappa}_{\lambda,\eta}f(z)\right)'}{H^{\gamma,\kappa}_{\lambda,\eta}\psi(z)}\in P_{(m,\beta)}[A,B],\textit{where}\quad \psi(z)\in S^{\Lambda}_{\beta}[A,B,\gamma,\eta]\},$$

where $\eta,\gamma\in C$, $\Re(\lambda)>max\{0,\Re(k)-1\}$ and $\Re(k)>0$. Also, $\Re(\lambda)=0$ when $\Re(k)=1;\eta\neq 0$. It can easily be noted that there exists Alexander relation between the classes $V_{(m,\beta)}[A,B;\gamma,\eta]$ and $R_{(m,\beta)}[A,B;\gamma,\eta]$, i.e.,

$$f\in V_{(m,\beta)}[A,B;\gamma,\eta]\Longleftrightarrow zf'\in R_{(m,\beta)}[A,B;\gamma,\eta]. \tag{1.12}$$

Throughout this investigation, $-1\leq B<A\leq 1$, $m\geq 2$ and $0<\beta\leq 1$ unless otherwise stated.

2. Preliminaries

Lemma 2.1. *([13]) Let $\nu(z)$ as defined in* (1.3) *be in $P_{(m,\beta)}[A,B]$. Then $\nu(z)\in P_m(\varrho)$, where $0\leq\varrho=\left(\frac{1-A}{1-B}\right)^{\beta}<1$.*

Lemma 2.2. *([8]) Let $\hbar(z)$ be convex univalent in Δ with $h(0)=1$ and $\Re(\zeta\hbar(z)+\alpha)>0\quad(\zeta\in C)$. Let $p(z)$ be analytic in Δ with $p(0)=1$, which satisfy the following subordination relation*

$$p(z)+\frac{zp'(z)}{\zeta p(z)+\alpha}\prec\hbar(z),$$

then

$$p(z)\prec\hbar(z).$$

Lemma 2.3. *([10]) Let $\hbar(z)\in P$. Then for $|z|<r$, $\frac{1-r}{1+r}\leq\Re(\hbar(z))\leq\ |\hbar(z)|\leq\frac{1+r}{1-r}$, and $|h'(z)|\leq\frac{2r\Re\hbar(z)}{1-r^2}$.*

3. Main results

Theorem 3.1. *Let $\varrho=\left(\frac{1-A}{1-B}\right)^{\beta}$. Then for $\Re(\frac{\gamma}{\kappa})>-\varrho$,*

$$R_{(m,\beta)}[A,B,\gamma+1,\eta]\subset R_{(m,\beta)}[A,B,\gamma,\eta].$$

Proof. Let $f(z)\in R_{(m,\beta)}[A,B,\gamma+1,\eta]$. Set

$$\varphi(z)=\frac{z\left(H^{\gamma+1,\kappa}_{\lambda,\eta}f(z)\right)'}{H^{\gamma+1,\kappa}_{\lambda,\eta}f(z)}, \tag{3.1}$$

then $\varphi(z) \in P_{(m,\beta)}[A, B]$. Now, Assume that

$$\psi(z) = \frac{z\left(H_{\lambda,\eta}^{\gamma,\kappa} f(z)\right)'}{H_{\lambda,\eta}^{\gamma,\kappa} f(z)}. \tag{3.2}$$

Plugging (1.10) in (3.2), we get

$$\psi(z) = \frac{\left(\frac{\gamma+\kappa}{\kappa}\right)\left(H_{\lambda,\eta}^{\gamma+1,\kappa} f(z)\right) - \left(\frac{\gamma}{\kappa}\right)\left(H_{\lambda,\eta}^{\gamma,\kappa} f(z)\right)}{H_{\lambda,\eta}^{\gamma,\kappa} f(z)}.$$

It follows that

$$H_{\lambda,\eta}^{\gamma,\kappa} f(z)\left(\frac{\kappa}{\gamma+\kappa}\right)\left(\psi(z) + \frac{\gamma}{\kappa}\right) = H_{\lambda,\eta}^{\gamma+1,\kappa} f(z).$$

After performing logarithmic differentiation and simple computation, we get

$$\psi(z) + \frac{z\psi'(z)}{\psi(z) + \frac{\gamma}{\kappa}} = \varphi(z). \tag{3.3}$$

Now, for $m \geq 2$, consider

$$\psi(z) = (\frac{m}{4} + \frac{1}{2})\psi_1(z) - (\frac{m}{4} - \frac{1}{2})\psi_2(z). \tag{3.4}$$

Combining (3.3) and (3.4) with the similar technique as used in Theorem 3.1 of [20], we get

$$\varphi(z) = (\frac{m}{4} + \frac{1}{2})\varphi_1(z) - (\frac{m}{4} - \frac{1}{2})\varphi_2(z),$$

where

$$\varphi_i(z) = \psi_i(z) + \frac{z\psi_i'(z)}{\psi_i(z) + \frac{\gamma}{\kappa}},$$

for $i = 1, 2$. Since $\varphi(z) \in P_{(m,\beta)}[A, B]$, therefore

$$\varphi_i(z) = \psi_i(z) + \frac{z\psi_i'(z)}{\psi_i(z) + \frac{\gamma}{\kappa}} \prec \left(\frac{1+Az}{1+Bz}\right)^{\beta},$$

for $i = 1, 2$. By using Lemma 2.1 and the condition $\Re(\frac{\gamma}{\kappa}) > -\varrho$, we have

$$\Re\left(\frac{\gamma}{\kappa} + \left(\frac{1+Az}{1+Bz}\right)^{\beta}\right) > 0,$$

where $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$. Hence, in view of Lemma 2.2, we have

$$\psi_i(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\beta},$$

for i=1,2. This implies $\psi(z) \in P_{(m,\beta)}[A, B]$, so

$$f(z) \in R_{(m,\beta)}[A, B, \gamma, \eta],$$

which is required to prove. □

Theorem 3.2. *If* $\Re(\frac{\lambda}{\eta}) > -\varrho$*, where* $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$*,then*

$$R_{(m,\beta)}[A, B, \gamma, \eta] \subset R_{(m,\beta)}[A, B, \gamma, \eta + 1].$$

Proof. Let $f(z) \in R_{(m,\beta)}[A, B, \gamma, \eta]$. Taking

$$\varphi(z) = \frac{z\left(H^{\gamma,\kappa}_{\lambda,\eta} f(z)\right)'}{H^{\gamma,\kappa}_{\lambda,\eta} f(z)}, \tag{3.5}$$

we have $\varphi(z) \in P_{(m,\beta)}[A, B]$. Now, suppose that

$$\psi(z) = \frac{z\left(H^{\gamma,\kappa}_{\lambda,\eta+1} f(z)\right)'}{H^{\gamma,\kappa}_{\lambda,\eta+1} f(z)}. \tag{3.6}$$

