Coupled fixed point theorems on $C^*$-algebra valued bipolar metric spaces

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Abstract: In the present paper, we introduce the notion of a $C^*$-algebra valued bipolar metric space and prove coupled fixed point theorems. Some of the well-known outcomes in the literature are generalized and expanded by the results shown. An example and application to support our result is presented.

Keywords: $C^*$-algebra; $C^*$-algebra valued bipolar metric space; coupled fixed point

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1. Introduction

In 2016, Mutlu and Grdal [1] introduced the concepts of bipolar metric space and proved fixed point theorems.

Definition 1.1. [1] Let $\Gamma$ and $\Psi$ be two non-void sets and $\varphi: \Gamma \times \Psi \to \mathbb{R}^+$ be a function, such that
(a) \( \varphi(\sigma, \zeta) = 0 \) iff \( \sigma = \zeta \), for all \( (\sigma, \zeta) \in \Gamma \times \Psi \).

(b) \( \varphi(\sigma, \zeta) = \varphi(\zeta, \sigma) \), for all \( \sigma, \zeta \in \Gamma \cap \Psi \).

(c) \( \varphi(\sigma, \zeta) \leq \varphi(\sigma, \omega) + \varphi(\sigma_1, \omega) + \varphi(\sigma_1, \zeta) \), for all \( \sigma, \sigma_1 \in \Gamma \) and \( \omega, \zeta \in \Psi \).

The pair \((\Gamma, \Psi, \varphi)\) is called a bipolar metric space.

In bipolar metric spaces, a lot of significant work has been done (see [2–10]). In 2014, Ma, Jiang and Sun [11] introduced the notion of a \( C^* \)-algebra-valued metric space and proved fixed point theorems. In 2015, Batul and Kamran [12] proved fixed theorems on \( C^* \)-algebra-valued metric space. In the present paper, we introduce a new notion of \( C^* \)-algebra valued bipolar metric space and proved coupled fixed point theorems. The details on \( C^* \)-algebra are available in [13–17].

An algebra \( \mathbb{A} \), together with a conjugate linear involution map \( \vartheta \mapsto \vartheta^* \), is called a \( \star \)-algebra if \( (\vartheta \varphi)^* = \varphi^* \vartheta^* \) and \( (\vartheta^*)^* = \vartheta \) for all \( \vartheta, \varphi \in \mathbb{A} \). Moreover, the pair \((\mathbb{A}, \star)\) is called a unital \( \star \)-algebra if \( \mathbb{A} \) contains the identity element \( 1_\mathbb{A} \). By a Banach \( \star \)-algebra we mean a complete normed unital \( \star \)-algebra \((\mathbb{A}, \star)\) such that the norm on \( \mathbb{A} \) is submultiplicative and satisfies \( \| \vartheta \varphi \| = \| \vartheta \| \| \varphi \| \) for all \( \vartheta \in \mathbb{A} \).

Further, if for all \( \vartheta \in \mathbb{A} \), we have \( \| \vartheta \varphi \vartheta \| = \| \vartheta \| \| \varphi \| \) in a Banach \( \star \)-algebra \((\mathbb{A}, \star)\), then \( \mathbb{A} \) is known as a \( C^* \)-algebra. A positive element of \( \mathbb{A} \) is an element \( \vartheta \in \mathbb{A} \) such that \( \vartheta = \vartheta^* \) and its spectrum \( \sigma(\vartheta) \subset \mathbb{R}_+ \), where \( \sigma(\vartheta) = \{ v \in \mathbb{R} : v1_\mathbb{A} - \vartheta \text{ is noninvertible} \} \). The set of all positive elements will be denoted by \( \mathbb{A}_+ \). Such elements allow us to define a partial ordering \( \geq \) on the elements of \( \mathbb{A} \). That is,

\[
\vartheta \geq \varphi \text{ if and only if } \vartheta - \varphi \in \mathbb{A}_+.
\]

If \( \vartheta \in \mathbb{A} \) is positive, then we write \( \vartheta \geq 0_\mathbb{A} \), where \( 0_\mathbb{A} \) is the zero element of \( \mathbb{A} \). Each positive element \( \vartheta \) of a \( C^* \)-algebra \( \mathbb{A} \) has a unique positive square root. From now on, by \( \mathbb{A} \) we mean a unital \( C^* \)-algebra with identity element \( 1_\mathbb{A} \). Further, \( \mathbb{A}_+ = \{ \vartheta \in \mathbb{A} : \vartheta \geq 0_\mathbb{A} \} \) and \( (\vartheta^* \vartheta)^{1/2} = \| \vartheta \| \).

2. Preliminaries

In this section, we extend Definition 1.1 to introduce the notion bipolar metric space in the setting of \( C^* \)-algebra as follows.

**Definition 2.1.** Let \( \mathbb{A} \) be a \( C^* \)-algebra, and \( \Gamma, \Psi \) be two non-void sets. A mapping \( \varphi : \Gamma \times \Psi \to \mathbb{A}_+ \) be a function such that

(a) \( \varphi(\sigma, \zeta) = 0 \) iff \( \sigma = \zeta \), for all \( (\sigma, \zeta) \in \Gamma \times \Psi \).

(b) \( \varphi(\sigma, \zeta) = \varphi(\zeta, \sigma) \), for all \( \sigma, \zeta \in \Gamma \cap \Psi \).

(c) \( \varphi(\sigma, \zeta) \leq \varphi(\sigma, \gamma) + \varphi(\sigma_1, \omega) + \varphi(\sigma_1, \zeta) \), for all \( \sigma, \sigma_1 \in \Gamma \) and \( \gamma, \zeta \in \Psi \).

The 4-tuple \((\Gamma, \Psi, \mathbb{A}, \varphi)\) is called a \( C^* \)-algebra valued bipolar metric space.

**Lemma 2.2** ([14, 17]). Suppose that \( \mathbb{A} \) is a unital \( C^* \)-algebra with a unit 1.

(A1) For any \( \sigma \in \mathbb{A}_+ \), we have \( \sigma \leq 1 \) iff \( \| \sigma \| \leq 1 \).

(A2) If \( \vartheta \in \mathbb{A}_+ \) with \( \| \vartheta \| < \frac{1}{2} \), then \((I - \vartheta)\) is invertible and \( \| \vartheta(I - a^{-1}) \| < 1 \).
(A3) Suppose that $\vartheta, \varpi \in A$ with $\vartheta \varpi \geq 0_{A}$ and $\vartheta \varpi = \varpi \vartheta$, then $\varpi \vartheta \geq 0_{A}$.

(A4) By $A'$ we denote the set $\{ \vartheta \in A : \vartheta \varpi = \varpi \vartheta, \forall \varpi \in A \}$. Let $\vartheta \in A'$, if $\varpi, \epsilon \in A$ with $\varpi \geq \epsilon \geq 0_{A}$, and $I - \vartheta \in A'_{+}$ is an invertible operator, then

$$(I - \vartheta)^{-1} \varpi \geq (I - \vartheta)^{-1} \epsilon.$$  

(A5) If $\varpi, \epsilon \in A_{\neq} = \{ \sigma \in A : \sigma = \sigma^{*} \}$ and $\vartheta \in A$, then $\varpi \leq \epsilon$ implies that $\vartheta^{*} \varpi \vartheta \leq \vartheta^{*} \epsilon \vartheta$.

(A6) $0_{A} \leq \vartheta \leq \varpi$, then $\| \vartheta \| \leq \| \varpi \|$.

