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Research article

Discussion on boundary controllability of nonlocal fractional neutral integrodifferential evolution systems

Yong-Ki Ma[1], Kamalendra Kumar[2], Rakesh Kumar[3], Rohit Patel[4], Anurag Shukla[4] and Velusamy Vijayakumar[5,*]

[1] Department of Applied Mathematics, Kongju National University, Chungcheongnam-do 32588, Korea

[2] Department of Mathematics, SRMS College of Engineering and Technology, Bareilly-243001, India

[3] Department of Mathematics, Hindu College, Moradabad-244001, India

[4] Department of Applied Science, Rajkiya Engineering College Kannauj, Kannauj-209732, India

[5] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore-632014, Tamil Nadu, India

* **Correspondence:** Email: vijaysarovel@gmail.com.

Abstract: In the present work, we have established sufficient conditions for boundary controllability of nonlocal fractional neutral integrodifferential evolution systems with time-varying delays in Banach space. The outcomes are obtained by applying the fractional theory and Banach fixed point theorem. At last, we give an application for the validation of the theoretical results.

Keywords: integrodifferential system; Banach fixed point theorem; nonlocal condition; boundary controllability; time varying delays; semigroup theory
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1. Introduction

Most of the dynamical systems in science and engineering have the intrinsic nature of time delay to some degree. Time delays occur in many dynamical systems such as chemical or process control systems, biological systems, network control systems, etc. As per the practical requirement, the delays may be time-varying or constant. Many systems, for example, sample data control, aircraft control, etc., are modelled as time-varying delays, which are the principal cause of instability and weak control performance. Neutral systems with time delay are the systems in which uncertainty is present in the state and derivatives. These systems arise in many areas like population ecology, distributed networks, etc. Due to various applications in engineering and science, many researchers worked on the delay differential equation. Besides the above, in control theory, the problem of controllability is of specific importance. Controllability of the system can handle various problems like stabilization, pole assignment, and optimal control.

Fattorini [1] introduced the controllability condition on the boundary control system of first and second-order replacing boundary controls by distributed controls. Park et al. [2, 4] gave the results for the boundary controllability of semilinear nonlocal control systems with the help of the Banach fixed point theorem. Ahmed et al. [3] derived results for approximate boundary controllability of stochastic control system of fractional order having Poisson jump and fractional Brownian motion. Balachandran et al. [5] investigated sufficient conditions for controllability of abstract neutral integrodifferential control systems having infinite delay using Nussbaum fixed point theorem and analytic semigroup theory. Radhakrishnan et al. [7] and Kumar et al. [10] studied the controllability for the integrodifferential control system in Banach space with the help of the Schaefer fixed point theorem. Cheng et al. [11] presented the exact controllability of fractional control system having time-varying delay using the theory of propagation family and Leray-Schauder theorem. Applications of fractional calculus in neural networks can be found in [45–54].

In [12–16, 23] researchers investigated the existence and uniqueness of mild solution for certain classes of a fractional evolution equation with the help of fixed point theorem, fractional power operators, semigroup theory, the measure of non-compactness, etc. Guo et al. [21] presented null boundary controllability and proved nonlinear Cauchy-Kowalevski theorem for 1-D semilinear heat equations. In [8, 9, 17–20, 24–39, 44], researchers studied the approximate controllability for the semilinear control system of fractional order $\alpha \in (1, 2]$, first and second-order using fixed-point theorem, cosine and sine theory of operators, fractional calculus, Lipschitz continuity and sequential approach.

Under Lion boundary conditions, the 3D Navier-Stokes system is approximately controllable, as discussed by Phan et al. [40]. Meraj et al. [41] derived the results for approximate controllability for a non-autonomous control system having nonlocal initial conditions using Krasnoselski theorem and evolution system. In modern era, fixed point theory is employed in many areas of mathematics and engineering. Keeping these utilizations of fixed point theory in mind, some fixed point results are proved in the structure of partial b-metric spaces and allied abstract spaces which emphasize primarily the applications for existence of the solution of various functional equations occurring in dynamic programming, integral equations, boundary value problems, equations representing LCR circuits and simple harmonic motion. Fixed point theory is a powerful tool to determine uniqueness of solutions to dynamical systems and is widely used in theoretical and applied analysis.

The main contributions of our paper are:

- We have derived the results for the boundary controllability of integrodifferential neutral control system having time-varying delays of fractional order.
- The primary outcomes for the systems (2.1)–(2.3) are derived by employing Banach fixed point theorem, semigroup theory and fractional calculus.
- By considering a set of assumptions $(A1)$–$(A10)$ results can be obtained.
- The research focused on the boundary controllability of proposed systems (2.1)–(2.3) under consideration are not addressed in the literature to our knowledge, and it supports the current

findings.

This article's structure will now be presented as follows:

(1) Section 2 discusses some definitions and assumptions.
(2) Results for boundary controllability are presented in section 3.
(3) An example is demonstrated to validate the theoretical result.
(4) Finally we have given a conclusion section.