Applying the relation (1.11) in the Eq (3.6), we have

$$\psi(z) = \frac{\left(\frac{\lambda+\eta}{\lambda}\right)\left(H^{\gamma,\kappa}_{\lambda,\eta} f(z)\right) - \left(\frac{\eta}{\lambda}\right)\left(H^{\gamma,\kappa}_{\lambda,\eta+1} f(z)\right)}{H^{\gamma,\kappa}_{\lambda,\eta+1} f(z)}.$$

arrives at

$$H^{\gamma,\kappa}_{\lambda,\eta+1} f(z)\left(\frac{\lambda}{\eta+\lambda}\right)\left(\psi(z) + \frac{\eta}{\lambda}\right) = H^{\gamma,\kappa}_{\lambda,\eta} f(z).$$

So by the logarithmic differentiation and simple computation we get,

$$\psi(z) + \frac{z\psi'(z)}{\psi(z) + \frac{\eta}{\lambda}} = \varphi(z). \tag{3.7}$$

Therefore, for $m \geq 2$, take

$$\psi(z) = (\frac{m}{4} + \frac{1}{2})\psi_1(z) - (\frac{m}{4} - \frac{1}{2})\psi_2(z). \tag{3.8}$$

Combining Eqs (3.6) and (3.7) using the similar technique as in Theorem 3.1 of [20], we get

$$\varphi(z) = (\frac{m}{4} + \frac{1}{2})\varphi_1(z) - (\frac{m}{4} - \frac{1}{2})\varphi_2(z),$$

where

$$\varphi_i(z) = \psi_i(z) + \frac{z\psi_i'(z)}{\psi_i(z) + \frac{\eta}{\lambda}},$$

for $i = 1, 2$. Since $\varphi(z) \in P_{(m,\beta)}[A, B]$, therefore

$$\varphi_i(z) = \psi_i(z) + \frac{z\psi_i'(z)}{\psi_i(z) + \frac{\eta}{\lambda}} \prec \left(\frac{1+Az}{1+Bz}\right)^{\beta},$$

for $i = 1, 2$. Applying Lemma 2.1 and the condition $\Re(\frac{\eta}{\lambda}) > -\varrho$, we get

$$\Re\left(\frac{\eta}{\lambda} + \left(\frac{1+Az}{1+Bz}\right)^{\beta}\right) > 0,$$

where $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$. Hence, by Lemma 2.2, we have

$$\psi_i(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\beta},$$

for i=1,2. This implies $\psi(z) \in P_{(m,\beta)}[A,B]$, so

$$f(z) \in R_{(m,\beta)}[A,B,\gamma,\eta+1],$$

which completes the proof. □

Corollary 3.1. *For $m=2$, if $\Re(\frac{\gamma}{\kappa}) > -\varrho$, where $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$. Then*

$$S_{\beta}^{\Lambda}[A,B,\gamma+1,\eta] \subset S_{\beta}^{\Lambda}[A,B,\gamma,\eta].$$

Moreover, if $\Re(\frac{\lambda}{\eta}) > -\varrho$, then

$$S_{\beta}^{\Lambda}[A,B,\gamma,\eta] \subset S_{\beta}^{\Lambda}[A,B,\gamma,\eta+1].$$

Theorem 3.3. *Let $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$. Then for $\Re(\frac{\gamma}{\kappa}) > -\varrho$,*

$$V_{(m,\beta)}[A,B,\gamma+1,\eta] \subset V_{(m,\beta)}[A,B,\gamma,\eta].$$

Proof. By means of theorem 3.1 and Alexander relation defined in (1.12), we get

$$\begin{aligned} f \in V_{(m,\beta)}[A,B,\gamma+1,\eta] &\Longleftrightarrow zf' \in R_{(m,\beta)}[A,B,\gamma+1,\eta] \\ &\Longleftrightarrow zf' \in R_{(m,\beta)}[A,B,\gamma,\eta] \\ &\Longleftrightarrow f \in V_{(m,\beta)}[A,B,\gamma,\eta]. \end{aligned}$$

Hence the result. □

Analogously, we can prove the following theorem.

Theorem 3.4. *If $\Re(\frac{\lambda}{\eta}) > -\varrho$, where $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$,then*

$$V_{(m,\beta)}[A,B,\gamma,\eta] \subset V_{(m,\beta)}[A,B,\gamma,\eta+1].$$

Corollary 3.2. *For $m=2$, if $\Re(\frac{\gamma}{\kappa}) > -\varrho$, where $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$. Then*

$$C_{\beta}[A,B,\gamma+1,\eta] \subset C_{\beta}[A,B,\gamma,\eta].$$

Moreover, if $\Re(\frac{\lambda}{\eta}) > -\varrho$, then

$$C_{\beta}[A,B,\gamma,\eta] \subset C_{\beta}[A,B,\gamma,\eta+1].$$

Theorem 3.5. *Let $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$, and $\Re(\frac{\gamma}{\kappa}) > -\varrho$. Then*

$$T_{(m,\beta)}[A,B;\gamma+1,\eta] \subset T_{(m,\beta)}[A,B;\gamma,\eta].$$

Proof. Let $f(z) \in T_{(m,\beta)}[A, B, \gamma+1, \eta]$. Then there exist $\psi(z) \in S^{\Lambda}_{\beta}[A, B, \gamma+1, \eta]$ such that

$$\varphi(z) = \frac{z\left(H^{\gamma+1,\kappa}_{\lambda,\eta} f(z)\right)'}{H^{\gamma+1,\kappa}_{\lambda,\eta}\psi(z)} \in P_{(m,\beta)}[A, B]. \tag{3.9}$$

Now consider

$$\phi(z) = \frac{z\left(H^{\gamma,\kappa}_{\lambda,\eta} f(z)\right)'}{H^{\gamma,\kappa}_{\lambda,\eta}\psi(z)}. \tag{3.10}$$

Since $\psi(z) \in S^{\Lambda}_{\beta}[A, B, \gamma+1, \eta]$ and $\Re(\frac{\gamma}{\kappa}) > -\varrho$, therefore by Corollary 3.3, $\psi(z) \in S^{\Lambda}_{\beta}[A, B, \gamma, \eta]$. So

$$q(z) = \frac{z(H^{\gamma,\kappa}_{\lambda,\eta}\psi(z))'}{H^{\gamma,\kappa}_{\lambda,\eta}\psi(z)} \in P_{\beta}[A, B]. \tag{3.11}$$