Notice that in a $C^{*}$-algebra, if $0_{A} \leq \vartheta, \varpi$, one cannot conclude that $0_{A} \leq \vartheta \varpi$.

**Definition 2.3.** Let $(\Gamma_{1}, \varPsi_{1}, A, \varphi)$ and $(\Gamma_{2}, \varPsi_{2}, A, \varphi)$ be pairs of sets and given a function $\Phi : \Gamma_{1} \cup \varPsi_{1} \to \Gamma_{2} \cup \varPsi_{2}$.

(B1) If $\Phi(\Gamma_{1}) \subseteq \Gamma_{2}$ and $\Phi(\varPsi_{1}) \subseteq \varPsi_{2}$, then $\Phi$ is called a covariant map, or a map from $(\Gamma_{1}, \varPsi_{1}, A, \varphi_{1})$ to $(\Gamma_{2}, \varPsi_{2}, A, \varphi_{2})$ and this is written as

$$\Phi : (\Gamma_{1}, \varPsi_{1}, A, \varphi_{1}) \Rightarrow (\Gamma_{2}, \varPsi_{2}, A, \varphi_{2}).$$

(B2) If $\Phi(\Gamma_{1}) \subseteq \varPsi_{2}$ and $\Phi(\varPsi_{1}) \subseteq \Gamma_{2}$, then $\Phi$ is called a contravariant map from $(\Gamma_{1}, \varPsi_{1}, A, \varphi_{1})$ to $(\Gamma_{2}, \varPsi_{2}, A, \varphi_{2})$ and this is denoted as

$$\Phi : (\Gamma_{1}, \varPsi_{1}, A, \varphi_{1}) \Leftarrow (\Gamma_{2}, \varPsi_{2}, A, \varphi_{2}).$$

**Definition 2.4.** Let $(\Gamma, \varPsi, A, \varphi)$ be a $C^{*}$-algebra valued bipolar metric space.

(C1) A point $\sigma \in \Gamma \cup \varPsi$ is said to be a left point if $\sigma \in \Gamma$, a right point if $\sigma \in \varPsi$ and a central point if both hold. Similarly, a sequence $\{ \sigma_{n} \}$ on the set $\Gamma$ and a sequence $\{ \zeta_{n} \}$ on the set $\varPsi$ are called a left and right sequence with respect to $A$, respectively.

(C2) A sequence $\{ \sigma_{n} \}$ converges to a point $\zeta$ with respect to $A$ iff $\{ \sigma_{n} \}$ is a left sequence, $\zeta$ is a right point and $\lim_{a \to \infty} \varphi(\sigma_{n}, \zeta) = 0_{A}$ or $\{ \sigma_{n} \}$ is a right sequence, $\zeta$ is a left point and $\lim_{a \to \infty} \varphi(\zeta, \sigma_{n}) = 0_{A}$.

(C3) A bisquence $\{(\sigma_{n}, \zeta_{n})\}$ is a sequence on the set $\Gamma \times \varPsi$. If the sequence $\{ \sigma_{n} \}$ and $\{ \zeta_{n} \}$ are convergent with respect to $A$, then the sequence $\{(\sigma_{n}, \zeta_{n})\}$ is said to be convergent with respect to $A$. $\{(\sigma_{n}, \zeta_{n})\}$ is a Cauchy bisquence with respect to $A$ if $\lim_{a, b \to \infty} \varphi(\sigma_{a}, \zeta_{b}) = 0_{A}$, hence biconvergent with respect to $A$.

(C4) $(\Gamma, \varPsi, A, \varphi)$ is complete, if every Cauchy bisquence with respect to $A$ is convergent in $\Gamma \times \varPsi$.

3. Main results

**Theorem 3.1.** Let $(\Gamma, \varPsi, A, \varphi)$ be a complete $C^{*}$-algebra valued bipolar metric space. Suppose

$$\Phi : (\Gamma^{2}, \varPsi^{2}, A, \varphi) \Rightarrow (\Gamma, \varPsi, A, \varphi)$$

is a covariant mapping such that

$$
\varphi(\Phi(\sigma, \zeta), \Phi(u, v)) \leq v^* \varphi(\sigma, u)u + v^* \varphi(\zeta, v)v \quad \text{for all } \sigma, \zeta \in \Gamma, u, v \in \Psi,
$$

where \( v \in A \) with \( 2 ||v||^2 < 1 \). Then the function

$$
\Phi : \Gamma^2 \cup \Psi^2 \to \Gamma \cup \Psi
$$

has a unique coupled fixed point.

**Proof.** Let \( \sigma_0, \zeta_0 \in \Gamma \) and \( u_0, v_0 \in \Psi \). For each \( \alpha \in \mathbb{N} \), define

$$
\Phi(\sigma_\alpha, \zeta_\alpha) = \sigma_{\alpha+1},
\Phi(\zeta_\alpha, \sigma_\alpha) = \zeta_{\alpha+1},
\Phi(u_\alpha, v_\alpha) = u_{\alpha+1}
$$

and

$$
\Phi(v_\alpha, u_\alpha) = v_{\alpha+1}.
$$

Then \( (\{\sigma_\alpha\}, \{\zeta_\alpha\}) \) and \( (\{u_\alpha\}, \{v_\alpha\}) \) are bisequences on \((\Gamma, \Psi, A, \varphi)\). Then, for each \( \alpha \in \mathbb{N} \),

$$
\varphi(\sigma_\alpha, u_\alpha) = \varphi(\Phi(\sigma_{\alpha-1}, \zeta_{\alpha-1}), \Phi(u_{\alpha-1}, v_{\alpha-1})) \\
\leq v^* \varphi(\sigma_{\alpha-1}, u_{\alpha-1})u + v^* \varphi(\zeta_{\alpha-1}, v_{\alpha-1})v \\
= v^* M_\alpha u,
$$

where

$$
M_\alpha = \varphi(\sigma_{\alpha-1}, u_{\alpha-1}) + \varphi(\zeta_{\alpha-1}, v_{\alpha-1}).
$$

Similarly, we get

$$
\varphi(\zeta_\alpha, v_\alpha) = \varphi(\Phi(\zeta_{\alpha-1}, \sigma_{\alpha-1}), \Phi(v_{\alpha-1}, u_{\alpha-1})) \\
\leq v^* \varphi(\zeta_{\alpha-1}, v_{\alpha-1})v + v^* \varphi(\sigma_{\alpha-1}, u_{\alpha-1})u \\
= v^* M_\alpha v.
$$

Now,

$$
M_{\alpha+1} = \varphi(\sigma_\alpha, u_{\alpha+1}) + \varphi(\zeta_\alpha, v_{\alpha+1}) \\
\leq v^* [\varphi(\sigma_{\alpha-1}, u_{\alpha}) + \varphi(\zeta_{\alpha-1}, v_{\alpha})]v \\
+ v^* [\varphi(\zeta_{\alpha-1}, v_{\alpha}) + \varphi(\sigma_{\alpha-1}, u_{\alpha})]v \\
\leq (\sqrt{2}v)^* M_\alpha (\sqrt{2}v).
$$