2. Preliminaries

Let us assume that V and U are two real Banach spaces having norms $\|\cdot\|$ and $|\cdot|$ respectively. Consider that ρ be a linear, closed and densely defined operator with $D(\rho) \subseteq V$. Consider $\Theta \subseteq X$ be linear operator with $D(\rho)$ and $R(\Theta) \subseteq X$, a Banach space. Now define the nonlocal fractional integrodifferential boundary control system having time varying delay as:

$$\begin{aligned} \frac{d^\alpha}{d\varsigma^\alpha}\Big[s(\varsigma) + \eta(\varsigma, s(\varsigma), s(\gamma_1(\varsigma)))\Big] =& \rho s(\varsigma) + g(\varsigma, s(\varsigma), s(\gamma_2(\varsigma))) \\ &+ \int_0^\varsigma \Psi(\varsigma, \varrho, s(\gamma_3(\varrho)))d\varrho, \varsigma \in J = [0, b], \quad (2.1) \\ \Theta s(\varsigma) =& E_1 u(\varsigma), \quad (2.2) \\ s(0) + \phi(\varsigma_1, \varsigma_2, ..., \varsigma_p, s(\cdot)) =& s_0, \quad (2.3) \end{aligned}$$

where $\varsigma_1 > 0$, $\varsigma_p \leq b$ and $\varsigma_i < \varsigma_{i+1}$ $i = 1, 2, ...p-1$ and $\alpha \in (0, 1]$. The continuous linear operator E_1 is defined from U to X. The control function $u \in L^2(J, U)$, a Banach space of admissible control functions with U. The delays functions γ_1, γ_2 and γ_3 are continuous. The state $s(.) \in V$, the functions g, ϕ, Ψ and η are defined as $g : J\times V\times V \to V$, $\phi : J^p \times V \to V$, $\Psi : \Delta\times V \to V$ and $\eta : J\times V\times V \to V$, and $\Delta = \{(\varsigma, \varrho), 0 \leq \varrho \leq \varsigma \leq b\}$.

Suppose that the linear operator $A : D(A) \to V$ where $D(A) \subseteq V$ be defined as: $D(A) = \{s \in D(\rho) : \Theta s = 0\}, As = \rho s$, for $s \in D(A)$. Consider

$$E_r = \{y : \| y\| \leq r\} \subset V.$$

We consider the following assumptions:

(A_1) $D(\rho) \subset D(\Theta)$ where ρ is closed linear operator having domain $D(\rho)$ and Θ is a boundary operator having domain $D(\Theta)$. Also the restriction of Θ to $D(\rho)$is continuous (w.r.t. graph norm).

(A_2) $T(\varsigma)$ is a compact semigroup having infinitesimal generator A. Also there exist $L \in (0, \infty)$ satisfying
$$\|T(\varsigma)\| \leq L.$$

(A_3) There exists an operator E defined from U to V which is continuous and linear, with $\rho E \in L(U, V)$, $\Theta(EU) = E_1 U, \forall u \in U$. In addition, there exist a constant $C > 0$ such that $\| Eu\| \leq C\| E_1 u\|$ for every $u \in U$ and continuous differentiability is satisfied by $Eu(\varsigma)$.

(A_4) For $u \in U$ and $0 < \varsigma \leq b$, $T(\varsigma)Eu \in D(A)$. Also, there is a constant $M \in (0, \infty)$ satisfying $\| AT(\varsigma)\| \leq M$.

(A_5) The operator $W : L^2(J, U) \to V$ is linear and explained in the following way

$$Wu = \alpha \int_0^b \int_0^\infty \varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)\Big[T((b-\varrho)^\alpha\varpi)\rho - AT((b-\varrho)^\alpha\varpi)\Big]Eu(\varrho)d\varpi d\varrho,$$

where $\xi_\alpha(\varpi)$ is called p.d.f, $0 < \varpi < \infty$, see [13, 14]. The operator $\tilde{W}^{-1}$ induced by $\xi_\alpha(\varpi)$, is invertible and is defined on $L^2(J, U)/kerW$. There exist $L_1, L_2 \in (0, \infty)$ satisfying

$$\|E\| \le L_1,$$

and

$$\| \tilde{W}^{-1}\| \le L_2.$$

(A_6) The function η, g is continuous in ς. Also, for every $s_i, y_i, u_i, v_i \in E_r$, $\; i = 1, 2$ and $\varsigma \in J$, there exist $C_1, M_1 \in (0, \infty)$

$$\| \eta(\varsigma, u_1, v_1) - \eta(\varsigma, u_2, v_2)\| \le C_1\Big[\| u_1 - u_2\| + \| v_1 - v_2\|\Big].$$
$$\| g(\varsigma, s_1, y_1) - g(\varsigma, s_2, y_2)\| \le M_1\Big[\| s_1 - s_2\| + \| y_1 - y_2\|\Big].$$

(A_7) The function Ψ is continuous in ς and ϱ, there exist $N_1 \in (0, \infty)$ such that for every $u, v \in E_r$ and $\varsigma, \varrho \in \Delta$,

$$\| \Psi(\varsigma, \varrho, u) - \Psi(\varsigma, \varrho, v)\| \le N_1\| u - v\|.$$

(A_8) There exists a constant $H \in (0, \infty)$ such that for $z_1, z_2 \in C(J, E_r)$, we have

$$\| \phi(\varsigma_1, \varsigma_2, ..., \varsigma_p, z_1(\cdot)) - \phi(\varsigma_1, \varsigma_2, ..., \varsigma_p, z_2(\cdot))\| \le H \sup_{\varsigma\in[0,b]} \| z_1(\varsigma) - z_2(\varsigma)\|.$$

(A_9) $\forall s_1, s_2 \in V$ there is a constant $p > 0$ such that

$$\| s_1(\gamma_i(\varsigma)) - s_2(\gamma_i(\varsigma))\| \le p\| s_1(\varsigma) - s_2(\varsigma)\|, \quad for \;\; i = 1, 2.$$

(A_{10}) There exists a constant $r > 0$ such that

$$\Big[\Big(L\| s_0\|+LH_1+LC_3+C_4+M(2rC_1+C_2)b^\alpha+b^\alpha L(2rM_1+M_2)+b^\alpha L(N_1rb+N_2b)\Big)\Big(1+b^\alpha\{M+L\|\rho\|\}L_1L_2\Big)$$
$$+\| a\|L_1L_2b^\alpha(M + L\|\rho\|)\Big] \le r,$$

where the constants M_2, C_2, C_3, C_4 and N_2 will be defined later.