By doing some simple calculations on (3.11), we get

$$(\kappa q(z) + \gamma) H^{\gamma,\kappa}_{\lambda,\eta}\psi(z) = (\gamma+\kappa) H^{\gamma+1,\kappa}_{\lambda,\eta}\psi(z). \tag{3.12}$$

Now applying the relation (1.10) on (3.10), we get

$$\phi(z) H^{\gamma,\kappa}_{\lambda,\eta}\psi(z) = \frac{\gamma+\kappa}{\kappa} H^{\gamma+1,\kappa}_{\lambda,\eta} f(z) - \frac{\gamma}{\kappa} H^{\gamma,\kappa}_{\lambda,\eta} f(z). \tag{3.13}$$

Differentiating both sides of (3.13), we have

$$\phi(z)(H^{\gamma,\kappa}_{\lambda,\eta}\psi(z))' + \phi'(z) H^{\gamma,\kappa}_{\lambda,\eta}\psi(z) = \frac{\gamma+\kappa}{\kappa}(H^{\gamma+1,\kappa}_{\lambda,\eta} f(z))' - \frac{\gamma}{\kappa}(H^{\gamma,\kappa}_{\lambda,\eta} f(z))'.$$

By using (3.12) and with some simple computations, we get

$$\phi(z) + \frac{z\phi'(z)}{q(z) + \frac{\gamma}{\kappa}} = \varphi(z) \in P_{(m,\beta)}[A, B], \tag{3.14}$$

with $\Re(q(z) + \frac{\gamma}{\kappa}) > 0$, since $q(z) \in P_{\beta}[A, B]$, so by Lemma 2.1, $\Re(q(z) > \varrho$ and $\Re(\frac{\gamma}{\kappa}) > -\varrho$. Now consider

$$\phi(z) = \left(\frac{m}{4} + \frac{1}{2}\right)\phi_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)\phi_2(z). \tag{3.15}$$

Combining (3.14) and (3.15) with the similar technique as used in Theorem 3.1 of [20], we get

$$\varphi(z) = \left(\frac{m}{4} + \frac{1}{2}\right)\varphi_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)\varphi_2(z), \tag{3.16}$$

where

$$\varphi_i(z) = \phi(z) + \frac{z\phi' z}{q(z) + \frac{\gamma}{\kappa}},$$

for $i = 1, 2$. Since $\varphi(z) \in P_{(m,\beta)}[A, B]$, therefore

$$\varphi_i(z) \prec \left(\frac{1 + Az}{1 + Bz}\right)^{\beta}, \quad i = 1, 2.$$

Using the fact of Lemma 2.2, we can say that

$$\phi_i(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\beta}, \quad i = 1, 2.$$

So, $\phi(z) \in P_{(m,\beta)}[A, B]$. Hence we get the required result. □

Theorem 3.6. *If* $\Re(\frac{\lambda}{\eta}) > -\varrho$, *where* $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$, *then*

$$T_{(m,\beta)}[A, B, \gamma, \eta] \subset T_{(m,\beta)}[A, B, \gamma, \eta + 1].$$

Let $f(z) \in T_{(m,\beta)}[A, B, \gamma, \eta]$. Then there exist $\psi(z) \in S^{\Lambda}_{\beta}[A, B, \gamma, \eta]$ such that

$$\varphi(z) = \frac{z\left(H^{\gamma,\kappa}_{\lambda,\eta} f(z)\right)'}{H^{\gamma,\kappa}_{\lambda,\eta}\psi(z)} \in P_{(m,\beta)}[A, B]. \tag{3.17}$$

Taking

$$\phi(z) = \frac{z\left(H^{\gamma,\kappa}_{\lambda,\eta+1} f(z)\right)'}{H^{\gamma,\kappa}_{\lambda,\eta+1}\psi(z)}. \tag{3.18}$$

As we know that, $\psi(z) \in S^{\Lambda}_{\beta}[A, B, \gamma, \eta]$ and $\Re(\frac{\eta}{\lambda}) > -\varrho$, therefore by Corollary 3.3, $\psi(z) \in S^{\Lambda}_{\beta}[A, B, \gamma, \eta + 1]$. So

$$q(z) = \frac{z(H^{\gamma,\kappa}_{\lambda,\eta+1}\psi(z))'}{H^{\gamma,\kappa}_{\lambda,\eta+1}\psi(z)} \in P_{\beta}[A, B]. \tag{3.19}$$

By doing some simple calculations on (3.19) with the help of (1.11), we get

$$(\lambda q(z) + \eta)H^{\gamma,\kappa}_{\lambda,\eta+1}\psi(z) = (\eta + \lambda)H^{\gamma,\kappa}_{\lambda,\eta}\psi(z). \tag{3.20}$$

Now, applying the relation (1.11) on (3.18), we get

$$\phi(z)H^{\gamma,\kappa}_{\lambda,\eta+1}\psi(z) = \frac{\eta + \lambda}{\lambda} H^{\gamma,\kappa}_{\lambda,\eta} f(z) - \frac{\eta}{\lambda} H^{\gamma,\kappa}_{\lambda,\eta+1} f(z). \tag{3.21}$$

Differentiating both sides of Eq (3.21), we have

$$\phi(z)(H^{\gamma,\kappa}_{\lambda,\eta+1}\psi(z))' + \phi'(z)H^{\gamma,\kappa}_{\lambda,\eta+1}\psi(z) = \frac{\eta + \lambda}{\lambda}(H^{\gamma,\kappa}_{\lambda,\eta} f(z))' - \frac{\eta}{\lambda}(H^{\gamma,\kappa}_{\lambda,\eta+1} f(z))',$$

some simple calculations along with using (3.20) give us

$$\phi(z) + \frac{z\phi'(z)}{q(z) + \frac{\eta}{\lambda}} = \varphi(z) \in P_{(m,\beta)}[A, B], \tag{3.22}$$

with $\Re(q(z) + \frac{\eta}{\lambda}) > 0$. Since $q(z) \in P_{\beta}[A, B]$, so applying Lemma 2.1, we have $\Re(q(z) > \varrho$ and $\Re(\frac{\eta}{\lambda}) > -\varrho$.
Assume that

$$\phi(z) = \left(\frac{m}{4} + \frac{1}{2}\right)\phi_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)\phi_2(z). \tag{3.23}$$

Combining (3.22) and (3.23), along with using the similar technique as in Theorem 3.1 of [20], we get

$$\varphi(z) = \left(\frac{m}{4} + \frac{1}{2}\right)\varphi_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)\varphi_2(z), \tag{3.24}$$

where

$$\varphi_i(z) = \phi(z) + \frac{z\phi' z}{q(z) + \frac{\eta}{\lambda}},$$

for $i = 1, 2$. Since $\varphi(z) \in P_{(m,\beta)}[A, B]$, therefore

$$\varphi_i(z) \prec \left(\frac{1 + Az}{1 + Bz}\right)^{\beta}, \quad i = 1, 2.$$

Applying the fact of Lemma 2.2, we have

$$\phi_i(z) \prec \left(\frac{1 + Az}{1 + Bz}\right)^{\beta}, \quad i = 1, 2.$$

So $\phi(z) \in P_{(m,\beta)}[A, B]$. Which gives us the required result.