By Lemma 2.2 (A5), we have

$$
0_A \leq M_{\alpha+1} \leq (\sqrt{2}v)^* M_\alpha (\sqrt{2}v) \leq \cdots \leq ((\sqrt{2}v)^*)^n M_1 (\sqrt{2}v)^n.
$$
On the other hand,

$$\varphi(\sigma_{\alpha+1}, u_{\alpha}) = \varphi(\Phi(\sigma_{\alpha}, \zeta_{\alpha}), \Phi(u_{\alpha-1}, v_{\alpha-1}))$$

$$\leq \nu^* \varphi(\sigma_{\alpha}, u_{\alpha-1}) + \nu^* \varphi(\zeta_{\alpha}, v_{\alpha-1})$$

$$= \nu^* S_{\alpha} u,$$

where

$$S_{\alpha} = \varphi(\sigma_{\alpha}, u_{\alpha-1}) + \varphi(\zeta_{\alpha}, v_{\alpha-1}).$$

Similarly, we get

$$\varphi(\zeta_{\alpha+1}, v_{\alpha}) = \varphi(\Phi(\zeta_{\alpha}, \sigma_{\alpha}), \Phi(v_{\alpha-1}, u_{\alpha-1}))$$

$$\leq \nu^* \varphi(\zeta_{\alpha}, v_{\alpha-1}) + \nu^* \varphi(\sigma_{\alpha}, u_{\alpha-1})$$

$$= \nu^* S_{\alpha} v.$$

Now,

$$S_{\alpha+1} = \varphi(\sigma_{\alpha+1}, u_{\alpha}) + \varphi(\zeta_{\alpha+1}, v_{\alpha})$$

$$\leq \nu^* [\varphi(\sigma_{\alpha}, u_{\alpha-1}) + \varphi(\zeta_{\alpha}, v_{\alpha-1})] + \nu^* [\varphi(\sigma_{\alpha}, u_{\alpha-1}) + \varphi(\zeta_{\alpha}, v_{\alpha-1})] u$$

$$\leq (\sqrt{2}\nu^*)^* S_{\alpha} (\sqrt{2}\nu^*).$$

By Lemma 2.2 (A5), we have

$$0 \leq S_{\alpha+1} \leq (\sqrt{2}\nu^*)^* S_{\alpha} (\sqrt{2}\nu^*) \leq \cdots \leq ((\sqrt{2}\nu^*)^*)^* S_{1} (\sqrt{2}\nu^*).$$

Moreover,

$$\varphi(\sigma_{\alpha}, u_{\alpha}) = \varphi(\Phi(\sigma_{\alpha-1}, \zeta_{\alpha-1}), \Phi(u_{\alpha-1}, v_{\alpha-1}))$$

$$\leq \nu^* \varphi(\sigma_{\alpha-1}, u_{\alpha-1}) + \nu^* \varphi(\zeta_{\alpha-1}, v_{\alpha-1})$$

$$= \nu^* R_{\alpha} u,$$

where

$$R_{\alpha} = \varphi(\sigma_{\alpha-1}, u_{\alpha-1}) + \varphi(\zeta_{\alpha-1}, v_{\alpha-1}).$$

Similarly, we get

$$\varphi(\zeta_{\alpha}, v_{\alpha}) = \varphi(\Phi(\zeta_{\alpha-1}, \sigma_{\alpha-1}), \Phi(v_{\alpha-1}, u_{\alpha-1}))$$

$$\leq \nu^* \varphi(\zeta_{\alpha-1}, v_{\alpha-1}) + \nu^* \varphi(\sigma_{\alpha-1}, u_{\alpha-1})$$

$$= \nu^* R_{\alpha} v.$$

Now,
\( R_{\alpha+1} = \varphi(\sigma_\alpha, u_\alpha) + \varphi(\zeta_\alpha, v_\alpha) \)
\[
\leq u^* [\varphi(\sigma_{\alpha-1}, u_{\alpha-1}) + \varphi(\zeta_{\alpha-1}, v_{\alpha-1})] v \\
+ u^* [\varphi(\zeta_{\alpha-1}, v_{\alpha-1}) + \varphi(\sigma_{\alpha-1}, u_{\alpha-1})] v \\
\leq (\sqrt{2}v)^* R_\alpha(\sqrt{2}v).
\]

By Lemma 2.2 (A5), we have
\[
0_A \leq R_{\alpha+1} \leq (\sqrt{2}v)^* R_\alpha(\sqrt{2}v) \leq \cdots \leq ((\sqrt{2}v)^*)^n R_1(\sqrt{2}v)^n.
\]

Now,
\[
\varphi(\sigma_\alpha, u_\beta) \leq \varphi(\sigma_\alpha, u_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, u_{\alpha+1}) + \cdots + \varphi(\sigma_{\beta-1}, u_\beta),
\]
\[
\varphi(\zeta_\alpha, v_\beta) \leq \varphi(\zeta_\alpha, v_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, v_{\alpha+1}) + \cdots + \varphi(\zeta_{\beta-1}, v_\beta),
\]
and
\[
\varphi(\sigma_\beta, u_\alpha) \leq \varphi(\sigma_\beta, u_{\beta-1}) + \varphi(\sigma_{\beta-1}, u_{\beta-1}) + \cdots + \varphi(\sigma_{\alpha+1}, u_\alpha),
\]
\[
\varphi(\zeta_\beta, v_\alpha) \leq \varphi(\zeta_\beta, v_{\beta-1}) + \varphi(\zeta_{\beta-1}, v_{\beta-1}) + \cdots + \varphi(\zeta_{\alpha+1}, v_\alpha),
\]
for each \( \alpha, \beta \in \mathbb{N}, \alpha < \beta \). Then,
\[
\varphi(\sigma_\alpha, u_\beta) + \varphi(\zeta_\alpha, v_\beta) \leq (\varphi(\sigma_\alpha, u_{\alpha+1}) + \varphi(\zeta_\alpha, v_{\alpha+1})) + (\varphi(\sigma_{\alpha+1}, u_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, v_{\alpha+1})) \\
+ \cdots + (\varphi(\sigma_{\beta-1}, u_\beta) + \varphi(\zeta_{\beta-1}, v_\beta)) \\
= M_{\alpha+1} + R_{\alpha+2} + M_{\alpha+2} + \cdots + R_\beta + M_\beta \\
\leq ((\sqrt{2}v)^*)^m M_1(\sqrt{2}v)^m + ((\sqrt{2}v)^*)^{m+1} R_1(\sqrt{2}v)^{m+1} + \cdots \\
+ ((\sqrt{2}v)^*)^{n-1} R_1(\sqrt{2}v)^{n-1} + ((\sqrt{2}v)^*)^{n} M_1(\sqrt{2}v)^n \\
= \sum_{i=\alpha}^{\beta-1} ((\sqrt{2}v)^*)^i M_1(\sqrt{2}v)^i + \sum_{i=\alpha+1}^{\beta-1} ((\sqrt{2}v)^*)^i R_1(\sqrt{2}v)^i \\
= \sum_{i=\alpha}^{\beta-1} ((\sqrt{2}v)^*)^i M_1^i(\sqrt{2}v)^i + \sum_{i=\alpha+1}^{\beta-1} ((\sqrt{2}v)^*)^i R_1^i(\sqrt{2}v)^i \\
= \sum_{i=\alpha}^{\beta-1} |M_1^i(\sqrt{2}v)^i|^2 + \sum_{i=\alpha+1}^{\beta-1} |R_1^i(\sqrt{2}v)^i|^2 \\
\leq \sum_{i=\alpha}^{\beta-1} |M_1^i(\sqrt{2}v)^i|^2 1_A + \sum_{i=\alpha+1}^{\beta-1} |R_1^i(\sqrt{2}v)^i|^2 1_A \\
\leq \sum_{i=\alpha}^{\beta-1} \|M_1^i\|^2 \|\sqrt{2}v\|^2 1_A + \sum_{i=\alpha+1}^{\beta-1} \|R_1^i\|^2 \|\sqrt{2}v\|^2 1_A.
\]
\[
\leq \|M_i^\frac{\alpha}{\beta}\| \sum_{i=\alpha}^{\beta-1} \| (\sqrt{2}u)^2 \| 1_A + \|R_i^\frac{\alpha}{\beta}\| \sum_{i=\alpha+1}^{\beta-1} \| (\sqrt{2}v)^2 \| 1_A \\
= \|M_i^\frac{\alpha}{\beta}\| \sum_{i=\alpha}^{\beta-1} (2\|v\|^2) 1_A + \|R_i^\frac{\alpha}{\beta}\| \sum_{i=\alpha+1}^{\beta-1} (2\|v\|^2) 1_A \\
\rightarrow 0_A \quad \text{(as } \beta, \alpha \rightarrow \infty) \]