Let us assume the solutions of (2.1)–(2.3) as $s(\varsigma)$. Now, describe $z(\varsigma) = -Eu(\varsigma) + s(\varsigma)$ and by above assumption, $z(\varsigma) \in D(A)$.

We can express (2.1)–(2.3) as:

$$\frac{d^\alpha}{d\varsigma^\alpha}\Big[s(\varsigma) + \eta(\varsigma, s(\varsigma), s(\gamma_1(\varsigma)))\Big] = Az(\varsigma) + \rho Eu(\varsigma) + g(\varsigma, s(\varsigma), s(\gamma_2(\varsigma))) + \int_0^\varsigma \Psi(\varsigma, \varrho, s(\gamma_3(\varrho)))d\varrho, \quad \varsigma \in J = [0, b], \tag{2.4}$$

$$s(\varsigma) = z(\varsigma) + Eu(\varsigma), \tag{2.5}$$

$$s(0) + \phi(\varsigma_1, \varsigma_2, ..., \varsigma_p, s(\cdot)) = s_0. \tag{2.6}$$

If continuous differentiability on $[0, b]$ is satisfied by u, then we can express z as a mild solution of given below problem:

$$\begin{aligned} \frac{d^\alpha}{d\varsigma^\alpha}\Big[z(\varsigma) + \eta(\varsigma, s(\varsigma), s(\gamma_1(\varsigma)))\Big] =& Az(\varsigma) + \rho Eu(\varsigma) - E\frac{d^\alpha u(\varsigma)}{d\varsigma^\alpha} \\ &+ g(\varsigma, s(\varsigma), s(\gamma_2(\varsigma))) + \int_0^\varsigma \Psi(\varsigma, \varrho, s(\gamma_3(\varrho)))d\varrho, \\ z(0) =& s_0 - Eu(0) - \phi(\varsigma_1, \varsigma_2, ..., \varsigma_p, s(0)). \end{aligned}$$

The solutions of (2.1)–(2.3) is stated as:

$$\begin{aligned} s(\varsigma) =& \int_0^\infty \xi_\alpha(\varpi)T(\varsigma^\alpha\varpi)[s_0 - \phi(\varsigma_1, \varsigma_2, ..., \varsigma_p, s(\cdot)) - Eu(0)]d\varpi + Eu(\varsigma) \\ &- \alpha\int_0^\varsigma\int_0^\infty \varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)T((\varsigma-\varrho)^\alpha\varpi)\frac{d^\alpha}{d\varrho^\alpha}\Big[\eta(\varrho, s(\varrho), s(\gamma_1(\varrho)))\Big]d\varpi d\varrho \\ &+ \alpha\int_0^\varsigma\int_0^\infty \varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)T((\varsigma-\varrho)^\alpha\varpi)\Big[\rho Eu(\varrho) - E\frac{d^\alpha u(\varrho)}{d\varrho^\alpha}\Big]d\varpi d\varrho \\ &+ \alpha\int_0^\varsigma\int_0^\infty \varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)T((\varsigma-\varrho)^\alpha\varpi)\Big[g(\varrho, s(\varrho), s(\gamma_2(\varrho))) \\ &+ \int_0^\varrho \Psi(\varrho, \sigma, s(\gamma_3(\sigma)))d\sigma\Big]d\varpi d\varrho. \end{aligned} \tag{2.7}$$

Now, integrating (2.7), we get

$$\begin{aligned} s(\varsigma) =& \int_0^\infty \xi_\alpha(\varpi)T(\varsigma^\alpha\varpi)[s_0 - \phi(\varsigma_1, \varsigma_2, ..., \varsigma_p, s(\cdot)) + \eta(0, s(0), s(\gamma_1(0)))]d\varpi \\ &- \eta(\varsigma, s(\varsigma), s(\gamma_1(\varsigma))) \\ &- \alpha\int_0^\varsigma\int_0^\infty \varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)AT((\varsigma-\varrho)^\alpha\varpi)\eta(\varrho, s(\varrho), s(\gamma_1(\varrho)))d\varpi d\varrho \\ &- \alpha\int_0^\varsigma\int_0^\infty \varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)AT((\varsigma-\varrho)^\alpha\varpi)Eu(\varrho)d\varpi d\varrho \\ &+ \alpha\int_0^\varsigma\int_0^\infty \varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)T((\varsigma-\varrho)^\alpha\varpi)\Big[\rho Eu(\varrho) + g(\varrho, s(\varrho), s(\gamma_2(\varrho))) \\ &+ \int_0^\varrho \Psi(\varrho, \sigma, s(\gamma_3(\sigma)))d\sigma\Big]d\varpi d\varrho. \end{aligned} \tag{2.8}$$

Consequently, (2.8) is clearly defined and it is called a mild solution of the systems (2.1)–(2.3).

Definition 2.1. The systems (2.1)–(2.3) is called controllable on the interval J if for all $s_0, a \in V$, there exists a control $u \in L^2(J, U)$ such that the solutions $s(\cdot)$ of (2.1)–(2.3) meets $s(b) = a$ where $b > 0$ is a final point of J.

The main objective is to transfer the systems (2.4)–(2.6) from $s(0) = s_0 - \phi(\varsigma_1, \varsigma_2, ..., \varsigma_p, s(\cdot))$ to $s(b) = a$, see [4].