Corollary 3.3. *If $\varrho > -min\{\Re\left(\frac{\gamma}{\kappa}\right), \Re\left(\frac{\lambda}{\eta}\right)\}$, where $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$, then we have the following inclusion relations:*
(i) $R_{(m,\beta)}[A, B, \gamma + 1, \eta] \subset R_{(m,\beta)}[A, B, \gamma, \eta] \subset R_{(m,\beta)}[A, B, \gamma, \eta + 1]$.
(ii)$V_{(m,\beta)}[A, B, \gamma + 1, \eta] \subset V_{(m,\beta)}[A, B, \gamma, \eta] \subset V_{(m,\beta)}[A, B, \gamma, \eta + 1]$.
(iii)$T_{(m,\beta)}[A, B, \gamma + 1, \eta] \subset T_{(m,\beta)}[A, B, \gamma, \eta] \subset T_{(m,\beta)}[A, B, \gamma, \eta + 1]$.

Now, we will discuss some radius results for our defined classes.

Theorem 3.7. *Let $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$, and $\Re\left(\frac{\gamma}{\kappa}\right) > -\varrho$. Then*

$$R_{(m,\beta)}[A, B, \gamma, \eta] \subset R_{(m,\beta)}[\varrho, \gamma + 1, \eta]$$

whenever

$$|z| < r_o = \frac{1 - \varrho}{2 - \varrho + \sqrt{3 - 2\varrho}}, \quad \text{where} \quad 0 \leq \varrho < 1.$$

Proof. Let $f(z) \in R_{(m,\beta)}[A, B, \gamma, \eta]$. Then

$$\psi(z) = \frac{z\left(H^{\gamma,\kappa}_{\lambda,\eta} f(z)\right)'}{H^{\gamma,\kappa}_{\lambda,\eta} f(z)} \in P_{(m,\beta)}[A, B]. \tag{3.25}$$

In view of Lemma 2.1 $P_{(m,\beta)}[A, B] \subset P_m(\varrho)$, for $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$, therefore $\psi(z) \in P_m(\varrho)$. So by the Definition of $P_m(\varrho)$ given in [22], there exist two functions $\psi_1(z), \psi_2(z) \in P(\varrho)$ such that

$$\psi(z) = \left(\frac{m}{4} + \frac{1}{2}\right)\psi_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)\psi_2(z), \tag{3.26}$$

with $m \geq 2$ and $\Re(\psi_i(z)) > \varrho, i = 1, 2$. We can write

$$\psi_i(z) = (1 - \varrho)h_i(z) + \varrho, \tag{3.27}$$

where $h_i(z) \in P$ and $\Re(h_i(z) > 0$, for $i = 1, 2$. Now, let

$$\phi(z) = \frac{z\left(H_{\lambda,\eta}^{\gamma+1,\kappa} f(z)\right)'}{H_{\lambda,\eta}^{\gamma+1,\kappa} f(z)}. \tag{3.28}$$

We have to check when $\phi(z) \in P_m(\varrho)$. Using relation (1.10) in (3.25), we get

$$\psi(z) H_{\lambda,\eta}^{\gamma+1,\kappa} f(z) = \left(\frac{\gamma+\kappa}{\kappa}\right)\left(H_{\lambda,\eta}^{\gamma+1,\kappa}(f(z))\right) - \left(\frac{\gamma}{\kappa}\right)\left(H_{\lambda,\eta}^{\gamma,\kappa}(f(z))\right).$$

So, by simple calculation and logarithmic differentiation, we get

$$\psi(z) + \frac{z\psi' z}{\psi(z) + \frac{\gamma}{\kappa}} = \phi(z). \tag{3.29}$$

Now, consider

$$\phi(z) = \left(\frac{m}{4} + \frac{1}{2}\right)\phi_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)\phi_2(z),$$

where

$$\phi_i(z) = \psi_i(z) + \frac{z\psi_i' z}{\psi_i(z) + \frac{\gamma}{\kappa}}, \quad i = 1, 2.$$

To derive the condition for $\phi_i(z)$ to be in $P(\varrho)$, consider

$$\Re(\phi_i(z) - \varrho) = \Re\left(\psi_i(z) + \frac{z\psi_i' z}{\psi_i(z) + \frac{\gamma}{\kappa}} - \varrho\right).$$

In view of (3.27), we have

$$\begin{aligned} \Re(\phi_i(z) - \varrho) &= \Re\left((1-\varrho)h_i(z) + \varrho + \frac{z(1-\varrho)h_i'(z)}{\frac{\gamma}{\kappa} + \varrho + (1-\varrho)h_i(z)} - \varrho\right) \\ &\geq (1-\varrho)\Re(h_i(z)) - (1-\varrho)\frac{|zh_i'(z)|}{\Re(\frac{\gamma}{\kappa} + \varrho) + (1-\varrho)\Re(h_i(z))}. \end{aligned} \tag{3.30}$$

We have, $\Re(\frac{\gamma}{\kappa} + \varrho) > 0$ since $\Re(\frac{\gamma}{\kappa}) > -\varrho$. Since $h_i(z) \in P$, hence by using Lemma 2.3 in inequality (3.30), we have

$$\begin{aligned} \Re(\phi_i(z) - \varrho) &\geq (1-\varrho)\Re(h_i(z)) - \frac{1 - \varrho\frac{2r}{1-r^2}\Re(h_i(z))}{(1-\varrho)(\frac{1-r}{1+r})} \\ &= (1-\varrho)\Re(h_i(z))\left[\frac{(1-r)^2(1-\varrho) - 2r}{(1-r)^2(1-\varrho)}\right] \\ &\geq (1-\varrho)\left(\frac{1-r}{1+r}\right)\left[\frac{(1-r)^2(1-\varrho) - 2r}{(1-r)^2(1-\varrho)}\right] \\ &= \frac{r^2(1-\varrho) - 2r(2-\varrho) + (1-\varrho)}{1-r^2}. \end{aligned} \tag{3.31}$$