and

\[
\varphi(\sigma_\beta, u_\alpha) + \varphi(\zeta_\beta, v_\alpha) \leq \varphi(\sigma_\beta, u_{\beta-1}) + \varphi(\zeta_\beta, v_{\beta-1}) + \varphi(\sigma_{\beta-1}, u_{\beta-1}) + \varphi(\zeta_{\beta-1}, v_{\beta-1}) \\
+ \cdots + \varphi(\sigma_{\alpha+1}, u_\alpha) + \varphi(\zeta_{\alpha+1}, v_\alpha) \\
= S_\beta + R_\beta + S_{\beta-1} + \cdots + R_{\alpha+1} + S_{\alpha+1} \\
\leq ((\sqrt{2}u)^*\varphi(\sigma_\beta, u_{\beta-1}) + ((\sqrt{2}v)^*\varphi(\zeta_\beta, v_{\beta-1}) + ((\sqrt{2}u)^*\varphi(\sigma_{\beta-1}, u_{\beta-1}) + ((\sqrt{2}v)^*\varphi(\zeta_{\beta-1}, v_{\beta-1}) \\
+ \cdots + ((\sqrt{2}u)^*\varphi(\sigma_{\alpha+1}, u_\alpha) + ((\sqrt{2}v)^*\varphi(\zeta_{\alpha+1}, v_\alpha) \\
\leq \sum_{i=\beta}^{\alpha+1} ((\sqrt{2}u)^* S_1(\sqrt{2}u)^i + \sum_{i=\beta}^{\alpha+2} ((\sqrt{2}v)^* R_1(\sqrt{2}v)^i \\
= \sum_{i=\beta}^{\alpha+1} S_1^\frac{i}{\beta}(\sqrt{2}u)^i + \sum_{i=\beta}^{\alpha+2} R_1^\frac{i}{\beta}(\sqrt{2}v)^i \\
= \sum_{i=\beta}^{\alpha+1} |S_1^\frac{i}{\beta}(\sqrt{2}u)^i|^2 + \sum_{i=\beta}^{\alpha+2} |R_1^\frac{i}{\beta}(\sqrt{2}v)^i|^2 \\
\leq \|S_i^\frac{i}{\beta}(\sqrt{2}u)^i\|^2 1_A + \|R_i^\frac{i}{\beta}(\sqrt{2}v)^i\|^2 1_A \\
\leq \sum_{i=\beta}^{\alpha+1} |S_1^\frac{i}{\beta}(\sqrt{2}u)^i|^2 \|1_A + \|R_i^\frac{i}{\beta}(\sqrt{2}v)^i\|^2 \|1_A \\
\leq \sum_{i=\beta}^{\alpha+1} |S_1^\frac{i}{\beta}(\sqrt{2}u)^i|^2 \|1_A + \|R_i^\frac{i}{\beta}(\sqrt{2}v)^i\|^2 \|1_A \\
\leq |S_i^\frac{i}{\beta}|^2 \|1_A + \|R_i^\frac{i}{\beta}\|^2 \sum_{i=\beta}^{\alpha+2} |(|\sqrt{2}u|^2) 1_A \\
= |S_i^\frac{i}{\beta}|^2 \sum_{i=\beta}^{\alpha+1} (2\|v\|^2) 1_A + \|R_i^\frac{i}{\beta}\|^2 \sum_{i=\beta}^{\alpha+2} (2\|v\|^2) 1_A \\
\rightarrow 0_A \quad \text{(as } \beta, \alpha \rightarrow \infty). \]

Therefore, \([[\sigma_\beta], [u_\alpha]]\) and \([[\zeta_\beta], [v_\alpha]]\) are Cauchy bisequences in \(\Gamma \times \Psi\) with respect to \(A\). By completeness of \((\Gamma, \Psi, A, \varphi)\), there exist \(\sigma, \zeta \in \Gamma\) and \(u, v \in \Psi\) with

\[
\lim_{\alpha \to \infty} \sigma_\alpha = u, \quad \lim_{\alpha \to \infty} \zeta_\alpha = v, \quad \lim_{\alpha \to \infty} u_\alpha = \sigma \text{ and } \lim_{\alpha \to \infty} v_\alpha = \zeta.
\]
Since \((\{\sigma_a\}, \{u_a\})\) and \((\{\zeta_a\}, \{v_a\})\) are Cauchy bisquences, we derive that
\[
\varphi(\sigma_a, u_a) < \frac{\epsilon}{2} \quad \text{and} \quad \varphi(\zeta_a, v_a) < \frac{\epsilon}{2}.
\]
Then,
\[
\varphi(\Phi(\sigma, \zeta), u) \leq \varphi(\Phi(\sigma, \zeta), u_{a+1}) + \varphi(\sigma_{a+1}, u_{a+1}) + \varphi(\sigma_{a+1}, u_a)
\]
\[
= \varphi(\Phi(\sigma, \zeta), \Phi(u_a, v_a)) + \varphi(\sigma_{a+1}, u_{a+1}) + \varphi(\sigma_{a+1}, u_a)
\]
\[
\leq \nu^* \varphi(\sigma, u_a)u + \nu^* \varphi(\zeta, v_a)v + \varphi(\sigma_{a+1}, u_{a+1}) + \varphi(\sigma_{a+1}, u_a).
\]
As \(\alpha \to \infty\), we have
\[
\varphi(\Phi(\sigma, \zeta), u) < \epsilon.
\]
Then,
\[
\varphi(\Phi(\sigma, \zeta), u) = 0.
\]
Hence, \(\Phi(\sigma, \zeta) = u\). Similarly, we can derive \(\Phi(\zeta, \sigma) = v\), \(\Phi(u, v) = \sigma\) and \(\Phi(v, u) = \zeta\). On the other hand, we derive that
\[
\varphi(\sigma, u) = \varphi(\lim_{a \to \infty} u_a, \lim_{a \to \infty} \sigma_a) = \lim_{a \to \infty} \varphi(\sigma_a, u_a) = 0
\]
and
\[
\varphi(\zeta, v) = \varphi(\lim_{a \to \infty} v_a, \lim_{a \to \infty} \zeta_a) = \lim_{a \to \infty} \varphi(\zeta_a, v_a) = 0.
\]
So, \(\sigma = u\) and \(\zeta = v\). Therefore, \((\sigma, \zeta) \in \Gamma^2 \cap \Psi^2\) is a coupled fixed point of \(\Phi\). Let \((e, \bar{f}) \in \Gamma^2 \cup \Psi^2\) is a another coupled fixed point \(\Phi\). If \((e, \bar{f}) \in \Gamma^2\), then
\[
0_{\lambda} \leq \varphi(e, \sigma) = \varphi(\Phi(e, \bar{f}), \Phi(\sigma, \zeta)) \leq \nu^* \varphi(e, \sigma)v + \nu^* \varphi(\bar{f}, \zeta)v
\]
and
\[
0_{\lambda} \leq \varphi(\bar{f}, \zeta) = \varphi(\Phi(\bar{f}, e), \Phi(\zeta, \sigma)) \leq \nu^* \varphi(\bar{f}, \zeta)v + \nu^* \varphi(e, \sigma)u,
\]
which implies that
\[
0_{\lambda} \leq \varphi(e, \sigma) + \varphi(\bar{f}, \zeta) \leq (\sqrt{2}u)^* (\varphi(\bar{f}, \zeta) + \varphi(e, \sigma)) (\sqrt{2}v).
\]
Then,
\[
0 \leq ||\varphi(e, \sigma) + \varphi(\bar{f}, \zeta)|| \leq ||\sqrt{2}u||^2 ||\varphi(\bar{f}, \zeta) + \varphi(e, \sigma)||.
\]
Since \(2||u||^2 < 1\), we derive that
\[
\varphi(\bar{f}, \zeta) + \varphi(e, \sigma) = 0.
\]
Therefore, \(e = \sigma\) and \(\bar{f} = \zeta\). Similarly, if \((e, \bar{f}) \in \Psi^2\), then \(e = \sigma\) and \(\bar{f} = \zeta\). Then \((\sigma, \zeta)\) is a unique coupled fixed point of \(\Phi\). \(\square\)
Example 3.2. Let $\Gamma = [0, 1]$, $\Psi = \{0\} \cup \mathbb{N} \setminus \{1\}$, $\mathbb{A}_+ = M_2(\mathbb{C})$ and the map $\varphi : \Gamma \times \Psi \to \mathbb{A}_+$ is defined by