3. Main results

Theorem 3.1. *If the assumptions* (A_1)–(A_{10}) *are fulfilled, then the boundary control of the evolution systems* (2.1)–(2.3) *is controllable on J provided*

$$\Big[\Big(LH + LC_1(1+p) + C_1(1+p) + b^{\alpha}MC_1(1+p) + b^{\alpha}LM_1(1+p) + b^{\alpha}L(bN_1 p)\Big)$$
$$\Big(1 + b^{\alpha}\{M + L\|\rho\|\}L_1L_2\Big)\Big] \leq \Pi, \quad 0 \leq \Pi < 1.$$

Proof. By applying the invertible operator $\tilde{W}^{-1}$, for arbitrary function $s(\cdot)$, we present

$$\begin{aligned} u(\varsigma) = \tilde{W}^{-1}\Bigg[& a - \int_0^{\infty} \xi_{\alpha}(\varpi)T(b^{\alpha}\varpi)\Big[s_0 - \phi(\varsigma_1, \varsigma_2, ..., \varsigma_p, s(\cdot)) + \eta(0, s(0), s(\gamma_1(0)))\Big]d\varpi \\ &+ \eta(b, s(b), s(\gamma_1(b))) \\ &+ \alpha\int_0^b\int_0^{\infty} \varpi(b-\varrho)^{\alpha-1}\xi_{\alpha}(\varpi)AT((b-\varrho)^{\alpha}\varpi)\eta(\varrho, s(\varrho), s(\gamma_1(\varrho)))d\varpi d\varrho \\ &- \alpha\int_0^b\int_0^{\infty} \varpi(b-\varrho)^{\alpha-1}\xi_{\alpha}(\varpi)T((b-\varrho)^{\alpha}\varpi)g(\varrho, s(\varrho), s(\gamma_2(\varrho)))d\varpi d\varrho \\ &- \alpha\int_0^b\int_0^{\infty} \varpi(b-\varrho)^{\alpha-1}\xi_{\alpha}(\varpi)T((b-\varrho)^{\alpha}\varpi)\Big[\int_0^{\varrho}\Psi(\varrho, \sigma, s(\gamma_3(\sigma)))d\sigma\Big]d\varpi d\varrho\Bigg](\varsigma). \end{aligned} \tag{3.1}$$

Now, we shall observe that, when applying this control $u(\varsigma)$, the operator

$$F : C(J; E_r) \to C(J; E_r)$$

given as

$$\begin{aligned} (Fx)(\varsigma) = &\int_0^{\infty} \xi_{\alpha}(\varpi)T(\varsigma^{\alpha}\varpi)\Big[s_0 - \phi(\varsigma_1, \varsigma_2, ..., \varsigma_p, s(\cdot)) + \eta(0, s(0), s(\gamma_1(0)))\Big]d\varpi \\ &- \eta(\varsigma, s(\varsigma), s(\gamma_1(\varsigma))) \\ &- \alpha\int_0^{\varsigma}\int_0^{\infty} \varpi(\varsigma-\varrho)^{\alpha-1}\xi_{\alpha}(\varpi)AT((\varsigma-\varrho)^{\alpha}\varpi)\eta(\varrho, s(\varrho), s(\gamma_1(\varrho)))d\varpi d\varrho \\ &- \alpha\int_0^{\varsigma}\int_0^{\infty} \varpi(\varsigma-\varrho)^{\alpha-1}\xi_{\alpha}(\varpi)\Big[AT((\varsigma-\varrho)^{\alpha}\varpi) - T((\varsigma-\varrho)^{\alpha}\varpi)\rho\Big] \\ &\qquad \times E\tilde{W}^{-1}\Big[a - \int_0^{\infty} \xi_{\alpha}(\varpi)T(b^{\alpha}\varpi)\Big[s_0 - \phi(\varsigma_1, \varsigma_2, ..., \varsigma_p, s(\cdot)) \\ &+ \eta(0, s(0), s(\gamma_1(0)))\Big]d\varpi + \eta(b, s(b), s(\gamma_1(b))) \\ &+ \alpha\int_0^b\int_0^{\infty} \varpi(b-\sigma)^{\alpha-1}\xi_{\alpha}(\varpi)AT((b-\sigma)^{\alpha}\varpi)\eta(\sigma, s(\sigma), s(\gamma_1(\sigma)))d\varpi d\sigma \\ &- \alpha\int_0^b\int_0^{\infty} \varpi(b-\sigma)^{\alpha-1}\xi_{\alpha}(\varpi)T((b-\sigma)^{\alpha}\varpi)g(\sigma, s(\sigma), s(\gamma_2(\sigma)))d\varpi d\sigma \\ &- \alpha\int_0^b\int_0^{\infty} \varpi(b-\sigma)^{\alpha-1}\xi_{\alpha}(\varpi)T((b-\sigma)^{\alpha}\varpi) \times \Big[\int_0^{\sigma}\Psi(\sigma, \mu, s(\gamma_3(\mu)))d\mu\Big]d\varpi d\sigma\Big](\varrho)d\varpi d\varrho \end{aligned}$$

$$+\alpha\int_0^{\varsigma}\int_0^{\infty}\varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)T((\varsigma-\varrho)^\alpha\varpi)g(\varrho,s(\varrho),s(\gamma_2(\varrho)))d\varpi d\varrho$$
$$+\alpha\int_0^{\varsigma}\int_0^{\infty}\varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)T((\varsigma-\varrho)^\alpha\varpi)\Big[\int_0^{\varrho}\Psi(\varrho,\sigma,s(\gamma_3(\sigma)))d\sigma\Big]d\varpi d\varrho,$$

has a fixed point. This fixed point is then the solutions of the (2.1)–(2.3). $Fx(b)=a$, this means that the control u steers the nonlocal fractional neutral integrodifferential evolution systems with time varying delays from the initial state s_0 to a in time b provided we can achieve a fixed point of the operator F.