Since $1 - r^2 > 0$, letting $T(r) = r^2(1 - \varrho) - 2r(2 - \varrho) + (1 - \varrho)$. It is easy to note that $T(0) > 0$ and $T(1) < 0$. Hence, there is a root of $T(r)$ between 0 and 1. Let r_o be the root then by simple calculations, we get

$$r_o = \frac{1 - \varrho}{2 - \varrho + \sqrt{3 - 2\varrho}}.$$

Hence $\phi(z) \in P_m(\varrho)$ for $|z| < r_o$. Thus for this radius r_o the function $f(z)$ belongs to the class $R_{(m,\beta)}[\varrho, \gamma + 1, \eta]$, which is required to prove. □

Theorem 3.8. *Let* $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$, *and* $\Re\left(\frac{\lambda}{\eta}\right) > -\varrho$. *Then*

$$R_{(m,\beta)}[A, B, \gamma, \eta + 1] \subset R_{(m,\beta)}[\varrho, \gamma, \eta],$$

whenever

$$|z| < r_o = \frac{1 - \varrho}{2 - \varrho + \sqrt{3 - 2\varrho}}, \quad \textit{where} \quad 0 \le \varrho < 1.$$

Proof. Let $f(z) \in R_{(m,\beta)}[A, B, \gamma, \eta + 1]$. Then

$$\psi(z) = \frac{z\left(H^{\gamma,\kappa}_{\lambda,\eta+1} f(z)\right)'}{H^{\gamma,\kappa}_{\lambda,\eta+1} f(z)} \in P_{(m,\beta)}[A, B]. \tag{3.32}$$

By applying of Lemma 2.1, we get $P_{(m,\beta)}[A, B] \subset P_m(\varrho)$, for $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$, therefore $\psi(z) \in P_m(\varrho)$. Hence, the Definition of $P_m(\varrho)$ given in [22], there exist two functions $\psi_1(z), \psi_2(z) \in P(\varrho)$ such that

$$\psi(z) = \left(\frac{m}{4} + \frac{1}{2}\right)\psi_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)\psi_2(z), \tag{3.33}$$

with $m \ge 2$ and $\Re(\psi_i(z)) > \varrho, i = 1, 2$. We can say that

$$\psi_i(z) = (1 - \varrho)h_i(z) + \varrho, \tag{3.34}$$

where $h_i(z) \in P$ and $\Re(h_i(z) > 0$, for $i = 1, 2$. Now, assume

$$\phi(z) = \frac{z\left(H^{\gamma,\kappa}_{\lambda,\eta} f(z)\right)'}{H^{\gamma,\kappa}_{\lambda,\eta} f(z)}. \tag{3.35}$$

Here, We have to obtain the condition for which $\phi(z) \in P_m(\varrho)$. Using relation (1.11) in (3.51), we get

$$\psi(z)H^{\gamma,\kappa}_{\lambda,\eta} f(z) = \left(\frac{\eta + \lambda}{\lambda}\right)\left(H^{\gamma,\kappa}_{\lambda,\eta}(f(z))\right) - \left(\frac{\eta}{\lambda}\right)\left(H^{\gamma,\kappa}_{\lambda,\eta+1}(f(z))\right).$$

Thus, by simple calculation and logarithmic differentiation, we have

$$\psi(z) + \frac{z\psi' z}{\psi(z) + \frac{\eta}{\lambda}} = \phi(z). \tag{3.36}$$

Now, consider

$$\phi(z) = \left(\frac{m}{4} + \frac{1}{2}\right)\phi_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)\phi_2(z),$$

where

$$\phi_i(z) = \psi_i(z) + \frac{z\psi_i' z}{\psi_i(z) + \frac{\eta}{\lambda}}, \quad i = 1, 2.$$

To derive the condition for $\phi_i(z)$ to be in $P(\varrho)$, consider

$$\Re(\phi_i(z) - \varrho) = \Re\left(\psi_i(z) + \frac{z\psi_i' z}{\psi_i(z) + \frac{\eta}{\lambda}} - \varrho\right).$$

In view of (3.34), we have

$$\begin{aligned}\Re(\phi_i(z) - \varrho) &= \Re\left((1-\varrho)h_i(z) + \varrho + \frac{z(1-\varrho)h_i'(z)}{\frac{\eta}{\lambda} + \varrho + (1-\varrho)h_i(z)} - \varrho\right)\\ &\geq (1-\varrho)\Re(h_i(z)) - (1-\varrho)\frac{|zh_i'(z)|}{\Re(\frac{\eta}{\lambda} + \varrho) + (1-\varrho)\Re(h_i(z))}.\end{aligned} \tag{3.37}$$

Here, $\Re(\frac{\eta}{\lambda} + \varrho) > 0$ since $\Re(\frac{\eta}{\lambda}) > -\varrho$. We know that $h_i(z) \in P$, therefore by using Lemma 2.3 in inequality (3.37), we have

$$\begin{aligned}\Re(\phi_i(z) - \varrho) &\geq (1-\varrho)\Re(h_i(z)) - \frac{1 - \varrho\frac{2r}{1-r^2}\Re(h_i(z))}{(1-\varrho)(\frac{1-r}{1+r})}\\ &= (1-\varrho)\Re(h_i(z))\left[\frac{(1-r)^2(1-\varrho) - 2r}{(1-r)^2(1-\varrho)}\right]\\ &\geq (1-\varrho)\left(\frac{1-r}{1+r}\right)\left[\frac{(1-r)^2(1-\varrho) - 2r}{(1-r)^2(1-\varrho)}\right]\\ &= \frac{r^2(1-\varrho) - 2r(2-\varrho) + (1-\varrho)}{1-r^2}.\end{aligned} \tag{3.38}$$

Since $1 - r^2 > 0$, letting $T(r) = r^2(1-\varrho) - 2r(2-\varrho) + (1-\varrho)$. It can easily be seen that $T(0) > 0$ and $T(1) < 0$. Hence, there is a root of $T(r)$ between 0 and 1. Let r_o be the root then by simple calculations, we get

$$r_o = \frac{1-\varrho}{2-\varrho+\sqrt{3-2\varrho}}.$$

Hence $\phi(z) \in P_m(\varrho)$ for $|z| < r_o$. Thus for this radius r_o the function $f(z)$ belongs to the class $R_{(m,\beta)}[\varrho, \gamma, \eta]$, which is required to prove. □

Corollary 3.4. *Let* $\varrho = \left(\frac{1-A}{1-B}\right)^\beta$. *Then, for* $m = 2$, *and* $|z| < r_o = \frac{1-\varrho}{2-\varrho+\sqrt{3-2\varrho}}$,
(i) If $\Re\left(\frac{\gamma}{\kappa}\right) > -\varrho$, *then* $S_\beta^\Lambda[A, B, \gamma, \eta] \subset S_\beta^\Lambda[\varrho, \gamma+1, \eta]$.
(ii) If $\Re\left(\frac{\lambda}{\eta}\right) > -\varrho$, *then* $S_\beta^\Lambda[A, B, \gamma, \eta+1] \subset S_\beta^\Lambda[\varrho, \gamma, \eta]$.