$$\varphi(\sigma, u) = \begin{bmatrix} |\sigma - u| & 0 \\ 0 & k|\sigma - u| \end{bmatrix}$$

for all $\sigma \in \Gamma$ and $u \in \Psi$, where $k \geq 0$ is a constant. Let $\leq$ be the partial order on $\mathbb{A}$ given by

$$(\vartheta_1, \sigma_1) \leq (\vartheta_2, \sigma_2) \iff \vartheta_1 \leq \vartheta_2 \text{ and } \sigma_1 \leq \sigma_2.$$  

Then $(\Gamma, \Psi, \mathbb{A}, \varphi)$ is a complete $C^*$-algebra-valued bipolar metric space. Define

$$\Phi : \Gamma^2 \cup \Psi^2 \Rightarrow \Gamma \cup \Psi$$

by

$$\Phi(\sigma, \zeta) = \frac{\sigma + \zeta}{3},$$

$\forall \sigma, \zeta \in \Gamma^2 \cup \Psi^2$. Then

$$\varphi(\Phi(\sigma, \zeta), \Phi(u, v)) \leq \begin{bmatrix} |\Phi(\sigma, \zeta) - \Phi(u, v)| & 0 \\ 0 & k|\Phi(\sigma, \zeta) - \Phi(u, v)| \end{bmatrix}$$

$$= \begin{bmatrix} \frac{|\sigma + \zeta| - |u + v|}{3} & 0 \\ 0 & k\frac{|\sigma + \zeta| - |u + v|}{3} \end{bmatrix}$$

$$\leq \frac{1}{3} \left( \begin{bmatrix} |\sigma - u| & 0 \\ 0 & k|\sigma - u| \end{bmatrix} + \begin{bmatrix} |\zeta - v| & 0 \\ 0 & k|\zeta - v| \end{bmatrix} \right)$$

$$= \nu^* \varphi(\sigma, u)\nu + \nu^* \varphi(\zeta, v)\nu,$$

for all $\sigma, \zeta \in \Gamma$, $u, v \in \Psi$, where

$$\nu = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

and $\|\nu\| = \frac{1}{3} < \frac{1}{\sqrt{2}}$. All the conditions of Theorem 3.1 are fulfilled and $\Phi$ has a unique fixed point $(0, 0)$.

Theorem 3.3. Let $(\Gamma, \Psi, \mathbb{A}, \varphi)$ be a complete $C^*$-algebra valued bipolar metric space. Suppose

$$\Phi : (\Gamma \times \Psi, \Psi \times \Gamma, \mathbb{A}, \varphi) \Rightarrow (\Gamma, \Psi, \mathbb{A}, \varphi)$$

is a covariant mapping such that

$$\varphi(\Phi(\sigma, u), \Phi(\zeta, v)) \leq \nu^* \varphi(\sigma, v)\nu + \nu^* \varphi(\zeta, u)\nu$$

for all $\sigma, \zeta \in \Gamma$, $u, v \in \Psi$, where $\nu \in \mathbb{A}$ with $2\|\nu\|^2 < 1$. Then the function

$$\Phi : (\Gamma \times \Psi) \cup (\Psi \times \Gamma) \to \Gamma \cup \Psi$$

has a unique coupled fixed point.
Proof. Let $\sigma_0, \zeta_0 \in \Gamma$ and $u_0, v_0 \in \Psi$. For each $\alpha \in \mathbb{N}$, define $\Phi(\sigma_\alpha, u_\alpha) = \sigma_{\alpha+1}$, $\Phi(\zeta_\alpha, v_\alpha) = \zeta_{\alpha+1}$, $\Phi(u_\alpha, \sigma_\alpha) = u_{\alpha+1}$ and $\Phi(v_\alpha, \zeta_\alpha) = v_{\alpha+1}$. Then $(\{\sigma_\alpha\}, \{u_\alpha\})$ and $(\{\zeta_\alpha\}, \{v_\alpha\})$ are bisequences on $(\Gamma, \Psi, A, \varphi)$. Then, for each $\alpha \in \mathbb{N}$,

$$\varphi(\sigma_\alpha, v_{\alpha+1}) = \varphi(\Phi(\sigma_{\alpha-1}, u_{\alpha-1}), \Phi(v_\alpha, \zeta_\alpha))$$

$$\leq v^* \varphi(\sigma_{\alpha-1}, v_{\alpha-1}) + v^* \varphi(\zeta_{\alpha-1}, u_{\alpha-1})v,$$

$$\varphi(\sigma_{\alpha+1}, v_\alpha) = \varphi(\Phi(\sigma_\alpha, u_\alpha), \Phi(v_{\alpha-1}, \zeta_{\alpha-1}))$$

$$\leq v^* \varphi(\sigma_\alpha, v_{\alpha-1}) + v^* \varphi(\zeta_{\alpha-1}, u_\alpha)v,$$

$$\varphi(\zeta_\alpha, u_{\alpha+1}) = \varphi(\Phi(\zeta_{\alpha-1}, v_{\alpha-1}), \Phi(u_\alpha, \sigma_\alpha))$$

$$\leq v^* \varphi(\zeta_{\alpha-1}, u_{\alpha-1}) + v^* \varphi(\sigma_\alpha, v_\alpha)v,$$

$$\varphi(\zeta_{\alpha+1}, u_\alpha) = \varphi(\Phi(\zeta_\alpha, v_\alpha), \Phi(u_{\alpha-1}, \sigma_{\alpha-1}))$$

$$\leq v^* \varphi(\zeta_\alpha, u_{\alpha-1}) + v^* \varphi(\sigma_{\alpha-1}, v_\alpha)v.$$