Initially, we demonstrate that F maps $C(J;E_r)$ into itself. We can select $M_2, C_2, C_3, C_4, N_2>0$ such that

$$M_2=\max_{\varrho\in J}\|g(\varrho,0,0)\|,$$
$$C_2=\max_{\varrho\in[0,T]}\|\eta(\varrho,0,0)\|,$$
$$C_3=\max_{s\in C([0,\varsigma],E_r)}\|\eta(0,s(0),s(\gamma_1(0)))\|,$$
$$C_4=\max_{s\in C([0,\varsigma],E_r)}\|\eta(0,s(b),s(\gamma_1(b)))\|,$$
$$N_2=\max_{\varsigma,\varrho\in\Delta}\|\Psi(\varsigma,\varrho,0)\|.$$

Moreover, in view of the fact $s(\cdot)$ in ϕ is continuous on J, we consider

$$H_1=\max_{s\in C(J,E_r)}\|\phi(\varsigma_1,\varsigma_2,...,\varsigma_p,s(\cdot))\|.$$

From Eq (3.1), we have

$$\|(Fx)(\varsigma)\|\le\Big\|\int_0^{\infty}\xi_\alpha(\varpi)T(\varsigma^\alpha\varpi)s_0d\varpi\Big\|+\Big\|\int_0^{\infty}\xi_\alpha(\varpi)T(\varsigma^\alpha\varpi)\phi(\varsigma_1,\varsigma_2,...,\varsigma_p,s(\cdot))d\varpi\Big\|$$
$$+\Big\|\int_0^{\infty}\xi_\alpha(\varpi)T(\varsigma^\alpha\varpi)\eta(0,s(0),s(\gamma_1(0)))d\varpi\Big\|+\|\eta(\varsigma,s(\varsigma),s(\gamma_1(\varsigma)))\|$$
$$+\alpha\int_0^{\varsigma}\Big\|\int_0^{\infty}\varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)AT((\varsigma-\varrho)^\alpha\varpi)\eta(\varrho,s(\varrho),s(\gamma_1(\varrho)))d\varpi\Big\|d\varrho$$
$$+\alpha\int_0^{\varsigma}\Big\|\int_0^{\infty}\varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)\Big[AT((\varsigma-\varrho)^\alpha\varpi)-T((\varsigma-\varrho)^\alpha\varpi)\rho\Big]d\varpi\Big\|\|E\|\Big\|\tilde{W}^{-1}\Big\|$$
$$\Big[\|a\|+\Big\|\int_0^{\infty}\xi_\alpha(\varpi)T(b^\alpha\varpi)\Big[s_0-\phi(\varsigma_1,\varsigma_2,...,\varsigma_p,s(\cdot))+\eta(0,s(0),s(\gamma_1(0)))\Big]d\varpi\Big\|$$
$$+\|\eta(b,s(b),s(\gamma_1(b)))\|$$
$$+\alpha\int_0^{b}\Big\|\int_0^{\infty}\varpi(b-\sigma)^{\alpha-1}\xi_\alpha(\varpi)AT((b-\sigma)^\alpha\varpi)\eta(\sigma,s(\sigma),s(\gamma_1(\sigma)))d\varpi\Big\|d\sigma$$
$$+\alpha\int_0^{b}\Big\|\int_0^{\infty}\varpi(b-\sigma)^{\alpha-1}\xi_\alpha(\varpi)T((b-\sigma)^\alpha\varpi)g(\sigma,s(\sigma),s(\gamma_2(\sigma)))d\varpi\Big\|d\sigma$$
$$+\alpha\int_0^{b}\Big\|\int_0^{\infty}\varpi(b-\sigma)^{\alpha-1}\xi_\alpha(\varpi)T((b-\sigma)^\alpha\varpi)\Big[\int_0^{\sigma}\Psi(\sigma,\mu,s(\gamma_3(\mu)))d\mu\Big]d\varpi\Big\|d\sigma\Big](\varrho)d\varrho$$