Theorem 3.9. *Let* $\varrho = \left(\frac{1-A}{1-B}\right)^\beta$. *Then for* $|z| < r_o = \frac{1-\varrho}{2-\varrho+\sqrt{3-2\varrho}}$, *we have*
(1)$V_{(m,\beta)}[A, B, \gamma, \eta] \subset V_{(m,\beta)}[\varrho, \gamma+1, \eta]$, *if* $\Re\left(\frac{\gamma}{\kappa}\right) > -\varrho$.
(2)$V_{(m,\beta)}[A, B, \gamma, \eta+1] \subset V_{(m,\beta)}[\varrho, \gamma, \eta]$, *if* $\Re\left(\frac{\lambda}{\eta}\right) > -\varrho$.

Proof. The above results can easily be proved by using Theorem 3.10, Theorem 3.11 and the Alexander relation defined in (1.12). □

Theorem 3.10. *Let* $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$, *and* $\Re\left(\frac{\gamma}{\kappa}\right) > -\varrho$. *Then*

$$T_{(m,\beta)}[A, B, \gamma, \eta] \subset T_{(m,\beta)}[\varrho, \gamma + 1, \eta],$$

whenever

$$|z| < r_o = \frac{1-\varrho}{2-\varrho+\sqrt{3-2\varrho}}, \quad \text{where} \quad 0 \leq \varrho < 1.$$

Proof. Let $f \in T_{(m,\beta)}[A, B, \gamma, \eta]$, then there exist $\psi(z) \in S_{\beta}^{\Lambda}[A, B, \gamma, \eta]$ such that

$$\varphi(z) = \frac{z\left(H_{\lambda,\eta}^{\gamma,\kappa} f(z)\right)'}{H_{\lambda,\eta}^{\gamma,\kappa}\psi(z)} \in P_{(m,\beta)}[A, B]. \tag{3.39}$$

Since by Lemma 2.1 we know that $P_{(m,\beta)}[A, B] \subset P_m(\varrho)$, where $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$, therefore $\varphi(z) \in P_m(\varrho)$. So by using the Definition of $P_m(\varrho)$ defined in [22], there exist two functions $\varphi_1(z)$ and $\varphi_2(z)$ such that

$$\varphi(z) = \left(\frac{m}{4}+\frac{1}{2}\right)\varphi_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)\varphi_2(z), \tag{3.40}$$

where $\varphi_i(z) \in P(\varrho)$, $i = 1, 2$. We can write

$$\varphi_i(z) = \varrho + (1-\varrho)h_i(z), \tag{3.41}$$

where $h_i(z) \in P$. Now, let

$$\phi(z) = \frac{z\left(H_{\lambda,\eta}^{\gamma+1,\kappa} f(z)\right)'}{H_{\lambda,\eta}^{\gamma+1,\kappa}\psi(z)}.$$

Since $\psi(z) \in S_{\beta}^{\Lambda}[A, B, \gamma, \eta]$, therefore

$$q(z) = \frac{z\left(H_{\lambda,\eta}^{\gamma,\kappa}\psi(z)\right)'}{H_{\lambda,\eta}^{\gamma,\kappa}\psi(z)} \in P_{\beta}[A, B], \tag{3.42}$$

then by using relation (1.10) and doing some simple computation on Eq (3.42), we have

$$(\kappa q(z) + \gamma)H_{\lambda,\eta}^{\gamma,\kappa}\psi(z) = (\gamma+\kappa)H_{\lambda,\eta}^{\gamma+1,\kappa}\psi(z). \tag{3.43}$$

Now, using relation (1.10) in (3.39), we get

$$\varphi(z) = \frac{\left(\frac{\gamma+\kappa}{\kappa}\right)\left(H_{\lambda,\eta}^{\gamma+1,\kappa} f(z)\right) - \left(\frac{\gamma}{\kappa}\right)\left(H_{\lambda,\eta}^{\gamma,\kappa} f(z)\right)}{H_{\lambda,\eta}^{\gamma,\kappa}\psi(z)}. \tag{3.44}$$

By some simple calculations along with differentiation of both sides of (3.44) and then applying (3.43) we get the following relation

$$\varphi(z) + \frac{z\varphi'(z)}{q(z)+\left(\frac{\gamma}{\kappa}\right)} = \phi(z).$$

Let us consider

$$\phi(z)=\left(\frac{m}{4}+\frac{1}{2}\right)\phi_1(z)-\left(\frac{m}{4}-\frac{1}{2}\right)\phi_2(z),$$

where

$$\phi_i(z)=\varphi_i(z)+\frac{z\varphi_i'(z)}{q(z)+\left(\frac{\gamma}{\kappa}\right)},$$

$i=1,2$. Since $q(z)\in P_\beta[A,B]\subset P(\varrho)$. Therefore, we can write

$$q(z)=\varrho+(1-\varrho)q_o(z), \tag{3.45}$$

where $q_o(z)\in P$. We have to check when $\phi_i(z)\in P_m(\varrho)$. For this consider

$$\Re(\phi_i(z)-\varrho)=\Re\left(\varphi_i(z)+\frac{z\varphi_i'(z)}{q(z)+\left(\frac{\gamma}{\kappa}\right)}-\varrho\right).$$

Using (3.41) and (3.45), we have

$$\Re(\phi_i(z)-\varrho)=\Re\left(\varrho+(1-\varrho)h_i(z)+\frac{(1-\varrho)zh_i'(z)}{\varrho+(1-\varrho)q_o(z)+\left(\frac{\gamma}{\kappa}\right)}-\varrho\right),$$

where $h_i(z),q_o(z)\in P$.

$$\Re(\phi_i(z)-\varrho)=(1-\varrho)\Re(h_i(z))-\frac{(1-\varrho)|zh_i'(z)|}{\Re\left(\varrho+\frac{\gamma}{\kappa}\right)+(1-\varrho)\Re q_o(z)}.$$