Let

$$M_\alpha = \varphi(\sigma_\alpha, v_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, u_\alpha),$$

for all $\alpha \in \mathbb{N}$. Then

$$M_\alpha = \varphi(\sigma_\alpha, v_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, u_\alpha)$$

$$\leq v^* [\varphi(\sigma_{\alpha-1}, v_{\alpha-1}) + \varphi(\zeta_{\alpha-1}, u_{\alpha-1})]v$$

$$+ v^* [\varphi(\zeta_{\alpha-1}, u_{\alpha-1}) + \varphi(\sigma_{\alpha-1}, v_\alpha)]v$$

$$\leq (\sqrt{2}v)^* M_{\alpha-1}(\sqrt{2}v).$$

By Lemma 2.2 (A5), we have

$$0_\lambda \leq M_\alpha \leq (\sqrt{2}v)^* M_{\alpha-1}(\sqrt{2}v) \leq \cdots \leq ((\sqrt{2}v)^*)^\alpha M_0(\sqrt{2}v)^\alpha.$$

Let

$$S_\alpha = \varphi(\sigma_{\alpha+1}, v_\alpha) + \varphi(\zeta_{\alpha}, u_{\alpha+1})$$

for all $\alpha \in \mathbb{N}$. Then

$$S_\alpha = \varphi(\sigma_{\alpha+1}, v_{\alpha-1}) + \varphi(\zeta_{\alpha}, u_{\alpha+1})$$

$$\leq v^* [\varphi(\sigma_{\alpha-1}, v_{\alpha-1}) + \varphi(\zeta_{\alpha-1}, u_{\alpha-1})]v$$

$$+ v^* [\varphi(\zeta_{\alpha-1}, u_{\alpha-1}) + \varphi(\sigma_{\alpha-1}, v_\alpha)]v$$

$$\leq (\sqrt{2}v)^* S_{\alpha-1}(\sqrt{2}v).$$
By Lemma 2.2 (A5), we have
\[ 0_A \leq S_\alpha \leq (\sqrt{2}v)^* S_{\alpha-1} (\sqrt{2}v) \leq \cdots \leq ((\sqrt{2}v)^*)^n S_0 (\sqrt{2}v)^n. \]

On the other hand,
\[
\varphi(\sigma_\alpha, v_{\alpha+1}) = \varphi(\Phi(\sigma_{\alpha-1}, u_{\alpha-1}), \Phi(v_{\alpha}, \zeta_{\alpha})) \\
\leq v^* \varphi(\sigma_{\alpha-1}, v_{\alpha}) + v^* \varphi(\zeta_{\alpha}, u_{\alpha-1})u.
\]
\[
\varphi(\sigma_\alpha, v_\alpha) = \varphi(\Phi(\sigma_{\alpha-1}, u_{\alpha-1}), \Phi(v_{\alpha-1}, \zeta_{\alpha-1})) \\
\leq v^* \varphi(\sigma_{\alpha-1}, v_{\alpha-1}) + v^* \varphi(\zeta_{\alpha-1}, u_{\alpha-1})u
\]
and
\[
\varphi(\zeta_\alpha, u_\alpha) = \varphi(\Phi(\zeta_{\alpha-1}, u_{\alpha-1}), \Phi(u_{\alpha-1}, \sigma_{\alpha-1})) \\
\leq v^* \varphi(\zeta_{\alpha-1}, u_{\alpha-1}) + v^* \varphi(\sigma_{\alpha-1}, v_{\alpha-1})u
\]
for all \( \alpha \in \mathbb{N} \). Let
\[
R_\alpha = \varphi(\sigma_\alpha, v_\alpha) + \varphi(\zeta_\alpha, u_\alpha),
\]
for all \( \alpha \in \mathbb{N} \). Then
\[
R_\alpha = \varphi(\sigma_\alpha, v_\alpha) + \varphi(\zeta_\alpha, u_\alpha) \\
\leq v^* [\varphi(\sigma_{\alpha-1}, v_{\alpha-1}) + \varphi(\zeta_{\alpha-1}, u_{\alpha-1})]u \\
+ v^* [\varphi(\zeta_{\alpha-1}, u_{\alpha-1}) + \varphi(\sigma_{\alpha-1}, v_{\alpha-1})]u \\
\leq (\sqrt{2}v)^* R_{\alpha-1} (\sqrt{2}v).
\]
By Lemma 2.2 (A5), we have
\[ 0_A \leq R_\alpha \leq (\sqrt{2}v)^* R_{\alpha-1} (\sqrt{2}v) \leq \cdots \leq ((\sqrt{2}v)^*)^n R_0 (\sqrt{2}v)^n. \]
Now,
\[
\varphi(\sigma_\alpha, v_\beta) \leq \varphi(\sigma_\alpha, v_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, v_{\alpha+1}) + \cdots + \varphi(\sigma_{\beta-1}, v_\beta), \\
\varphi(\zeta_\alpha, u_\beta) \leq \varphi(\zeta_\alpha, u_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, u_{\alpha+1}) + \cdots + \varphi(\zeta_{\beta-1}, u_\beta),
\]
and
\[
\varphi(\sigma_\beta, v_\alpha) \leq \varphi(\sigma_\beta, v_{\beta-1}) + \varphi(\sigma_{\beta-1}, v_{\beta-1}) + \cdots + \varphi(\sigma_{\alpha+1}, v_\alpha), \\
\varphi(\zeta_\beta, u_\alpha) \leq \varphi(\zeta_\beta, u_{\beta-1}) + \varphi(\zeta_{\beta-1}, u_{\beta-1}) + \cdots + \varphi(\zeta_{\alpha+1}, u_\alpha),
\]
for each \( \alpha, \beta \in \mathbb{N}, \alpha < \beta \). Then,
\[ \varphi(\sigma, v) + \varphi(\zeta, u) \leq (\varphi(\sigma, v, u) + \varphi(\zeta, u, v)) + (\varphi(\sigma, v, u) + \varphi(\zeta, u, v)) + \cdots + (\varphi(\sigma, v, u) + \varphi(\zeta, u, v)) = M_{\alpha} + R_{\alpha+1} + M_{\alpha+1} + \cdots + R_{\beta-1} + M_{\beta-1} \]
\[ \leq ((\sqrt{2}v)^*)^\beta M_{0}(\sqrt{2}v)^\beta + ((\sqrt{2}v)^*)^{\beta+1} R_{0}(\sqrt{2}v)^{\beta+1} + \cdots + ((\sqrt{2}v)^*)^{\beta-1} M_{0}(\sqrt{2}v)^{\beta-1} + ((\sqrt{2}v)^*)^{\beta-1} M_{0}(\sqrt{2}v)^{\beta-1} \]
\[ = \sum_{i=\alpha}^{\beta-1} ((\sqrt{2}v)^*)^i M_{0}(\sqrt{2}v)^i + \sum_{i=\alpha+1}^{\beta-1} ((\sqrt{2}v)^*)^i R_{0}(\sqrt{2}v)^i \]
\[ = \sum_{i=\alpha}^{\beta-1} ((\sqrt{2}v)^*)^i M_{0}^{-1} M_{0}^{i}(\sqrt{2}v)^i + \sum_{i=\alpha+1}^{\beta-1} ((\sqrt{2}v)^*)^i R_{0}^{-1} R_{0}^{i}(\sqrt{2}v)^i \]
\[ = \sum_{i=\alpha}^{\beta-1} |M_{0}^{-1} M_{0}^{i}(\sqrt{2}v)^i|^2 + \sum_{i=\alpha+1}^{\beta-1} |R_{0}^{-1} R_{0}^{i}(\sqrt{2}v)^i|^2 \]
\[ \leq \| \sum_{i=\alpha}^{\beta-1} |M_{0}^{-1} M_{0}^{i}(\sqrt{2}v)^i||^2 1_A + \| \sum_{i=\alpha+1}^{\beta-1} |R_{0}^{-1} R_{0}^{i}(\sqrt{2}v)^i||^2 1_A \]
\[ \leq \sum_{i=\alpha}^{\beta-1} \| M_{0}^{-1} M_{0}^{i}(\sqrt{2}v)^i ||^2 1_A + \sum_{i=\alpha+1}^{\beta-1} \| R_{0}^{-1} R_{0}^{i}(\sqrt{2}v)^i ||^2 1_A \]
\[ \leq \| M_{0}^{-1} M_{0}^{i} \|^2 \sum_{i=\alpha}^{\beta-1} \| (\sqrt{2}v)^i \|^2 1_A + \| R_{0}^{-1} R_{0}^{i} \|^2 \sum_{i=\alpha+1}^{\beta-1} \| (\sqrt{2}v)^i \|^2 1_A \]
\[ = \| M_{0}^{-1} \|^2 \sum_{i=\alpha}^{\beta-1} (2\| v \|^2) 1_A + \| R_{0}^{-1} \|^2 \sum_{i=\alpha+1}^{\beta-1} (2\| v \|^2) 1_A \]
\[ \to 0_A \quad (\text{as } \beta, \alpha \to \infty) \]