$$+ \alpha \int_0^{\varsigma} \left\| \int_0^{\infty} \varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)T((\varsigma-\varrho)^\alpha\varpi)d\varpi \right\| \| g(\varrho, s(\varrho), s(\gamma_2(\varrho)))\| d\varrho$$
$$+ \alpha \int_0^{\varsigma} \left\| \int_0^{\infty} \varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)T((\varsigma-\varrho)^\alpha\varpi)d\varpi \right\| \left\| \int_0^{\varrho} \Psi(\varrho,\sigma,s(\gamma_3(\sigma)))d\sigma \right\| d\varrho$$
$$\leq L\| s_0\| + LH_1 + LC_3 + C_4 + M\alpha \int_0^{\varsigma} (\varsigma-\varrho)^{\alpha-1}\Big[\| \eta(\varrho, s(\varrho), s(\gamma_1(\varrho))) - \eta(\varrho,0,0)\|$$
$$+ \| \eta(\varrho,0,0)\|\Big]d\varrho + \alpha \int_0^{\varsigma} (\varsigma-\varrho)^{\alpha-1}\{M + L\|\rho\|\}L_1L_2\Big[\| a\| + L(\| s_0\| + H_1 + C_3) + C_4$$
$$+ \alpha \int_0^{b} (b-\sigma)^{\alpha-1}M\{\| \eta(\sigma, s(\sigma), s(\gamma_1(\sigma))) - \eta(\sigma,0,0)\| + \| \eta(\sigma,0,0)\|\}d\sigma$$
$$+ \alpha \int_0^{b} (b-\sigma)^{\alpha-1}L\{\| g(\sigma, s(\sigma), s(\gamma_2(\sigma))) - g(\sigma,0,0)\| + \| g(\sigma,0,0)\|\}d\sigma$$
$$+ \alpha \int_0^{b} (b-\sigma)^{\alpha-1}L\Big[\Big\| \int_0^{\sigma} \Big(\Psi(\sigma,\mu,s(\gamma_3(\mu))) - \Psi(\sigma,\mu,0)\Big)d\mu\Big\| + \Big\| \int_0^{\sigma} \Psi(\sigma,\mu,0)d\mu\Big\|\Big]d\sigma\Big]d\varrho$$
$$+ \alpha \int_0^{\varsigma} (\varsigma-\varrho)^{\alpha-1}L\{\| g(\varrho, s(\varrho), s(\gamma_2(\varrho))) - g(\varrho,0,0)\| + \| g(\varrho,0,0)\|\}d\varrho$$
$$+ \alpha \int_0^{\varsigma} (\varsigma-\varrho)^{\alpha-1}L\Big[\Big\| \int_0^{\varrho} \Big(\Psi(\varrho,\sigma,s(\gamma_3(\sigma))) - \Psi(\varrho,\sigma,0)\Big)d\sigma\Big\| + \Big\| \int_0^{\varrho} \Psi(\varrho,\sigma,0)d\sigma\Big\|\Big]d\varrho$$
$$\leq L\| s_0\| + LH_1 + LC_3 + C_4 + M(2rC_1 + C_2)b^\alpha + b^\alpha\{M + L\|\rho\|\}L_1L_2$$
$$\times \Big[\| a\| + L(\| s_0\| + H_1 + C_3) + C_4 + Mb^\alpha(2rC_1 + C_2) + b^\alpha L(2rM_1 + M_2)$$
$$+ b^\alpha L(N_1rb + N_2b)\Big] + b^\alpha L(2rM_1 + M_2) + b^\alpha L(N_1rb + N_2b)$$
$$\leq \Big[L\| s_0\| + LH_1 + LC_3 + C_4 + M(2rC_1 + C_2)b^\alpha + b^\alpha L(2rM_1 + M_2)$$
$$+ b^\alpha L(N_1rb + N_2b)\Big]\Big[1 + b^\alpha\{M + L\|\rho\|\}L_1L_2\Big] + \| a\|L_1L_2b^\alpha\{M + L\|\rho\|\}$$
$$\leq r.$$

Hence, F maps $C(J;E_r)$ into itself.

Next, we demonstrate that F is a contraction on $C(J;E_r)$. Indeed,

$$\| Fx_1(\varsigma) - Fx_2(\varsigma)\| \leq \left\| \int_0^{\infty} \xi_\alpha(\varpi)T(\varsigma^\alpha\varpi)\Big[\phi(\varsigma_1,...,\varsigma_p,s_1(\cdot)) - \phi(\varsigma_1,...,\varsigma_p,s_2(\cdot))\Big]d\varpi \right\|$$
$$+ \left\| \int_0^{\infty} \xi_\alpha(\varpi)T(\varsigma^\alpha\varpi)\Big[\eta(0,s(0),s_1(\gamma_1(0))) - \eta(0,s(0),s_2(\gamma_1(0)))\Big]d\varpi \right\|$$
$$+ \Big\| \eta(\varsigma, s_1(\varsigma), s_1(\gamma_1(\varsigma))) - \eta(\varsigma, s_2(\varsigma), s_2(\gamma_1(\varsigma)))\Big\|$$
$$+ \Big\|\alpha \int_0^{\varsigma}\int_0^{\infty} \varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)AT((\varsigma-\varrho)^\alpha\varpi)[\eta(\varrho, s_1(\varrho), s_1(\gamma_1(\varrho)))$$
$$- \eta(\varrho, s_2(\varrho), s_2(\gamma_1(\varrho)))]d\varpi d\varrho\Big\| + \Big\|\alpha \int_0^{\varsigma}\int_0^{\infty} \varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)\{AT((\varsigma-\varrho)^\alpha\varpi)$$
$$- T((\varsigma-\varrho)^\alpha\varpi)\rho\}E\tilde{W}^{-1}\Big[\int_0^{\infty} \xi_\alpha(\varpi)T(b^\alpha\varpi)\Big[\phi(\varsigma_1,...,\varsigma_p,s_1(\cdot)) - \phi(\varsigma_1,...,\varsigma_p,s_2(\cdot))\Big]d\varpi$$