Since $\Re(\frac{\gamma}{\kappa})>-\varrho$, so $\Re\left(\varrho+\frac{\gamma}{\kappa}\right)>0$. Now by using the distortion results of Lemma 2.3, we have

$$\begin{aligned}\Re(\phi_i(z)-\varrho)&=\Re\left((1-\varrho)h_i(z)+\varrho+\frac{z(1-\varrho)h_i'(z)}{\frac{\gamma}{\kappa}+\varrho+(1-\varrho)h_i(z)}-\varrho\right)\\&\geq(1-\varrho)\Re(h_i(z))-(1-\varrho)\frac{|zh_i'(z)|}{\Re(\frac{\gamma}{\kappa}+\varrho)+(1-\varrho)\Re(h_i(z))}.\end{aligned} \tag{3.46}$$

Since $h_i(z)\in P$, so $\Re(h_i(z))>0$ and $\Re(\frac{\gamma}{\kappa}+\varrho)>0$ for $\Re(\frac{\gamma}{\kappa})>-\varrho$. Hence, by using Lemma 2.3 in inequality (3.46), we have

$$\begin{aligned}\Re(\phi_i(z)-\varrho)&\geq(1-\varrho)\Re(h_i(z))-\frac{1-\varrho\frac{2r}{1-r^2}\Re(h_i(z))}{(1-\varrho)(\frac{1-r}{1+r})}\\&\geq\frac{r^2(1-\varrho)-2r(2-\varrho)+(1-\varrho)}{1-r^2}.\end{aligned}$$

Since $1-r^2>0$, taking $T(r)=r^2(1-\varrho)-2r(2-\varrho)+(1-\varrho)$. Let r_o be the root then by simple calculations, we get

$$r_o=\frac{1-\varrho}{2-\varrho+\sqrt{3-2\varrho}}.$$

Hence $\phi(z)\in P_m(\varrho)$ for $|z|<r_o$. Thus for this radius r_o the function $f(z)$ belongs to the class $T_{(m,\beta)}[\varrho,\gamma+1,\eta]$, which is required to prove. □

Using the analogous approach used in Theorem 3.14, one can easily prove the following theorem.

Theorem 3.11. *Let* $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$, *and* $\Re\left(\frac{\eta}{\lambda}\right) > -\varrho$. *Then*

$$T_{(m,\beta)}[A, B, \gamma, \eta + 1] \subset T_{(m,\beta)}[\varrho, \gamma, \eta]$$

whenever

$$|z| < r_o = \frac{1-\varrho}{2-\varrho+\sqrt{3-2\varrho}}, \quad \textit{where} \quad 0 \le \varrho < 1.$$

Integral Preserving Property: Here, we will discuss some integral preserving properties of our aforementioned classes. The generalized Libera integral operator I_σ introduced and discussed in [6, 14] is defined by:

$$I_\sigma(f)(z) = \frac{\sigma+1}{z^\sigma}\int_0^z t^{\sigma-1} f(t)dt, \tag{3.47}$$

where $f(z) \in A$ and $\sigma > -1$.

Theorem 3.12. *Let* $\sigma > -\varrho$, *where* $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$. *If* $f \in R_{(m,\beta)}[A, B, \gamma, \eta]$ *then* $I_\sigma(f) \in R_{(m,\beta)}[A, B, \gamma, \eta]$.

Proof. Let $f \in R_{(m,\beta)}[A, B, \gamma, \eta]$, and set

$$\psi(z) = \frac{z\left(H^{\gamma,\kappa}_{\lambda,\eta} I_\sigma(f)(z)\right)'}{H^{\gamma,\kappa}_{\lambda,\eta} I_\sigma(f)(z)}, \tag{3.48}$$

where $\psi(z)$ is analytic and $\psi(0) = 1$. From definition of $H^{\gamma,\kappa}_{\lambda,\eta}(f)$ given by [1] and using Eq (3.47), we have

$$z\left(H^{\gamma,\kappa}_{\lambda,\eta} I_\sigma(f)(z)\right)' = (\sigma+1)H^{\gamma,\kappa}_{\lambda,\eta} f(z) - \sigma H^{\gamma,\kappa}_{\lambda,\eta} I_\sigma(f)(z). \tag{3.49}$$

Then by using Eqs (3.48) and (3.49), we have

$$(\sigma+1)\frac{H^{\gamma,\kappa}_{\lambda,\eta} f(z)}{H^{\gamma,\kappa}_{\lambda,\eta} I_\sigma(f)(z)} = \psi(z) + \sigma.$$

Logarithmic differentiation and simple computation results in

$$\phi(z) = \psi(z) + \frac{z\psi'(z)}{\psi(z)+\sigma} = \frac{z\left(H^{\gamma,\kappa}_{\lambda,\eta} f(z)\right)'}{H^{\gamma,\kappa}_{\lambda,\eta} f(z)} \in P_{(m,\beta)}[A, B], \tag{3.50}$$

with $\Re(\psi(z) + \sigma) > 0$, since $\Re(\sigma) > -\varrho$. Now, consider

$$\psi(z) = \left(\frac{m}{4} + \frac{1}{2}\right)\psi_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)\psi_2(z). \tag{3.51}$$

Combining (3.50) and (3.51), we get

$$\phi(z) = \left(\frac{m}{4} + \frac{1}{2}\right)\phi_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)\phi_2(z),$$

where $\phi_i(z) = \psi_i(z) + \frac{z\psi_i'(z)}{\psi_i(z)+\sigma}$, $i = 1, 2$. Since $\phi(z) \in P_{(m,\beta)}[A, B]$, therefore

$$\phi_i(z) \prec \left(\frac{1 + Az}{1 + Bz}\right)^{\beta},$$

which implies

$$\psi_i(z) + \frac{z\psi_i'(z)}{\psi_i(z) + \sigma} \prec \left(\frac{1 + Az}{1 + Bz}\right)^{\beta} \quad i = 1, 2.$$

Therefore, using Lemma 2.2 we get

$$\psi_i(z) \prec \left(\frac{1 + Az}{1 + Bz}\right)^{\beta},$$

or $\psi(z) \in P_{(m,\beta)}[A, B]$. Hence the result. □

Corollary 3.5. *Let $\sigma > -\varrho$. Then for $m = 2$, if $f \in S^{\Lambda}_{\beta}[A, B, \gamma, \eta]$ then $I_\sigma(f) \in S^{\Lambda}_{\beta}[A, B, \gamma, \eta]$, where $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$.*

Theorem 3.13. *Let $\sigma > -\varrho$, where $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$. If $f \in V_{(m,\beta)}[A, B, \gamma, \eta]$ then $I_\sigma(f) \in V_{(m,\beta)}[A, B, \gamma, \eta]$.*