and

\[ \varphi(\sigma, v) + \varphi(\zeta, u) \leq (\varphi(\sigma, v, u) + \varphi(\zeta, u, v)) + (\varphi(\sigma, v, u) + \varphi(\zeta, u, v)) + \cdots + (\varphi(\sigma, v, u) + \varphi(\zeta, u, v)) = S_{\beta-1} + R_{\beta-1} + S_{\beta-1} + \cdots + R_{\alpha+1} + S_{\alpha} \]
\[ \leq ((\sqrt{2}v)^*)^\beta S_{0}(\sqrt{2}v)^\beta + ((\sqrt{2}v)^*)^{\beta-1} R_{0}(\sqrt{2}v)^{\beta-1} + \cdots + ((\sqrt{2}v)^*)^{\alpha+1} R_{0}(\sqrt{2}v)^{\alpha+1} + ((\sqrt{2}v)^*)^{\alpha} S_{0}(\sqrt{2}v)^\alpha \]
\[ = \sum_{i=\alpha}^{\beta-1} ((\sqrt{2}v)^*)^i S_{0}(\sqrt{2}v)^i + \sum_{i=\alpha+1}^{\beta-1} ((\sqrt{2}v)^*)^i R_{0}(\sqrt{2}v)^i \]
\[ = \sum_{i=\alpha}^{\beta-1} ((\sqrt{2}v)^*)^i S_{0}^{-1} S_{0}^{i}(\sqrt{2}v)^i + \sum_{i=\alpha+1}^{\beta-1} ((\sqrt{2}v)^*)^i R_{0}^{-1} R_{0}^{i}(\sqrt{2}v)^i \]
\[
\begin{align*}
&= \sum_{i=\alpha}^{\beta-1} (S_0^i(\sqrt{2}u)) + \sum_{i=\alpha+1}^{\beta-1} (R_0^i(\sqrt{2}v)) \\
&= \sum_{i=\alpha}^{\beta-1} |S_0^i(\sqrt{2}u)|^2 + \sum_{i=\alpha+1}^{\beta-1} |R_0^i(\sqrt{2}v)|^2 \\
&\leq \| S_0^i(\sqrt{2}u) \|^2_1 + \| R_0^i(\sqrt{2}v) \|^2_1 \\
&\leq \| S_0^i \|^2_{\beta-1} \sum_{i=\alpha}^{\beta-1} (2|v|)^2 + \| R_0^i \|^2 \sum_{i=\alpha+1}^{\beta-1} (2|v|)^2 \\
&\rightarrow 0_\mathbb{A} \quad \text{(as } \beta, \alpha \rightarrow \infty). 
\end{align*}
\]

Therefore, \((\sigma_\alpha, \{v_\alpha\})\) and \((\zeta_\alpha, \{u_\alpha\})\) are Cauchy bisequences in \(\Gamma \times \Psi\) with respect to \(\mathbb{A}\). By completeness of \((\Gamma, \Psi, \mathbb{A}, \phi)\), there exist \(\sigma, \zeta \in \Gamma\) and \(u, v \in \Psi\) with

\[
\lim_{\alpha \to \infty} \sigma_\alpha = v, \quad \lim_{\alpha \to \infty} \zeta_\alpha = u, \quad \lim_{\alpha \to \infty} u_\alpha = \zeta \quad \text{and} \quad \lim_{\alpha \to \infty} v_\alpha = \sigma. 
\]

Then for given \(\epsilon > 0\), there exists \(\alpha_1 \in \mathbb{N}\) with \(\phi(\sigma_\alpha, v) < \frac{\epsilon}{2}\) for all \(\alpha \geq \alpha_1\). Since \((\sigma_\alpha, \{v_\alpha\})\) and \((\zeta_\alpha, \{u_\alpha\})\) are Cauchy bisequences, we derive that

\[
\phi(\sigma_\alpha, v_\alpha) < \frac{\epsilon}{2}. 
\]

Then,

\[
\phi(\Phi(\sigma, u), v) \leq \phi(\Phi(\sigma, u), v_{\alpha+1}) + \phi(\sigma_{\alpha+1}, v_{\alpha+1}) + \phi(\sigma_{\alpha+1}, v) \\
= \phi(\Phi(\sigma, u), \Phi(u_{\alpha}, \zeta_\alpha)) + \phi(\sigma_{\alpha+1}, v_{\alpha+1}) + \phi(\sigma_{\alpha+1}, v) \\
\leq u^* \phi(\sigma, u_\alpha)u + u^* \phi(\zeta_\alpha, u)u + \phi(\sigma_{\alpha+1}, v_{\alpha+1}) + \phi(\sigma_{\alpha+1}, v). 
\]

As \(\alpha \to \infty\), we have

\[
\phi(\Phi(\sigma, u), v) < \epsilon. 
\]

Then,

\[
\phi(\Phi(\sigma, u), v) = 0. 
\]

Hence, \(\Phi(\sigma, u) = v\). Similarly, we can derive \(\Phi(u, \sigma) = \zeta\, \Phi(\zeta, u) = u\) and \(\Phi(v, \zeta) = \sigma\). On the other hand, we derive that

\[
\phi(\sigma, v) = \phi(\lim_{\alpha \to \infty} v_\alpha, \lim_{\alpha \to \infty} \sigma_\alpha) = \lim_{\alpha \to \infty} \phi(\sigma_\alpha, v_\alpha) = 0
\]
and
\[ \varphi(\zeta, u) = \varphi(\lim_{a \to \infty} u_a, \lim_{a \to \infty} \zeta_a) = \lim_{a \to \infty} \varphi(\zeta_a, u_a) = 0. \]

So, \( \sigma = v \) and \( \zeta = u \). Therefore, \((\sigma, u) \in (\Gamma \times \Psi) \cap (\Psi \times \Gamma)\) is a coupled fixed point of \( \Phi \). As in the proof of the Theorem 3.1, one can easily prove uniqueness part. \( \square \)