$$
\begin{aligned}
&+ \int_0^\infty \xi_\alpha(\varpi)T(b^\alpha\varpi)\Big[\eta(0, s(0), s_1(\gamma_1(0))) - \eta(0, s(0), s_2(\gamma_1(0)))\Big]d\varpi \\
&+ \eta(b, s_1(b), s_1(\gamma_1(b))) - \eta(b, s_2(b), s_2(\gamma_1(b)))] + \alpha\int_0^b\int_0^\infty \varpi(b-\sigma)^{\alpha-1}\xi_\alpha(\varpi) \\
&\qquad \times AT((b-\sigma)^\alpha\varpi)[\eta(\sigma, s_1(\sigma), s_1(\gamma_1(\sigma))) - \eta(\sigma, s_2(\sigma), s_2(\gamma_1(\sigma)))]d\varpi d\sigma \\
&- \alpha\int_0^b\int_0^\infty \varpi(b-\sigma)^{\alpha-1}\xi_\alpha(\varpi)T((b-\sigma)^\alpha\varpi)[g(\sigma, s_1(\sigma), s_1(\gamma_2(\sigma))) \\
&- g(\sigma, s_2(\sigma), s_2(\gamma_2(\sigma)))]d\varpi d\sigma - \alpha\int_0^b\int_0^\infty \varpi(b-\sigma)^{\alpha-1}\xi_\alpha(\varpi)T((b-\sigma)^\alpha\varpi) \\
&\qquad \times\Big[\int_0^\sigma \Psi(\sigma,\mu, s_1(\gamma_3(\mu))) - \Psi(\sigma,\mu, s_2(\gamma_3(\mu)))d\mu\Big]d\varpi d\sigma\Big\|\Big\|(\varrho)d\varpi d\varrho \\
&+ \Big\|\alpha\int_0^\varsigma\int_0^\infty \varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)T((\varsigma-\varrho)^\alpha\varpi) \\
&\qquad \times[g(\varrho, s_1(\varrho), s_1(\gamma_2(\varrho))) - g(\varrho, s_2(\varrho), s_2(\gamma_2(\varrho)))]d\varpi d\varrho\Big\| \\
&+ \Big\|\alpha\int_0^\varsigma\int_0^\infty \varpi(\varsigma-\varrho)^{\alpha-1}\xi_\alpha(\varpi)T((\varsigma-\varrho)^\alpha\varpi) \\
&\qquad \times\Big[\int_0^\varrho \Psi(\varrho,\sigma, s_1(\gamma_3(\sigma))) - \Psi(\varrho,\sigma, s_2(\gamma_3(\sigma)))d\sigma\Big]d\varpi d\varrho\Big\| \\
&\le LH\sup_{\varsigma\in J}\|s_1(\varsigma) - s_2(\varsigma)\| + LC_1\{\| s_1(0) - s_2(0)\| + \| s_1(\gamma_1(0)) - s_2(\gamma_1(0))\|\} \\
&+ C_1\{\| s_1(\varsigma) - s_2(\varsigma)\| + \| s_1(\gamma_1(\varsigma)) - s_2(\gamma_1(\varsigma))\|\} + b^\alpha MC_1\{\| s_1(\varrho) - s_2(\varrho)\| \\
&+ \| s_1(\gamma_1(\varrho)) - s_2(\gamma_1(\varrho))\|\} + b^\alpha(M + L\|\rho\|)L_1L_2\Big[LH\sup_{\varsigma\in J}\|s_1(\varsigma) - s_2(\varsigma)\| \\
&+ LC_1\{\| s_1(0) - s_2(0)\| + \| s_1(\gamma_1(0)) - s_2(\gamma_1(0))\|\} + C_1\{\| s_1(b) - s_2(b)\| \\
&+ \| s_1(\gamma_1(b)) - s_2(\gamma_1(b))\|\} + b^\alpha MC_1\{\| s_1(\sigma) - s_2(\sigma)\| + \| s_1(\gamma_1(\sigma)) - s_2(\gamma_1(\sigma))\|\} \\
&+ b^\alpha LM_1\{\| s_1(\sigma) - s_2(\sigma)\| + \| s_1(\gamma_2(\sigma)) - s_2(\gamma_2(\sigma))\|\} \\
&+ b^\alpha L\{bN_1(s_1(\gamma_3(\mu)) - s_2(\gamma_3(\mu)))\}\Big] + b^\alpha LM_1\{\| s_1(\varrho) - s_2(\varrho)\| \\
&+ \| s_1(\gamma_2(\varrho)) - s_2(\gamma_2(\varrho))\|\} + b^\alpha L\{bN_1(s_1(\gamma_3(\sigma)) - s_2(\gamma_3(\sigma)))\} \\
&\le \Big[LH + LC_1(1+p) + C_1(1+p) + b^\alpha MC_1(1+p) + b^\alpha(M + L\|\rho\|)L_1L_2 \\
&\qquad \times\{LH + LC_1(1+p) + C_1(1+p) + b^\alpha MC_1(1+p) \\
&+ b^\alpha LM_1(1+p) + b^\alpha L(bN_1p)\} + b^\alpha LM_1(1+p) + b^\alpha L(bN_1p)\Big]\sup_{\varsigma\in J}\| s_1(\varsigma) - s_2(\varsigma)\| \\
&\le [LH + LC_1(1+p) + C_1(1+p) + b^\alpha MC_1(1+p) + b^\alpha LM_1(1+p) \\
&\quad + b^\alpha L(bN_1p)][1 + b^\alpha(M + L\|\rho\|)L_1L_2]\| s_1(\varsigma) - s_2(\varsigma)\| \\
&\le \Pi\| s_1(\varsigma) - s_2(\varsigma)\|,
\end{aligned}
$$

where

$$\Pi = [LH + LC_1(1+p) + C_1(1+p) + b^\alpha MC_1(1+p) + b^\alpha LM_1(1+p) + b^\alpha L(bN_1 p)]$$
$$[1 + b^\alpha(M + L\|\rho\|)L_1 L_2].$$

Since $0 \le \Pi < 1$, F is a contraction on $C([0,b]; E_r)$. Implementing the Banach fixed point theorem we obtain a unique fixed point for F in $C(J; E_r)$ and which is the mild solutions of (2.1)–(2.3) that fulfills $s(b) = a$. Thus, the systems (2.1)–(2.3) is controllable on J. □

4. Application

Let Ω be a bounded subset of $\mathbb{R}^n$ and let boundary of Ω be Γ, which is smooth. Consider the boundary control nonlocal neutral fractional differential system with time varying delay

$$\frac{\partial^\alpha}{\partial\varsigma^\alpha}\Big[z(\varsigma, s) + \eta(\varsigma, z(\varsigma, s), z(\gamma_1(\varsigma), s))\Big] = \Delta z(\varsigma, s) + g(\varsigma, z(\varsigma, s), z(\gamma_2(\varsigma), s)) + \int_0^\varsigma \Psi(\varsigma, \varrho, z(\gamma_3(\varsigma), s))d\varrho, \text{ in } Q = (a,b)\times\Omega, \tag{4.1}$$

$$z(\varsigma, 0) = u(\varsigma, 0) \text{ on } \Sigma = (0,b)\times\Gamma, \quad \varsigma \in [0,b], \tag{4.2}$$

$$z(0, s) + \phi(z(b^*, s)) = z_0(s), \text{ for } s \in \Omega, \quad b^* \in [0,b], \tag{4.3}$$

in which $u \in L^2(\Sigma), z_0 \in L^2(\Omega), g, \Psi \in L^2(Q)$.