Proof. Let $f \in V_{(m,\beta)}[A, B, \gamma, \eta]$. Then by using relation (1.12), we have

$$zf'(z) \in R_{(m,\beta)}[A, B, \gamma, \eta],$$

so by using Theorem 3.16, we can say that

$$I_\sigma(zf'(z)) \in R_{(m,\beta)}[A, B, \gamma, \eta],$$

equivalently

$$z(I_\sigma(f(z)))' \in R_{(m,\beta)}[A, B, \gamma, \eta],$$

so again by using the relation (1.12), we get

$$I_\sigma(f) \in V_{(m,\beta)}[A, B, \gamma, \eta].$$

□

Theorem 3.14. *Let $\sigma > -\varrho$, where $\varrho = \left(\frac{1-A}{1-B}\right)^{\beta}$. If $f \in T_{(m,\beta)}[A, B, \gamma, \eta]$ then $I_\sigma(f) \in T_{(m,\beta)}[A, B, \gamma, \eta]$.*

Proof. Let $f \in T_{(m,\beta)}[A, B, \gamma, \eta]$. Then there exists $\psi(z) \in S^{\Lambda}_{\beta}[A, B, \gamma, \eta]$, such that

$$\varphi(z) = \frac{z(H^{\gamma,\kappa}_{\lambda,\eta} f(z))'}{(H^{\gamma,\kappa}_{\lambda,\eta}\psi(z)} \in P_{(m,\beta)}[A, B]. \tag{3.52}$$

Consider

$$\phi(z) = \frac{z\left(H^{\gamma,\kappa}_{\lambda,\eta} I_\sigma(f)(z)\right)'}{H^{\gamma,\kappa}_{\lambda,\eta} I_\sigma(\psi)(z)}. \tag{3.53}$$

Since $\psi(z) \in S_{\beta}^{\Lambda}[A, B, \gamma, \eta]$, then by Corollary 3.17, $I_{\sigma}(\psi)(z) \in S_{\beta}^{\Lambda}[A, B, \gamma, \eta]$. Therefore

$$q(z) = \frac{z\left(H_{\lambda,\eta}^{\gamma,\kappa} I_{\sigma}(\psi)(z)\right)'}{H_{\lambda,\eta}^{\gamma,\kappa} I_{\sigma}(\psi)(z)} \in P_{\beta}[A, B]. \tag{3.54}$$

By using (3.47) and Definition of $H_{\lambda,\eta}^{\gamma,\kappa}$, we get

$$q(z) H_{\lambda,\eta}^{\gamma,\kappa} I_{\sigma}(\psi)(z) = (\sigma + 1) H_{\lambda,\eta}^{\gamma,\kappa}(\psi)(z) - \sigma H_{\lambda,\eta}^{\gamma,\kappa} I_{\sigma}(\psi)(z),$$

or we can write it as

$$H_{\lambda,\eta}^{\gamma,\kappa} I_{\sigma}(\psi)(z) = \frac{\sigma + 1}{q(z) + \sigma} H_{\lambda,\eta}^{\gamma,\kappa}(\psi)(z). \tag{3.55}$$

Now using the relation (3.47) and the Definition of $H_{\lambda,\eta}^{\gamma,\kappa}$, in (3.53), we have

$$\phi(z) H_{\lambda,\eta}^{\gamma,\kappa} I_{\sigma}(\psi)(z) = (\sigma + 1) H_{\lambda,\eta}^{\gamma,\kappa}(f)(z) - \sigma H_{\lambda,\eta}^{\gamma,\kappa} I_{\sigma}(f)(z). \tag{3.56}$$

Differentiating both sides of (3.56), we have

$$\phi'(z) H_{\lambda,\eta}^{\gamma,\kappa} I_{\sigma}(\psi)(z) + \phi(z)(H_{\lambda,\eta}^{\gamma,\kappa} I_{\sigma}(\psi)(z))' = (\sigma + 1)(H_{\lambda,\eta}^{\gamma,\kappa}(f)(z))' - \sigma(H_{\lambda,\eta}^{\gamma,\kappa} I_{\sigma}(f)(z))',$$

then by simple computations and using (3.53)–(3.55), we get

$$\phi(z) + \frac{z\phi'(z)}{q(z) + \sigma} = \varphi(z), \tag{3.57}$$

with $\Re(\sigma) > -\varrho$, so $\Re(q(z) + \sigma) > 0$, since $q(z) \in P_{\beta}[A, B] \subset P(\varrho)$. Consider

$$\phi(z) = \left(\frac{m}{4} + \frac{1}{2}\right)\phi_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)\phi_2(z), \tag{3.58}$$

Combining Eqs (3.57) and (3.58), we have

$$\varphi(z) = \left(\frac{m}{4} + \frac{1}{2}\right)\varphi_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)\varphi_2(z), \tag{3.59}$$

where $\varphi_i(z) = \phi_i(z) + \frac{z\phi_i'(z)}{q(z)+\sigma}$, $i = 1, 2$.

Since $\varphi(z) \in P_{(m,\beta)}[A, B]$, thus we have

$$\varphi_i(z) \prec \left(\frac{1 + Az}{1 + Bz}\right)^{\beta},$$

then

$$\phi_i(z) + \frac{z\phi_i'(z)}{q(z) + \sigma} \prec \left(\frac{1 + Az}{1 + Bz}\right)^{\beta}, \quad i = 1, 2.$$

Since $\Re(q(z) + \sigma) > 0$, therefore using Lemma 2.2 we get

$$\phi_i(z) \prec \left(\frac{1 + Az}{1 + Bz}\right)^{\beta}, \quad i = 1, 2,$$

thus $\phi(z) \in P_{(m,\beta)}[A, B]$. Hence the result. □

4. Conclusions

Due to their vast applications, Mittag-Leffler functions have captured the interest of a number of researchers working in different fields of science. The present investigation may help researchers comprehend some stimulating consequences of the special functions. In the present article, we have used generalized Mittag-Leffler functions to define some novel classes related to bounded boundary and bounded radius rotations. Several inclusion relations and radius results for these classes have been discussed. Moreover, it has been proved that these classes are preserved under the generalized Libera integral operator. Finally, we can see that the projected solution procedure is highly efficient in solving inclusion problems describing the harmonic analysis. It is hoped that our investigation and discussion will be helpful in cultivating new ideas and applications in different fields of science, particularly in mathematics.

List of Notations

Δ Open Unit Disc.
Ω Class of normalized analytic functions.
$\Re$ Real part of complex number.
Γ Gamma function.
$\chi(z)$ Schwartz function.

Conflict of interest

The authors declare that they have no competing interests.

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