**Example 3.4.** Let \( \Gamma = \{0, 1, 2, 7\}, \Psi = \{0, \frac{1}{2}, \frac{1}{4}, 3\}, \mathbb{A}_+ = \mathcal{M}_2(\mathbb{C}) \) and the map \( \varphi : \Gamma \times \Psi \rightarrow \mathbb{A}_+ \) is defined by
\[ \varphi(\sigma, u) = \begin{bmatrix} |\sigma - u| & 0 \\ 0 & k|\sigma - u| \end{bmatrix}, \]
for all \( \sigma \in \Gamma \) and \( u \in \Psi \), where \( k \geq 0 \) is a constant. Let \( \leq \) be the partial order on \( \mathbb{A} \) given by
\[ (\vartheta_1, \varpi_1) \leq (\vartheta_2, \varpi_2) \iff \vartheta_1 \leq \vartheta_2 \text{ and } \varpi_1 \leq \varpi_2. \]

Then \((\Gamma, \Psi, \mathbb{A}, \varphi)\) is a complete \( C^* \)-algebra-valued bipolar metric space. Define \( \Phi : (\Gamma \times \Psi) \cup (\Psi \times \Gamma) \rightarrow \Gamma \cup \Psi \)
by
\[ \Phi(\sigma, \zeta) = \frac{\sigma + \zeta}{5}, \]
for all \( \sigma, \zeta \in (\Gamma \times \Psi) \cup (\Psi \times \Gamma) \). Then
\[ \varphi(\Phi(\sigma, u), \Phi(\zeta, v)) \leq \frac{1}{5} \begin{bmatrix} |\sigma - v| & 0 \\ 0 & k|\sigma - v| \end{bmatrix} + \begin{bmatrix} |\zeta - u| & 0 \\ 0 & k|\zeta - u| \end{bmatrix} = \nu^* \varphi(\sigma, v)\nu + \nu^* \varphi(\zeta, u)\nu, \]
for all \( \sigma, \zeta \in \Gamma \) and \( u, v, \varpi \in \Psi \), where
\[ \nu = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \]
and \( \|\nu\| = \frac{1}{5} < \frac{1}{\sqrt{2}} \). All the conditions of Theorem 3.3 are fulfilled and \( \Phi \) has a unique fixed point \((0, 0)\).

### 4. Applications

As an application of Theorem 3.1, we find an existence and uniqueness result for a type of following system of Fredholm integral equations.

**Theorem 4.1.** Let us consider the system of Fredholm integral equations
\[ \sigma(\mu) = \int_{E_1 \cup E_2} G(\mu, p, \sigma(p), \zeta(p))dp + \delta(\mu), \mu, p \in E_1 \cup E_2, \]
\[ \zeta(\mu) = \int_{E_1 \cup E_2} G(\mu, p, \sigma(p), \zeta(p))dp + \delta(\mu), \mu, p \in E_1 \cup E_2, \]
where \( E_1 \cup E_2 \) is a Lebesgue measurable set. Suppose
(T1) $G : (E_1^2 \cup E_2^2) \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ and $\delta \in L^\infty(E_1) \cup L^\infty(E_2)$.

(T2) There exists a continuous function $\kappa : E_1^2 \times E_2^2 \to \mathbb{R}$ and $\theta \in (0, 1)$, such that

$$|G(\mu, p, \sigma(p), \zeta(p)) - G(\mu, p, \upsilon(p), \upsilon(p))|$$

$$\leq \theta |\kappa(\mu, p)|(\sigma(p) - u(p)) + |\zeta(p) - \upsilon(p)| + 1 - \theta^{-1}I,$$

for all $\mu, p \in E_1 \cup E_2$.

(T3) $\sup_{p \in E_1 \cup E_2} \int_{E_1 \cup E_2} \kappa(\mu, p)dp \leq 1$.

Then the integral equation has a unique solution in $L^\infty(E_1) \cup L^\infty(E_2)$.

Proof. Let $\Gamma = L^\infty(E_1)$ and $\Psi = L^\infty(E_2)$ be two normed linear spaces, where $E_1, E_2$ are Lebesgue measurable sets and $m(E_1 \cup E_2) < \infty$. Let $H = L^2(E_1) \cup L^2(E_2)$. Consider $\varphi : \Gamma \times \Psi \to L(H)$ defined by $\varphi(\sigma, \zeta) = \pi(\sigma - \zeta)$, where $\pi : H \to H$ is the multiplication operator defined by $\pi(\omega) = h \omega$ for $\omega \in H$. Then $(\Gamma, \Psi, A, \varphi)$ is a complete C*-algebra valued bipolar metric space.

Define the covariant mapping $\Phi : \Gamma^2 \cup \Psi^2 \to \Gamma \cup \Psi$ by

$$\Phi(\sigma, \zeta)(\mu) = \int_{E_1 \cup E_2} G(\mu, p, \sigma(p), \zeta(p))dp + \delta(\mu), \forall \mu, p \in E_1 \cup E_2.$$

Set $\tau = \theta I$, then $\tau \in L(H)$, and $||\tau|| = \theta < 1$. For any $\omega \in H$, we have

$$||\varphi(\Phi(\sigma, \zeta), \Phi(u, v))|| = \sup_{|\omega|=1} \int_{E_1 \cup E_2} (||\Phi(\sigma, \zeta) - \Phi(u, v)|| + I)\omega(\mu)\bar{\omega}(\mu)d\mu$$

$$\leq \sup_{|\omega|=1} \int_{E_1 \cup E_2} \int_{E_1 \cup E_2} G(\mu, p, \sigma(p), \zeta(p))$$

$$- G(\mu, p, u(p), \upsilon(p))dp||\omega(\mu)||^2d\mu$$

$$+ \sup_{|\omega|=1} \int_{E_1 \cup E_2} \int_{E_1 \cup E_2} dp||\omega(\mu)||^2d\mu I$$

$$\leq \sup_{|\omega|=1} \int_{E_1 \cup E_2} \int_{E_1 \cup E_2} \theta |\kappa(\mu, p)|(\sigma(p) - u(p))$$

$$+ |\zeta(p) - \upsilon(p)| + I - \theta^{-1}I dp||\omega(\mu)||^2d\mu + I$$

$$\leq \theta \sup_{\mu \in E_1 \cup E_2} \int_{E_1 \cup E_2} |\kappa(\mu, p)|dp \sup_{|\omega|=1} \int_{E_1 \cup E_2} ||\omega(\mu)||^2d\mu(||\sigma - u||_\infty$$

$$+ ||\zeta - \upsilon||_\infty)$$

$$\leq \theta \sup_{\mu \in E_1 \cup E_2} \int_{E_1 \cup E_2} |\kappa(\mu, p)|dp \sup_{|\omega|=1} \int_{E_1 \cup E_2} ||\omega(\mu)||^2d\mu(||\sigma - u||_\infty$$

$$+ ||\zeta - \upsilon||_\infty)$$

$$\leq \theta ||\sigma - u||_\infty + ||\zeta - \upsilon||_\infty$$

$$= ||\tau||||\Phi(\sigma, u)|| + ||\Phi(\zeta, v)||.$$

Therefore, all the conditions of Theorem 3.1 are fulfilled. Hence, the integral equation (4.1) has a unique solution. □
5. Conclusions

In this paper, we introduced the notion of a $C^*$-algebra valued bipolar metric space and proved coupled fixed point theorems. An illustrative example is provided that show the validity of the hypothesis and the degree of usefulness of our findings.

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Conflict of interest

The authors declare no conflicts of interest.

References


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