Moreover, consider that the functions η, g, Ψ and ϕ are fulfilled the conditions given as:

$$\|\eta(\varsigma, s, y) - \eta(\varsigma, u, v)\| \le a_1[\|s-u\| + \|y-v\|], \varsigma \in J,$$

$$\|g(\varsigma, s, y) - g(\varsigma, u, v)\| \le a_2[\|s-u\| + \|y-v\|], \varsigma \in J,$$

$$\|\Psi(\varsigma, \varrho, s) - \Psi(\varsigma, \varrho, y)\| \le a_3\|s-y\|, \varsigma, \varrho \in \Delta,$$

and

$$\|\phi(s_1(b^*, s)) - \phi(s_2(b^*, s))\| \le a_4 \sup_{\varsigma\in[0,b]} \|s_1(\varsigma) - s_2(\varsigma)\|,$$

where a_1, a_2, a_3, a_4 are positive constants, $s, y, z, u, v \in E_r$ and $s_1, s_2 \in C(J; E_r)$. Now, we can take operators ρ, E_1 and Θ, the space U, X, V as below, so we can formulate (4.1)–(4.3) as a boundary control problems of (2.1)–(2.3).

$$D(\rho) = \{z \in L^2(\Omega); \Delta z \in L^2(\Omega)\},$$

$\rho = \Delta$, E_1 is equal to I, where I denotes identity operator. Θ is called trace operator and is defined as $\Theta z = z|_\Gamma$. Clearly Θ is well defined and $\Theta \in H^{-\frac{1}{2}}(\Gamma)$.

Also, $L^2(\Gamma) = U$, $X = H^{-\frac{1}{2}}(\Gamma)$ and $V = L^2(\Omega)$. A is defined as $A = \Delta, D(A) = H_0^1(\Omega)UH^2(\Omega)$, where $H^k(\Omega), H^\chi(\Gamma)$ and $H_0^1(\Omega)$ are usual Sobolev space on Ω, Γ.

Then A can be represented as:

$$A(z) = \sum_{n=1}^{\infty} -(n)^2(z, z_n)z_n \quad z \in D(A),$$

where $z_n(y) = \sqrt{2}\sin ny,\ n = 1, 2, ...$ is the orthogonal set of eigenvectors of A.

Again, for $z \in U$,

$$T(\varsigma)z = \sum_{n=1}^{\infty} e^{-\frac{n^2}{1+n^2}\varsigma}(z, z_n)z_n.$$

It is easy to see that A generates a strongly continuous semigroup $\{T(\varsigma) : \varsigma \geq 0\}$ in U. Hence, the hypothesis $(A1), (A2)$ are satisfied.

We express the operator E defined from $L^2(\Gamma)$ to $L^2(\Omega)$ as $Eu = w_u$, w_u is the unique solution to the subsequent system,

$$\Delta w_u = 0 \quad \text{in} \quad \Omega,$$

$$w_u = u \quad \text{in} \quad \Gamma.$$

It is proved in [22] that for every $u \in H^{-\frac{1}{2}}(\Gamma)$, the above system has a unique solution $w_u \in L^2(\Omega)$ satisfying $\|Eu\|_{L^2(\Omega)} = \|wu\|_{L^2(\Omega)} = c_1\|u\|_{H^{-\frac{1}{2}}(\Gamma)}$. This shows that (A3) is satisfied. From the above estimates, it follows by interpolation argument [6] that

$$\|AT(\varsigma)E\|_{L(L^2(\Gamma),L^2(\Gamma))} \leq C'\varsigma^{\frac{-3}{4}}, \quad \forall\varsigma > 0,$$

with $\nu(\varsigma) = C'\varsigma^{\frac{-3}{4}}$, where c_1, C' are positive constant independent of u. Therefore, hypothesis (A4) is satisfied. The boundary controllability of the systems (4.1)–(4.3) is discussed in detailed in [1,5]. For condition (A5), the detailed discussion is presented in [42,43]. Clearly, the functions η, g, Ψ, ϕ satisfies the assumptions (A6)–(A9).

Select b and the remaining constants such that (A_1)–(A_{10}) are fulfilled, see [22]. Thus, Theorem 3.1 can be applied for (4.1)–(4.3) and so the systems (4.1)–(4.3) is controllable on $[0, b]$.

5. Conclusions

To sum up it, we explored the sufficient conditions for boundary controllability of neutral integrodifferential evolution system of fractional order with time varying delay and nonlocal condition in Banach spaces. To establish the result, we apply the Banach fixed point theorem. In the end, we stated an application to validate the abstract result. The above researches of (2.1)–(2.3) provides only an analytic result. This result which is given in (2.1)–(2.3), may be applied to get the numerical solution of such kind of equation. In future we will try to obtain some results on stability, asymptotic behavior and some numerical method like HAM. Also researchers can extend boundary controllability to boundary communication.

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Conflict of interest

This work does not have any conflict of interest.

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